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Converse mean value theorems on trees and symmetric spaces

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Abstract. Harmonic functions satisfy the mean value property with respect to all integrable radial weights: if f is harmonic then $h{*}f{=}f\int h$ for any such weight h. But need a function f that satisfies this relation with a given (non-negative) h be harmonic? By a classical result of Furstenberg the answer is positive for every bounded f on a Riemannian symmetric space, but if the boundedness condition is relaxed then the answer turns out to depend on the weight h.

In this paper various types of weights are investigated on Euclidean and hyperbolic spaces as well as on homogeneous and semi-homogeneous trees. If h decays faster than exponentially then the mean value property $h{*}f{=}f\int h$ does not imply harmonicity of f. For weights decaying slower than exponentially, at least a weak converse mean value property holds: the eigenfunctions of the Laplace operator which satisfy $h{*}f{=}f\int h$ are harmonic. The critical case is that of exponential decay. In this class we exhibit weights that characterize harmonicity and others that do not.
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1. Introduction

A harmonic function f on (a domain of) $\mathbb{R}^n$ satisfies the *mean value property (MVP)*: the average of f on every ball or sphere equals the value at the center. Conversely, if the MVP holds for every ball or sphere then the function is harmonic. If f satisfies the MVP for only one sphere centered at each point then f need not be harmonic unless some kind of boundedness of f (e.g., positivity) and some regularity of the radius function are also assumed [2], [4], [5], [16], [25],

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[24], [35], [36]. The situation is similar on hyperbolic spaces (which are, together with $\mathbb{R}^n$, the main examples of non-compact harmonic spaces) as well as on homogeneous and semi-homogeneous trees (which can be regarded as discrete analogues of hyperbolic spaces); there are also similarities with non-homogeneous trees [33]. Harmonic functions on trees have been studied intensely in the recent literature, both for their intrinsic interest (e.g., Poisson and Martin boundaries, asymptotic behavior: see [23], [8], [33], and references therein) and for their applications to Hardy spaces [31] and to unitary representations of free groups [9], [20], although we will not pursue these applications in the present paper. In general, trees can also be a good testing ground for theorems that often extend, even with similar proofs, to hyperbolic spaces or manifolds with non-positive curvature. Besides the present paper, this principle has yielded several results in the past (cf., e.g., [11] or [7]).

(The reader is referred to §2 below, especially Remark 2.1 and Definition 2.2, for the notation and terminology used in the remainder of this introduction.)

On $\mathbb{R}^n$, the characteristic function h of a ball does not characterize harmonicity, by a well-known result of Delsarte [18]: at least two different balls are necessary (for a related question on homogeneous trees see [15]). But if f is assumed positive or bounded there are positive results in a number of cases [13], [34]: condition $(*)$ implies harmonicity for bounded f in $\mathbb{R}^n$ [12] and all non-compact Riemannian symmetric spaces [22], [6]. Thus $(*)$ does imply harmonicity if suitable boundedness assumptions are made on f, but does not in the general case. It is proved in [30], inspired by [1], that a specific weight h of exponential decay on $\mathbb{R}^n$ and hyperbolic spaces does or does not characterize harmonicity depending on whether the dimension of the space is small or large. This result does not contradict the one recalled above, because $(*)$ makes sense only if $f \in L^1_h$, a restriction on the growth of f. The effect of such restriction depends on the dimension, whence the converse to the MVP also does.

The above seems to suggest the following general principle: a non-negative radial weight h which decreases at infinity fast enough (faster than any exponential) does not characterize harmonicity; one, however, which decreases slowly enough (slower than any exponential) does. For intermediate rates of decay (exponential decay), the behavior depends on the exact nature of h. (Exponential decay is understood with respect to the distance from o in the ambient space; note that on hyperbolic spaces and trees the volume of spheres grows exponentially.) The purpose of this paper is to prove some instances of this principle for $\mathbb{R}^n$, hyperbolic spaces and (homogeneous and semi-homogeneous) trees. The analysis turns out to be not only easier but also more effective in this last environment, for which we shall prove suitable spectral synthesis results.

We show in Theorem 4.5 for trees and in Theorem 7.5 for hyperbolic spaces and $\mathbb{R}^n$ that any weight h which satisfies $h(r) \leq Ce^{-A|r|^{1+\alpha}}$ for some positive constants C, A, α does not (with a trivial exception) characterize harmonicity. On the other hand, if h is non-negative and decays slower than any exponential then it characterizes harmonicity in the weak sense (Theorem 4.4 and Theorem 7.4).

In the remainder of the paper exponentially decaying weights are considered. We construct examples with arbitrarily fast (exponential) decay which nevertheless characterize harmonicity, and others with arbitrarily slow (exponential) decay which do not (some of these were already known for $\mathbb{R}^n$ and hyperbolic spaces). Our results are more complete for trees—all results here were in fact obtained on trees first, thereby providing inspiration for the continuous setup. This is reflected in the structure of the paper: throughout §6 we present the results for trees, and in the remainder we prove some of their counterparts on hyperbolic spaces and $\mathbb{R}^n$. Notation, e.g., as concerns the parameters for spherical transforms (cf. Remark 7.3), is so chosen as to stress the similarities among the various settings.

In view of [30] one might expect that pure exponential weights on trees would characterize harmonicity only if the homogeneity degree of the tree is sufficiently low, but instead it turns out that they always do (Theorem 3.2). The idea of the proof is quite general, as it only uses the fact that the exponential function is the resolvent of the discrete Laplacian Δ. The same approach applies (Theorem 3.3) in a variety of discrete homogeneous spaces (whose groups of isometries are free groups and free products), and to Laplacians which are not necessarily isotropic. Of particular interest are semi-homogeneous trees, studied in detail in §6. Their interest is due to the fact that, together with homogeneous trees, they arise (for suitable degrees) as the Bruhat-Tits buildings associated to rank-1 linear algebraic reductive groups on non-Archimedean local fields. The group of automorphisms of a semi-homogeneous tree is not transitive, therefore convolution is somewhat clumsy, and we replace it by *semi-convolution*, the summation against an automorphism-invariant kernel. In extending Theorem 3.2 we prove that a slightly more general (even in the homogeneous case) exponential-type invariant kernel characterizes harmonicity. It appears worth remarking (cf. Remark 6.6) that the expression of this kernel resembles closely that of Theorem 5.5 (described below), both having an oscillatory behavior, but the latter does not characterize harmonicity.

In §5 we consider the closed convex cone generated by positive exponentials on homogeneous trees (that is, the set of their integral averages with respect to a positive measure). Every h in this cone characterizes harmonicity in the weak sense; moreover, if h is actually a finite linear combination of positive exponentials with positive coefficients then it characterizes harmonicity in the strong sense (Theorem 5.4). Then we provide examples of linear combinations of exponentials which do not characterize harmonicity (Theorem 5.5); unfortunately this does not seem to provide inspiration for a continuous analogue.

In §§7,8,9 attention is focused on the continuous setting. Here it is natural to replace exponential weights with the resolvent of Δ at an eigenvalue for which it is positive and summable. On $\mathbb{R}^n$ this leads to considering the MVP with respect to a class of exponentially decaying functions related to Bessel functions of the second kind, while on hyperbolic spaces we have MVPs related to spherical functions. The positive result of §3 again holds here, with essentially the same proof (except for minor technical details): this MVP characterizes harmonic functions also in the continuous setup. The weight h can be chosen here with an arbitrarily fast exponential rate of decay at infinity. On the other hand, on $\mathbb{R}^n$, if $n \geq 9$, the radial exponential $a^{|x|}$ does not characterize harmonicity for any

$0 < a < 1$, by a straightforward adaptation of the proof of [30, Proposition 2.1]. Consequently the feature of an exponentially decaying weight which determines whether or not it characterizes harmonicity is not its rate of decrease.

The continuous picture is not so complete as the discrete one when we consider MVPs with respect to convex combinations of resolvents. Following the approach for trees, we prove that such weights characterize harmonicity in the weak sense, but it is not clear whether they do in the strong sense. This question occurs often in this paper, because, in order to pass from weak to strong, some spectral synthesis result is needed. So far, in the continuous case such a result has only been proved on hyperbolic spaces for one specific weight [30]. Some considerations suggesting that spectral synthesis should hold more generally are presented in §10.

Remark 1.1. Harmonicity on higher-rank symmetric spaces or buildings cannot be expected to be characterized by any single weight h, unless f is assumed to be bounded. Indeed in those cases the spherical Fourier transform $\hat{h}$ of h is an analytic function of several variables. The origin is among the points at which it attains the value $\int h$ (the corresponding eigenfunction of Δ is a constant, therefore in L^1_h), and there are others arbitrarily close. The eigenfunctions corresponding to these are counterexamples in L^1_h (cf. [3]). ■

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2. Basic notation and terminology

A *tree* is a connected graph without loops. A tree is *homogeneous* if each vertex has the same *degree* (number of neighbors) $q+1$. Each such tree is a homogeneous space of a free group or a certain free product, in either case contained in the full automorphism group of the tree itself. In the rest of the paper we shall not make explicit reference to the group structure.

On a tree assume fixed a reference vertex o. Let d be the natural integer-valued distance, and set $|x| = d(o,x)$ for every vertex x. By a function on a tree we mean a function on the set of its vertices. A function h is *radial* if $h(x)$ only depends on $|x|$; if so, by abuse of notation we write $h(|x|) = h(x)$. The basic instance is the normalized equidistributed measure μ_n on the sphere of radius $n \geq 0$ around o. In particular $\mu_0 = \delta_o$, where δ_x denotes the Dirac delta at x, i.e., the characteristic function of $\{x\}$.

If f, g are functions, set $\langle f, g\rangle = \sum f\bar{g}$, summation being performed over the whole tree. With regard to asymptotics for $|x| \to \infty$, write $f \preceq g$ or $f \prec g$ if $f = O(|g|)$ or $f = o(|g|)$, respectively, and $f \sim g$ if $f \preceq g \preceq f$.

Let Δ denote the isotropic Laplacian $M_1 - \mathrm{id}$ on a tree, where id is the identity and M_1 is the equidistributed nearest-neighbor transition operator. A function is *harmonic* if it is annihilated by Δ—equivalently, the value at each point is the average of values at its neighbors. Harmonic functions enjoy the MVP for balls or spheres, as in $\mathbb{R}^n$.

Let w_n be the number of vertices of the sphere $\{|x| = n\}$. If the tree is homogeneous then

$$w_n = \begin{cases} 1 & \text{if } n = 0, \\ (q+1)q^{n-1} & \text{if } n > 0, \end{cases}$$

and, since the operator M_1 is the convolution with μ_1, then Δ is the convolution with $\mu_1 - \delta_o$, so a function f is harmonic if and only if $\mu_1 * f = f$.

Several of the above conventions and considerations extend to hyperbolic spaces and $\mathbb{R}^n$. On each of these spaces fix a reference point o (the origin in $\mathbb{R}^n$). If d is the distance in the space, set $|x| = d(o,x)$ for every point x. A function (or measure) h is *radial* if $h(x)$ only depends on $|x|$, and write $h(|x|) = h(x)$ in this case. One example is the normalized equidistributed measure μ_r on the sphere of radius $r \geq 0$ around o. Note that $\mu_0 = \delta_o$, where δ_x denotes the Dirac delta at x (not a function, in these settings). For f, g functions, set $\langle f, g\rangle = \int f\bar{g}$, integration being performed over the whole space with respect to the standard isometry-invariant measure. For $|x| \to \infty$, write $f \preceq g$ or $f \prec g$ if $f = O(|g|)$ or $f = o(|g|)$, respectively, and $f \sim g$ if $f \preceq g \preceq f$. With Δ denote the Laplace operator on $\mathbb{R}^n$, and the Laplace-Beltrami operator on a hyperbolic space. Also in the latter setting harmonic functions enjoy the MVP for balls or spheres.

Remark 2.1. Assume the ambient space is homogeneous (semi-homogeneous trees will be dealt with separately in §6), so that its automorphism group induces a convolution product. If f is harmonic and h a summable radial weight (function or measure) then the MVP on spheres and a straightforward integration in polar coordinates yield the *MVP with respect to* h, namely

$$(*) \qquad \begin{aligned} h * f &= f \sum h && \text{on homogeneous trees,} \\ h * f &= f \int h && \text{on } \mathbb{R}^n \text{ or hyperbolic spaces,} \end{aligned}$$

provided that, for each x, the left-hand side of the equality exists, i.e., the function $y \mapsto h(d(x,y))f(y)$ is summable; with a slight abuse of notation we write $f \in L^1_h$ in this case. The MVP on spheres thus reads $\mu_r * f = f$ for all $r \geq 0$. ∎

Conversely, fixing an h as above, we may ask whether $(*)$ implies that f is harmonic. We shall frequently make the further assumption that $h \geq 0$, although several of the results also hold without it.

Definition 2.2. We shall say that the summable weight h *characterizes harmonicity (in the strong sense)* if every $f \in L^1_h$ that satisfies $(*)$ is harmonic, and that it *characterizes harmonicity in the weak sense* if every $f \in L^1_h$ that satisfies $(*)$ and is an eigenfunction of Δ is harmonic. ∎

These two notions clearly coincide whenever L^1_h has spectral synthesis, in the sense that every closed translation-invariant subspace is generated by the eigenfunctions of Δ contained in it.

Remark 2.3. The weight δ_o does not characterize harmonicity, since $(*)$ holds for every f. But if h characterizes harmonicity (in the strong or weak sense), then so does $\tilde{h} = c_0\delta_o + c_1 h$ for all constants c_0, c_1, if $c_1 \neq 0$, because condition $(*)$ for h immediately follows from condition $(*)$ for $\tilde{h}$. Thus the set of weights that characterize harmonicity is not a linear space. ∎

3. Exponential weights on homogeneous trees

The natural definition of a radial exponential weight on a homogeneous tree is $h_a(x) = a^{|x|}$ (i.e., $h_a = \sum_{n=0}^{\infty} w_n a^n \mu_n$) for a real, where we stipulate that $t^0 = 1$ for any real t. Observe that $h_0 = \delta_o$, and that h_a is summable for $|a| < 1/q$.

Lemma 3.1. *For $0 \neq |a| < 1/q$ we have*

$$\mu_1 * h_a = (\lambda_a + 1)h_a + c\delta_o, \tag{3.1}$$

where

$$\lambda_a = \frac{qa + 1/a}{q+1} - 1 \tag{3.2}$$

and $c = -\lambda_a \sum h_a$. Up to a constant factor, h_a is thus the resolvent of μ_1 at the eigenvalue $\lambda_a + 1$ (or, equivalently, of Δ at λ_a) for the convolution product. Furthermore

$$\sum h_a = \frac{a+1}{1-qa}.$$

Proof. We have

$$\mu_1 * h_a(x) = \frac{1}{q+1} \sum_{d(y,x)=1} a^{|y|} = \begin{cases} a & \text{if } x = o, \\ (\lambda_a + 1)a^{|x|} & \text{if } x \neq o, \end{cases}$$

whence (3.1) holds for some constant c. Applying to both sides of (3.1) the convolution homomorphism $L^1 \to \mathbb{C}$ given by $g \mapsto \sum g$ we obtain $\sum h_a = (\lambda_a + 1)\sum h_a + c$. Finally

$$\sum h_a = \sum_{n=0}^{\infty} w_n a^n = \frac{q+1}{q} \sum_{n=0}^{\infty} (qa)^n - \frac{1}{q} = \frac{a+1}{1-qa}.$$ ∎

For our first result we do not need any spherical transform calculus.

Theorem 3.2. *On a homogeneous tree of degree $q+1$, for $0 \neq |a| < 1/q$ the weight $h = h_a$ characterizes harmonicity.*

Proof. If $f \in L^1_{h_a}$ satisfies $(*)$ with $h = h_a$, then

$$\begin{aligned} \mu_1 * f \sum h_a &= \mu_1 * (h_a * f) = (\mu_1 * h_a) * f = ((\lambda_a + 1)h_a + c\delta_o) * f \\ &= \Big((\lambda_a + 1)\sum h_a + c\Big) f = f \sum h_a \end{aligned}$$

(associativity holds because μ_1 is finitely supported and $f \in L^1_{h_a}$). ∎

The above argument has a wider range of application—e.g., to other discrete setups—in that it only requires general properties of convolution operators. The general statement is as follows. (For convenience, λ will denote the eigenvalue with respect to μ, instead of to the associated Laplacian $\mu - \delta_e \sum \mu$ as above.)

Theorem 3.3. *Let μ be a finitely supported function on a discrete group G, and assume that its resolvent (for the convolution product) $R_{\mu,\lambda}$ at some eigenvalue λ exists and is summable. Then $h = R_{\mu,\lambda}$ characterizes μ-harmonicity, that is, $h * f = f \sum h$ implies $\mu * f = f \sum \mu$.*

Proof. As before, we have $\sum \mu \cdot \sum R_{\mu,\lambda} = \lambda \sum R_{\mu,\lambda} + 1$, whence $\sum R_{\mu,\lambda} \neq 0$. The remainder of the argument is the same. ■

Note that Remark 2.3 also holds in this setting, where the Dirac delta is at the identity e of G. Assume μ is non-negative. The summability condition for the resolvent is fulfilled if the eigenvalue is greater than $\sum \mu$ (as is the case for $\lambda_a + 1$ in Theorem 3.2) and if G is a free group or a free product of two finite groups and μ is supported on words of distance 1 from e (cf. [20], [21], [10]), or if G is the free product of finitely many copies of the same finite group, and μ is equidistributed on the set of words of *block length* 1 (that is, on the union of the factor groups, see [28]). Therefore, the theorem applies to homogeneous or semi-homogeneous trees with a (not necessarily radial) "Laplace operator" μ supported on words of length 0 or 1, or to *symmetric graphs* (in the terminology of [28]) which arise as Cayley graphs of free products of two finite groups, or of several copies of the same finite group, with μ equidistributed on the factor groups; the latter case will be considered in detail in a forthcoming paper. In §§8,9 we shall show how the same argument can be adapted to Euclidean spaces and rank-1 symmetric spaces.

4. Decay of weights and their spherical transforms on homogeneous trees

A crucial role in the sequel will be played by spherical functions, so we briefly recall their asymptotics. Our main source for background is [20, Chapter 3], to which the reader is referred for further details. Consider a homogeneous tree of degree $q+1$. If μ is a finitely supported radial function then there exists a polynomial P such that $\mu = P(\mu_1 - \delta_o)$ (powers with respect to convolution are intended here, the unit constant being consequently δ_o). In fact for each n there exists a polynomial P_n of degree n such that $\mu_n = P_n(\mu_1 - \delta_o)$, recursively given by

$$P_n(t) = \begin{cases} 1 & \text{if } n = 0, \\ t+1 & \text{if } n = 1, \\ \dfrac{q+1}{q}(t+1)P_{n-1}(t) - \dfrac{1}{q}P_{n-2}(t) & \text{if } n > 1 \end{cases}$$

(the recurrence relation also holds for $n = 1$ if we set $P_{-1} = P_1$). Replacing t

with $\mu_1 - \delta_o$ we obtain

$$\mu_n * \mu_1 = \begin{cases} \mu_1 & \text{for } n = 0, \\ \dfrac{q\mu_{n+1} + \mu_{n-1}}{q+1} & \text{for } n > 0. \end{cases} \tag{4.1}$$

For $\lambda \in \mathbb{C}$ denote by ϕ_λ (a *spherical function*) the unique radial eigenfunction of Δ of eigenvalue λ (i.e., $\mu_1 * \phi_\lambda = (\lambda+1)\phi_\lambda$) such that $\phi_\lambda(o) = 1$. Recalling §2, for each n one has

$$\begin{aligned} \phi_\lambda(n) &= \sum_{|y|=n} \phi_\lambda(y)/w_n = \sum_y \mu_n(y)\phi_\lambda(y) = \mu_n * \phi_\lambda(o) \\ &= P_n(\mu_1 - \delta_o) * \phi_\lambda(o) = P_n(\lambda)\phi_\lambda(o) = P_n(\lambda). \end{aligned}$$

Spherical functions are indexed in [20] by the parameter z that satisfies $\lambda = \lambda_{q^{-z}} = \gamma(z) - 1$, where

$$\gamma(z) = \frac{q^z + q^{1-z}}{q+1} = \frac{2\sqrt{q}}{q+1}\cosh((z-1/2)\log q). \tag{4.2}$$

(Each of λ, z will be used frequently in the sequel, but λ, the eigenvalue of Δ, appears to be a better choice since γ is not one-to-one.) By [20, Theorem 3.3.3] the spectral radius of μ_1 in L^2 is $\rho = 2\sqrt{q}/(q+1)$. In the parameter z, from [20, Theorem 3.2.2] we have a closed expression for $P_n(\lambda)$ which turns out to be a linear combination of the exponentials $h_{q^{-z}}, h_{q^{z-1}}$, namely

$$\phi_\lambda(n) = \begin{cases} c(z)q^{-nz} + c(1-z)q^{n(z-1)} & \text{if } \gamma(z) \neq \pm\rho, \\ (\pm 1)^n \left(1 + n\dfrac{q-1}{q+1}\right) q^{-n/2} & \text{if } \gamma(z) = \pm\rho, \end{cases} \tag{4.3}$$

where $c(z) = (q^{1-z} - q^{z-1})/(q+1)(q^{-z} - q^{z-1})$. The asymptotic behavior for fixed λ as $n \to \infty$ (see also [14, Theorem 3]) is

$$\phi_\lambda(n) \preceq \begin{cases} nq^{-n/2} & \text{if } \operatorname{Re} z = 1/2, \\ q^{-n\min\{\operatorname{Re} z, 1-\operatorname{Re} z\}} = q^{-n(1/2 - |\operatorname{Re} z - 1/2|)} & \text{if } \operatorname{Re} z \neq 1/2, \end{cases} \tag{4.4}$$

uniformly on $\{z \in \mathbb{C}\colon |\operatorname{Re} z - 1/2| \geq c\}$ for each $c > 0$. For real values of the parameter λ we obtain

$$|P_n(\lambda)| \preceq \begin{cases} nq^{-n/2} & \text{if } |\lambda+1| \leq \rho, \\ \left(\dfrac{(q+1)\,|\lambda+1| + \sqrt{(q+1)^2\,|\lambda+1|^2 - 4q}}{2q}\right)^n & \text{if } |\lambda+1| > \rho. \end{cases} \tag{4.5}$$

The above inequalities are the so-called *majorization principle* [20, §3.5]. In particular, the following asymptotic monotonicity properties hold for real $\lambda, \lambda', \lambda''$:

$$\begin{aligned} &|\lambda+1| < \rho < |\lambda'+1| < |\lambda''+1| \\ &\qquad \text{implies} \quad P_n(\lambda) \prec P_n(\rho-1) \prec P_n(\lambda') \prec P_n(\lambda''). \end{aligned} \tag{4.6}$$

The majorization principle generalizes to non-real eigenvalues. The value of λ is not real unless either $\operatorname{Im} z$ is an integer multiple of $\pi/\log q$ or $\operatorname{Re} z = 1/2$; in general

$$|P_n(\gamma(z)-1)| \leq |P_n(\gamma(\operatorname{Re} z)-1)|.$$

(This follows immediately [20, Remark 3.2.4] from the integral expression

$$\phi_\lambda(x) = \int_\Omega K(x,\omega)^z \, d\omega,$$

where $d\omega$ is the measure on the Poisson boundary Ω of the tree with respect to o, and K is the corresponding Poisson kernel.) Since the map γ is onto $\mathbb{C}$, this gives an asymptotic estimate of $P_n(\lambda)$ for every complex λ. The inequality means that for every $\zeta \in \mathbb{R}$ the values of λ on the ellipse ∂E_ζ, boundary of

$$E_\zeta = \{\lambda \in \mathbb{C} \colon (\operatorname{Re}\lambda + 1)^2 + (\operatorname{Im}\lambda)^2 \coth^2((\zeta - 1/2)\log q) \leq \gamma(\zeta)^2\} \tag{4.7}$$

(cf. [20, Proposition 3.3.1]) satisfy $|P_n(\lambda)| \leq |P_n(\gamma(\zeta)-1)|$. Observe that the greater of the two real points of ∂E_ζ is $\gamma(\zeta)-1$.

If h is a function on the tree write $\hat{h}(\lambda) = \langle h, \bar{\phi}_\lambda\rangle$ whenever the right-hand side is defined. For $h \in L^1$ the value $\hat{h}(\gamma(z)-1)$ coincides with the *spherical transform* of h at z in [20, Chapter 3]. The map $h \mapsto \hat{h}$ is an injective homomorphism of the commutative Banach convolution algebra of radial L^1 functions on the tree into the algebra of analytic functions, and takes $\mu_1 - \delta_o$ to the identity, because

$$\hat{\mu}_1(\lambda) = \langle \mu_1, \phi_\lambda\rangle = \mu_1 * \phi_\lambda(o) = (\lambda+1)\phi_\lambda(o) = \lambda + 1.$$

Note that if h is a summable radial weight then $\hat{h}(0)$ is the constant $\sum h$ of $(*)$.

Definition 4.1. Let h be a summable function on a homogeneous tree, and set $h(n) = \langle h, \mu_n\rangle$ for all n (this extends the common abuse of notation for radial functions). We say that, for $n \to \infty$, the function h decays:

(1) *faster than exponentially* if $|h(n)| \prec a^n$ for every $a > 0$;
(2) *exponentially* (*like* a^n) if $0 < a < 1/q$ is such that $b^n \prec |h(n)| \prec c^n$ whenever $0 < b < a < c$;
(3) *slower than exponentially* if $a^n \prec |h(n)|$ whenever $0 < a < 1/q$. ∎

Remark 4.2. The requirement that a summable function decay slower than exponentially is rather restrictive, since the measure of the spherical surface of radius n grows as the reciprocal q^n of the 'critical' decay. ∎

As customary for Fourier transforms, a function will be said to be *analytic* in a subset of $\mathbb{C}$ if it is analytic in its interior and continuous up to the boundary.

Proposition 4.3. *Let h be a summable function on a homogeneous tree. Then:*

(1) if h decays faster than exponentially then $\hat{h}$ is entire;
(2) if h decays exponentially like a^n then $\hat{h}$ is analytic on its domain $E_{1/qa}$ of (4.7)*;*
(3) if h decays slower than exponentially then $\hat{h}$ is analytic on E_1. ∎

Theorem 4.4. *A non-negative summable radial weight h on a homogeneous tree that decays slower than exponentially (in the sense of Definition 4.1) characterizes harmonicity in the weak sense.*

Proof. By Proposition 4.3(3), the spherical transform $\hat{h}$ is defined only in E_1, and $\hat{h}(\lambda) = \sum_{n=0}^{\infty} h(n) w_n P_n(\lambda)$. If $(*)$ holds for some non-zero $f \in L^1_h$ such that $\Delta f = \lambda f$ for some λ, then: first, translating if necessary, we can assume that $f(o) \neq 0$; then, radializing and normalizing at o, we may assume that $f = \phi_\lambda$ (because the radializing operator M given by $Mf(x) = \langle f, \mu_{|x|}\rangle$ satisfies $h * Mf = M(h*f)$). Then $h*\phi_\lambda = \hat{h}(\lambda)\phi_\lambda$ implies $\hat{h}(\lambda) = \sum h = \hat{h}(0)$. But every $\lambda \in \partial E_1$ can be written as $\gamma(it)-1$ (or, equivalently, $\gamma(it+1)-1$) for some real t, hence $|P_n(\lambda)| \leq P_n(0) = 1$ by the majorization principle. On the other hand, by (4.3), for each $\lambda \in \partial E_1$ the sequence $|P_n(\lambda)|$ is not constantly 1 on E_1 unless $\lambda = 0$. Observe that if $P_n(\lambda) = 1$ for every n then ϕ_λ, being constant, is harmonic, so $\lambda = 0$. By the maximum principle $|\hat{h}(\lambda)| < \hat{h}(0)$ in E_1 except at $\lambda = 0$. ■

For functions decaying sufficiently fast we have the following result.

Theorem 4.5. *Let h be a radial weight on a homogeneous tree such that for some positive constants A, α we have $|h(n)| \preceq e^{-An^{1+\alpha}}$ for $n \to \infty$. Then h does not characterize harmonicity, unless $h = c_0\delta_o + c_1\mu_1$ with constants c_0, c_1, and $c_1 \neq 0$.*

Proof. Since the change of parameter γ in (4.2) is periodic with imaginary period (namely $2\pi i/\log q$), then so is the spherical transform $\hat{h}(\gamma(z)-1)$, which is therefore bounded on $\{z \in \mathbb{C}\colon u_z \leq 1\}$, where $u_z = 1/2 + |\operatorname{Re} z - 1/2|$. On the other hand, on $\{u_z > 1\}$ we have (replacing the constant A with $A\log q$) that

$$|\hat{h}(\gamma(z)-1)| \leq C\sum_{n=0}^{\infty} w_n q^{-An^{1+\alpha}}\,|P_n(\gamma(z)-1)| \leq C'\sum_{n=0}^{\infty} q^{-n(An^\alpha - u_z)}$$

by estimates (4.4) and the fact that $w_n \sim q^n$. Split the last sum as $I_1 + I_2$, where I_1 is the sum over the index set $D_1 = \{n \in \mathbb{N}\colon An^\alpha > 2u_z\}$, and I_2 over its complement D_2. On D_1 one has $An^\alpha - u_z \geq u_z \geq 1$, thus

$$I_1 \leq C'\sum_{n\in D_1} q^{-n} \leq C'\sum_{n=0}^{\infty} q^{-n} = C_q.$$

On D_2 one has

$$I_2 \leq C'\sum_{n\in D_2} q^{nu_z} \leq C' q^{u_z \max D_2}\max D_2 \leq C'' u_z^{1/\alpha} Q^{u_z^{1+1/\alpha}},$$

where $C'' > 0$ and $Q > 1$.

Since $u_z \leq |z| + 1$, then $\hat{h}\circ(\gamma - 1)$ is of finite order (in the variable z), and this is immediately seen to imply that $\hat{h}$ itself is (in the variable λ). If $\hat{h}(\lambda) = \hat{h}(0)$ only at $\lambda = 0$, Hadamard's product theorem [32, Theorem 1.13] yields $\hat{h}(\lambda) - \hat{h}(0) = B\lambda^k e^{P(\lambda)}$ for all $\lambda \in \mathbb{C}$ with some constant $B \neq 0$, some $k \geq 1$, and some polynomial P of degree not larger than the order of $\hat{h}$. If P is not constant (i.e., if the support of h is infinite) then $e^{P\circ(\gamma-1)}$ does not have finite order, and neither does $\hat{h}\circ(\gamma-1)$. Therefore P is constant, that is, h has finite support. Hadamard's product theorem is not really needed for this case, since every finitely supported weight equals $Q(\mu_1)$ for some polynomial Q. Up to a constant factor and to addition of a multiple of δ_o (cf. Remark 2.3) we may assume that $\hat{h}(\lambda) = \lambda^k$, that is, $h = (\mu_1 - \delta_o)^k$. This weight characterizes harmonicity if and only if $k = 1$. Indeed f given by $f(x) = (|x|\,(q^2-1) + 2q^{1-|x|})/(q-1)^2$ satisfies $\Delta f \equiv 1$, whence it is not harmonic, but $\Delta^k f \equiv 0$ for $k \geq 2$. Note that, since the support of h is finite, every function belongs to L^1_h. ■

Remark 4.6. The above argument fails if we only require $h(n) \leq e^{-An\log n}$. In this case $\hat{h}(\lambda)$ is still of finite order, but $\hat{h}(\gamma(z)-1)$ in general is not, so one cannot conclude that the polynomial P is constant. ■

Remark 4.7. The simplest examples of non-constant weights that do not characterize harmonicity on a homogeneous tree are μ_1^2 and $\mu_1 * \mu_2 = (q\mu_3 + \mu_1)/(q+1)$. For instance in the former case the eigenfunctions of μ_1 with eigenvalue -1 (that is, of Δ with eigenvalue -2) are non-harmonic functions which nevertheless fulfill $(*)$. The spherical function ϕ_{-2} is the alternating function $(-1)^{|x|}$, which is bounded. This shows that Furstenberg's result quoted in the introduction (that every weight characterizes harmonicity for bounded functions) has no analogue on trees. ■

5. The convex cone generated by exponentials on homogeneous trees

In order to examine more general exponentially decaying radial weights than $h_a(x) = a^{|x|}$ let us first compute its spherical transform. We follow the notation of §3; in particular λ_a is given by (3.2).

Lemma 5.1. *For $0 \neq |a| < 1/q$, the spherical transform of the radial exponential h_a on a homogeneous tree is*

$$\hat{h}_a(\lambda) = \frac{a - 1/a}{(q+1)(\lambda - \lambda_a)}. \tag{5.1}$$

Consequently

$$\hat{h}_a(\lambda) - \hat{h}_a(0) = \lambda \frac{a+1}{(qa-1)(\lambda - \lambda_a)}. \tag{5.2}$$

If $\lambda \in \mathbb{R}$ and $|\lambda + 1| < \lambda_{|a|}+1$ (i.e., for $\lambda = \gamma(z)-1$, if $|a| < q^{-z} < 1/q|a|$) then the series defining $\hat{h}_a(\lambda)$ converges absolutely to the expression given in (5.1)*; otherwise, it diverges.*

Proof. From Lemma 3.1 we get $\lambda\hat{h}_a(\lambda) = \lambda_a\hat{h}_a(\lambda) + c$, with $c = -\lambda_a(a+1)/(1-qa) = (a-1/a)/(q+1)$. We can then solve for $\hat{h}_a$ and infer (5.1), provided $\lambda \neq \lambda_a$ and the series defining $\hat{h}_a(\lambda)$ converges.

Since $a \mapsto qa + 1/a$ is decreasing in $(0, 1/q)$, we have $\lambda_{|a|} > 0$. By the monotonicity properties (4.6) and by (4.5), if $|\lambda+1| < \lambda_{|a|}+1$ (notice that $\lambda_{-a}+1 = -(\lambda_a+1)$) then asymptotically $|P_n(\lambda)| < |P_n(\lambda_{|a|})| \sim (q\,|a|\,)^{-n}$, and the series $\sum_{n=0}^{\infty} w_n\,|a|^{\,n}\,|P_n(\lambda)|$ converges if and only if $|\lambda+1| < \lambda_{|a|}+1$. ■

Consider now the convex cone generated by h_a for $0 < a < 1/q$ and closed with respect to pointwise convergence.

Lemma 5.2. *Let ν be a positive measure on $(0, 1/q)$ such that the function $a \mapsto \sum h_a = (a+1)/(1-qa)$ is integrable. Then the radial weight $h = \int_0^{1/q} h_a\,d\nu(a)$ is summable, and its spherical transform is*

$$\hat{h}(\lambda) = \int_0^{1/q} \frac{a - 1/a}{(q+1)(\lambda-\lambda_a)}\,d\nu(a).$$

The spherical function ϕ_λ is in L^1_h if and only if $|\lambda+1| < \lambda_{a_0}+1$, where $a_0 = \max \operatorname{supp}\nu$.

Proof. The result follows from Lemma 5.1, because h_{a_0} is the slowest decreasing in the one-parameter family of functions h_a for $a \in \operatorname{supp}\nu$, so it is the one that determines summability against ϕ_λ; indeed $\lambda_{a_0} = \min\{\lambda_a : a \in \operatorname{supp}\nu\}$, since $a \mapsto qa + 1/a$ (hence $a \mapsto \lambda_a$) is decreasing in $(0, 1/q)$. It may be worth showing this directly, however. We have $h(n)\phi_\lambda(n) = w_n P_n(\lambda)\int_0^{a_0} a^n\,d\nu(a)$, so

$$\sum |h\phi_\lambda| = \sum_{n=0}^{\infty} w_n\,|P_n(\lambda)| \int_0^{a_0} a^n\,d\nu(a) \le \frac{q+1}{q}\nu([0,a_0]) \sum_{n=0}^{\infty} (qa_0)^n\,|P_n(\lambda)|\,.$$

On the other hand, if $0 < a' < a_0$ then

$$\sum_{n=0}^{\infty} w_n\,|P_n(\lambda)| \int_0^{a_0} a^n\,d\nu(a) \ge C \sum_{n=0}^{\infty} (qa')^n\,|P_n(\lambda)|$$

for some positive constant C, since $\int_{a'}^{a_0} d\nu(a) > 0$. ■

Proposition 5.3. *The radial weight h of Lemma 5.2 characterizes harmonicity in the weak sense.*

Proof. As in Theorem 4.4 we can assume $f = \phi_\lambda$ (which belongs to L^1_h by assumption). Then evaluating $(*)$ at o yields $\hat{h}(\lambda) = \hat{h}(0)$. We have to prove that $\lambda = 0$ is the only root of this equation satisfying $|\lambda+1| < \lambda_{a_0}+1$. Suppose $\lambda \neq 0$ is another such root. From (5.2) we have

$$0 = \hat{h}(\lambda) - \hat{h}(0) = -\lambda \int_0^{1/q} \frac{a+1}{(1-qa)(\lambda-\lambda_a)}\,d\nu(a).$$

The imaginary part of the integrand is $(a+1)(\operatorname{Im}\lambda)/(1-qa)\,|\lambda-\lambda_a|^{\,2}$, whose factors (except possibly $\operatorname{Im}\lambda$) are all positive; therefore λ must be real. Hence the integrand itself is real, and its only factor that can change sign is $\lambda - \lambda_a$; in fact it must do so, in order for the integral to vanish. Therefore for some $a \in \operatorname{supp}\nu$ this factor is positive, contradicting $|\lambda+1| < \lambda_{a_0}+1 \le \lambda_a+1$. ■

Proposition 5.3 shows that if h belongs to the convex cone generated by exponentials then its $\hat{h}(0)$-eigenspace does not contain any eigenfunction of Δ except harmonic functions. For finite convex combinations of exponentials we can improve this by showing that the $\hat{h}(0)$-eigenspace of h and the kernel of Δ coincide:

Theorem 5.4. *On a homogeneous tree of degree $q+1$, the radial weight $h = \sum_{j=1}^{k} c_j h_{a_j}$, where $c_1, \dots, c_k$ are positive constants and $0 < a_1, \dots, a_k < 1/q$, characterizes harmonicity.*

Proof. Since

$$\hat{h}(\lambda) - \hat{h}(0) = \lambda \sum_{j=1}^{k} c_j \frac{a_j + 1}{(qa_j - 1)(\lambda - \lambda_{a_j})},$$

if $P(\lambda) = \prod_{j=1}^{k}(\lambda - \lambda_{a_j})$ then $P(\lambda)(\hat{h}(\lambda) - \hat{h}(0))$ is a polynomial of degree k in λ, with roots $\lambda_{(1)} = 0$ and $\lambda_{(2)}, \dots, \lambda_{(k)} > 0$ (cf. the end of the proof of Proposition 5.3), thus it equals $C\lambda \prod_{j=2}^{k}(\lambda - \lambda_{(j)})$ for some non-zero constant C. Suppose $f \in L^1_h$ satisfies $(*)$. As before, we can assume it radial. We have $0 = P(\mu_1 - \delta_o) * (h - \delta_o \sum h) * f = C \prod_{j=2}^{k}(\mu_1 - (\lambda_{(j)} + 1)\delta_o) * \Delta f$. If $\Delta f \neq 0$, let k_0 be the smallest integer (with $2 \leq k_0 \leq k$) such that $\prod_{j=2}^{k_0}(\mu_1 - (\lambda_{(j)} + 1)\delta_o) * \Delta f = 0$. Then $\prod_{j=2}^{k_0 - 1}(\mu_1 - (\lambda_{(j)} + 1)\delta_o) * \Delta f$ is a radial eigenfunction of Δ of non-zero eigenvalue $\lambda_{(k_0)}$, does not vanish identically, and belongs to L^1_h. Yet it satisfies $(*)$, a contradiction by Proposition 5.3. ■

Until now we have only produced exponentially decaying weights which characterize harmonicity (in the weak or strong sense). We now exhibit a linear combination with positive coefficients of two exponentials that does not.

Theorem 5.5. *On a homogeneous tree of degree $q+1$, for $0 < a < 1/q$ and $0 < c \leq 1$ the non-negative radial weight $h = h_a + ch_{-a}$ does not characterize harmonicity.*

Proof. From Lemma 5.1 we have

$$\hat{h}(\lambda) - \hat{h}(0) = -\lambda\left(\frac{1+a}{(1-qa)(\lambda - \lambda_a)} + c\frac{1-a}{(1+qa)(\lambda - \lambda_{-a})}\right).$$

This quantity (using the equality $\lambda_{-a} + 1 = -(\lambda_a + 1)$) can be seen to vanish if λ equals either $\lambda_{(1)} = 0$ or

$$\lambda_{(2)} = -\frac{(1+a)(1+qa) - c(1-a)(1-qa)}{(1+a)(1+qa) + c(1-a)(1-qa)}(\lambda_a + 1) - 1.$$

For $c > 0$ the absolute value of the coefficient of $\lambda_a + 1$ is less than 1, that is, $|\lambda_{(2)} + 1| < \lambda_a + 1$. Thus $\phi_{\lambda_{(2)}}$ is in $L^1_{h_a} = L^1_{h_{-a}}$, whence also in L^1_h, and satisfies $(*)$, but is not harmonic, because $\lambda_{(2)} + 1 < 0$. ■

The same proof works for $c > 1$—although h is not positive—except for $c = (1+a)^2(1+qa)^2/(1-a)^2(1-qa)^2$ (since in this case $\lambda_{(2)} = 0$). Theorem 5.5 shows that Theorem 5.4 does not generalize to negative exponentials.

As an immediate consequence of Theorem 5.5 we see that the property of characterizing harmonicity is unstable under small perturbations of the weight in the L^1 norm:

Corollary 5.6. *For $c \to 0^+$ the weights $h_a + ch_{-a}$, which do not characterize harmonicity, tend in L^1 to h_a, which, instead, does.* ∎

6. Exponential-type bi-weights on semi-homogeneous trees

On a tree, let ϵ be the *parity* function with respect to o, given by $\epsilon(x) = (-1)^{|x|}$. A function f is *alternating* if it factors through ϵ. In this case, letting $c_\pm$ (or $c^\pm$) equal $f(o_\pm)$, where $o_+ = o$, and o_- is a fixed neighbor of o, we have $f(x) = c_{\epsilon(x)}$, so we write c_ϵ instead of f. Obviously f is constant if $c_+ = c_-$. Since ϵ is radial, we will write $\epsilon(n) = (-1)^n$; thus $\epsilon(x) = \epsilon(d(x,y))\epsilon(y)$ for any two vertices x, y.

If h is a summable radial weight on a homogeneous tree, we have set $h * f(x) = \sum_y k(x,y)f(y)$ whenever the integrand is summable for each x (i.e., if $f \in L^1_h$), where the summation kernel $k(x,y) = h(d(x,y))$ is invariant under the diagonal action of the group of automorphisms, i.e., $k(g \cdot x, g \cdot y) = k(x,y)$ for any two vertices x, y and any automorphism g. In fact, the group acts transitively on the tree, and, for every x, the isotropy group acts transitively on each sphere centered at x. Conversely, given such a kernel k the corresponding radial weight is determined by $h(n) = k(x,y)$ for any x, y such that $d(x,y) = n$.

A tree is *semi-homogeneous* if its degree $q + 1 = q_\epsilon + 1$ is alternating. Summation against a kernel, automorphism-invariant in the sense explained above, can substitute for convolution on a semi-homogeneous, but not homogeneous tree, as is typical in non-homogeneous settings. Since the group of automorphisms in this case has exactly two orbits and the isotropy subgroups are transitive on spheres, the kernel k is determined by a pair (called *summable radial bi-weight*) $h^\epsilon = (h^+, h^-)$ of independent, summable weights on the tree—namely: h^+, radial around $o_+ = o$; and h^-, radial around a fixed neighbor o_- of o—via the relation $k(x,y) = h^{\epsilon(x)}(d(x,y))$; conversely, k determines the bi-weight h^ϵ through $h^\pm(n) = k(x,y)$ for any two vertices x, y such that $\epsilon(x) = \pm 1$ and $d(x,y) = n$. The *semi-convolution* of the bi-weight h^ϵ with a function f is given by

$$h^\epsilon *' f(x) = \sum_y h^{\epsilon(x)}(d(x,y))f(y),$$

if the integrand is summable for every x, which we shall indicate by $f \in L^1_{h^\epsilon}$. If a tree is homogeneous, in particular it is also semi-homogeneous, hence semi-convolution with a bi-weight h^ϵ makes sense. The corresponding summation kernel $k(x,y)$ is now invariant under parity-preserving automorphisms only. The space $L^1_{h^\epsilon}$ contains $L^1_{h^+} \cap L^1_{h^-}$. This convolution coincides with ordinary convolution by h^+ if and only if $h^+(n) = h^-(n)$ for all n.

On a semi-homogeneous tree, using semi-convolution define the *MVP with respect to the bi-weight* h^ϵ as

$$(*') \qquad h^\epsilon *' f(x) = f(x) \sum h^{\epsilon(x)} \qquad \text{for all } x,$$

for $f \in L^1_{h^\epsilon}$. Note that the function $x \mapsto \sum h^{\epsilon(x)}$ is alternating (non-constant, in general). Terminology of Definition 2.2 carries over in the obvious way.

To prove the analogue of Remark 2.1 we need to show that harmonic functions enjoy the (ordinary) MVP with respect to spheres of any radius. Let $w_n^\pm$ be the number of vertices at distance n from a vertex of parity ± 1. Setting $q = \sqrt{q_+ q_-}$, it is easy to verify that

$$w_n^\pm = \begin{cases} 1 & \text{if } n = 0, \\ (q_\pm + 1) q^{n-1} \sqrt{q_\mp / q_{\pm\epsilon(n)}} & \text{if } n > 0. \end{cases} \tag{6.1}$$

Let $\mu_n^\pm$ be the normalized equidistributed measure on the sphere of radius n centered at $o_\pm$, and let M_n be the operator of semi-convolution with μ_n^ϵ, so that $M_n f(x) = (1/w_n^{\epsilon(x)}) \sum_{d(y,x)=n} f(y)$. Now, f is harmonic if $M_1 f = \mu_1^\epsilon *' f = f$. It turns out that for every odd n the operator M_n does not preserve the L^1 norm of positive functions, that is, it is not an L^1-isometry when restricted to the cone. Instead it is an isometry in this cone for the weighted L^1 space with alternating weight $(q_\epsilon + 1)/(q_{-\epsilon} + 1)$. As a generalization of relations (4.1), one gets

Lemma 6.1. *We have*

$$M_n M_1 = \begin{cases} M_1 & \text{if } n = 0, \\ \dfrac{q_{\epsilon(n)\epsilon} M_{n+1} + M_{n-1}}{q_{\epsilon(n)\epsilon} + 1} & \text{if } n > 0. \end{cases} \tag{6.2}$$

Therefore if f is harmonic then $M_n f = f$ for all n.

Proof. The procedure for (6.2) is similar to that followed for (3.1). If y is a vertex at distance n from x, then the number of neighbors of y farther away from x is

$$v_n(x) = \begin{cases} q_{\epsilon(x)} + 1 & \text{if } n = 0, \\ q_{\epsilon(n)\epsilon(x)} & \text{if } n > 0. \end{cases}$$

Thus for $n > 0$

$$\begin{aligned} M_n M_1 f(x) &= \frac{1}{w_n^{\epsilon(x)}} \sum_{d(y,x)=n} \frac{1}{q_{\epsilon(y)} + 1} \sum_{d(z,y)=1} f(z) \\ &= \frac{1}{(q_{\epsilon(n)\epsilon(x)} + 1) w_n^{\epsilon(x)}} \Bigg(\sum_{d(z,x)=n+1} + v_{n-1}(x) \sum_{d(z,x)=n-1} \Bigg) f(z) \\ &= \frac{w_{n+1}^{\epsilon(x)} M_{n+1} + v_{n-1}(x) w_{n-1}^{\epsilon(x)} M_{n-1}}{(q_{\epsilon(n)\epsilon(x)} + 1) w_n^{\epsilon(x)}} f(x). \end{aligned}$$

Now replace each factor $w_m^{\epsilon(x)}$ with its expression as given in (6.1).

The second statement holds for $n = 0$ with any f, and for $n = 1$ by hypothesis. Proceed by induction on n, using (6.2). ∎

Corollary 6.2. *For any summable radial bi-weight h^ϵ, if $f \in L^1_{h^\epsilon}$ is harmonic then $(*')$ holds.*

Proof. Integrating in polar coordinates around x we have

$$h^\epsilon *' f(x) = \sum_{n=0}^\infty h^{\epsilon(x)}(n) \sum_{d(y,x)=n} f(y) = f(x) \sum_{n=0}^\infty h^{\epsilon(x)}(n) w_n^{\epsilon(x)} = f(x) \sum h^{\epsilon(x)},$$

the middle equality following from Lemma 6.1. ∎

Remark 6.3. Exactly as in Remark 2.3, the bi-weight $\delta_{o_\epsilon} = (\delta_{o_+}, \delta_{o_-})$ does not characterize harmonicity, and, if the bi-weight h^ϵ characterizes harmonicity (in the strong or weak sense), then so does $\tilde{h}^\epsilon = c_0^\epsilon \delta_{o_\epsilon} + c_1^\epsilon h^\epsilon$ for all constants $c_0^+, c_0^-, c_1^+, c_1^-$, with $c_1^+, c_1^- \neq 0$. ■

A natural analogue of the exponential weight h_a is the bi-weight h_a^ϵ given by $h_a^\pm(n) = a^n$. We shall consider one which is slightly more general, given by

$$h_{a,b}^\pm(n) = a^n b^{\pm\epsilon(n)} \tag{6.3}$$

for $0 \neq |a| < 1/q$ (so $h_{a,b}^\pm$ is summable) and $b \neq 0$; the corresponding summation kernel is therefore $k(x,y) = a^{d(x,y)} b^{\epsilon(y)}$. Corresponding to Lemma 3.1 we have:

Lemma 6.4. *The operator H of semi-convolution with $h^\epsilon = h_{a,b}^\epsilon$ satisfies the relation*

$$M_1 \frac{H}{\sum h^\epsilon} = \lambda_\epsilon \frac{H}{\sum h^\epsilon} + c_\epsilon \,\mathrm{id}, \tag{6.4}$$

with

$$\lambda_\pm = \frac{(q_\pm a + 1/a) \sum h^\pm}{(q_\pm + 1) \sum h^\mp}.$$

and $c_\pm = 1 - \lambda_\pm$. Explicitly

$$\sum h^\pm = \frac{(q_\mp a + 1/a) b^{\pm 1} + (q_\pm + 1) b^{\mp 1}}{1/a - q^2 a}.$$

Proof. We have

$$\begin{aligned} M_1 \frac{H}{\sum h^\epsilon} f(x) &= \frac{1}{q_{\epsilon(x)} + 1} \sum_{d(y,x)=1} \frac{1}{\sum h^{\epsilon(y)}} \sum_z a^{d(y,z)} b^{\epsilon(z)} f(z) \\ &= \sum_z \frac{b^{\epsilon(z)} f(z) \sum_{d(y,x)=1} a^{d(y,z)}}{(q_{\epsilon(x)} + 1) \sum h^{-\epsilon(x)}} = \lambda_{\epsilon(x)} \frac{H}{\sum h^\epsilon} f(x) + c(x) f(x) \end{aligned}$$

for some function c, because

$$\frac{\sum_{d(y,x)=1} a^{d(y,z)}}{q_{\epsilon(x)} + 1} = \begin{cases} a & \text{if } z = x, \\ \dfrac{q_{\epsilon(x)} a + 1/a}{q_{\epsilon(x)} + 1} a^{d(x,z)} & \text{if } z \neq x. \end{cases}$$

To finish the proof of (6.4) we are only left to show that c is itself alternating: applying the identity to a non-zero constant function we indeed obtain $\lambda_\epsilon + c \equiv 1$.

Finally

$$\sum h^\pm = \sum_{n=0}^\infty a^n b^{\pm\epsilon(n)} w_n^\pm = \sum_{n=0}^\infty a^n b^{\pm\epsilon(n)} (q_\pm + 1) q^{n-1} \sqrt{q_\mp / q_{\pm\epsilon(n)}} - b^{\pm 1}/q_\pm.$$

Splitting the series according to the parity of the index n, one has two geometric series with ratio qa; the remainder is an immediate algebraic verification. ■

We are now in a position to prove the generalization of Theorem 3.2.

Theorem 6.5. *On a semi-homogeneous tree, the bi-weight $h^\epsilon = h^\epsilon_{a,b}$ characterizes harmonicity if $0 \neq |a| < 1/q$ and $b \neq 0$.*

Proof. If f satisfies $(*')$, then by (6.4)

$$M_1 f = M_1 \frac{H}{\sum h^\epsilon} f = \lambda_\epsilon \frac{H}{\sum h^\epsilon} f + c_\epsilon f = (\lambda_\epsilon + c_\epsilon) f = f. \qquad \blacksquare$$

Observe that, although $\sum h^\epsilon$ appears in relation (6.4) and in the expression of the 'bi-eigenvalue' λ_ϵ, its explicit value is never needed for the purpose of proving Theorem 6.5.

Remark 6.6. On a homogeneous tree, the weight of Theorem 5.5 equals, up to a multiplicative constant, the bi-weight $h^\pm(n) = a^n b^{\epsilon(n)}$ for some $b > 1$. In spite of the similarity with the expression (6.3) of $h^\epsilon_{a,b}$, the former does not characterize harmonicity, whereas the latter does. The respective semi-convolution kernels $k(x,y)$ are $a^{d(x,y)} b^{\epsilon(d(x,y))}$ and $a^{d(x,y)} b^{\epsilon(x)\epsilon(d(x,y))} = a^{d(x,y)} b^{\epsilon(y)}$. Observe that $a^{d(x,y)} b^{\epsilon(x)}$ is equivalent to $a^{d(x,y)}$ in the sense of Remark 6.3. $\blacksquare$

7. Decay of weights in the continuous setting

Let us recall the asymptotics of spherical functions on hyperbolic spaces, i.e., rank-1 symmetric spaces of non-compact type. The picture is essentially identical to that of trees given in §4, and well known; notation is as in [30] and references therein. On a hyperbolic space $\mathbb{H}$ there are intrinsic polar coordinates as follows: let K be the stabilizer subgroup of o in the connected component G of the isometry group of $\mathbb{H}$ containing the identity e. Then $\mathbb{H} = G/K$, and every point of $\mathbb{H}$ can be uniquely written as $ka_t \cdot o$, where $k \in K$ and $\{a_t\}$ is a one-parameter group of transvections based at o. Therefore functions on $\mathbb{H}$ can be lifted to K-invariant functions on G, and we shall do so whenever appropriate without further notice. A quantity that occurs often is $\rho = p/2 + q$, where p, q are the multiplicities of short, respectively, long roots. If dk denotes the normalized Haar measure on K then the intrinsic volume element on $\mathbb{H}$ (with respect to which the scalar product of functions is defined—cf. §2) is $dv = (2\sinh r)^{p+q}(2\cosh r)^q \, dr \, dk$. The K-invariant surface measure w_r of the spherical surface of radius r grows as $e^{2\rho r}$ for $r \to \infty$.

The radial component of the Laplace-Beltrami operator Δ is

$$\partial^2/\partial r^2 + ((p+q)\coth r + q \tanh r)\partial/\partial r.$$

The spherical function ϕ_λ is the bi-K-invariant (i.e., radial) eigenfunction of Δ of eigenvalue λ such that $\phi_\lambda(e) = 1$, and, using the parameter z that satisfies $\tilde\gamma(z) = \lambda$, where $\tilde\gamma(z) = z(z + 2\rho)$, is given by

$$\phi_\lambda(g) = \int_K P(g,k)^z \, dk, \tag{7.1}$$

where $P(g,k) = e^{r(g^{-1}k)}$ is the Poisson kernel, and $r(g^{-1}k)$ stands for the value of r such that $g^{-1}k = k'a_r$ for some $k' \in K$. As for trees, it follows that

$$|\phi_{\tilde{\gamma}(z)}| \leq |\phi_{\tilde{\gamma}(\operatorname{Re} z)}|. \tag{7.2}$$

The *spherical transform* $\hat{h}$ of a function h is defined on $\{\lambda \in \mathbb{C}\colon \phi_\lambda \in L^1_h\}$ as $\hat{h}(\lambda) = \langle h, \bar{\phi}_\lambda \rangle$.

The asymptotic behavior of spherical functions is almost the same as on trees (deeper analogies between these two setups are described in [9]; see also [20]). With our choice of parameters, for $r \to \infty$ we have

$$|\phi_\lambda(r)| \begin{cases} \preceq e^{(|\sqrt{\lambda+\rho^2}|-\rho)r}(1+r) & \text{if } \mathbb{R} \ni \lambda > -\rho^2, \\ \sim e^{(|\operatorname{Re}\sqrt{\lambda+\rho^2}|-\rho)r} & \text{otherwise.} \end{cases}$$

As done on trees in Definition 4.1, we define the rate of decay on hyperbolic spaces (as well as Euclidean spaces).

Definition 7.1. Let h be a summable function on a hyperbolic space $\mathbb{H}$, and write $h(r) = \langle h, \mu_r \rangle = \int_K h(ka_r \cdot o)\, dk$ for all $r \geq 0$. We say that, for $r \to \infty$, the function h decays:

(1) *faster than exponentially* if $|h(r)| \prec a^r$ for every $a > 0$;
(2) *exponentially* (*like* a^r) if $0 < a < e^{-2\rho}$ is such that $b^r \prec |h(r)| \prec c^r$ whenever $0 < b < a < c$;
(3) *slower than exponentially* if $a^r \prec |h(r)|$ whenever $0 < a < e^{-2\rho}$.

While, if h is a summable function on $\mathbb{R}^n$, set $h(r) = \langle h, \mu_r \rangle$ for all $r \geq 0$. We say that, for $r \to \infty$, the function h decays:

(1) *faster than exponentially* if $|h(r)| \prec a^r$ for every $a > 0$;
(2) *exponentially* (*like* a^r) if $a > 0$ is such that $b^r \prec |h(r)| \prec c^r$ whenever $0 < b < a < c$;
(3) *slower than exponentially* if $a^r \prec |h(r)|$ for every $a > 0$. ■

As observed for trees in Remark 4.2, the requirement that a summable function on a hyperbolic space decay slower than exponentially is rather restrictive, because the measure of the spherical surface of radius r grows itself as $e^{2\rho r}$. For the domain of the spherical transform we have the following (with the terminologic convention before Proposition 4.3 about analyticity on non-open sets).

Proposition 7.2. *Let h be a summable function on a hyperbolic space $\mathbb{H}$, and set $S_t = \{\lambda \in \mathbb{C}\colon |\operatorname{Re}\sqrt{\lambda+\rho^2}| \leq -\rho + t\}$. Then:*

(1) if h decays faster than exponentially then $\hat{h}$ is entire;
(2) if h decays exponentially like a^r then $\hat{h}$ is analytic on its domain $S_{-\log a}$;
(3) if h decays slower than exponentially then $\hat{h}$ is analytic on its domain $S_{2\rho}$. ■

Remark 7.3. The parameter λ is chosen, for all the three ambient spaces, to be the eigenvalue of Δ. Consequently, if h is a summable radial weight then the constant $\int h$ of $(*)$ equals $\hat{h}(0)$ in any of these setups. ∎

The results of Theorem 4.4 and Theorem 4.5 for slowly as well as for fast decaying weights carry over to hyperbolic spaces and Euclidean spaces.

Theorem 7.4. *A non-negative summable radial weight h on a hyperbolic space, or on $\mathbb{R}^n$, that decays slower than exponentially (in the sense of Proposition 7.2) characterizes harmonicity in the weak sense.*

Proof. The proof for hyperbolic spaces is the same as for trees given in Theorem 4.4. By the majorization principle (7.2), for $\lambda \in S_{2\rho}$ we have the inequality $|\hat{h}(\lambda)| \leq \int h = \hat{h}(0)$, which is strict in the interior of $S_{2\rho}$ by the maximum principle, and is an equality for $\lambda = 0$. On the rest of $\partial S_{2\rho}$ the inequality is also strict since $|\phi_\lambda(x)| < \phi_0(x) = 1$ for x in a set of positive measure, because of cancellations in the integral $\langle h, \phi_\lambda \rangle$ due to the fact that the corresponding exponent z in (7.1) has a non-zero imaginary part.

Finally, on $\mathbb{R}^n$ the statement is obvious: the Fourier transform $\hat{h}(\lambda)$ converges only for real λ, and $|\hat{h}(\lambda)| < \int h = \hat{h}(0)$ for $\lambda \neq 0$, because h is a non-atomic positive measure. ∎

Theorem 7.5. *Let h be a radial weight on a hyperbolic space, or on $\mathbb{R}^n$, such that for some positive constants A, α we have $|h(r)| \preceq e^{-Ar^{1+\alpha}}$ for $r \to \infty$. Assume in addition that $\int h \neq 0$. Then h does not characterize harmonicity.*

Proof. We first give the proof for $\mathbb{R}^n$. It is enough to show that $\hat{h}(\lambda) = \hat{h}(0)$ for some $\lambda \neq 0$ in $\mathbb{C}^n$. Note first that $\hat{h}(\lambda) \to 0$ as $|\lambda| \to \infty$ in $\{|\operatorname{Im}\lambda| < 1\}$. In fact, given $\epsilon > 0$ one can find $R > 0$ and $\phi \in C^\infty(\mathbb{R}^n)$ with support $\{|x| \leq R\}$ such that

$$\int_{|x|>R} |h(x)|\, e^{|x|}\, dx < \epsilon/3,$$

$$\int_{|x|\leq R} |h(x) - \phi(x)|\, dx < \epsilon/3e^R.$$

Now, by the Paley-Wiener theorem, when $|\lambda|$ is sufficiently large and $|\operatorname{Im}\lambda| < 1$ we have $|\hat{\phi}(\lambda)| < \epsilon/3$. But also, if $|\operatorname{Im}\lambda| < 1$ then

$$\begin{aligned} |\hat{h}(\lambda) - \hat{\phi}(\lambda)| &= \left| \int_{|x|>R} h(x) e^{i\lambda\cdot x}\, dx + \int_{|x|\leq R} (h(x) - \phi(x)) e^{i\lambda\cdot x}\, dx \right| \\ &\leq \int_{|x|>R} |h(x)|\, e^{|x|}\, dx + \int_{|x|\leq R} |h(x) - \phi(x)|\, e^{|x|}\, dx \leq 2\epsilon/3, \end{aligned}$$

thus $|\hat{h}(\lambda)| < \epsilon$.

Next we show that the entire function $\hat{h}$ has finite order, that is, as $|\lambda| \to \infty$ we have $|\hat{h}(\lambda)| \preceq e^{|\lambda|^B}$ for some B. Indeed,

$$|\hat{h}(\lambda)| \leq C \int_{\mathbb{R}^n} e^{-|x|(A|x|^\alpha - |\operatorname{Im}\lambda|)}\, dx.$$

Split the integral as I_1+I_2, where I_1 is the value of the integral over the domain $D_1=\{x\in\mathbb{R}^n\colon A\,|x|^{\,\alpha}>2\,|\operatorname{Im}\lambda|\,\}$, and I_2 over its complement D_2; it is enough to consider $|\operatorname{Im}\lambda|\geq 1$. On D_1 one has $A\,|x|^{\,\alpha}-|\operatorname{Im}\lambda|\geq|\operatorname{Im}\lambda|\geq 1$, hence

$$I_1\leq\int_{D_1}e^{-|x|}\,dx\leq\int_{\mathbb{R}^n}e^{-|x|}\,dx=C_n.$$

On D_2 one has

$$\begin{aligned}I_2&\leq\int_{D_2}e^{|x||\operatorname{Im}\lambda|}\,dx\leq e^{(2/A)^{1/\alpha}|\operatorname{Im}\lambda|^{1+1/\alpha}}\operatorname{Vol}D_2\\&\leq C_n(2\,|\operatorname{Im}\lambda|\,/A)^{n/\alpha}e^{(2/A)^{1/\alpha}|\operatorname{Im}\lambda|^{1+1/\alpha}}.\end{aligned}$$

Thus $\hat h$, hence $\hat h-\hat h(0)$, is of finite order. If $\hat h(\lambda)=\hat h(0)$ only at $\lambda=0$, then, for any fixed unit vector $v\in\mathbb{R}^n$, again Hadamard's product theorem yields $\hat h(sv)-\hat h(0)=B's^ke^{P(s)}$ for all $s\in\mathbb{C}$ with some $B'\neq 0$, some $k\geq 1$, and some polynomial P of degree not larger than the order of $\hat h$. It follows that $|\hat h(sv)-\hat h(0)|$ tends to 0 or ∞ as $s\to\infty$ through real numbers. This contradicts the Riemann-Lebesgue lemma, which implies that the limit must be $|\hat h(0)|$.

An obvious modification of the proof for $\mathbb{R}^n$ works for hyperbolic spaces, adapting the argument used in Theorem 4.5 for trees, the bounds of whose spherical functions are essentially the same. The Riemann-Lebesgue lemma needed here is [26, Exercise B.6 of the Introduction, with solution], with the condition $|\operatorname{Re}\sqrt{\lambda+\rho^2}|<\rho$ (under which the spherical function ϕ_λ is bounded). ■

8. The resolvent of the Laplacian on Euclidean spaces

In order to find a positive radial weight h of exponential decay on $\mathbb{R}^n$ which characterizes harmonicity, let us proceed as in §3, and look for a fundamental solution R_ζ of the equation $\Delta R_\zeta=\zeta R_\zeta-\delta_o$. For $r>0$ the equation $\Delta R_\zeta=\zeta R_\zeta$ in polar coordinates is the Bessel equation, hence for $\zeta>0$ a positive radial fundamental solution exists, given by

$$R_\zeta(x)=c_n\,|x|^{\,1-n/2}K_{n/2-1}(\sqrt{\zeta}\,|x|\,),\tag{8.1}$$

where $K_{n/2-1}$ is the Bessel function of the second kind, and c_n is a suitable constant. For $r=|x|\to 0$, the solution R_ζ has the characteristic singularity cr^{2-n} if n is odd, and $cr^{2-n}+d\log r$ if n is even, for $c,d\neq 0$ [17]. Positivity follows from the integral representations of K_ν [19, (7.12.21–23)].

Lemma 8.1. *We have* $\int R_\zeta=1/\zeta$.

Proof. Formally this is immediate: $0=\langle R_\zeta,\Delta 1\rangle=\langle\Delta R_\zeta,1\rangle=\zeta\langle R_\zeta,1\rangle-\langle\delta_o,1\rangle=\zeta\int R_\zeta-1$. To make this rigorous, denote by B_r the ball of radius r centered at o. Then, for $0<r_1<r_2$, Green's identity yields

$$\zeta\int_{B_{r_2}\setminus B_{r_1}}R_\zeta=\int_{B_{r_2}\setminus B_{r_1}}\Delta R_\zeta=\int_{\partial B_{r_2}}\frac{\partial R_\zeta}{\partial r}-\int_{\partial B_{r_1}}\frac{\partial R_\zeta}{\partial r}.$$

On the right-hand side the first integral tends to 0 as $r_2 \to \infty$, because $\partial R_\zeta/\partial r$ decays exponentially, while the second tends to -1 for $r_1 \to 0$, due to the characteristic singularity of R_ζ. ∎

The argument of Theorem 3.2 applies to this continuous setup:

Theorem 8.2. *On $\mathbb{R}^n$, for $\zeta > 0$ the resolvent $h = R_\zeta$ of Δ at the eigenvalue ζ characterizes harmonicity.*

Proof. The fact that $f \in L^1_h$ implies that $f * h(x)$ exists as a Lebesgue integral for all x. Also, $f*h$ is locally integrable. Hence for any $\phi \in \mathcal{D}(\mathbb{R}^n)$ the function $\phi(x+y)f(x)h(y)$ is in $L^1(\mathbb{R}^{2n})$, and [27, Condition (Γ)] is satisfied. So by [27, Proposition 18] and $(*)$ we have in the distribution sense $\Delta f/\zeta = \Delta(h*f) = \Delta h * f = (\zeta h - \delta_o) * f = \zeta h * f - f = 0$. ∎

This shows that the resolvent of the Laplacian in $\mathbb{R}^n$ characterizes harmonicity in much the same way as on trees. For the convex cone generated by resolvents we can prove the analogue of Proposition 5.3.

Proposition 8.3. *Let ν be a positive measure on $(0,\infty)$ such that the radial weight $h = \int_0^\infty R_\zeta \, d\nu(\zeta)$ on $\mathbb{R}^n$ is finite and summable. Then h characterizes harmonicity in the weak sense.*

Proof. We follow the approach of [30, Proposition 2.1]. Let f be such that $(*)$ and $\Delta f = \lambda f$ hold, with $\lambda \in \mathbb{C}$. As before, f may be assumed radial. Radial eigenfunctions of Δ with eigenvalue λ which are finite (and non-zero) at the origin are multiples, for $z = \pm\sqrt{\lambda}/2\pi i$, of a function j_z (related to the Bessel function of the first kind), given by

$$j_z(x) = \int_K e^{2\pi z \langle k\cdot x, v\rangle} \, dk,$$

where dk is the normalized Haar measure on $K = SO(n)$, $v \in \mathbb{R}^n$ is a fixed unit vector, and $\langle\cdot,\cdot\rangle$ is here the standard scalar product in $\mathbb{R}^n$. The asymptotic behavior for $|x| \to \infty$ is $j_z(x) \sim |x|^{-(n+1)/2} e^{2\pi |\operatorname{Im} z||x|}$ (see [30] for references). On the other hand, if $\zeta_0 = \min\operatorname{supp}\nu$ we see, as for Lemma 5.2, that (8.1) implies $h(x) \sim |x|^{(1-n)/2} e^{-\sqrt{\zeta_0}|x|}$. Therefore the condition $j_z \in L^1_h$ amounts to

$$2\pi\,|\operatorname{Im} z| < \sqrt{\zeta_0}. \tag{8.2}$$

As in [30], $h * j_z = \hat{h}(zv) j_z$, hence $(*)$ gives $\hat{h}(zv) = \int h = \hat{h}(0)$.

Since $\Delta R_\zeta = \zeta R_\zeta - \delta_o$, for every $\lambda \in \mathbb{R}^n$ the Fourier transform of R_ζ satisfies $-|\lambda|^2 \hat{R}_\zeta(\lambda) = \zeta \hat{R}_\zeta(\lambda) - 1$, so that $\hat{R}_\zeta(\lambda) = 1/(\zeta + |\lambda|^2)$. Therefore

$$0 = \hat{h}(zv) - \hat{h}(0) = -z^2 \int_0^\infty \frac{1}{(\zeta + z^2)\zeta} \, d\nu(\zeta).$$

The imaginary part of the integrand is $-(\operatorname{Im} z^2)/\zeta\,|\zeta + z^2|^2$, whence z^2 is real. If $z \neq 0$ is a solution of the equation $\hat{h}(zv) = \hat{h}(0)$ then the integral vanishes, therefore at some $\zeta \in \operatorname{supp}\nu$ the integrand must be negative, so $z^2 < -\zeta \leq -\zeta_0 \leq 0$. Then $|\operatorname{Im} z| = |z| > \sqrt{\zeta_0}$, contradicting (8.2). So the only possible solution is $z = 0$, corresponding to the function j_0, a constant, hence harmonic. ∎

9. The resolvent of the Laplacian on hyperbolic spaces

In this section we extend to hyperbolic spaces the results proved for $\mathbb{R}^n$ in the previous section. With notation as in §7, the spherical function ϕ_λ (the radial eigenfunction of Δ with eigenvalue λ such that $\phi_\lambda(o)=1$) can be expressed as a linear combination of $\Phi_\lambda^+, \Phi_\lambda^-$, where $\Phi_\lambda^\pm(r)$ is singular at $r=0$, behaves as $e^{(\pm\sqrt{\lambda+\rho^2}-\rho)r}$ for $r\to\infty$, and is real-valued if and only if $\mathbb{R}\ni\lambda>-\rho^2$.

For $\zeta>0$ the resolvent R_ζ of Δ at the eigenvalue ζ (i.e., the Green kernel of $\Delta-\zeta\,\mathrm{id}$) is $\min\{\Phi_\zeta^+,\Phi_\zeta^-\}$, cf. [29] (observe the analogy with trees, where, as briefly outlined in §4, spherical functions decompose as linear combinations of two exponentials, the smaller of which is the resolvent of Δ which was used in Theorem 3.2—see [20, Chapter 3] for more details). Hence for $r\to\infty$

$$R_\zeta(r)\sim e^{-(\rho+\sqrt{\zeta+\rho^2})r}.$$

Lemma 9.1. *We have* $\int_{\mathbb{H}} R_\zeta = 1/\zeta$.

Proof. The argument is the same as for Lemma 8.1. The only difference is that for $r_2\to\infty$ the volume of the sphere of radius r_2 now grows exponentially, namely as $e^{2\rho r_2}$, but this is compensated by $(\partial R_\zeta/\partial r)(r_2)\sim e^{-(\rho+\sqrt{\zeta+\rho^2})r_2}\prec e^{-2\rho r_2}$. Hence in Green's identity the integral over such sphere tends to 0. ∎

As in §8, the relevant part of [27] works in this setting, so we can infer

Theorem 9.2. *On a hyperbolic space, for $\zeta>0$ the resolvent $h=R_\zeta$ of Δ at the eigenvalue ζ characterizes harmonicity.* ∎

Also the analogue of Proposition 8.3 holds here.

Proposition 9.3. *Let ν be a positive measure on $(0,\infty)$ such that the radial weight $h=\int_0^\infty R_\zeta\,d\nu(\zeta)$ on a hyperbolic space is finite and summable. Then h characterizes harmonicity in the weak sense.*

Proof. We proceed as for Proposition 8.3. By Proposition 7.2(2), for given $\zeta>0$ the spherical transform $\hat R_\zeta(\lambda)=\int R_\zeta\phi_\lambda$ is analytic in $|\mathrm{Re}\sqrt{\lambda+\rho^2}|<\sqrt{\zeta+\rho^2}$, therefore $\hat h(\lambda)$ is analytic in the strip

$$|\mathrm{Re}\sqrt{\lambda+\rho^2}|<\sqrt{\zeta_0+\rho^2}, \tag{9.1}$$

where $\zeta_0=\min\operatorname{supp}\nu$. In order that ϕ_λ be in L_h^1, inequality (9.1) must hold.

Taking the spherical Fourier transform of the resolvent identity $\Delta R_\zeta=\zeta R_\zeta-\delta_o$ we obtain $\hat R_\zeta(\lambda)=1/(\zeta-\lambda)$, so

$$0=\hat h(\lambda)-\hat h(0)=\lambda\int_0^\infty\frac{1}{(\zeta-\lambda)\zeta}\,d\nu(\zeta)$$

(cf. Remark 7.3). Taking the imaginary part of the integrand, one sees that λ is real. Unless λ itself vanishes, we get $\lambda>\zeta\ge\zeta_0\ge 0$ for some $\zeta\in\operatorname{supp}\nu$, as long as $\hat h(\lambda)=\hat h(0)$. This implies $|\mathrm{Re}\sqrt{\lambda+\rho^2}|=\sqrt{\lambda+\rho^2}>\sqrt{\zeta_0+\rho^2}$, contradicting (9.1). ∎

10. A conjecture on the spectral resolution of the Laplace-Beltrami operator

In the continuous setting ($\mathbb{R}^n$ and $\mathbb{H}$) we have shown that the convolution eigenspace $M_h = \{f \in L^1_h\colon f \text{ satisfies } (*)\}$ of an exponentially decaying weight h in a large class does not contain any eigenfunction of Δ except its kernel. We have also shown that $\Delta(h * f) = \Delta h * f$. Then, under mild assumptions on f, convolution by h commutes with Δ: in fact $\Delta h * f = h * \Delta f$ by integration by parts, provided that $f \operatorname{grad} h$ and $h \operatorname{grad} f$ vanish at infinity. But in our classes it is easy to see that $\operatorname{grad} h \sim h$, therefore it is enough to assume that hf vanishes at infinity (because then so does $h \operatorname{grad} f$, by integration by parts); in this case M_h is an invariant subspace for Δ. In order to show that h characterizes harmonicity, it would thus be sufficient to prove that M_h is a direct sum of eigenspaces of $\Delta|_{M_h}$.

In the discrete setting of trees, a similar result is required to prove that weights in the closed convex cone generated by exponentials characterize harmonicity. The statement for finite positive combinations was proved in Theorem 5.4 by identifying the spectrum of $\Delta|_{M_h}$ with the set of roots of a suitable polynomial. Here no vanishing at infinity is needed to ensure that M_h is invariant for the Laplacian, because it is finitely supported and commutes with every radial convolutor. A similar approach has been followed in [30] with a specific weight h for which the equation $\hat{h} = \int h$ has finitely many solutions, showing that the spectrum of $\Delta|_{M_h}$ consists exactly of such solutions.

It seems reasonable to believe that M_h is indeed a direct sum of eigenspaces of $\Delta|_{M_h}$ in all the cases considered. For assume that we have a spectral resolution $\Delta|_{M_h} = \int_\sigma \lambda \, dE(\lambda)$, with σ denoting the spectrum, and $\int_\sigma dE(\lambda) = \mathrm{id}$. If the radial eigenvector ϕ_λ of $\Delta|_{M_h}$ with eigenvalue λ is in L^1_h, then $h * \phi_\lambda = \hat{h}(\lambda)\phi_\lambda$. This suggests that the operator H of convolution by h satisfies $H = \int_\sigma \hat{h}(\lambda) \, dE(\lambda)$. On the other hand, H on M_h equals $\hat{h}(0)$ times the identity, hence its spectral decomposition with respect to the operator-valued measure $E(\lambda)$ should be $\int_\sigma \hat{h}(0) \, dE(\lambda)$. The support of $E(\lambda)$ should thus be the set $\{\lambda\colon \hat{h}(\lambda) = \hat{h}(0)\}$, discrete because $\hat{h}$ is analytic.

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