Journal of Lie Theory
Volume **9** (1999) 491–506

Spectra of self-gradients on spheres

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Communicated by B. Ørsted

Abstract. We give a general formula for the spectral resolution of a class of first-order differential operators on the sphere S^n which includes, among the most elementary cases, the Dirac and Rarita-Schwinger operators.

1. Introduction

There is a long literature on spectra of natural second-order differential operators on vector bundles over homogeneous spaces; see, e.g., [11, 9, 10, 3]. Recently, there has been some interest in explicit spectra of natural first-order operators like the Dirac and Rarita-Schwinger operators. The purpose of this paper is to show, by theorem and example in the case of the sphere S^n, that such explicit spectra are computable using general results of [6] on the spectra of intertwining operators for group representations, and results like those of [4] which employ intertwinors to get spectral information on other operators which do not enjoy quite as much symmetry.

All spaces of eigensections on the sphere treated below are expressed as irreducible $\mathrm{Spin}(n+1)$-modules, or direct sums of such. As a result their dimensions, and thus the multiplicities of all eigenvalues, are computable via Weyl's dimension formula.

2. Parameterization by dominant weights

Irreducible representations of $\mathrm{Spin}(n)$, and thus irreducible associated $\mathrm{Spin}(n)$-bundles, are parameterized by *dominant weights* $(\lambda_1, \dots, \lambda_\ell) \in \mathbb{Z}^\ell \cup (\frac{1}{2} + \mathbb{Z})^\ell$, $\ell = [n/2]$, satisfying the inequality constraint

$$\begin{aligned} &\lambda_1 \geq \ldots \geq \lambda_\ell \geq 0, && n \text{ odd}, \\ &\lambda_1 \geq \ldots \geq \lambda_{\ell-1} \geq |\lambda_\ell|, && n \text{ even}. \end{aligned}$$

The dominant weight λ is the highest weight of the corresponding representation. The representations which factor through $\mathrm{SO}(n)$ are exactly those with $\lambda \in \mathbb{Z}^\ell$.

ISSN 0949–5932 / $2.50

We shall denote by $V(\lambda)$ the representation with highest weight λ. If M is an n-dimensional smooth manifold with $\mathrm{Spin}(n)$ structure and $\mathcal{F}$ is the bundle of spin frames, we denote by $\mathbb{V}(\lambda)$ the vector bundle $\mathcal{F}\times_\lambda V(\lambda)$. When spin structure is not involved (i.e. when λ is integral), we may use the orthonormal frame bundle in constructing $\mathbb{V}(\lambda)$. We shall denote by $\chi(n)$ the set of dominant $\mathrm{Spin}(n)$ weights.

3. The selection rule

We shall discuss several familiar examples of bundles and identify their highest weights below. One important highest weight is that of the defining representation $V(1,0,\dots,0)$ of $\mathrm{SO}(n)$. The classical *selection rule* describes the $\mathrm{Spin}(n)$ decomposition of $V(1,0,\dots,0)\otimes V(\lambda)$ for an arbitrary dominant λ:

$$V(1,0,\dots,0)\otimes V(\lambda)\cong_{\mathrm{Spin}(n)} V(\sigma_1)\oplus\dots\oplus V(\sigma_{N(\lambda)}), \tag{1}$$

where the σ_u are distinct ($\sigma_u\cong_{\mathrm{Spin}(n)}\sigma_v\Rightarrow u=v$), and a given σ appears if and only if σ is a dominant weight and

$$\sigma=\lambda\pm e_a\,,\ \text{some } a\in\{1,\dots,\ell\}, \qquad \underline{\text{or}} \tag{2}$$

$$n \text{ is odd},\ \lambda_\ell\neq 0,\ \sigma=\lambda. \tag{3}$$

Here e_a is the $a^{\underline{\text{th}}}$ standard basis vector in $\mathbb{R}^\ell$. (1) implicitly defines a numerical invariant $N(\lambda)$, the number of selection rule "targets" of $V(\lambda)$. We shall use the notation

$$\lambda\leftrightarrow\sigma$$

for the selection rule: $\lambda\leftrightarrow\sigma$ if and only if $V(\sigma)$ appears as a summand in $V(1,0,\dots,0)\otimes V(\lambda)$. The notation $\leftrightarrow$ is justified because the relation is symmetric. In fact, one can see *a priori* that the relation must be symmetric: the defining representation of $\mathrm{SO}(n)$ is real, and thus self-contragredient.

4. The branching rule

The classical *branching rule* governs the restriction of a $\mathrm{Spin}(n+1)$ representation to a copy of $\mathrm{Spin}(n)$ which is embedded in the standard way, as follows. By changing n to $n+1$ above we have a parameterization of the irreducible representations of $\mathrm{Spin}(n+1)$. For a dominant $\mathrm{Spin}(n+1)$-weight α, let $\mathcal{V}(\alpha)$ be the corresponding irreducible representation. The branching rule says that $\dim\mathrm{Hom}_{\mathrm{Spin}(n)}(V(\lambda),\mathcal{V}(\alpha)|_{\mathrm{Spin}(n)})\in\{0,1\}$, with $\dim\mathrm{Hom}_{\mathrm{Spin}(n)}(V(\lambda),\mathcal{V}(\alpha)|_{\mathrm{Spin}(n)})$ $=1$ if and only if

$$\alpha_1-\lambda_1\in\mathbb{Z}\quad\text{and}\quad\begin{cases}\alpha_1\ge\lambda_1\ge\alpha_2\ge\dots\ge\lambda_\ell\ge|\alpha_{\ell+1}|, & n\text{ odd},\\ \alpha_1\ge\lambda_1\ge\alpha_2\ge\dots\ge\lambda_{\ell-1}\ge\alpha_\ell\ge|\lambda_\ell|, & n\text{ even}.\end{cases} \tag{4}$$

We use $\alpha\downarrow\lambda$ or $\lambda\uparrow\alpha$ as an abbreviation for (4).

5. Stein-Weiss operators (gradients)

An interesting concept related to the selection rule is that of *generalized gradients*, or *Stein-Weiss operators* [13]. The covariant derivative ∇ carries sections of $\mathbb{V}(\lambda)$

to sections of

$$\begin{aligned} T^*M \otimes \mathbb{V}(\lambda) &\cong_{\mathrm{Spin}(n)} \mathbb{V}(1,0\ldots,0) \otimes \mathbb{V}(\lambda) \\ &\cong_{\mathrm{Spin}(n)} \mathbb{V}(\sigma_1) \oplus \ldots \oplus \mathbb{V}(\sigma_{N(\lambda)}). \end{aligned}$$

Since the selection rule is multiplicity free, we may project onto the unique σ_u summand; the result is our gradient:

$$G_u = G_{\lambda\sigma_u} = \mathrm{Proj}_u \circ \nabla.$$

Up to normalization, some examples of gradients, or direct sums of gradients, are the exterior derivative d, its formal adjoint δ, the conformal Killing operator S, the Dirac operator, the twistor operator, and the Rarita-Schwinger operator. In fact, every first-order $\mathrm{Spin}(n)$-equivariant differential operator is a direct sum of gradients [8].

6. Proper self-gradients

The exceptional case (3) in the selection rule, in which $\lambda \leftrightarrow \lambda$, together with the above concept of gradient, provides an interesting series of first-order operators. In this case, $G_{\lambda\lambda}$ carries sections of $\mathbb{V}(\lambda)$ to sections of a copy of $\mathbb{V}(\lambda)$ which lives as a subbundle in $T^*M \otimes \mathbb{V}(\lambda)$. Examples are, up to constant multiples, the Dirac and Rarita-Schwinger operators (the cases $\lambda = (\frac{1}{2},\ldots,\frac{1}{2})$ and $(\frac{3}{2},\frac{1}{2},\ldots,\frac{1}{2})$ respectively), and the operator $\star d$ on $(n-1)/2$-forms, in odd dimensions $n \geq 3$. Given any tensor-spinor realization $\mathbb{V}$ of $\mathbb{V}(\lambda)$, the target of the corresponding realization of $G_{\lambda\lambda}$ is a subbundle $\mathbb{W}$ of $T^*M \otimes \mathbb{V}$, with $\mathbb{W} \cong_{\mathrm{Spin}(n)} \mathbb{V}$. If we would like to use the same realization $\mathbb{V}$ as both source and target bundle for a realization $D_\lambda^{\mathrm{self}}$ of $G_{\lambda\lambda}$, we need a choice of normalization. First, normalize the Hermitian inner product on $T^*M \otimes \mathbb{V}$ so that

$$|\xi \otimes v|^2 = |\xi|^2 |v|^2; \tag{5}$$

then normalize $D_\lambda^{\mathrm{self}}$ so that $(D_\lambda^{\mathrm{self}})^2 = G_{\lambda\lambda}^* G_{\lambda\lambda}$. This determines $D_\lambda^{\mathrm{self}}$ only up to multiplication by ± 1. It is important to note that this ambiguity is in the nature of things: it is analogous to the ambiguity in the naming of the complex units $\pm\sqrt{-1}$. Indeed, this is more than an analogy: gradients generalize the Cauchy-Riemann equations [13], which are sensitive to the renaming of $\pm\sqrt{-1}$. In our examples, the ambiguity may be viewed as residing in a choice of fundamental tensor-spinor (or more generally, a Clifford structure) γ. The Clifford relations and spin connection (in particular the relation $\nabla\gamma = 0$) are invariant under interchange of γ and $-\gamma$, but the Dirac operator $\gamma^a \nabla_a$ undergoes a sign change. As is standard in complex variables and spinor theory, we shall implicitly make a choice in each example. When examples interact, as in the series of spinor-form bundles treated in Sec. 10., it is of course important to make a consistent choice.

7. Frobenius reciprocity

Let $\lambda \in \chi(n)$ and $\alpha \in \chi(n+1)$ be given. By *Frobenius reciprocity* (see, e.g., [7]),

$$\mathrm{Hom}_{\mathrm{Spin}(n+1)}(\mathcal{V}(\alpha), \mathrm{Ind}_{\mathrm{Spin}(n)}^{\mathrm{Spin}(n+1)} V(\lambda)) \cong_{\mathrm{Spin}(n+1)} \mathrm{Hom}_{\mathrm{Spin}(n)}(\mathcal{V}(\alpha)|_{\mathrm{Spin}(n)}, V(\lambda)).$$

The *induced representation* $\mathrm{Ind}_{\mathrm{Spin}(n)}^{\mathrm{Spin}(n+1)} V(\lambda)$ is carried by $\mathcal{E}(\lambda)$, the space of $\mathrm{Spin}(n+1)$-finite sections of $\mathbb{V}(\lambda)$ on $\mathrm{Spin}(n+1)/\mathrm{Spin}(n) = S^n$. As a result,

$$\mathcal{E}(\lambda) \cong_{\mathrm{Spin}(n+1)} \bigoplus_{\alpha\downarrow\lambda} \mathcal{V}(\alpha). \tag{6}$$

For $\alpha \downarrow \lambda$, let $\mathcal{V}(\alpha;\lambda)$ be the copy of $\mathcal{V}(\alpha)$ appearing in this decomposition; then

$$\mathcal{E}(\lambda) = \bigoplus_{\alpha\downarrow\lambda} \mathcal{V}(\alpha;\lambda).$$

$\mathrm{Spin}(n)$-equivariant differential operators are $\mathrm{Spin}(n+1)$-invariant in the induced representation. Thus by Schur's Lemma, every $\mathrm{Spin}(n)$-equivariant differential operator D takes on an eigenvalue on each $\mathcal{V}(\alpha;\lambda)$:

$$D|_{\mathcal{V}(\alpha;\lambda)} = \mathrm{eig}(D,\alpha)\,\mathrm{id}_{\mathcal{V}(\alpha;\lambda)}\,.$$

8. Eigenvalue formulas

In [6], formulas for eigenvalues of intertwining operators on G/P (G a semisimple Lie group and P a maximal parabolic) are developed. In the present case, these results apply with $G = \mathrm{Spin}_0(n+1,1)$; then $G/P = S^n$ (as a conformal manifold). This special case was developed earlier, in [2], Sec. 5. A technical condition for the validity of the eigenvalue formulas involves the effect of multiplication by the *cocycle* of the principal series. Luckily, in the present setting, this condition has an elementary statement:

$$\alpha \leftrightarrow \beta,\ \alpha \neq \beta,\ \alpha,\beta \downarrow \lambda\ \Rightarrow\ \mathrm{Proj}_{\mathcal{V}(\beta;\lambda)}(x\mathcal{V}(\alpha;\lambda)) \neq 0$$

for x any of the homogeneous coordinate functions on S^n (i.e. the coordinates in the ambient $\mathbb{R}^{n+1}$). This condition was established for general λ in [4], Sec. 6. For ease of notation, we write the result here only for operators of nonnegative order; in particular, this covers all differential intertwinors.

Let $L := [(n+1)/2]$; this is the number of entries in a $\mathrm{Spin}(n+1)$-weight.

Theorem 8.1. *Suppose it is* **not** *the case that*

$$n \textit{ is even and } \lambda_\ell \neq 0. \tag{7}$$

Let $\mathrm{Re}(r) \geq 0$. *Then there exists a nonzero intertwinor* $\mathrm{Ind}_P^G \lambda \otimes (-r\nu_0) \otimes 1 \to \mathrm{Ind}_P^G \lambda \otimes r\nu_0 \otimes 1$, *and any such operator must have eigenvalue*

$$c \prod_{a=1}^{L} \frac{\Gamma(\tilde{\alpha}_a + \frac{1}{2} + r)}{\Gamma(\tilde{\alpha}_a + \frac{1}{2} - r)} \tag{8}$$

on $\mathcal{V}(\alpha;\lambda)$. *In particular, for* $r = \frac{1}{2}$, *the eigenvalue is*

$$c \prod_{a=1}^{L} \tilde{\alpha}_a. \tag{9}$$

Here c is a nonzero constant which is independent of α, and $\tilde{\alpha}$ is the *rho-shift* of α:

$$\tilde{\alpha} = \alpha + \rho_{n+1}, \qquad (\rho_{n+1})_a = (n+1-2a)/2.$$

ν_0 is the unique positive $(\mathfrak{g}, \mathfrak{a})$ root, MAN being the Langlands decomposition of the parabolic subgroup P; see [6] for details. Note that if poles of the Gamma function come into play in (8), the order of the pole of the denominator at r is at least the order of the pole of the numerator; thus the expression (8) is analytic in r in at least the half-plane $\mathrm{Re}(r) \geq -1/2$.

By the results of [8] on first-order differential intertwinors and the conformal covariance of gradients, (9) is not the spectrum of a differential operator unless we are in case (3). (If neither (7) nor (3) holds, the intertwinor for which (9) is the spectrum is nonlocal.) In case (3), by the uniqueness theorem of [4], Sec. 6, we have:

Corollary 8.2. *In case (3),*

$$\mathrm{eig}(D_\lambda^{\mathrm{self}}, \alpha) = c_\lambda \prod_{a=1}^{L} \tilde{\alpha}_a,$$

for some nonzero constant c_λ which is independent of α.

Using results of [4], we may determine the constant c_λ in Corollary 8.2 up to a factor of ± 1 (Theorem 8.4 below). By Sec. 6. above, this specifies the normalization to the full extent possible.

Given $\lambda \in \chi(n)$, let $\tilde{\lambda}$ be the rho-shift of λ:

$$\tilde{\lambda} = \lambda + \rho_n\,, \qquad (\rho_n)_a = n - 2a,$$

and let $\mathcal{T}(\lambda)$ be the set of component labels $a \in \{1, \ldots, L\}$ for which $\tilde{\alpha}_a^2$ is allowed, by the condition $\alpha \downarrow \lambda$, to take on more than one value.

Theorem 8.3. ([4], Theorems 4.1 and 5.2.) *Let $\lambda \in \chi(n)$ be arbitrary. The cardinality of $\mathcal{T}(\lambda)$ is $t(\lambda) = [(N(\lambda)+1)/2]$, and the eigenvalue of $G_u^* G_u$ on the $\mathcal{V}(\alpha)$ summand of $\mathcal{E}(\lambda)$ is*

$$\mu(\alpha; \lambda, \sigma_u) := \mathrm{eig}(G_u^* G_u, \alpha) = \tilde{c}_{\lambda\sigma_u} \prod_{a \in \mathcal{T}(\lambda)} \left(\tilde{\alpha}_a^2 - s_u^2\right), \tag{10}$$

where

$$s_u = \frac{1}{2}(|\tilde{\lambda}|^2 - |\tilde{\sigma}_u|^2),$$

and $\tilde{c}_{\lambda\sigma_u} =$

$$\frac{(-1)^{t(\lambda)+1}}{\prod_{\substack{1 \leq v \leq N(\lambda) \\ v \neq u}} (s_v - s_u)} \qquad \textit{if } N(\lambda) \textit{ is odd;} \tag{11}$$

$$\frac{(-1)^{t(\lambda)+1}}{2\prod_{u=1}^{N(\lambda)-2} (s_u + \frac{1}{2})} \qquad \textit{if } n \textit{ is even, } \lambda_\ell = 0 \neq \lambda_{\ell-1}\,,\ \sigma_u = \lambda \pm e_\ell\,; \tag{12}$$

$$\frac{(-1)^{t(\lambda)} \left(s_u + \frac{1}{2}\right)}{\prod_{\substack{1 \leq v \leq N(\lambda) \\ v \neq u}} (s_v - s_u)} \qquad \textit{otherwise.} \tag{13}$$

This shows that in case (3),

$$\mathrm{eig}(D_\lambda^{\mathrm{self}},\alpha)^2=\mathrm{eig}((D_\lambda^{\mathrm{self}})^2,\alpha)=\mu(\alpha;\lambda,\lambda)=\tilde{c}_{\lambda\lambda}\prod_{a\in\mathcal{T}_\lambda}\tilde{\alpha}_a^2.$$

This determines the constant in Corollary 8.2 up to sign:

$$c_\lambda^2=\tilde{c}_{\lambda\lambda}\prod_{a\notin\mathcal{T}_\lambda}\alpha_a^2.$$

(This is independent of α because for $a\notin\mathcal{T}_\lambda$, α_a^2 takes on only one value.) We have shown:

Theorem 8.4. *In case (3),*

$$\mathrm{eig}(D_\lambda^{\mathrm{self}},\alpha)=(\mathrm{sgn}\,\alpha_L)\sqrt{\mu(\alpha;\lambda,\lambda)}=(\mathrm{sgn}\,\alpha_L)\sqrt{\tilde{c}_{\lambda\lambda}}\prod_{a\in\mathcal{T}(\lambda)}|\tilde{\alpha}_a|. \tag{14}$$

In particular, for each real number η, the multiplicities of η and $-\eta$ as eigenvalues of $D_\lambda^{\mathrm{self}}$ are equal.

Remark 8.5. In the statement of the theorem, we have made a choice of sign in the definition of $D_\lambda^{\mathrm{self}}$, making $\mathrm{eig}(D_\lambda^{\mathrm{self}},\alpha)>0$ on the sphere when $\alpha_L>0$. In particular, we have chosen one of the two possible leading symbols for $D_\lambda^{\mathrm{self}}$ on the sphere. This choice may be transferred to arbitrary $\mathrm{Spin}(n)$-manifolds, as it is simply a choice of a connected component in the punctured real line of $\mathrm{Spin}(n)$ maps $V(1,0,\dots,0)\otimes V(\lambda)\to V(\lambda)$.

Note that we cannot write simply $\sqrt{\tilde{c}_{\lambda\lambda}}\prod_{a\in\mathcal{T}(\lambda)}\tilde{\alpha}_a$ on the right in (14), because of the case where $L\notin\mathcal{T}(\lambda)$; this happens exactly when $\lambda_\ell=\frac{1}{2}$. The statement about the multiplicities of $\pm\eta$ follows from the equality of $\dim\mathcal{V}(\alpha)$ and $\dim\mathcal{V}(\bar{\alpha})$, where

$$\bar{\alpha}=(\alpha_1,\dots,\alpha_{L-1},-\alpha_L).$$

Note that the eigenvalue formula (14) predicts that $\tilde{c}_{\lambda\lambda}$ should be positive. We can also obtain this sign directly, by a counting argument. Suppose (3) holds; then $s_{\mathrm{self}}=0$. If $N(\lambda)$ is odd, then half of the other s_v , that is $(N(\lambda)-1)/2=t(\lambda)-1$ of them, are negative; this and (11) show that $\tilde{c}_{\lambda\lambda}>0$. If $N(\lambda)$ is even, then $N(\lambda)/2=t(\lambda)$ of the other s_v are negative; this and (13) show that $\tilde{c}_{\lambda\lambda}>0$.

Remark 8.6. Equation (14), together with Corollary 8.2, shows that

$$c_\lambda=\begin{cases}\dfrac{\sqrt{\tilde{c}_{\lambda\lambda}}}{\prod_{a\notin\mathcal{T}(\lambda)}\tilde{\alpha}_a}, & \lambda_\ell\neq\frac{1}{2},\\[2ex] 2\dfrac{\sqrt{\tilde{c}_{\lambda\lambda}}}{\prod_{a\notin\mathcal{T}(\lambda)\cup\{L\}}\tilde{\alpha}_a}, & \lambda_\ell=\frac{1}{2},\end{cases}$$

since $\mathrm{sgn}\,\alpha_L=2\alpha_L$ in case $\lambda_\ell=\frac{1}{2}$. Note that all terms in the denominator on the right are necessarily nonzero: if $\tilde{\alpha}_a$ appears and vanishes, then $a=L$, and the condition $\alpha\downarrow\lambda$ constrains α_L to vanish; this can only happen if $\lambda_\ell=0$.

9. Reducible self-gradients

It is sometimes useful to expand the notion of self-gradient to reducible bundles. For example, in even dimensions, the Dirac operator interchanges the bundles $\Sigma_\pm$ of positive and negative spinors. Thus to speak of the *spectrum* of the Dirac operator, one should work on $\Sigma_+ \oplus \Sigma_-$.

If $\lambda \leftrightarrow \sigma \in \chi(n)$, let $\mathcal{G}_{\lambda\sigma}$ be the operator on sections of $\mathbb{V}(\lambda) \oplus \mathbb{V}(\sigma)$ given in block form by

$$\mathcal{G}_{\lambda\sigma} = \begin{pmatrix} 0 & G^*_{\lambda\sigma} \\ G_{\lambda\sigma} & 0 \end{pmatrix}.$$

This operator is formally self-adjoint, with square $\mathrm{diag}(G^*_{\lambda\sigma}G_{\lambda\sigma}, G_{\lambda\sigma}G^*_{\lambda\sigma})$.

Now specialize to the sphere S^n, where the $\mathrm{Spin}(n+1)$-finite section space of $\mathbb{V}(\lambda) \oplus \mathbb{V}(\sigma)$ is

$$\mathcal{E}(\lambda) \oplus \mathcal{E}(\sigma) = \left(\bigoplus_{\alpha \downarrow \lambda} \mathcal{V}(\alpha; \lambda) \right) \oplus \left(\bigoplus_{\beta \downarrow \sigma} \mathcal{V}(\beta; \sigma) \right). \tag{15}$$

Given $\alpha \downarrow \lambda$, either $G_{\lambda\sigma}|_{\mathcal{V}(\alpha;\lambda)} = 0$, or $\mu(\alpha; \lambda, \sigma) > 0$. In the latter case, for each $\varphi \in \mathcal{V}(\alpha; \lambda)$, we get $\mathcal{G}_{\lambda\sigma}$-eigensections with eigenvalues $\pm\sqrt{\mu(\alpha; \lambda, \sigma)}$, namely

$$\begin{pmatrix} \varphi \\ \pm\mu(\alpha; \lambda, \sigma)^{-1/2} G_{\lambda\sigma}\varphi \end{pmatrix} \tag{16}$$

In fact, by [4], Lemma 4.7, or inspection of the formula (10), $G_{\lambda\sigma}|_{\mathcal{V}(\alpha;\lambda)} = 0$ if and only if $\alpha \not\downarrow \sigma$ *or*

$$n \text{ is odd}, \ \alpha_L = 0, \ \lambda \text{ is integral}, \ \sigma = \lambda, \text{ and } \lambda_\ell \neq 0.$$

The zero eigenvalue when $\alpha \not\downarrow \sigma$ is required by Schur's Lemma, as is a zero eigenvalue on $\mathcal{V}(\beta; \sigma)$ when $\beta \not\downarrow \lambda$. With this collection of remarks, we have proved:

Theorem 9.1. *Let $\lambda \leftrightarrow \sigma$ in $\chi(n)$. A complete spectral resolution of $\mathcal{G}_{\lambda\sigma}$ is given by:*

- *Eigenvalue $\sqrt{\mu(\alpha; \lambda, \sigma)}$ on one summand* $\mathrm{Spin}(n+1)$*-isomorphic to $\mathcal{V}(\alpha)$ when $\alpha \downarrow \lambda$ and $\alpha \downarrow \sigma$;*
- *Eigenvalue $-\sqrt{\mu(\alpha; \lambda, \sigma)}$ on the $\mathcal{V}(\alpha)$ summand complementary to that above when $\alpha \downarrow \lambda$ and $\alpha \downarrow \sigma$;*
- *Eigenvalue* 0 *on summands $\mathcal{V}(\alpha; \lambda)$ when $\alpha \not\downarrow \sigma$ and on summands $\mathcal{V}(\beta; \sigma)$ when $\beta \not\downarrow \lambda$.*

In particular, for each real number η, the multiplicities of η and $-\eta$ as eigenvalues of $\mathcal{G}_{\lambda\sigma}$ are equal.

Note that the first two cases cover some of the zero eigenvalue situations. The $\mathrm{Spin}(n+1)$-types α with $\alpha \downarrow \lambda$ and $\alpha \downarrow \sigma$ occur with multiplicity 2 in the decomposition (15); there are infinitely many splittings of this isotypic submodule as $\mathcal{V}(\alpha) \oplus \mathcal{V}(\alpha)$; (16) is the unique choice of such a splitting with projections that commute with $\mathcal{G}_{\lambda\sigma}$. The nonzero spectrum of $\mathcal{G}_{\sigma\lambda}$ is a constant multiple of the nonzero spectrum of $\mathcal{G}_{\lambda\sigma}$; in fact, the constant is $\sqrt{\tilde{c}_{\sigma\lambda}/\tilde{c}_{\lambda\sigma}}$. ($\tilde{c}_{\lambda\sigma}$ may be positive or negative, but (10) shows that $\operatorname{sgn}\tilde{c}_{\sigma\lambda} = \operatorname{sgn}\tilde{c}_{\lambda\sigma}$.)

An aspect of Theorem 9.1 that is inconvenient in some applications is that it calls for a realization of $\mathbb{V}(\sigma)$ as a subbundle of $T^*M \otimes \mathbb{V}(\lambda)$. If some other realization is more desirable, and if $\lambda \not\leftrightarrow \lambda$, $\sigma \not\leftrightarrow \sigma$, we may remedy this situation as follows. Choose realizations for $\mathbb{V}(\lambda)$ and $\mathbb{V}(\sigma)$, and choose an isometric, $\mathrm{Spin}(n)$-equivariant bundle injection

$$\iota : \mathbb{V}(\lambda) \oplus \mathbb{V}(\sigma) \to T^*M \otimes (\mathbb{V}(\lambda) \oplus \mathbb{V}(\sigma)).$$

By Schur's Lemma, the range of ι is the (unique, by $\lambda \not\leftrightarrow \lambda$ and $\sigma \not\leftrightarrow \sigma$) $(\mathbb{V}(\lambda) \oplus \mathbb{V}(\sigma))$-isomorphic summand of the target. The operator

$$D^{\mathrm{self}}_{\mathbb{V}(\lambda)\oplus\mathbb{V}(\sigma)} := \begin{pmatrix} 0 & G^*_{\lambda\sigma}\circ\iota \\ \iota^*\circ G_{\lambda\sigma} & 0 \end{pmatrix} \tag{17}$$

is then a realization of $\mathcal{G}_{\lambda\sigma}$ which acts on the "original" realization of $\mathbb{V}(\lambda)\oplus\mathbb{V}(\sigma)$. If we choose the same isometric injection ι for the construction of $D^{\mathrm{self}}_{\mathbb{V}(\lambda)\oplus\mathbb{V}(\sigma)}$ as for $D^{\mathrm{self}}_{\mathbb{V}(\sigma)\oplus\mathbb{V}(\lambda)}$, we get the same operator, after the appropriate permutation of blocks in (17); this justifies the notation. We have:

Theorem 9.2. *Suppose $\lambda \not\leftrightarrow \lambda \leftrightarrow \sigma \not\leftrightarrow \sigma$ in $\chi(n)$. A complete spectral resolution of $D^{\mathrm{self}}_{\mathbb{V}(\lambda)\oplus\mathbb{V}(\sigma)}$ is given by:*

- *Eigenvalue $\sqrt{\mu(\alpha;\lambda,\sigma)}$ on one summand* $\mathrm{Spin}(n+1)$*-isomorphic to $\mathcal{V}(\alpha)$ when $\alpha \downarrow \lambda$ and $\alpha \downarrow \sigma$;*
- *Eigenvalue $-\sqrt{\mu(\alpha;\lambda,\sigma)}$ on the $\mathcal{V}(\alpha)$ summand complementary to that above when $\alpha \downarrow \lambda$ and $\alpha \downarrow \sigma$;*
- *Eigenvalue 0 on $\mathcal{V}(\alpha)$ summands when $\alpha \downarrow \lambda$ and $\alpha \not\downarrow \sigma$, or $\alpha \downarrow \sigma$ and $\alpha \not\downarrow \lambda$.*

In particular, for each real number η, the multiplicities of η and $-\eta$ as eigenvalues of $\mathcal{G}_{\lambda\sigma}$ are equal.

10. Examples

Example 10.1. Let $n \geq 3$ be odd, and consider the self-gradient on $\Lambda^{(n-1)/2} \cong_{\mathrm{Spin}(n)} \mathbb{V}(1,\dots,1)$. By the branching rule, on the sphere S^n,

$$\mathcal{E}(\Lambda^{(n-1)/2}) \cong_{\mathrm{Spin}(n+1)} \bigoplus_{\substack{j\in\mathbb{N} \\ q\in\{0,\pm1\}}} \mathcal{V}(\alpha_{j,q}),$$

where

$$\alpha_{j,q} := (1+j, 1, \dots, 1, q).$$

There are three gradient target bundles with source bundle $\Lambda^{(n-1)/2}$, namely $\Lambda^{(n-1)/2}$ itself ($\cong_{\mathrm{Spin}(n)} \Lambda^{(n+1)/2}$), $\Lambda^{(n-3)/2}$, and $\mathbf{F} \cong_{\mathrm{Spin}(n)} \mathbb{V}(2,1,\dots,1)$. We have

$$s_{\Lambda^{(n-1)/2}\Lambda^{(n-1)/2}} = 0, \qquad s_{\Lambda^{(n-1)/2}\Lambda^{(n-3)/2}} = 1, \qquad s_{\Lambda^{(n-1)/2}\mathbf{F}} = -(n+1)/2,$$

from which

$$\tilde{c}_{\Lambda^{(n-1)/2}\Lambda^{(n-1)/2}} = \frac{2}{n+1}.$$

By Theorem 8.4,

$$\mathrm{eig}(D^{\mathrm{self}}_{\Lambda^{(n-1)/2}}, \alpha_{j,q}) = q\sqrt{\frac{2}{n+1}}\left(j + \frac{n+1}{2}\right). \tag{18}$$

The operator $(\star d)^2$ on $\Lambda^{(n-1)/2}$ agrees with $(-1)^{n(n+1)/2+1}\delta d$, where d is the exterior derivative, and δ is the formal adjoint of d in the usual form metric

$$(\varphi, \psi)_{\Lambda^k} = \frac{1}{k!} g^{a_1 b_1} \dots g^{a_k b_k} \varphi_{a_1 \dots a_k} \psi_{b_1 \dots b_k}. \tag{19}$$

Thus $\sqrt{-1}^{n(n+1)/2+1} \star d$ is a real nonzero multiple of $D^{\mathrm{self}}_{\Lambda^k}$. Comparing the normalizations (19) and (5), we find that

$$d|_{\Lambda^k} = (k+1) G_{\Lambda^k \Lambda^{k+1}}, \qquad \delta|_{\Lambda^{k+1}} = G^*_{\Lambda^k \Lambda^{k+1}},$$

so that

$$(\delta d)_{\Lambda^{(n-1)/2}} = \frac{n+1}{2} (D^{\mathrm{self}}_{\Lambda^{(n-1)/2}})^2.$$

This explains the factor of $\sqrt{2/(n+1)}$ in (18), and gives

$$\mathrm{eig}(\sqrt{-1}^{n(n+1)/2+1} (\star d)_{\Lambda^{(n-1)/2}}, \alpha_{j,q}) = q\left(j + \frac{n+1}{2}\right).$$

Theorem 9.2 is of special interest in the case in which n is even, $\lambda_\ell = \frac{1}{2}$, $\sigma_\ell = -\frac{1}{2}$, and $\sigma_a = \lambda_a$ for $a < \ell$. Examples of $\mathcal{G}_{\lambda\sigma}$ in this case are the even-dimensional Dirac and Rarita-Schwinger operators, and generalizations to certain bundles of spinor-forms first studied in [1]. When n is odd, the Dirac, Rarita-Schwinger, and spinor-form self-gradients are covered by Theorem 8.4.

Example 10.2. Let $n \geq 3$ be odd. The spinor bundle Σ is a copy of $\mathbb{V}(\frac{1}{2}, \dots, \frac{1}{2})$. On the sphere, by (6),

$$\mathcal{E}(\Sigma) \cong_{\mathrm{Spin}(n+1)} \bigoplus_{j=0}^{\infty} \left(\mathcal{V}(\tfrac{1}{2}+j, \tfrac{1}{2}, \dots, \tfrac{1}{2}, \tfrac{1}{2}) \oplus \mathcal{V}(\tfrac{1}{2}+j, \tfrac{1}{2}, \dots, \tfrac{1}{2}, -\tfrac{1}{2})\right).$$

The Dirac operator $\nabla\!\!\!/$ must be, up to a constant factor, the realization of the self-gradient D^{self}_Σ, which, by Theorem 8.4, takes the eigenvalue

$$\sqrt{\tilde{c}_{\Sigma\Sigma}}\tilde{\alpha}_1$$

on $\mathcal{V}(\alpha; \Sigma)$. (We abuse notation in the obvious way by writing Σ instead of its highest weight in expressions like $\tilde{c}_{\Sigma\Sigma}$, $\mathcal{V}(\alpha;\Sigma)$, etc.) There two gradient targets, Σ itself, and the twistor bundle $\mathbb{T} \cong_{\mathrm{Spin}(n)} \mathbb{V}(\frac{3}{2}, \frac{1}{2}, \dots, \frac{1}{2})$. Since $\tilde{c}_{\Sigma\Sigma} = 1/n$ by (13), we have

$$\mathrm{eig}(D^{\mathrm{self}}_\Sigma, (\tfrac{1}{2}+j, \tfrac{1}{2}, \dots, \tfrac{1}{2}, \pm\tfrac{1}{2})) = \pm\frac{1}{\sqrt{n}}\left(\frac{n}{2} + j\right).$$

By [3], Sec. 5, the eigenvalues of the classical Dirac operator $\nabla\!\!\!/ = \gamma^a\nabla_a$ (in abstract index notation) on the sphere are $\pm(\frac{n}{2}+j)$. The factor $1/\sqrt{n}$ which emerges above from the general results can also be seen classically, as follows. The $\mathbb{T}$ subbundle of $T^*M\otimes\Sigma$ consists of the spinor one-forms φ_a for which $\gamma^a\varphi_a=0$. For ψ a spinor, $\nabla_a\psi$ decomposes as

$$\nabla_a\psi = \left(\nabla_a\psi + \frac{1}{n}\gamma_a\gamma^b\nabla_b\psi\right) + \left(-\frac{1}{n}\gamma_a\gamma^b\nabla_b\psi\right) \tag{20}$$

into $\mathbb{T}$ and Σ parts; thus

$$(G_{\Sigma\Sigma}\psi)_a = -\frac{1}{n}\gamma_a\gamma^b\nabla_b\psi.$$

This gives

$$G^*_{\Sigma\Sigma}\varphi = \frac{1}{n}\gamma^b\gamma^a\nabla_b\varphi_a, \qquad G^*_{\Sigma\Sigma}G_{\Sigma\Sigma} = \frac{1}{n}\nabla\!\!\!/^{\,2},$$

since $\gamma^a\gamma_a=-n$. Thus

$$D^{\text{self}}_{\Sigma} = \nabla\!\!\!/ / \sqrt{n}. \tag{21}$$

Note that both sides of (21) are subject to a sign ambiguity (recall Sec. 6.).

Example 10.3. In even dimensions, the Dirac operator carries sections of

$$\Sigma_+ \cong_{\text{Spin}(n)} V(\tfrac{1}{2},\dots,\tfrac{1}{2})$$

to sections of

$$\Sigma_- \cong_{\text{Spin}(n)} V(\tfrac{1}{2},\dots,\tfrac{1}{2},-\tfrac{1}{2})$$

and *vice versa.* Let $\Sigma=\Sigma_+\oplus\Sigma_-$. To get the spectrum of D^{self}_{Σ} on S^n using Theorem 9.2, note that

$$\mathcal{E}(\Sigma_+) \cong_{\text{Spin}(n+1)} \mathcal{E}(\Sigma_-) \cong_{\text{Spin}(n+1)} \bigoplus_{j=0}^{\infty} \mathcal{V}(\tfrac{1}{2}+j,\tfrac{1}{2},\dots,\tfrac{1}{2}).$$

There are two gradient target bundles with source bundle Σ_+, namely Σ_- and the positive twistor bundle $\mathbb{T}_+ \cong_{\text{Spin}(n)} \mathbb{V}(\tfrac{3}{2},\tfrac{1}{2},\dots,\tfrac{1}{2})$. We have

$$s_{\Sigma_-}=0, \qquad s_{\mathbb{T}_+}=-n/2,$$

and by (13), $\tilde{c}_{\Sigma_+\Sigma_-}=1/n$. Setting

$$\alpha_j := (\tfrac{1}{2}+j,\tfrac{1}{2},\dots,\tfrac{1}{2}),$$

$\mathcal{G}_{\Sigma_+\Sigma_-}$ takes the eigenvalues

$$\pm\sqrt{\mu(\alpha_j;\Sigma_+,\Sigma_-)} = \pm\frac{1}{\sqrt{n}}\left(\frac{n}{2}+j\right)$$

on two complementary summands in the isotypic submodule $\mathcal{V}(\alpha_j;\Sigma_+)\oplus\mathcal{V}(\alpha_j;\Sigma_-)$. Switching the roles of Σ_+ and Σ_-, the calculation is the same. (20) gives the decomposition of a spinor (i.e. a section of $\Sigma_+\oplus\Sigma_-$) into $\mathbb{T}_+\oplus\mathbb{T}_-$ and $\Sigma_+\oplus\Sigma_-$ parts, and a calculation similar to the one in Example 10.2 explains the factor of $1/\sqrt{n}$.

Example 10.4. Let $n \geq 3$ be odd, and consider the twistor bundle

$$\mathbb{T} \cong_{\mathrm{Spin}(n)} \mathbb{V}(\tfrac{3}{2}, \tfrac{1}{2}, \ldots, \tfrac{1}{2})$$

defined above. If $n \geq 5$, there are four gradient targets: $\mathbb{T}$ itself, Σ, $\mathbf{Z} := \mathbb{V}(\frac{5}{2}, \frac{1}{2}, \ldots, \frac{1}{2})$, and $\mathbb{T}^2 := \mathbb{V}(\frac{3}{2}, \frac{3}{2}, \frac{1}{2}, \ldots, \frac{1}{2})$. (See Example 10.6 below for more on the series $\mathbb{T}^k$.) If $n = 3$, the last of these targets is missing. As mentioned in Example 10.2, in abstract index notation, $\mathbb{T}$ is realized in spinor-one-forms φ annihilated by interior multiplication by γ:

$$\gamma^a \varphi_a = 0.$$

After decomposing $\nabla\varphi$ into its components in these bundles, one gets

$$\begin{aligned}(G_{\mathbb{T}\mathbb{T}}\varphi)_{ab} &= \frac{1}{(n+2)(n-2)}\Big\{ -n\gamma_a\gamma^c\nabla_c\varphi_b + 2\gamma_b\gamma^c\nabla_c\varphi_a \\ &\qquad +2\gamma_a\gamma_b\nabla^c\varphi_c - \frac{4}{n}\gamma_b\gamma_a\nabla^c\varphi_c\Big\}.\end{aligned} \tag{22}$$

(See [5] for details of the calculation.)

The *Rarita-Schwinger operator* is the self-gradient $D_{\mathbb{T}}^{\mathrm{self}}$ corresponding to the gradient $G_{\mathbb{T}\mathbb{T}}$. To get a formula for $D_{\mathbb{T}}^{\mathrm{self}}$, we need an explicit identification of the $\mathbb{T}$ summand in $T^* \otimes \mathbb{T}$ with the "original" realization of $\mathbb{T}$ as spinor-one-forms φ_a with $\gamma^a\varphi_a = 0$. For this, consider the spinor-two-tensor $\gamma_a\varphi_b$, and project to the $\mathbb{T}$ summand in $T^*M \otimes \mathbb{T}$:

$$\mathrm{Proj}_{\mathbb{T}}(\gamma_a\varphi_b) = \frac{n}{(n+2)(n-2)}(n\gamma_a\varphi_b - 2\gamma_b\varphi_a) =: -\frac{n}{(n+2)(n-2)}(\tilde{\iota}\varphi)_{ab}.$$

Note that $\tilde{\iota}$ is injective, with adjoint

$$(\tilde{\iota}^*\varphi)_a = n\gamma^b\varphi_{ba}.$$

Since $\tilde{\iota}^*\tilde{\iota} = n(n+2)(n-2)$, the maps

$$\iota = \{n(n+2)(n-2)\}^{-1/2}\tilde{\iota}, \qquad \iota^* = \{n(n+2)(n-2)\}^{-1/2}\tilde{\iota}^*$$

are isometric bundle isomorphisms of the original $\mathbb{T}$ realization and the $\mathbb{T}$-isomorphic summand in $T^*M \otimes \mathbb{T}$. Thus we get a concrete realization of the Rarita-Schwinger operator as

$$D_{\mathbb{T}}^{\mathrm{self}} = \iota^* \circ G_{\mathbb{T}\mathbb{T}}.$$

This and the above formulas give

$$(D_{\mathbb{T}}^{\mathrm{self}}\varphi)_a = \{n(n+2)(n-2)\}^{-1/2}\left(n\gamma^b\nabla_b\varphi_a - 2\gamma_a\nabla^b\varphi_b\right). \tag{23}$$

For the spectrum of $D_{\mathbb{T}}^{\mathrm{self}}$ on the sphere S^n, note first that the branching rule gives

$$\mathcal{E}(\mathbb{T}) \cong_{\mathrm{Spin}(n+1)} \bigoplus_{\substack{j\in\mathbb{N}\\ q\in\{0,1\}}} \left(\mathcal{V}(\alpha_{j,q,+}) \oplus \mathcal{V}(\alpha_{j,q,-})\right),$$

where

$$\alpha_{j,q\pm} := \begin{cases} (\frac{3}{2}+j, \frac{1}{2}+q, \frac{1}{2}, \ldots, \frac{1}{2}, \pm\frac{1}{2}), & n > 3, \\ (\frac{3}{2}+j, \pm(\frac{1}{2}+q)), & n = 3. \end{cases}$$

We have

$$s_\Sigma = \frac{n}{2},\ s_{\mathbb{T}} = 0,\ s_{\mathbb{T}^2} = -\frac{n-2}{2}\ (n>3),\ s_{\mathbf{Z}} = -\frac{n+2}{2},\ \tilde{c}_{\mathbb{T}\mathbb{T}} = \frac{4}{n(n+2)(n-2)}.$$

By Theorem 8.4,

$$\mathrm{eig}(D^{\mathrm{self}}_{\mathbb{T}}, \alpha_{j,q,\pm}) = \frac{2}{\sqrt{n(n+2)(n-2)}}\left(j+\frac{n}{2}+1\right)\left(q+\frac{n}{2}-1\right).$$

Thus if $\mathcal{S}$ is the non-normalized Rarita-Schwinger operator

$$(\mathcal{S}\varphi)_a := n\gamma^b\nabla_b\varphi_a - 2\gamma_a\nabla^b\varphi_b, \tag{24}$$

we have

$$\mathrm{eig}(\mathcal{S}, \alpha_{j,q,\pm}) = \pm 2\left(j+\frac{n}{2}+1\right)\left(q+\frac{n}{2}-1\right).$$

Example 10.5. Let $n \geq 4$ be even. The twistor bundle splits as $\mathbb{T} = \mathbb{T}_+ \oplus \mathbb{T}_-$, where

$$\mathbb{T}_\pm \cong_{\mathrm{Spin}(n)} \mathbb{V}(\tfrac{3}{2}, \tfrac{1}{2}, \dots, \tfrac{1}{2}, \pm\tfrac{1}{2}).$$

The right side of (22) gives a formula for $G_{\mathbb{T}_+\mathbb{T}_-}$ (when φ is a section of $\mathbb{T}_+$) and for $G_{\mathbb{T}_-\mathbb{T}_+}$ (when φ is a section of $\mathbb{T}_-$). Equation (23) holds (with the meaning of $D^{\mathrm{self}}_{\mathbb{T}}$ given in (17)).

For the spectrum of $D^{\mathrm{self}}_{\mathbb{T}}$, note that on the sphere S^n, the branching rule gives

$$\mathcal{E}(\mathbb{T}_+) \cong_{\mathrm{Spin}(n+1)} \mathcal{E}(\mathbb{T}_-) \cong_{\mathrm{Spin}(n+1)} \bigoplus_{\substack{j\in\mathbb{N}\\ q\in\{0,1\}}} \mathcal{V}(\alpha_{j,q}),$$

where

$$\alpha_{j,q} := (\tfrac{3}{2}+j, \tfrac{1}{2}+q, \tfrac{1}{2}, \dots, \tfrac{1}{2}).$$

Let $\mathcal{I}_{j,q}$ be the multiplicity 2 isotypic component $\mathcal{V}(\alpha_{j,q};\mathbb{T}_+)\oplus\mathcal{V}(\alpha_{j,q};\mathbb{T}_-)$ of $\mathcal{E}(\mathbb{T})$. Theorem 9.2 applies, and gives

$$D^{\mathrm{self}}_{\mathbb{T}}|_{\mathcal{I}_{j,q}} = \frac{2}{\sqrt{n(n+2)(n-2)}}\left(j+\frac{n}{2}+1\right)\left(q+\frac{n}{2}-1\right)\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix},$$

relative to some $\mathrm{Spin}(n+1)$-invariant block decomposition of $\mathcal{I}_{j,q}$ as $\mathcal{V}(\alpha_{j,q}) \oplus \mathcal{V}(\alpha_{j,q})$. (Note that this is *not* the decomposition $\mathcal{V}(\alpha_{j,q};\mathbb{T}_+)\oplus\mathcal{V}(\alpha_{j,q};\mathbb{T}_-)$, but rather the one described in the argument leading to Theorem 9.1, as realized in the argument leading to Theorem 9.2.) For the non-normalized Rarita-Schwinger operator $\mathcal{S}$ of (24),

$$\mathcal{S}|_{\mathcal{I}_{j,q}} = 2\left(j+\frac{n}{2}+1\right)\left(q+\frac{n}{2}-1\right)\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix},$$

Example 10.6. This example contains Examples 10.2 and 10.4 as special cases. Let $n \geq 3$ be odd, and consider the bundle $\mathcal{S}^k$ of *spinor-k-forms* as in [1]. If $k < n/2$, this is a copy of

$$\mathbb{V}(\tfrac{1}{2}, \dots, \tfrac{1}{2}) \otimes \mathbb{V}(\underbrace{1, \dots, 1}_{k}, 0, \dots, 0).$$

This bundle is generally reducible, but the *Cartan summand*, i.e. the summand

$$\mathbb{V}(\underbrace{\tfrac{3}{2},\dots,\tfrac{3}{2}}_{k},\tfrac{1}{2},\dots,\tfrac{1}{2})$$

containing the highest weight vector (which necessarily occurs in the tensor product with multiplicity one) consists of spinor-forms $\varphi_{a_1\dots a_k}$ with $\gamma^{a_1}\varphi_{a_1\dots a_k}=0$. We denote this bundle by $\mathbb{T}^k$; note that $\mathbb{T}_0=\Sigma$ and $\mathbb{T}^1=\mathbb{T}$.

The following operators, defined in [1], are notationally convenient. Let φ be a section of $\mathcal{S}^k$, and put

$$\begin{aligned}
(\tilde{d}\varphi)_{a_0\dots a_k} &= \textstyle\sum_{i=0}^k(-1)^i\nabla_{a_i}\varphi_{a_0\dots a_{i-1}a_{i+1}\dots a_k},\\
(\tilde{\delta}\varphi)_{a_2\dots a_k} &= -\nabla^b\varphi_{ba_2\dots a_k},\\
(\varepsilon(\gamma)\varphi)_{a_0\dots a_k} &= \textstyle\sum_{i=0}^k(-1)^i\gamma_{a_i}\varphi_{a_0\dots a_{i-1}a_{i+1}\dots a_k},\\
(\iota(\gamma)\varphi)_{a_2\dots a_k} &= \gamma^b\varphi_{ba_2\dots a_k}.
\end{aligned}$$

Note that $\mathbb{T}^k$ as defined above is the null space of $\iota(\gamma)$. (The lower weight summands in $\mathcal{S}^k$ may be distinguished by computing the null spaces of powers $\iota(\gamma)^h$; see [12].) [1] shows that the operator

$$P_k:=\frac{n-2k+4}{2}\iota(\gamma)\tilde{d}+\frac{n-2k}{2}(\tilde{d}\iota(\gamma)-\tilde{\delta}\varepsilon(\gamma))-\frac{n-2k-4}{2}\varepsilon(\gamma)\tilde{\delta} \tag{25}$$

is conformally covariant: if we change the metric conformally, $\bar{g}=\Omega^2 g$ for $0<\Omega\in C^\infty(M)$, and make the compatible change in the fundamental tensor-spinor, $\bar{\gamma}=\Omega^{-1}\gamma$, then

$$P_k(\Omega^{(n-2k-1)/2}\varphi)=\Omega^{(n-2k+2)/2}\bar{P}_k\varphi. \tag{26}$$

By uniqueness of the first-order conformal covariant on irreducible bundles [8], the compression $\mathrm{Proj}_{\mathbb{T}^k}P_k|_{\mathbb{T}^k}$ of P_k to an operator on sections of $\mathbb{T}^k$ is a constant multiple of the self-gradient $D^{\mathrm{self}}_{\mathbb{T}_k}$. (This might *a priori* be the zero multiple, but it is not, as we shall show presently.) Moreover, since the conformal biweights of gradients from $\mathbb{T}^k$ to other bundles differ from that in (26) (see [8]), this compression is equal to the restriction: $P_k|_{\mathbb{T}^k}=a_kD^{\mathrm{self}}_{\mathbb{T}_k}$. The constant a_k is *universal*; i.e. independent of the particular n-dimensional Riemannian spin manifold on which we realize these operators.

On the sphere S^n,

$$\mathcal{E}(\mathbb{T}^k)\cong_{\mathrm{Spin}(n+1)}\bigoplus_{\substack{j\in\mathbb{N}\\ q\in\{0,1\}}}(\mathcal{V}(\alpha_{k,j,q,+})\oplus\mathcal{V}(\alpha_{k,j,q,-}))$$

where

$$\alpha_{k,j,q,\pm}=\begin{cases}(\tfrac{3}{2}+j,\tfrac{3}{2},\dots,\tfrac{3}{2},\underbrace{\tfrac{1}{2}+q}_{(k+1)^{\mathrm{st}}},\tfrac{1}{2},\dots,\tfrac{1}{2},\pm\tfrac{1}{2}), & k<(n-1)/2,\\[2ex] (\tfrac{3}{2}+j,\tfrac{3}{2},\dots,\tfrac{3}{2},\pm(\tfrac{1}{2}+q)), & k=(n-1)/2.\end{cases}$$

Let

$$(D\varphi)_{a_1\dots a_k}=\gamma^b\nabla_b\varphi_{a_1\dots a_k}.$$

Some short calculations with the definitions give

$$\begin{aligned} \iota(\gamma)\tilde{d} + \tilde{d}\iota(\gamma) &= D && (27) \\ \tilde{\delta}\iota(\gamma) + \iota(\gamma)\tilde{\delta} &= 0 && (28) \end{aligned}$$

Since

$$\tilde{d}^* = \tilde{\delta}, \qquad \varepsilon(\gamma)^* = -\iota(\gamma), \qquad D^* = D,$$

we also have, by taking formal adjoints in (27),

$$\tilde{\delta}\varepsilon(\gamma) + \varepsilon(\gamma)\tilde{\delta} = -D \tag{29}$$

As a result,

$$a_k D^{\rm self}_{\mathbb{T}^k} = P_k|_{\mathbb{T}^k} = (n-2k+2)D + 2\varepsilon(\gamma)\tilde{\delta}. \tag{30}$$

Compare this to (21) and (23); this shows in particular that

$$a_0 = (n+2)\sqrt{n}, \qquad a_1 = \sqrt{n(n+2)(n-2)}. \tag{31}$$

Since $\tilde{\delta}D - D\tilde{\delta}$ and $\tilde{\delta}\tilde{\delta}$ are $0^{\underline{\rm th}}$ order curvature terms, (29) and (28) (in particular, the fact that $\tilde{\delta}$ carries $\mathbb{T}^k$ to $\mathbb{T}^{k-1}$) give

$$\begin{aligned} \tilde{\delta}P_k &= (n-2k+2)\tilde{\delta}D\varphi + 2\tilde{\delta}\varepsilon(\gamma)\tilde{\delta} \\ &= (n-2k)D\tilde{\delta} + (0^{\underline{\rm th}} \text{ order curvature terms}) \\ &= \frac{n-2k}{n-2k+4}P_{k-1}\tilde{\delta} + (0^{\underline{\rm th}} \text{ order curvature terms}) \end{aligned}$$

on $\mathbb{T}^k$. But $\tilde{\delta}$, being $\mathrm{Spin}(n+1)$-invariant, carries $\mathcal{V}(\alpha_{k,j,0,+};\mathbb{T}^k)$ to $\mathcal{V}(\alpha_{k-1,j,1,+};\mathbb{T}^{k-1})$, since $\alpha_{k,j,0,+} = \alpha_{k-1,j,1,+}$. By [4], Lemma 4.7 the restriction of $\tilde{\delta}$ to $\mathcal{V}(\alpha_{k,j,0,+};\mathbb{T}^k)$ is nonzero. Thus

$$\mathrm{eig}(a_k D^{\rm self}_{\mathbb{T}^k}, \alpha_{k,j,0,+}) = \frac{n-2k}{n-2k+4}\,\mathrm{eig}(a_{k-1}D^{\rm self}_{\mathbb{T}^{k-1}}, \alpha_{k-1,j,1,+}) + O(j^0). \tag{32}$$

By (14), the $O(j^0)$ term must vanish.

If $k < (n-1)/2$, there are four gradient target bundles with source bundle $\mathbb{T}^k$, namely

$$\mathbb{T}^{k-1},\ \mathbb{T}^k, \mathbb{T}^{k+1},\ \mathbf{Z}^k \cong_{\mathrm{Spin}(n)} \mathbb{V}(\tfrac{5}{2}, \underbrace{\tfrac{3}{2}, \ldots, \tfrac{3}{2}}_{k-1}, \tfrac{1}{2}, \ldots, \tfrac{1}{2}).$$

If $k = (n-1)/2$, the summand $\mathbb{T}^{k+1}$ is missing. (If we define $\mathbb{T}^{k+1}$ as the spinor-$(k+1)$-forms φ with $\iota(\gamma)\varphi = 0$, we get another $\mathrm{Spin}(n)$-isomorphic copy of $\mathbb{T}^{k-1}$.) We have

$$\begin{aligned} s_{\mathbb{T}^{k-1}} &= (n-2k+2)/2, & s_{\mathbb{T}^k} &= 0, \\ s_{\mathbb{T}^{k+1}} &= -(n-2k)/2\ (k < (n-1)/2), & s_{\mathbf{Z}^k} &= -(n+2)/2. \end{aligned}$$

By (13) (or (11) if $k = (n-1)/2$),

$$\tilde{c}_{\mathbb{T}^k\mathbb{T}^k} = \frac{4}{(n-2k+2)(n+2)(n-2k)}.$$

By (14),

$$\operatorname{eig}(D^{\mathrm{self}}_{\mathbb{T}^k}, \alpha_{k,j,q,\pm}) = \pm\frac{2}{\sqrt{(n-2k+2)(n+2)(n-2k)}}\left(j+\tfrac{n}{2}+1\right)\left(q+\tfrac{n}{2}-k\right). \tag{33}$$

By (33) and (32),

$$a_k = \sqrt{\frac{n-2k}{n-2k+4}}a_{k-1}.$$

(Compare (31).) Computing inductively, we have

$$a_k = \sqrt{\frac{(n-2k+2)(n-2k)}{(n+2)n}}\,a_0 = \sqrt{(n-2k+2)(n+2)(n-2k)}.$$

This and (33) give

$$\operatorname{eig}(P_k|_{\mathbb{T}^k}, \alpha_{k,j,q,\pm}) = \pm 2\left(j+\frac{n}{2}+1\right)\left(q+\frac{n}{2}-k\right)$$

Example 10.7. This example contains Examples 10.3 and 10.5 as special cases. Let $n \geq 4$ be even. For $k < n/2$, the bundle of spinor-k-forms φ with $\iota(\gamma)\varphi = 0$ splits as $\mathbb{T}^k = \mathbb{T}^k_+ \oplus \mathbb{T}^k_-$, where

$$\mathbb{T}^k_\pm \cong_{\mathrm{Spin}(n)} \mathbb{V}(\underbrace{\tfrac{3}{2},\dots,\tfrac{3}{2}}_{k},\tfrac{1}{2},\dots,\tfrac{1}{2},\pm\tfrac{1}{2}).$$

(30), or (25) restricted to $\mathbb{T}^k$, gives a formula for a conformally covariant operator $\mathbb{T}^k_\pm \to \mathbb{T}^k_\mp$. By a calculation similar to that in Example 10.6,

$$P_k|_{\mathbb{T}^k} = \sqrt{(n-2k+2)(n+2)(n-2k)}D^{\mathrm{self}}_{\mathbb{T}^k},$$

where in this case, $D^{\mathrm{self}}_{\mathbb{T}^k}$ is defined by (17).

On the sphere S^n,

$$\mathcal{E}(\mathbb{T}^k_+) \cong_{\mathrm{Spin}(n+1)} \mathcal{E}(\mathbb{T}^k_-) \cong_{\mathrm{Spin}(n+1)} \bigoplus_{\substack{j\in\mathbb{N}\\ q\in\{0,1\}}} \mathcal{V}(\alpha_{k,j,q}),$$

where

$$\alpha_{k,j,q} := (\tfrac{3}{2}+j,\tfrac{3}{2},\dots,\tfrac{3}{2},\underbrace{\tfrac{1}{2}+q}_{(k+1)^{\underline{\mathrm{st}}}},\tfrac{1}{2},\dots,\tfrac{1}{2}).$$

Let $\mathcal{I}_{k,j,q}$ be the multiplicity 2 isotypic component $\mathcal{V}(\alpha_{k,j,q};\mathbb{T}_+) \oplus \mathcal{V}(\alpha_{k,j,q};\mathbb{T}_-)$ of $\mathcal{E}(\mathbb{T}^k)$. By Theorem 9.2,

$$D^{\mathrm{self}}_{\mathbb{T}^k}|_{\mathcal{I}_{k,j,q}} = \frac{2}{\sqrt{(n-2k+2)(n+2)(n-2k)}}\left(j+\frac{n}{2}+1\right)\left(q+\frac{n}{2}-k\right)\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix},$$

relative to some $\mathrm{Spin}(n+1)$-invariant block decomposition of $\mathcal{I}_{k,j,q}$ as $\mathcal{V}(\alpha_{k,j,q}) \oplus \mathcal{V}(\alpha_{k,j,q})$. As a result, relative to the same decomposition,

$$(P_k|_{\mathbb{T}^k})|_{\mathcal{I}_{k,j,q}} = 2\left(j+\frac{n}{2}+1\right)\left(q+\frac{n}{2}-k\right)\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}.$$

References

[1] Branson, T., *Intertwining differential operators for spinor-form representations of the conformal group*, Advances in Math. **54** (1984), 1–21.

[2] —, *Second order conformal covariants*, University of Copenhagen Mathematical Institute preprints #2,3 (1989).

[3] —, *Harmonic analysis in vector bundles associated to the rotation and spin groups*, J. Funct. Anal. **106** (1992), 314–328.

[4] —, *Stein-Weiss operators and ellipticity*, J. Funct. Anal. **151** (1997), 334–383.

[5] Branson, T. and O. Hijazi, *Eigenvalue estimates for the Rarita-Schwinger and related operators*, In preparation.

[6] Branson, T., G. Ólafsson and B. Ørsted, *Spectrum generating operators, and intertwining operators for representations induced from a maximal parabolic subgroup*, J. Funct. Anal. **135** (1996), 163–205.

[7] Bröcker, T. and T. tom Dieck, "Representations of Compact Lie Groups, " Springer-Verlag, Berlin, 1985.

[8] Fegan, H., *Conformally invariant first order differential operators*, Quart. J. Math. Oxford **27** (1976), 371–378.

[9] Gallot, S. et D. Meyer, *Opérateur de courbure et laplacian des formes différentielles d'une variété riemannienne*, J. Math. Pures Appl. **54** (1975), 259–284.

[10] Ikeda, A. and Y. Taniguchi, *Spectra and eigenforms of the Laplacian on* S^n *and* $P^n(\mathbb{C})$, Osaka J. Math. **15** (1978), 515–546.

[11] D. Ray, *Reidemeister torsion and the Laplacian on lens spaces*, Advances in Math. **4** (1970), 109–126.

[12] Delanghe, R., F. Sommen, and V. Soucek, "Clifford Algebra and Spinor-Valued Functions," Kluwer Academic Publishers, Dordrecht, 1992.

[13] Stein, E. and G. Weiss, *Generalization of the Cauchy-Riemann equations and representations of the rotation group*, Amer. J. Math. **90** (1968), 163–196.

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Received August 4, 1998
and in final form November 24, 1998