Journal of Lie Theory
Volume **9** (1999) 125–156
© 1999 Heldermann Verlag

On classification of metabelian Lie algebras

L. Yu. Galitski, D. A. Timashev

Abstract. We classify metabelian Lie algebras with successive dimensions of quotients of the lower central series $(m,n)=(5,5)$ and $(6,3)$. The problem is reduced to describing orbits of the linear group $\bigwedge^2 SL_m \otimes SL_n$, the latter being a θ-group in both cases. The results obtained in the paper allow to complete the classification of metabelian Lie algebras of dimension up to 9.

Introduction

The classification of finite-dimensional Lie algebras divides in three parts: (1) classification of nilpotent Lie algebras; (2) description of solvable Lie algebras with given nilradical; (3) description of Lie algebras with given radical. The third problem reduces to the description of semisimple subalgebras in the algebra of derivations of a given solvable algebra [8]. The second problem reduces to the description of orbits of certain unipotent linear groups [9]. The first problem is most complicated. Just recall that the classification of *all* Lie algebras over $\mathbb{C}$ is obtained in dimension up to 6, and nilpotent complex Lie algebras are classified only in dimension up to 7.

Two-step nilpotent, or metabelian, Lie algebras form the first non-trivial subclass of nilpotent algebras. However even the classification of metabelian algebras is a rather complicated problem. The solution in dimension up to 7 is given in [4]. In greater dimensions, only partial results are obtained. In this paper, we introduce an invariant-theoretic approach to this problem, which allows to complete the classification of complex metabelian Lie algebras in dimension up to 9.

Let m, n be the dimensions of successive quotients of the lower central series for a metabelian Lie algebra. We call the pair (m, n) the *signature* of the metabelian Lie algebra. The classification of metabelian Lie algebras with given signature reduces to the following problem of linear algebra: classify the orbits of $GL_m(\mathbb{C}) \times GL_n(\mathbb{C})$ acting naturally on $\bigwedge^2 \mathbb{C}^m \otimes \mathbb{C}^n$. Moreover, the solution of the latter problem yields the classification of all metabelian Lie algebras with signature (p, q), $p \le m$, $q \le n$. The problem of classifying the orbits of the linear group $\bigwedge^2 GL_m(\mathbb{C}) \otimes GL_n(\mathbb{C})$ belongs to Invariant Theory and may be attacked by invariant-theoretic methods. It is more convenient to consider the respective unimodular group $\bigwedge^2 SL_m(\mathbb{C}) \otimes SL_n(\mathbb{C})$, since $\bigwedge^2 GL_m(\mathbb{C}) \otimes GL_n(\mathbb{C})$ has no polynomial invariants, and homotheties are easily controlled.

An inspection of tables in [5] yields the list of pairs (m, n) such that the group $\bigwedge^2 SL_m(\mathbb{C}) \otimes SL_n(\mathbb{C})$ is visible: $(5, 2)$, $(6, 2)$, $(7, 2)$, $(8, 2)$, $(5, 3)$, $(6, 3)$,

$(5,4)$, $(5,5)$, $(4,n)$. (A linear algebraic group is called *visible* if each level variety for the polynomial invariants consists of finitely many orbits.) The description of orbits for the group $\bigwedge^2 SL_m(\mathbb{C}) \otimes SL_2(\mathbb{C})$, or classes of two-dimensional subspaces of skew-symmetric matrices, can be obtained in terms of the Weierstraß–Kronecker theory of pencils, see sect. 1 and references therein. Here we classify the orbits of $\bigwedge^2 SL_m(\mathbb{C}) \otimes SL_n(\mathbb{C})$ for $(m,n) = (5,5)$ and $(6,3)$. This yields the solution in all remaining visible cases. If a group is not visible, then there arise "conditional" continuous invariants on some "bad" level varieties of invariant polynomials, and the classification of orbits is much more complicated if at all possible.

Our linear groups in the cases $(m,n) = (5,5)$ and $(6,3)$ are not only visible, but they are θ-groups, i. e., they integrate the adjoint action of the degree-zero component of a periodically graded semisimple Lie algebra on its degree-one component. The invariant theory of θ-groups is very nice. It resembles the theory of the adjoint representation in many features. We use the theory of θ-groups developed in [13] and [14] (see sect. 2) in order to describe the invariants and classify the orbits of our linear groups.

We obtain the following results. There are three families of metabelian complex Lie algebras of signature $(5,5)$. The first one depends on one continuous parameter. The second family contains 7 algebras. The third family consists of those algebras, whose structure tensor can be contracted to zero by unimodular change of coordinates. It contains 43 algebras. Moreover, there are 38 algebras of signature $(5,4)$, 17 of $(5,3)$, 5 of $(5,2)$, 2 of $(5,1)$, 2 of $(4,5)$, 3 of $(4,4)$, 5 of $(4,3)$, 3 of $(4,2)$, 2 of $(4,1)$, 1 of $(3,3)$, 1 of $(3,2)$, 1 of $(3,1)$, and 1 of $(2,1)$. This reproduces the results of [4]. For the signature $(6,3)$ we have 7 families. The first one depends on 2 continuous parameters. The second and third families depend on one continuous and one arithmetic parameter. The fourth family contains 6 algebras, the fifth one contains 6 algebras, and the sixth family contains 15 algebras. The seventh family consists of those algebras, whose structure tensor can be contracted to zero by unimodular change of coordinates. It contains 61 algebras. Moreover, there are 11 algebras of signature $(6,2)$, and 3 of $(6,1)$. See sect. 3–4 and the tables therein.

The paper is organized as follows. In section 1 we recall the general theory of metabelian algebras from [4], whereas section 2 contains a brief review of θ-groups. In section 3 we study the linear θ-group $\bigwedge^2 SL_5(\mathbb{C}) \otimes SL_5(\mathbb{C})$, and section 4 deals with $\bigwedge^2 SL_6(\mathbb{C}) \otimes SL_3(\mathbb{C})$.

Acknowledgments. Both authors would like to express their gratitude to Professor E. B. Vinberg, whose encouraging help made possible to carry on the work. The second author thanks CRDF (grant RM1–206) and RFBR (grant 98–01–00598) for the support.

1. Metabelian Lie algebras

In this section we recall basic facts on metabelian Lie algebras and reduce their classification to a certain problem of linear algebra. The paper [4] will be our basic reference.

1.1. General definitions. All Lie algebras, algebraic varieties and groups are considered over $\mathbb{C}$. However all definitions and results remain valid over an arbitrary algebraically closed field of characteristic zero.

Definition 1.1. A finite-dimensional Lie algebra L is called *metabelian* if $[L,[L,L]]=0$. Its *signature* is a pair (m,n), where $m=\dim L/[L,L]$, $n=\dim[L,L]$.

A metabelian Lie algebra structure on L is completely determined by the commutator map $\bigwedge^2 U\to V$, where $V=[L,L]$ and U is its complement in L. Conversely, let U,V be two vector spaces of dimensions m,n. Then each skew-symmetric bilinear surjective map $b:\bigwedge^2 U\to V$ determines a metabelian Lie algebra structure on $L=U\oplus V$ such that $V=[L,L]$. Two such structures are isomorphic iff the respective bilinear maps differ by a transformation from the group $GL(U)\times GL(V)$ acting naturally on the space $\bigwedge^2 U^*\otimes V$ of all such maps.

Surjective maps, or tensors of full rank, form a Zariski open $GL(U)\times GL(V)$-stable subset in $\bigwedge^2 U^*\otimes V$. Thus the classification of metabelian Lie algebras of signature (m,n) is equivalent to the classification of orbits in an open subset of $\bigwedge^2 U^*\otimes V$ stable under the action of $GL(U)\times GL(V)$. Moreover, if $L_0=U_0\oplus V_0$ is a metabelian algebra of signature (p,q), $p\le m$, $q\le n$, then its structure map is an element of $\bigwedge^2 U_0^*\otimes V_0$, and embeddings $U_0^*\hookrightarrow U^*$, $V_0\hookrightarrow V$ induce an embedding $\bigwedge^2 U_0^*\otimes V_0\hookrightarrow\bigwedge^2 U^*\otimes V$ such that $GL(U)\times GL(V)$-orbits intersect the subspace in $GL(U_0)\times GL(V_0)$-orbits. Now it is clear that the description of $GL(U)\times GL(V)$-orbits on $\bigwedge^2 U^*\otimes V$ is the same thing as the classification of metabelian Lie algebras of signature (p,q), $p\le m$, $q\le n$. This is the problem we are interested in.

Remark 1.2. Each tensor of full rank in $\bigwedge^2 U_0^*\otimes V_0$, i. e., such that its contractions with all bivectors from $\bigwedge^2 U_0$ generate V_0, represents not only a metabelian Lie algebra of signature (p,q) but also a series of metabelian algebras of signature (l,q), $p<l\le m$ that are obtained by adding a $(l-p)$-dimensional central subalgebra.

Next we describe the automorphism group of a metabelian Lie algebra. Let $L=U\oplus V$ be a metabelian algebra defined by a structure map $b:\bigwedge^2 U\to V$. Then $\operatorname{Aut} L$ is a semidirect product of the subgroup of automorphisms normalizing U, which is exactly the stabilizer $G(b)$ of b in $GL(U)\times GL(V)$, and the normal unipotent subgroup of inner automorphisms (acting trivially on V and L/V), which is isomorphic to the vector group $\operatorname{Hom}_{\mathbb{C}}(U,V)$. The identity component $G(b)^0$ of $G(b)$ is a quasi-direct product of $\mathbb{C}^*$, or $(\mathbb{C}^*)^2$ if the $SL(U)\times SL(V)$-orbit of b is conical, and $S(b)^0$, the identity component of the stabilizer $S(b)$ of b in $SL(U)\times SL(V)$. The subgroups $S(b)^0$ are computed (in a sense) in sect. 3–4 for all orbits in $\bigwedge^2 U^*\otimes V$ whenever $(m,n)=(5,5)$ or $(6,3)$.

Furthermore, suppose a tensor b belongs to a subspace $\bigwedge^2 U_0^*\otimes V_0$ of $\bigwedge^2 U^*\otimes V$, and U_0^*,V_0 are minimal subspaces with this property. This means that U_0^* (resp. V_0) is generated by all contractions of b with a vector from U and a covector from V^* (resp. a bivector from $\bigwedge^2 U$), or equivalently, that b defines a metabelian algebra structure on $L_0=U_0\oplus V_0$ such that $Z(L_0)=[L_0,L_0]=V_0$.

Then $G(b)$ is a semidirect product of the stabilizer $G_0(b)$ of b in $GL(U_0)\times GL(V_0)$, and the subgroup of transformations from $GL(U)\times GL(V)$ acting trivially on U_0^* and V_0. Thus we may compute the dimension of $G_0(b)$ (or $\operatorname{Aut} L_0$) and the type of its Levi part if we know these data for $G(b)$, and conversely.

1.2. Duality. Another interesting thing is a natural duality on metabelian Lie algebras, cf. [4, §3]. To each surjective map $b:\bigwedge^2 U\to V$ we assign the projection map $b^\vee:\bigwedge^2 U^*\to\bigwedge^2 U^*/(\operatorname{Ker} b)^\perp$. (Here $^\perp$ denotes annihilator in the dual space.) This is a bijection between (isomorphism classes of) metabelian algebras of signature (m,n), and those of signature $\bigl(m,\binom{m}{2}-n\bigr)$. Therefore it suffices to classify the algebras in one of the two cases, say, for $2n\le\binom{m}{2}$.

1.3. Classification in simple cases. First we consider metabelian Lie algebras of signature $(m,1)$. It is clear from the above discussion that to classify them is the same thing as to classify skew-symmetric bilinear forms on U. These forms are classified by their rank $2r$. It now follows that each metabelian Lie algebra of signature $(m,1)$ is isomorphic to one of $L=H_r(\mathbb{C})\oplus\mathbb{C}^{m-2r}$, where $H_r(\mathbb{C})$ is the $(2r+1)$-dimensional Heisenberg algebra determined by commutator relations $[e_i,e_{r+i}]=e_{2r+1}$, $1\le i\le r$.

Now we consider the classification of metabelian algebras of signature $(m,2)$. This problem reduces to classifying pairs or, more precisely, two-dimensional subspaces of skew-symmetric bilinear forms on U. This problem can be solved in general with the aid of the Weierstraß–Kronecker theory of pencils, see [4, §6] and [3, Chap. 12].

Consider a pencil of skew-symmetric $(m\times m)$-matrices $\mathbf{x}A+\mathbf{y}B$ ($\mathbf{x}$ and $\mathbf{y}$ are indeterminates). We say that a pencil $\mathbf{x}A'+\mathbf{y}B'$ is *equivalent* (*congruent*) to $\mathbf{x}A+\mathbf{y}B$ if $\mathbf{x}A'+\mathbf{y}B'=C(\mathbf{x}A+\mathbf{y}B)C'$, where C,C' are non-degenerate complex matrices (respectively, $\mathbf{x}A'+\mathbf{y}B'=C(\mathbf{x}A+\mathbf{y}B)C^T$ for some non-degenerate complex matrix C).

Definition 1.3. *Invariant factors* of the pencil $\mathbf{x}A+\mathbf{y}B$ are the homogeneous polynomials $d_i(\mathbf{x},\mathbf{y})=\frac{\Delta_i}{\Delta_{i-1}}$, $i=1,\dots,r$, where $\Delta_i(\mathbf{x},\mathbf{y})$ is the g. c. d. of all $(i\times i)$-minors of $\mathbf{x}A+\mathbf{y}B$, and $r=\operatorname{rk}(\mathbf{x}A+\mathbf{y}B)$. Each invariant factor decomposes into a product of powers of prime (linear) polynomials in $\mathbf{x},\mathbf{y}$. These prime power factors are called *elementary divisors* of the pencil.

Minimal indices $\delta_1,\dots,\delta_{m-r}$ of the pencil are defined by induction. Consider polynomial solutions of a linear system $(\mathbf{x}A+\mathbf{y}B)X=0$. Each m-column X with polynomial entries satisfying the above equation is the sum of homogeneous components satisfying the equation, too. (A column X is said to be *homogeneous* if all its entries are homogeneous polynomials in $\mathbf{x},\mathbf{y}$ of the same degree denoted by $\deg X$.)

Suppose we have chosen homogeneous solutions X_i and defined the minimal indices $\delta_i=\deg X_i$ for $i<k$. Among all homogeneous solutions that are linearly independent of X_i $(i<k)$, choose one, say X_k, of the lowest degree and set $\delta_k=\deg X_k$. Clearly, we have $0\le\delta_1\le\ldots\le\delta_{m-r}$. It is easy to show that the sequence $\delta_1,\dots,\delta_{m-r}$ does not depend on the choice of the fundamental solutions X_i.

Theorem 1.4. [4, §6], [3, Chap. 12] *For two skew-symmetric pencils, the following assertions are equivalent:*

1. *The pencils are congruent.*
2. *The pencils are equivalent.*
3. *Their sets of elementary divisors and minimal indices coincide.*

For a skew-symmetric pencil, elementary divisors occur with even multiplicities. One can write out a canonical form for a skew-symmetric pencil [4, §6], which generalizes both the canonical form for one skew-symmetric matrix and Jordan normal form.

If we are interested rather in classifying two-dimensional subspaces than pairs of skew-symmetric matrices, then we should allow linear substitutions of $\mathbf{x},\mathbf{y}$. Minimal indices do not change under a linear substitution of $\mathbf{x},\mathbf{y}$ and elementary divisors transform in accordance with the change of variables. One can find a "canonical set" in each equivalence class of sets of elementary divisors by transforming three given linear forms to $\mathbf{x}$, $\mathbf{y}$, $\mathbf{x}+\mathbf{y}$. Thus we obtain a canonical form for a two-dimensional subspace of skew-symmetric matrices or skew bilinear forms. See [4] for precise formulations.

1.4. Invariant-theoretic approach. Now we return to our problem in its general setting: classify the orbits for the natural action $GL(U)\times GL(V):\bigwedge^2 U^*\otimes V$. It will be more convenient for us to consider the equivalent action $GL(U)\times GL(V):\bigwedge^2 U\otimes V$, so that each orbit determines a class of isomorphic metabelian algebra structures on $L=U^*\oplus V$. The respective linear group $\bigwedge^2 GL(U)\otimes GL(V)$ contains homotheties, hence this action has no non-constant polynomial invariants. Thus it is more convenient to consider the respective unimodular linear group $\bigwedge^2 SL(U)\otimes SL(V)$ and classify its orbits. Homotheties leave conical orbits stable and glue together one-parametric families of non-conical orbits.

The first approximation in classifying orbits of a linear algebraic group is to separate them by invariant polynomials. If each level variety of invariant polynomials contains already finitely many orbits, then a group is "good" from the invariant-theoretic point of view, and one may hope to complete the classification by introducing some arithmetic invariants on each level variety. Such algebraic linear groups are called *visible*. We give the precise definition.

Definition 1.5. Let $H\subseteq GL(M)$ be a reductive algebraic linear group, $\pi: M\to M/\!\!/H:=\operatorname{Spec}\mathbb{C}[M]^H$ the categorical quotient. The group H is *visible* if $\pi^{-1}(z)$ contains finitely many H-orbits for $\forall z\in M/\!\!/H$.

The visibility correlates with other good invariant-theoretic properties of a representation (equidimensionality, freeness of the algebra of invariants etc.). Visible semisimple irreducible linear groups are classified by Kac [5]. An easy inspection of tables in [5] gives the list of pairs (m,n) such that the semisimple irreducible linear group $\bigwedge^2 SL(U)\otimes SL(V)$ is visible: $(5,2)$, $(6,2)$, $(7,2)$, $(8,2)$, $(5,3)$, $(6,3)$, $(5,4)$, $(5,5)$, $(4,n)$. The classification of orbits in the first four cases is covered by 1.3. It is clear from the discussion in 1.1 that the five remaining cases reduce to the cases $(m,n)=(5,5)$ and $(6,3)$. (In the last case, we may assume $n\le 3$ by 1.2.)

In the latter two cases, the group $\bigwedge^2 SL(U)\otimes SL(V)$ is even a θ-group, and its invariant theory is especially nice. See sect. 2 for the definition and properties of θ-groups. The case $(m,n)=(5,5)$ is considered in sect. 3, and the case $(m,n)=(6,3)$ is considered in sect. 4.

Remark 1.6. Suppose $b\in\bigwedge^2 U\otimes V$ is degenerate, i. e., it lies in a subspace of the type $\bigwedge^2 U_0\otimes V_0$, $U_0\subseteq U$, $V_0\subseteq V$, and one of the inclusions is strict. It is easy to see that the intersection of the $SL(U)\times SL(V)$-orbit of b with $\bigwedge^2 U_0\otimes V_0$ is a $GL(U_0)\times GL(V_0)$-orbit. In particular, it contains λb, $\forall\lambda\in\mathbb{C}^\times$. Hence $(SL(U)\times SL(V))b=(GL(U)\times GL(V))b$ contains 0 in its closure, and b lies in the null-cone $\pi^{-1}(\pi(0))$. Elements of the null-cone are called *nilpotent.* Thus the classification of orbits for $\bigwedge^2 GL_p(\mathbb{C})\otimes GL_q(\mathbb{C})$, where $p\le m$, $q\le n$ and one of the inequalities is strict, reduces to the classification of nilpotent orbits for $\bigwedge^2 SL_m(\mathbb{C})\otimes SL_n(\mathbb{C})$.

It remains to note that each metabelian algebra of dimension ≤ 9 has the signature (m,n) such that either $n\le 2$, or $m,n\le 5$, or $m\le 6, n\le 3$. The same holds for 10-dimensional algebras with the two exceptions: $(m,n)=(7,3)$ and $(6,4)$. Thus we obtain a complete classification of metabelian Lie algebras in dimension up to 9.

1.5. Notation and terminology. It is a good time to introduce some notation and terminology that will be used in the sequel.

We fix bases ${}_1e,\dots,{}_me$ in U and $e_1,\dots,e_n$ in V, thus identifying U,V with $\mathbb{C}^m,\mathbb{C}^n$.

We denote dual bases of U^* and V^* by lifting indices: ${}^ie$ and e^k.

We denote respective tensor bases in various spaces of tensors by writing indices in a sequence, as in the "tensor notation". For example, ${}_{ij}e_k={}_ie\wedge{}_je\otimes e_k$, ${}^{ij}e^k={}^ie\wedge{}^je\otimes e^k$, etc. In tables 1–8, we use the shorthand notation: $(\,abc\ \dots\ ijk\,) := {}_{ab}e_c+\ldots+{}_{ij}e_k$.

Let $\mathfrak{t}\subset\mathfrak{sl}(U)\oplus\mathfrak{sl}(V)$ be the standard Cartan subalgebra of operators diagonal in the chosen bases, and $T\subset SL(U)\times SL(V)$ be the respective Cartan subgroup.

Let ${}_i\varepsilon$, ε_k be the weights of ${}_ie$, e_k with respect to $\mathfrak{t}$ or T. (We will not distinguish the notation.) We have $\sum{}_i\varepsilon=\sum\varepsilon_k=0$. Choose ${}_i\varepsilon-{}_j\varepsilon$, $\varepsilon_k-\varepsilon_l$ $(i<j,\ k<l)$ as positive roots of $\mathfrak{sl}(U)\oplus\mathfrak{sl}(V)$ relative to $\mathfrak{t}$.

For any vector space M with a fixed basis $e_1,\dots,e_n$, we use the following notation. Let $\langle\cdot,\cdot\rangle$ denote the natural pairing between tensor spaces $M^{\otimes p}$ and $(M^*)^{\otimes p}$. Denote by $\mathrm{C}(\gamma)$ the contraction of $\gamma\in M^{\otimes p}\otimes(M^*)^{\otimes p}$ in the first index. We use the "big" exterior product of skew-symmetric tensors, so that $\langle v_1^*\wedge\ldots\wedge v_p^*, v_1\wedge\ldots\wedge v_p\rangle=p!\det\langle v_i^*,v_j\rangle$ $(v_i^*\in M^*,\ v_j\in M)$. The isomorphism $\iota:\bigwedge^p M\xrightarrow{\sim}\bigwedge^{n-p}M^*$ is defined by $\alpha\wedge\beta=\langle\iota(\alpha),\beta\rangle\cdot e_1\wedge\ldots\wedge e_n$, $\forall\alpha\in\bigwedge^p M,\ \beta\in\bigwedge^{n-p}M^*$.

Suppose $H:M$ is a rational representation of a reductive algebraic group, $\pi:M\to M/\!\!/H$ is the categorical quotient. A vector $v\in M$ is called *nilpotent* if $v \in \pi(\pi^{-1}(0))$ (or, equivalently, if all homogeneous invariants of positive degree vanish at v). A vector $v \in M$ is called *semisimple* if the orbit $Hv \subseteq M$ is closed. A vector that is neither semisimple nor nilpotent is called *mixed.*

Suppose H is an algebraic group with the Lie algebra $\mathfrak{h}$. Then H^0 denotes its identity component, and H' ($\mathfrak{h}'$) the commutator subgroup (subalgebra) of H (resp. $\mathfrak{h}$). For any subset $\mathfrak{m} \subset \mathfrak{h}$, let $Z(\mathfrak{m})$ ($\mathfrak{z}(\mathfrak{m})$) be the centralizer of $\mathfrak{m}$ in H (resp. in $\mathfrak{h}$).

If X is an H-variety (H-module), then $H : X$ denotes the action of H on X, X^H ($X^{\mathfrak{h}}$) is the fixed point set of H (resp. the annihilator of $\mathfrak{h}$) in X, and H_x ($\mathfrak{h}_x$) denotes the stabilizer (subalgebra) of $x \in X$.

2. Generalities on θ-groups

In this section, we will give a brief review of the theory of θ-groups. For a detailed narration of this topic, see [13] and [14]. In [13], the theory of Cartan subspaces, of Weyl groups, and of invariants for a θ-group is considered. In [14], a method for classifying nilpotent elements w. r. t. the action of a θ-group is described. Papers [11] and [12] contain a shorthand exposition of main results of [13] and [14], respectively.

In [15], one of the most interesting examples of a θ-group, namely the group $\bigwedge^3 SL_9(\mathbb{C})$, is considered, and the orbits and the invariants are described in that case.

2.1. Definition of a θ-group. Let G be a semisimple algebraic group, $\mathfrak{g}$ its Lie algebra. Assume that $\mathfrak{g}$ is graded modulo m:

$$\mathfrak{g} = \bigoplus_{k \in \mathbf{Z}_m} \mathfrak{g}_k.$$

(Here m is any natural number or ∞; $\mathbf{Z}_m$ is the respective residue group; $\mathbf{Z}_\infty = \mathbb{Z}$. We also write $\mathfrak{g}_k = \mathfrak{g}_{k \bmod m}$ for any $k \in \mathbb{Z}$.) If $m < \infty$, then this gradation determines an automorphism $\theta \in \operatorname{Aut} \mathfrak{g}$:

$$\theta(x) = \omega^k x \qquad \text{for } \forall x \in \mathfrak{g}_k,$$

where $\omega = e^{\frac{2\pi i}{m}}$. If $m = \infty$, then this gradation determines a one-parameter subgroup $\theta_t \in \operatorname{Aut} \mathfrak{g}$, $t \in \mathbb{C}^\times$:

$$\theta_t(x) = t^k x \qquad \text{for } \forall x \in \mathfrak{g}_k.$$

Conversely, each automorphism of finite order m (or a one-parameter subgroup in $\operatorname{Aut} g$) determines a $\mathbf{Z}_m$- (or $\mathbb{Z}$-) gradation of $\mathfrak{g}$. We may (and will) assume that G is simply connected. Then θ (or θ_t) is the differential of an automorphism of G (or a one-parameter subgroup of them) denoted by the same letter.

Let G_0 be the connected reductive subgroup of G corresponding to the subalgebra $\mathfrak{g}_0 \subseteq \mathfrak{g}$. Since G is simply connected, $G_0 = G^\theta$. The restriction of the adjoint representation of G to G_0 leaves each g_k stable. Let $\rho : G_0 \to GL(\mathfrak{g}_1)$ be the restriction of the adjoint action of G_0 to $\mathfrak{g}_1$. The linear algebraic group $\rho(G_0)$ is called a *θ-group.* The representation $G_0 \underset{\rho}{:} \mathfrak{g}_1$ is said to be a *θ-representation.*

2.2. Jordan decomposition. Each element $x \in \mathfrak{g}$ has the (unique) Jordan decomposition $x = x_s + x_n$, where x_s is a semisimple element, x_n is a nilpotent element, and $[x_s, x_n] = 0$. Recall from the theory of the adjoint representation that adjoint orbits of semisimple elements are closed, and those of nilpotent elements contain 0 in their closure. The same thing holds for any θ-group.

Theorem 2.1. [13, §§1–2] 1. *Each element $x \in \mathfrak{g}_1$ has the Jordan decomposition $x = x_s + x_n$, where $x_s, x_n \in \mathfrak{g}_1$.*

2. *If $x \in \mathfrak{g}_1$ is semisimple, then the orbit $G_0 x$ is closed.*
3. *If $x \in \mathfrak{g}_1$ is nilpotent, then $\overline{G_0 x} \ni 0$.*
4. *Under the assumptions of 1, $G_0 x_s$ is the unique closed G_0-orbit in $\overline{G_0 x}$.*

Remark 2.2. Many representations with nice invariant-theoretic properties are θ-representations. The theorem gives a strong motivation for the following general terminology: a vector is called *semisimple* w. r. t. an algebraic linear group if its orbit is closed, and is called *nilpotent* if its orbit contains 0 in the closure.

2.3. The Cartan subspace, the Weyl group, and invariants. Any maximal abelian subspace $\mathfrak{c} \subseteq \mathfrak{g}$ that consists of commuting semisimple elements is called a *Cartan subspace.* This notion generalizes Cartan subalgebras. All Cartan subspaces are G_0-conjugated [13, §3], hence we can fix one of them, say $\mathfrak{c}$, for the sequel. Let $Z_0(\cdot)$ $(N_0(\cdot))$ denote the centralizer (resp. normalizer) in G_0.

Theorem 2.3. [13, §§3–4,6] 1. *Each semisimple element of $\mathfrak{g}_1$ is G_0-equivalent to some element of $\mathfrak{c}$.*

2. *The group $W = N_0(\mathfrak{c})/Z_0(\mathfrak{c})$ is a finite reflection group acting naturally on $\mathfrak{c}$. It is called the* Weyl group *of the θ-representation.*
3. *For $\forall x \in \mathfrak{c}$: $G_0 x \cap \mathfrak{c} = Wx$.*
4. *The restriction of functions on $\mathfrak{g}_1$ to $\mathfrak{c}$ induces an isomorphism*

$$\mathbb{C}[\mathfrak{g}_1]^{G_0} \xrightarrow{\sim} \mathbb{C}[\mathfrak{c}]^W.$$

Thus the classification of semisimple elements in $\mathfrak{g}_1$ reduces to the description of orbits for a finite reflection group.

Furthermore, each level variety of invariants, i. e., a fiber $\pi^{-1}(\pi(x))$ of the quotient morphism $\pi : \mathfrak{g}_1 \to \mathfrak{g}_1 /\!\!/ G_0$, has finitely many orbits (and exactly one closed orbit $G_0 x_s$). In other words, each θ-group is visible.

2.4. The covering loop algebra. The description of all $\mathbb{Z}$-gradations on a semisimple algebra $\mathfrak{g}$ may be obtained in the following way. Choose a maximal torus T in G containing a one-parameter subgroup θ_t related to the $\mathbb{Z}$-gradation. Its Lie algebra $\mathfrak{t}$ is a Cartan subalgebra of $\mathfrak{g}$ contained in $\mathfrak{g}_0$, and each root subspace $\mathfrak{g}_\alpha$ is contained in some $\mathfrak{g}_k$. Hence each root α receives the degree

$\deg\alpha = k$. The degree is an additive function on the root system, hence it is determined by its values on simple roots. We may choose simple roots so that their degrees are non-negative. Conversely, each non-negative integer labelling of the Dynkin diagram of $\mathfrak{g}$ determines a $\mathbb{Z}$-gradation on $\mathfrak{g}$.

In order to describe all $\mathbf{Z}_m$-gradations on $\mathfrak{g}$, it is convenient to pass to a "covering" infinite-dimensional Lie algebra. Assume $\mathfrak{g} = \bigoplus \mathfrak{g}_k$ is $\mathbf{Z}_m$-graded. Consider a twisted loop algebra

$$\mathfrak{G} = \bigoplus_{k\in\mathbb{Z}} \mathfrak{G}_k \subseteq \mathbb{C}[t, t^{-1}] \otimes \mathfrak{g}, \qquad \mathfrak{G}_k = t^k \mathfrak{g}_k,$$

so that $\mathfrak{g} = \mathfrak{G}/(t^m - 1)\mathfrak{G}$. In case, where $\mathfrak{g}$ is simple, $\mathfrak{G}$ is a simple codimension 2 subquotient of an affine Kac–Moody algebra. In general, $\mathfrak{G}$ is a direct sum of such algebras. For the rest of this subsection, we will assume that $\mathfrak{g}$ is simple.

See [6, Chap. 7–8] for a detailed exposition of the theory of loop algebras. It resembles the theory of finite-dimensional (semi)simple Lie algebras. In particular, $\mathfrak{G}$ is graded by a free abelian finitely-generated group Q (the *root lattice*) so that its component $\mathfrak{t}$ of degree zero is a Cartan subalgebra in $\mathfrak{G}_0 \cong \mathfrak{g}_0$ and each component $\mathfrak{G}_\alpha$ of degree $\alpha \in Q$ is contained in some $\mathfrak{G}_k$. Elements $\alpha \in Q$ such that $\mathfrak{G}_\alpha \neq 0$ are called *roots*. Thus each root α receives the degree $\deg\alpha = k$.

The period s of θ mod $\operatorname{Int}\mathfrak{g}$ is called the *index* of $\mathfrak{G}$. There exists a unique root $\nu \in Q$ such that $\forall\alpha \in Q : t^m\mathfrak{G}_\alpha = \mathfrak{G}_{\alpha+s\nu}$. There exists a homomorphism $Q \to \mathfrak{t}^*$, $\alpha \mapsto \bar{\alpha}$, such that $\forall h \in \mathfrak{t}$, $x \in \mathfrak{G}_\alpha$: $[h, x] = \bar{\alpha}(h)x$, and $\bar{\alpha} = 0$ iff α is a multiple of ν (such roots are called *imaginary*). For *real* roots (i. e., such α that $\bar{\alpha} \neq 0$) we have $\dim\mathfrak{G}_\alpha = 1$.

The set $\Delta = \{\, \bar{\alpha} \mid \alpha \text{ is a real root} \,\}$ is exactly the set of nonzero weights of $\mathfrak{g}$ relative to $\mathfrak{t}$. If $\mathfrak{t}$ is a Cartan subalgebra of $\mathfrak{g}$, not only of $\mathfrak{g}_0$, or equivalently, $s = 1$, then Δ is just the root system of $\mathfrak{g}$ relative to $\mathfrak{t}$.

There exists a base Π of Q consisting of real roots, and we may choose it so that $\deg\alpha \geq 0$ for $\forall\alpha \in \Pi$. One may associate a Cartan matrix and a Dynkin diagram with Π exactly in the same way as in the theory of semisimple algebras. The diagrams thus obtained are affine Dynkin diagrams $\mathbf{X}_l^{(s)}$, where $\mathbf{X}_l$ is the Dynkin diagram of $\mathfrak{g}$. We label the nodes of an affine Dynkin diagram by Kac labels a_α, which are the (integer mutually coprime) coefficients of the linear dependence among the columns of the respective Cartan matrix. We have

$$\nu = \sum_{\alpha\in\Pi} a_\alpha \alpha$$

The loop algebra is determined by its Dynkin diagram up to an isomorphism discarding the gradation. Since the degree function is additive on the roots, it is determined by its values $d_\alpha = \deg\alpha$ on $\alpha \in \Pi$. Clearly we have

$$\sum_{\alpha\in\Pi} a_\alpha d_\alpha = \frac{m}{s} \tag{1}$$

Conversely, each non-negative integer labelling of the affine Dynkin diagram satisfying (1) determines a $\mathbb{Z}$-gradation of $\mathfrak{G}$ and $\mathbf{Z}_m$-gradation of $\mathfrak{g}$. Simple roots α with zero labels (or, more precisely, their restrictions $\bar{\alpha}$ to $\mathfrak{t}$) form a base of the root system of $\mathfrak{g}_0$, whereas the functionals $\bar{\alpha}$ such that $d_\alpha = 1$ are exactly the lowest weights of irreducible components of the θ-representation $G_0 : \mathfrak{g}_1$.

Remark 2.4. If $\mathfrak{g}$ is not simple, then the respective θ-group is a direct product of θ-groups for simple graded algebras acting each on its own space, and the θ-representation is a direct sum of the respective θ-representations. The covering algebra $\mathfrak{G}$ is a direct sum of simple twisted loop algebras, and the whole theory generalizes with minor changes.

2.5. Classification of nilpotent elements. For θ-groups, we have the following generalization of the Morozov-Jacobson theorem.

Theorem 2.5. [14, §2] *Suppose $e \in \mathfrak{g}_1$ is a nonzero nilpotent element. Then there exists a nilpotent element $f \in \mathfrak{g}_{-1}$ and a semisimple element $h \in \mathfrak{g}_0$ such that $\{e, f, h\}$ is an $\mathfrak{sl}_2$-triple. The element h is called a* characteristic *of e. It is determined up to conjugation by $Z_0(e)$, whereas e is determined up to conjugation by $Z_0(h)$.*

Replacing a nilpotent element e by its conjugate, we may (and will) assume that a maximal torus of $N_0(e)$ lies in $\mathfrak{t}$. In particular, $h \in \mathfrak{t}$. Moreover, h lies in the rational form $\mathfrak{t}(\mathbb{Q})$ of $\mathfrak{t}$ defined by the condition that differentials of all T-characters have rational values, and we may assume that h lies in the positive Weyl chamber of $\mathfrak{t}(\mathbb{Q})$ w. r. t. a given base of the root system. Such h is determined uniquely by the orbit G_0e.

We will consider $\mathbb{Z}$-graded subalgebras $\mathfrak{s} \subseteq \mathfrak{g}$. (This means that $\mathfrak{s}_k \subseteq \mathfrak{g}_k$ for $\forall k \in \mathbb{Z}$.) A $\mathbb{Z}$-graded subalgebra is said to be *regular* if it is normalized by a maximal torus in G_0, and *complete* if it is maximal among all semisimple regular subalgebras of the same rank.

Definition 2.6. A *support* of a nilpotent element $e \in \mathfrak{g}_1$ is a minimal complete subalgebra containing e.

A support is determined up to conjugacy and may be constructed as follows. Consider a reductive $\mathbb{Z}$-graded subalgebra

$$\mathfrak{g}(h) = \sum_{k\in\mathbb{Z}} \mathfrak{g}_k(h), \qquad \mathfrak{g}_k(h) = \{\, x \in \mathfrak{g}_k \mid [h, x] = 2kx \,\}, \tag{2}$$

and let $\mathfrak{t}_e$ be the centralizer of e in $\mathfrak{t}$. Then $\mathfrak{s} = (\mathfrak{g}(h)^{\mathfrak{t}_e})'$ is the (T-regular) support of e [14, §4].

Let S_0 be the reductive subgroup of G_0 corresponding to $\mathfrak{s}_0$. Then all elements of $\mathfrak{s}_1$ are nilpotent, S_0 acts on $\mathfrak{s}_1$ with finitely many orbits, e lies in the dense orbit, and the stabilizer $(S_0)_e$ is finite. In particular, a nilpotent element is determined by its support up to conjugacy. Conversely, if $\mathfrak{s} \subseteq \mathfrak{g}$ is a complete subalgebra such that the stabilizer of the dense S_0-orbit in $\mathfrak{s}_1$ is finite, then $\mathfrak{s}$ is the support of any element e from the dense orbit. Semisimple $\mathbb{Z}$-graded algebras with the above property (or, equivalently, such that $\dim \mathfrak{s}_0 = \dim \mathfrak{s}_1$) are called *locally flat.* Simple locally flat algebras are listed in [14, §4], and semisimple ones are their direct sums.

The above discussion gives us the following method of classifying nilpotent elements in $\mathfrak{g}_1$.

1. *Classify all complete T-regular locally flat subalgebras of $\mathfrak{g}$ up to W_0-conjugacy, where $W_0 = N_0(T)/T$ is the Weyl group for G_0. Note that each $\mathbb{Z}$-graded subalgebra $\mathfrak{s} \subseteq \mathfrak{g}$ is canonically embedded in the covering algebra $\mathfrak{G}$ so that $\mathfrak{s}_k \subseteq \mathfrak{G}_k$ for $\forall k \in \mathbb{Z}$. Hence the root lattice of $\mathfrak{s}$ may be regarded as a sublattice $Q(s) \subseteq Q$. It is easy to see that $\mathfrak{s}$ is complete iff this sublattice is saturated or, in other words, the quotient group $Q/Q(\mathfrak{s})$ is torsion-free. Thus we have to classify all embeddings of the root systems $\Delta(\mathfrak{s})$ of locally flat algebras, or even of their systems of simple roots $\Pi(\mathfrak{s})$, in Δ which preserve the degrees and the invariant inner product, up to W_0-conjugacy.*

2. *For each complete locally flat subalgebra $\mathfrak{s} \subseteq \mathfrak{g}$, take an element e from the dense S_0-orbit in $\mathfrak{s}_1$. These e compose a full coset for the set of G_0-orbits of nilpotent elements in $\mathfrak{g}_1$.*

For the sequel, we need the third step of this procedure.

3. *Compute characteristics $h \in \mathfrak{t}(\mathbb{Q})$. Each h must lie in the Cartan subalgebra $\mathfrak{t}(\mathfrak{s})$ of $\mathfrak{s}$, hence it must be orthogonal to the annihilator of $Q(\mathfrak{s})$ in $\mathfrak{t}$ w. r. t. the invariant inner product. Moreover, h must have prescribed values on $\alpha \in \Pi(\mathfrak{s})$, since it defines the grading of $\mathfrak{s}$. These conditions determine h uniquely, and we may apply a transformation from W_0 to take h into the positive Weyl chamber of $\mathfrak{t}$.*

2.6. Classification of mixed elements. To classify vectors that are neither semisimple nor nilpotent, we first classify their semisimple parts, and then classify vectors $x = u + e$ with a given semisimple part u. We may assume $u \in \mathfrak{c}$. The nilpotent part e belongs to the semisimple $\mathbf{Z}_m$-graded subalgebra $\mathfrak{z}(u)'$. The problem is reduced to the classification of nilpotent elements in $\mathfrak{z}(u)'_1$ up to $Z_0(u)$-conjugacy. The classification up to $Z_0(u)^0$-conjugacy may be performed using the method of the previous subsection applied to the graded algebra $\mathfrak{z}(u)'$. The finite component group may glue some of these orbits together in one $Z_0(u)$-orbit. Taking this into consideration completes the classification.

2.7. Computing stabilizers. The centralizer of a semisimple element $u \in \mathfrak{g}_1$ is a graded reductive Levi subalgebra $\mathfrak{z}(u) \subseteq \mathfrak{g}$. We may assume $u \in \mathfrak{c}$, embed $\mathfrak{c}$ in a (graded) Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$, and determine the root system of $\mathfrak{z}(u)$ relative to $\mathfrak{h}$.

The stabilizer of a mixed element $x = u + e$ is the stabilizer of its nilpotent part e in the group $Z_0(u)$. If we are interested only in the stabilizer subalgebra, then we may pass to the identity component $Z_0(u)^0$. The center of $Z(u)$ stabilizes e, and it remains to determine the stabilizer of $e \in \mathfrak{z}(u)'_1$ in $(Z(u)')^0_0$. Thus the problem is reduced to computing the stabilizer of a nilpotent element.

The stabilizer subalgebra $\mathfrak{z}_0(e)$ of a nilpotent element $e \in \mathfrak{g}_1$ is spanned (as a linear space) by highest vectors of the 3-dimensional simple subalgebra $\langle e, f, h\rangle$ which lie in $\mathfrak{g}_0$. It follows from the theory of $\mathfrak{sl}_2$-representations that eigenvalues of $\operatorname{ad} h$ on $\mathfrak{z}_0(e)$ are non-negative, and $\operatorname{ad} e$ maps an eigenspace $\mathfrak{g}_{0,k}(h) \subset \mathfrak{g}_0$ of eigenvalue $k \geq 0$ onto an eigenspace $\mathfrak{g}_{1,k+2}(h) \subset \mathfrak{g}_1$ of eigenvalue $k+2$. Now it is easy to see [15, 4.5] that

$$(3)\qquad \dim \mathfrak{z}_0(e) = \sum_{k=0}^{\infty} \dim \mathfrak{g}_{0,k}(h) - \dim \mathfrak{g}_{1,k+2}(h).$$

The reductive subalgebra $\mathfrak{z}_0(e) \cap \mathfrak{z}_0(h)$ is a Levi part of $\mathfrak{z}_0(e)$. It coincides with the stabilizer subalgebra of general position for the θ-representation $\mathfrak{g}_0(h) : \mathfrak{g}_1(h)$ of the $\mathbb{Z}$-graded algebra $\mathfrak{g}(h)$ defined by (2). Indeed, the orbit of e in $\mathfrak{g}_1(h)$ is Zariski open [15, 4.5]. The algebra $\mathfrak{g}_0(h) = \mathfrak{z}_0(h)$ and its θ-representation in $\mathfrak{g}_1(h)$ are easy to find. Then the stabilizer subalgebra of general position can be found using Elashvili's tables [1], [2].

3. Metabelian algebras of signature $(5,5)$

This section deals with the θ-group $\bigwedge^2 SL_5(\mathbb{C}) \otimes SL_5(\mathbb{C})$. We apply the general theory of θ-groups to classifying the orbits of this linear group. This is the same thing as to classify metabelian Lie algebras of signature (m,n), $m,n \leq 5$.

3.1. Formulation of results. We retain the notation of 1.5 with $m = n = 5$. The classification of tensors in $\bigwedge^2 U \otimes V$ under the action of $SL(U) \times SL(V)$ is made as follows. Each tensor has the Jordan decomposition $x = u + e$ (see 2.2), where u is semisimple and e is nilpotent. Semisimple tensors fall into 3 families according to the type of stabilizer. For each family, we give a canonical form u of a semisimple tensor and classify all possible nilpotent parts e up to the action of the stabilizer of u. See tables 1–2 below. If we want to classify tensors under the action of $GL(U) \times GL(V)$, then the canonical form for u can be reduced by multiplying u by a nonzero scalar.

Thus the elements of $\bigwedge^2 U \otimes V$ are divided into 3 families according to the type of the semisimple part. All elements in the first family are semisimple, all elements in the third family are nilpotent. Tensors of the first two families and tensors of full rank in the third family represent metabelian algebras of signature $(5,5)$. All degenerate tensors lie in the third family (Remark 1.6), and each of them represents a series of metabelian algebras of signature (l,q), $p \leq l \leq 5$ (Remark 1.2). We indicate the "minimal" signature (p,q) in table 2.

The last two columns in each table contain information on the dimension of the stabilizer and the type of its (connected reductive) Levi part. Here $\mathbf{T}_l$ denotes an algebraic torus of dimension l. Making use of these data, one easily computes the dimension and the type of Levi part for the automorphism group (or algebra of derivations) of the respective metabelian algebra (cf. 1.1).

We indicate characteristics of nilpotent elements by their indices, i. e., values of simple roots. All other information contained in the tables is explained in section 2 and in 3.2–3.6.

Here is the classification.

Family 1. This family contains only semisimple tensors. The canonical form is

$$u = \lambda_1 u_1 + \lambda_2 u_2, \qquad \lambda_1\lambda_2(\lambda_1^{10} - 11\lambda_1^5\lambda_2^5 - \lambda_2^{10}) \neq 0,$$

where

$$u_1 = {}_{12}e_4 + {}_{34}e_1 + {}_{51}e_3 + {}_{23}e_5 + {}_{45}e_2,$$
$$u_2 = {}_{13}e_2 + {}_{52}e_1 + {}_{41}e_5 + {}_{35}e_4 + {}_{24}e_3.$$

The coefficients λ_i are determined up to the action of a certain finite group W described explicitly in 3.4. Two canonical forms are equivalent iff they have the same values of

$$\lambda_1^{20} + \lambda_2^{20} + 228\lambda_1^{15}\lambda_2^5 - 228\lambda_1^5\lambda_2^{15} + 494\lambda_1^{10}\lambda_2^{10} \quad \text{and}$$
$$\lambda_1^{30} + \lambda_2^{30} - 522\lambda_1^{25}\lambda_2^5 + 522\lambda_1^5\lambda_2^{25} - 10005\lambda_1^{20}\lambda_2^{10} - 10005\lambda_1^{10}\lambda_2^{20}.$$

The stabilizer subgroup of u is finite.

Family 2. The canonical form of a semisimple part is

$$u = \lambda u_1, \qquad \lambda \neq 0,$$

where λ is determined up to multiplication by a 10-th root of unity.

Table 1: Nilpotent parts of elements of family 2

No.	Canonical form	Support	Characteristic	Stabilizer	
				dim	Type
1	132 521 415 354	$\mathbf{A}_4$	2 2 2 2	0	0
2	132 521 415	$\mathbf{A}_3$	2 2 2 0	1	$\mathbf{T}_1$
3	132 521 354	$\mathbf{A}_2 + \mathbf{A}_1$	2 2 0 2	1	$\mathbf{T}_1$
4	132 521	$\mathbf{A}_2$	2 2 0 0	2	$\mathbf{T}_2$
5	132 415	$2\mathbf{A}_1$	2 0 2 0	2	$\mathbf{T}_2$
6	132	$\mathbf{A}_1$	2 0 0 0	3	$\mathbf{T}_3$
7	0			4	$\mathbf{T}_4$

Family 3. In this family, $u = 0$ and all elements are nilpotent.

Table 2: Elements of family 3

No.	Canonical form	Signature	Support	Characteristic	Stabilizer	
					dim	Type
1	125 144 153 234 243 252 342 351	$(5,5)$	$\mathbf{E}_8$	10 10 10 10 10 10 10 20	0	0
2	125 134 153 233 243 252 342 451	$(5,5)$	$\mathbf{E}_8(a_1)$	10 10 0 10 10 10 10 10	1	0
3	125 135 144 152 234 242 251 343	$(5,5)$	$\mathbf{E}_8(a_2)$	10 0 10 10 10 0 10 10	1	0
4	125 134 143 152 233 244 342 451	$(5,5)$	$\mathbf{E}_8(b)$	0 10 0 10 10 10 0 10	2	0
5	125 143 154 233 242 251 341 352	$(5,5)$	$\mathbf{E}_8(d_1)$	10 0 10 0 0 10 0 10	2	0
6	125 132 144 153 234 243 252 351	$(5,5)$	$\mathbf{E}_8(c_2)$	0 10 0 0 10 0 0 10	3	0
7	125 134 141 153 243 252 342 351	$(5,5)$	$\mathbf{E}_8(d_3)$	0 0 10 0 0 0 10 0	3	0
8	125 134 143 233 252 342 451	$(5,5)$	$\mathbf{E}_7$	1 9 1 9 10 10 1 9	3	$\mathbf{T}_1$
9	125 135 144 153 233 242 351	$(5,5)$	$\mathbf{E}_7(a_1)$	10 0 3 7 7 3 7 3	3	$\mathbf{T}_1$
10	135 144 152 234 243 251 341	$(5,5)$	$\mathbf{D}_7$	3 7 3 7 3 4 3 3	3	$\mathbf{T}_1$
11	121 144 153 234 243 252 342 451	$(5,4)$	$\mathbf{E}_8(d_5)$	0 0 0 0 0 0 0 10	4	0
12	125 134 144 152 232 243 351	$(5,5)$	$\mathbf{E}_7(a_2)$	1 9 0 1 10 0 1 9	4	$\mathbf{T}_1$
13	125 142 153 234 243 252 341	$(5,5)$	$\mathbf{E}_7(b)$	0 3 7 0 3 0 7 3	4	$\mathbf{T}_1$
14	135 143 152 234 242 251 342	$(5,5)$	$\mathbf{D}_7(a_1)$	2 0 8 2 2 2 6 2	4	$\mathbf{T}_1$
15	134 141 153 243 252 342 351	$(5,4)$	$\mathbf{E}_7(c_2)$	0 0 1 0 0 0 1 9	5	$\mathbf{T}_1$
16	125 133 144 152 233 251 342	$(5,5)$	$\mathbf{D}_7(a_2)$	0 5 0 5 0 5 0 5	5	$\mathbf{T}_1$

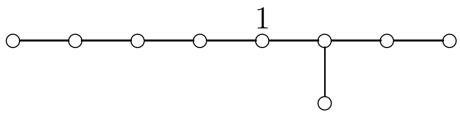
Table 2: (Continued)

No.	Canonical form	Signature	Support	Characteristic	Stabilizer	
					dim	Type
17	135 143 152 234 242 251 341	$(5,5)$	$\mathbf{A}_7$	3 3 4 3 3 3 1 3	5	$\mathbf{T}_1$
18	125 134 143 152 233 242 451	$(5,5)$	$\mathbf{E}_6+\mathbf{A}_1$	1 8 1 1 10 1 1 8	5	$\mathbf{T}_1$
19	125 134 143 153 233 251 342	$(5,5)$	$\mathbf{E}_6(a_1)+\mathbf{A}_1$	5 0 5 0 0 5 5 0	5	$\mathbf{T}_1$
20	125 143 152 234 242 351	$(5,5)$	$\mathbf{D}_6$	1 2 7 1 3 1 6 3	5	$\mathbf{T}_2$
21	124 153 233 241 252 342 351	$(5,4)$	$\mathbf{E}_6(b)+\mathbf{A}_1$	1 0 1 0 0 1 1 8	6	$\mathbf{T}_1$
22	125 144 152 233 242 251 341	$(5,5)$	$\mathbf{A}_6+\mathbf{A}_1$	3 3 1 3 3 1 2 4	6	$\mathbf{T}_1$
23	125 134 143 152 233 251 342	$(5,5)$	$\mathbf{D}_5+\mathbf{A}_2$	2 2 2 4 2 4 2 2	6	$\mathbf{T}_1$
24	125 134 143 153 251 342	$(5,5)$	$\mathbf{E}_6(a_1)$	6 0 4 0 0 6 4 0	6	$\mathbf{T}_2$
25	125 144 151 233 244 342	$(5,5)$	$\mathbf{D}_6(a_1)$	0 3 0 7 4 3 0 3	6	$\mathbf{T}_2$
26	133 152 233 244 251 341	$(5,4)$	$\mathbf{D}_6(a_2)$	0 1 0 1 0 1 0 9	6	$\mathbf{T}_2$
27	125 134 143 152 233 242 251 341	$(5,5)$	$2\mathbf{A}_4$	2 2 2 2 2 2 2 2	7	0
28	134 143 152 233 242 251 342	$(5,4)$	$\mathbf{D}_5(a_1)+\mathbf{A}_2$	1 0 1 1 1 1 1 7	7	$\mathbf{T}_1$
29	125 144 153 232 251 341	$(5,5)$	$\mathbf{A}_6$	4 2 2 2 4 0 2 4	7	$\mathbf{T}_2$
30	125 134 143 233 251 342	$(5,5)$	$\mathbf{D}_5+\mathbf{A}_1$	1 2 1 6 4 3 1 2	7	$\mathbf{T}_2$
31	144 152 233 242 251 341	$(5,4)$	$\mathbf{A}_5+\mathbf{A}_1$	1 1 0 1 1 0 1 8	7	$\mathbf{T}_2$
32	125 134 142 233 252 451	$(5,5)$	$\mathbf{E}_6$	0 8 2 0 10 2 0 8	7	$\mathbf{A}_1+\mathbf{T}_1$
33	135 141 152 234 243 251	$(5,5)$	$\mathbf{A}_6$	0 8 2 0 0 0 2 0	7	$\mathbf{A}_1+\mathbf{T}_1$
34	134 143 152 233 242 251 341	$(5,4)$	$\mathbf{A}_4+\mathbf{A}_3$	1 1 1 1 1 1 1 6	8	$\mathbf{T}_1$
35	124 132 153 233 242 451	$(5,4)$	$\mathbf{D}_4+\mathbf{A}_2$	0 2 0 0 2 0 2 6	8	$\mathbf{T}_2$
36	134 143 231 252 342 451	$(5,4)$	$\mathbf{E}_6(b)$	2 0 0 0 0 2 0 8	8	$\mathbf{A}_1+\mathbf{T}_1$
37	125 133 144 152 251 342	$(5,5)$	$\mathbf{D}_5+\mathbf{A}_1$	3 2 0 5 3 5 0 2	8	$\mathbf{A}_1+\mathbf{T}_1$
38	135 144 151 232 243 341	$(5,5)$	$\mathbf{A}_5+\mathbf{A}_1$	3 2 0 5 2 0 3 0	8	$\mathbf{A}_1+\mathbf{T}_1$
39	125 134 143 152 232 251 341	$(5,5)$	$\mathbf{A}_4+\mathbf{A}_3$	3 1 2 1 3 1 2 1	9	$\mathbf{T}_1$
40	125 134 143 151 233 242 341	$(5,5)$	$\mathbf{A}_4+\mathbf{A}_2+\mathbf{A}_1$	1 2 1 3 2 1 1 2	9	$\mathbf{T}_1$
41	124 143 152 233 242 251 341	$(5,4)$	$\mathbf{A}_4+\mathbf{A}_2+\mathbf{A}_1$	1 1 1 1 1 1 2 4	9	$\mathbf{T}_1$
42	123 135 152 234 242 251 341	$(5,5)$	$\mathbf{A}_4+\mathbf{A}_3$	0 0 5 0 0 5 0 0	9	$\mathbf{A}_1$
43	125 131 144 153 234 242 251	$(5,5)$	$\mathbf{A}_4+\mathbf{A}_2+\mathbf{A}_1$	0 5 0 0 0 0 0 5	9	$\mathbf{A}_1$
44	124 143 152 232 242 351	$(5,4)$	$\mathbf{D}_5(a_1)+\mathbf{A}_1$	1 1 0 1 2 1 1 6	9	$\mathbf{T}_2$
45	123 142 151 232 243 344	$(5,4)$	$\mathbf{D}_5(a_1)+\mathbf{A}_1$	0 0 0 3 3 0 0 7	9	$\mathbf{A}_1+\mathbf{T}_1$
46	125 133 144 251 342	$(5,5)$	$\mathbf{D}_5$	2 2 0 6 4 4 0 2	9	$\mathbf{A}_1+\mathbf{T}_2$
47	134 145 151 233 242	$(5,5)$	$\mathbf{A}_5$	3 3 0 4 1 0 3 0	9	$\mathbf{A}_1+\mathbf{T}_2$
48	144 153 232 241 351	$(5,4)$	$\mathbf{A}_5$	2 0 1 0 1 1 0 8	9	$\mathbf{A}_1+\mathbf{T}_2$
49	125 133 142 234 251 341	$(5,5)$	$\mathbf{A}_4+\mathbf{A}_2$	0 2 2 2 2 2 0 2	10	$\mathbf{T}_2$
50	134 143 152 232 251 341	$(5,4)$	$\mathbf{A}_4+\mathbf{A}_2$	2 0 2 0 2 0 2 4	10	$\mathbf{T}_2$
51	125 134 143 151 233 242	$(5,5)$	$\mathbf{A}_4+2\mathbf{A}_1$	1 3 1 1 0 1 1 3	10	$\mathbf{T}_2$
52	123 152 234 242 251 341	$(5,4)$	$\mathbf{A}_4+2\mathbf{A}_1$	0 0 3 0 0 3 0 4	10	$\mathbf{T}_2$
53	134 143 151 233 242 341	$(5,4)$	$\mathbf{A}_4+2\mathbf{A}_1$	1 1 1 2 1 1 1 4	10	$\mathbf{T}_2$
54	131 144 153 234 242 251	$(5,4)$	$\mathbf{A}_4+\mathbf{A}_2$	0 4 0 0 0 0 0 6	10	$\mathbf{A}_1+\mathbf{T}_1$
55	124 132 142 243 351	$(5,4)$	$\mathbf{D}_5(a_1)$	0 1 0 2 3 0 1 6	10	$\mathbf{T}_3$
56	124 133 152 242 251 341	$(5,4)$	$2\mathbf{A}_3$	1 1 1 1 1 2 1 3	11	$\mathbf{T}_2$
57	125 133 152 234 241	$(5,5)$	$\mathbf{A}_4+\mathbf{A}_1$	0 3 2 0 0 2 0 3	11	$\mathbf{T}_3$
58	134 143 151 233 242	$(5,4)$	$\mathbf{A}_4+\mathbf{A}_1$	1 2 1 1 0 1 1 4	11	$\mathbf{T}_3$
59	133 142 234 251 341	$(5,4)$	$\mathbf{A}_4+\mathbf{A}_1$	0 1 2 1 1 2 0 4	11	$\mathbf{T}_3$
60	124 132 143 232 251 341	$(5,4)$	$\mathbf{D}_4(a_1)+\mathbf{A}_2$	0 2 0 2 2 0 2 2	12	$\mathbf{T}_2$
61	125 134 143 151 232 241	$(5,5)$	$\mathbf{A}_3+\mathbf{A}_2+\mathbf{A}_1$	2 1 1 2 1 1 1 1	12	$\mathbf{T}_2$
62	121 153 233 252 342 451	$(5,3)$	$\mathbf{A}_3+\mathbf{A}_2+\mathbf{A}_1$	0 0 0 0 0 0 5 0	12	$2\mathbf{A}_1$
63	123 134 142 232 451	$(5,4)$	$\mathbf{D}_4+\mathbf{A}_1$	1 0 1 1 3 1 0 6	12	$\mathbf{A}_1+\mathbf{T}_2$
64	133 152 234 241	$(5,4)$	$\mathbf{A}_4$	0 2 2 0 0 2 0 4	12	$\mathbf{T}_4$
65	124 133 142 232 251 341	$(5,4)$	$\mathbf{A}_3+\mathbf{A}_2+\mathbf{A}_1$	1 1 1 1 2 1 1 2	13	$\mathbf{T}_2$
66	124 131 153 231 242	$(5,4)$	$\mathbf{D}_4(a_1)+\mathbf{A}_1$	0 3 0 0 0 0 3 1	13	$\mathbf{T}_3$
67	134 143 151 232 241	$(5,4)$	$\mathbf{A}_3+\mathbf{A}_2$	2 0 1 2 1 1 1 2	13	$\mathbf{T}_3$
68	141 152 233 242 351	$(5,3)$	$\mathbf{A}_3+\mathbf{A}_2$	0 0 1 0 0 1 4 0	13	$\mathbf{A}_1+\mathbf{T}_2$
69	122 134 145 153 251 341	$(5,5)$	$2\mathbf{A}_3$	5 0 0 0 5 0 0 0	14	$\mathbf{C}_2$
70	121 133 145 235 244 342	$(4,5)$	$2\mathbf{A}_2+2\mathbf{A}_1$	0 0 0 5 0 0 0 0	14	$\mathbf{C}_2$
71	124 143 151 232 241	$(5,4)$	$\mathbf{A}_3+2\mathbf{A}_1$	1 2 0 1 0 1 2 1	14	$\mathbf{T}_3$
72	132 143 232 251 341	$(5,3)$	$\mathbf{D}_4(a_1)+\mathbf{A}_1$	0 1 0 1 1 0 4 0	14	$\mathbf{A}_1+\mathbf{T}_2$
73	124 133 142 151 232 241	$(5,4)$	$2\mathbf{A}_2+2\mathbf{A}_1$	1 1 1 1 1 1 1 1	15	$\mathbf{T}_2$
74	124 132 143 251 341	$(5,4)$	$\mathbf{A}_3+\mathbf{A}_2$	2 1 0 1 3 0 1 2	15	$\mathbf{A}_1+\mathbf{T}_2$
75	123 152 232 251 341	$(5,3)$	$\mathbf{A}_3+2\mathbf{A}_1$	1 0 1 0 1 1 3 0	15	$\mathbf{A}_1+\mathbf{T}_2$
76	124 135 142 153 231	$(5,5)$	$\mathbf{A}_3+2\mathbf{A}_1$	3 0 2 0 1 0 2 0	15	$2\mathbf{A}_1+\mathbf{T}_1$
77	125 131 143 234 242	$(4,5)$	$2\mathbf{A}_2+\mathbf{A}_1$	0 1 0 4 0 0 0 1	15	$2\mathbf{A}_1+\mathbf{T}_1$
78	131 153 231 242	$(5,3)$	$\mathbf{D}_4(a_1)$	0 2 0 0 0 0 4 0	15	$\mathbf{A}_1+\mathbf{T}_3$
79	124 142 153 231	$(5,4)$	$\mathbf{A}_3+\mathbf{A}_1$	2 1 1 0 1 0 2 1	16	$\mathbf{A}_1+\mathbf{T}_3$
80	133 152 232 241	$(5,3)$	$\mathbf{A}_3+\mathbf{A}_1$	1 1 0 1 0 1 3 0	16	$\mathbf{A}_1+\mathbf{T}_3$
81	121 133 244 342	$(4,4)$	$2\mathbf{A}_2$	0 0 0 4 0 0 0 2	16	$2\mathbf{A}_1+\mathbf{T}_2$
82	122 133 234 451	$(5,4)$	$\mathbf{D}_4$	0 0 2 0 4 0 0 6	16	$\mathbf{A}_2+\mathbf{A}_1+\mathbf{T}_1$
83	124 132 151 233 241	$(5,4)$	$2\mathbf{A}_2+\mathbf{A}_1$	0 1 2 0 2 0 1 1	17	$\mathbf{A}_1+\mathbf{T}_2$

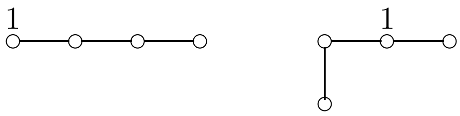
Table 2: (Continued)

No.	Canonical form	Signature	Support	Characteristic	Stabilizer	
					dim	Type
84	123 142 151 232 341	$(5,3)$	$2\mathbf{A}_2+\mathbf{A}_1$	1 0 1 1 1 1 2 0	17	$\mathbf{A}_1+\mathbf{T}_2$
85	124 131 143 233 242	$(4,4)$	$\mathbf{A}_2+3\mathbf{A}_1$	0 1 0 3 0 0 1 1	17	$\mathbf{A}_1+\mathbf{T}_2$
86	123 134 142 151 231	$(5,4)$	$\mathbf{A}_2+3\mathbf{A}_1$	2 0 1 1 1 1 0 1	18	$\mathbf{A}_1+\mathbf{T}_2$
87	122 153 251 341	$(5,3)$	$\mathbf{A}_3+\mathbf{A}_1$	2 0 0 0 2 0 3 0	18	$3\mathbf{A}_1+\mathbf{T}_1$
88	123 131 152 242 251	$(5,3)$	$\mathbf{A}_2+3\mathbf{A}_1$	0 2 0 0 0 2 1 0	19	$2\mathbf{A}_1+\mathbf{T}_1$
89	123 134 141 232	$(4,4)$	$\mathbf{A}_2+2\mathbf{A}_1$	1 0 1 2 0 1 0 1	19	$\mathbf{A}_1+\mathbf{T}_3$
90	122 133 251 341	$(5,3)$	$2\mathbf{A}_2$	0 0 2 0 2 0 2 0	19	$2\mathbf{A}_1+\mathbf{T}_2$
91	142 153 231	$(5,3)$	$\mathbf{A}_3$	2 0 1 0 1 0 3 0	19	$3\mathbf{A}_1+\mathbf{T}_2$
92	123 132 151 241	$(5,3)$	$\mathbf{A}_2+2\mathbf{A}_1$	1 1 0 1 1 1 1 0	20	$\mathbf{A}_1+\mathbf{T}_3$
93	121 133 243 342	$(4,3)$	$\mathbf{A}_2+2\mathbf{A}_1$	0 0 0 3 0 0 2 0	20	$3\mathbf{A}_1+\mathbf{T}_1$
94	123 131 242	$(4,3)$	$\mathbf{A}_2+\mathbf{A}_1$	0 1 0 2 0 1 1 0	21	$\mathbf{A}_1+\mathbf{T}_4$
95	131 152 242 251	$(5,2)$	$\mathbf{A}_2+2\mathbf{A}_1$	0 1 0 0 0 3 0 0	22	$\mathbf{A}_2+\mathbf{A}_1+\mathbf{T}_1$
96	122 133 141 231	$(4,3)$	$4\mathbf{A}_1$	1 0 1 1 1 0 1 0	23	$2\mathbf{A}_1+\mathbf{T}_2$
97	121 132 143 154	$(5,4)$	$4\mathbf{A}_1$	3 0 0 0 0 0 0 1	24	$\mathbf{A}_3+\mathbf{T}_1$
98	122 151 341	$(5,2)$	$\mathbf{A}_2+\mathbf{A}_1$	1 0 0 1 1 2 0 0	24	$\mathbf{A}_2+\mathbf{A}_1+\mathbf{T}_2$
99	121 132 143	$(4,3)$	$3\mathbf{A}_1$	2 0 0 1 0 0 1 0	26	$\mathbf{A}_2+\mathbf{A}_1+\mathbf{T}_2$
100	121 342	$(4,2)$	$\mathbf{A}_2$	0 0 0 2 0 2 0 0	26	$\mathbf{A}_2+2\mathbf{A}_1+\mathbf{T}_2$
101	122 131 241	$(4,2)$	$3\mathbf{A}_1$	0 1 0 1 1 1 0 0	27	$\mathbf{A}_2+\mathbf{A}_1+\mathbf{T}_2$
102	121 132 233	$(3,3)$	$3\mathbf{A}_1$	0 0 2 0 0 0 1 0	27	$\mathbf{A}_2+2\mathbf{A}_1+\mathbf{T}_1$
103	121 132	$(3,2)$	$2\mathbf{A}_1$	1 0 1 0 0 1 0 0	30	$\mathbf{A}_2+2\mathbf{A}_1+\mathbf{T}_2$
104	121 341	$(4,1)$	$2\mathbf{A}_1$	0 0 0 1 2 0 0 0	34	$\mathbf{C}_2+\mathbf{A}_3+\mathbf{T}_1$
105	121	$(2,1)$	$\mathbf{A}_1$	0 1 0 0 1 0 0 0	37	$\mathbf{A}_3+\mathbf{A}_2+\mathbf{A}_1+\mathbf{T}_1$
106	0				48	$2\mathbf{A}_4$

3.2. The $\mathbf{Z}_5$-graded algebra $\mathbf{E}_8$. Let $\mathfrak{g}$ be a simple Lie algebra of the type $\mathbf{E}_8$. We define its $\mathbf{Z}_5$-grading by the following labelling of the affine Dynkin diagram $\mathbf{E}_8^{(1)}$ (cf. 2.4):



(We omit zero labels.) From the discussion in 2.4, it is clear that the Lie algebra $\mathfrak{g}_0$ and its θ-representation in $\mathfrak{g}_1$ is determined by the following Dynkin diagram with numerical labels of the (unique) highest weight:



The precise description of the $\mathbf{Z}_5$-graded algebra $\mathfrak{g}$ may be obtained as follows. We consider a direct sum

$$\mathfrak{g} = (U \otimes \textstyle\bigwedge^2 V^*) \oplus (\textstyle\bigwedge^2 U^* \otimes V^*) \oplus (\mathfrak{sl}(U) \oplus \mathfrak{sl}(V)) \oplus (\textstyle\bigwedge^2 U \otimes V) \oplus (U^* \otimes \textstyle\bigwedge^2 V)$$

and introduce a Lie algebra structure on it so that the above direct sum is a $\mathbf{Z}_5$-grading

$$\mathfrak{g} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2.$$

The commutator map $\mathfrak{g}_0 \times \mathfrak{g}_i \to \mathfrak{g}_i$ is the natural action of $\mathfrak{sl}(U) \oplus \mathfrak{sl}(V)$ on the respective space of tensors. The commutator maps $\mathfrak{g}_i \times \mathfrak{g}_j \to \mathfrak{g}_{i+j}$ $(i, j \in \mathbf{Z}_5)$ are determined by their $\mathfrak{g}_0$-equivariance uniquely up to proportionality. The coefficients of proportionality are determined uniquely by choosing the identification of $\mathfrak{g}_i$ with the respective space of tensors and verifying the Jacobi identity.

The commutators between $\mathfrak{g}_i$ and $\mathfrak{g}_j$ are given by the formulas

$$\begin{aligned}
[\alpha^*\otimes v^*,\alpha\otimes v] &= (\langle\alpha^*,\alpha\rangle/5\cdot\mathbf{1}-\mathrm{C}(\alpha^*\otimes\alpha))\cdot\langle v^*,v\rangle \\
&\qquad \oplus\langle\alpha^*,\alpha\rangle/2\cdot(\langle v^*,v\rangle/5\cdot\mathbf{1}-v^*\otimes v)\\
[u\otimes\beta^*,u^*\otimes\beta] &= (u^*\otimes u-\langle u^*,u\rangle/5\cdot\mathbf{1})\cdot\langle\beta^*,\beta\rangle/2\\
&\qquad \oplus\langle u^*,u\rangle\cdot(\langle\beta^*,\beta\rangle/5\cdot\mathbf{1}-\mathrm{C}(\beta^*\otimes\beta))\\
[u_1\wedge u_2\otimes v_1,u_3\wedge u_4\otimes v_2] &= \iota(u_1\wedge u_2\wedge u_3\wedge u_4)\otimes v_1\wedge v_2\\
[u_1\wedge u_2\otimes v,u^*\otimes v_1\wedge v_2] &= -2\mathrm{C}(u^*\otimes u_1\wedge u_2)\otimes\iota(v\wedge v_1\wedge v_2)\\
[u_1^*\otimes v_1\wedge v_2,u_2^*\otimes v_3\wedge v_4] &= -u_1^*\wedge u_2^*\otimes\iota(v_1\wedge v_2\wedge v_3\wedge v_4)\\
[u_1^*\wedge u_2^*\otimes v^*,u^*\otimes v_1\wedge v_2] &= 2\iota(u_1^*\wedge u_2^*\wedge u^*)\otimes\mathrm{C}(v^*\otimes v_1\wedge v_2)
\end{aligned}$$

(where $\alpha\in\bigwedge^2U$; $\beta\in\bigwedge^2V$; $\alpha^*\in\bigwedge^2U^*$; $\beta^*\in\bigwedge^2V^*$; $u,u_i\in U$; $v,v_i\in V$; $u^*,u_i^*\in U^*$; $v^*,v_i^*\in V^*$), and the formulas dual to the last four. (That is, vectors and bivectors are replaced by those in dual spaces, and the terms in each commutator are swapped.)

Next we describe the root system Δ of $\mathfrak{g}$ relative to $\mathfrak{t}$. Denote by Δ_k the set of nonzero weights of $\mathfrak{t}$ in $\mathfrak{g}_k$. We have $\Delta_0=\{{}_i\varepsilon-{}_j\varepsilon,\varepsilon_k-\varepsilon_l\}$, $\Delta_1=\{{}_i\varepsilon+{}_j\varepsilon+\varepsilon_k\}$, $\Delta_2=\{-{}_i\varepsilon+\varepsilon_k+\varepsilon_l\}$, $\Delta_{-1}=\{-{}_i\varepsilon-{}_j\varepsilon-\varepsilon_k\}$, $\Delta_{-2}=\{{}_i\varepsilon-\varepsilon_k-\varepsilon_l\}$ $(i\neq j,\ k\neq l)$.

The Weyl group W_0 permutes ${}_i\varepsilon$ and ε_i arbitrarily. We choose ${}_i\varepsilon-{}_{i+1}\varepsilon$, $\varepsilon_i-\varepsilon_{i+1}$ $(i=1,\dots,4)$ as simple roots of $\mathfrak{g}_0$. Together with ${}_4\varepsilon+{}_5\varepsilon+\varepsilon_5$, they compose an extended system of simple roots of $\mathfrak{g}$.

An invariant inner product on $\mathfrak{t}^*$ can be determined by the formulas $({}_i\varepsilon,{}_j\varepsilon)=(\varepsilon_s,\varepsilon_t)=-\frac{1}{5}$, $({}_i\varepsilon,{}_i\varepsilon)=(\varepsilon_s,\varepsilon_s)=\frac{4}{5}$, $({}_i\varepsilon,\varepsilon_s)=0$ $(i\neq j,\ s\neq t)$. In particular, if $\sum{}_ix=\sum x_s=0$, then $(\sum{}_ix{}_i\varepsilon+\sum x_s\varepsilon_s,\sum{}_iy{}_i\varepsilon+\sum y_s\varepsilon_s)=\sum{}_ix{}_iy+\sum x_sy_s$, and the inner products of roots of degree 1 can be computed by the following rule:

$$({}_i\varepsilon+{}_j\varepsilon+\varepsilon_s,{}_k\varepsilon+{}_l\varepsilon+\varepsilon_t)=\begin{cases}-1 & \text{if } \delta=0,\\ 0 & \text{if } \delta=1,\\ 1 & \text{if } \delta=2,\\ 2 & \text{if } \delta=3,\end{cases} \tag{4}$$

where δ is δ_{st} plus the number of common elements in $\{i,j\}$ and $\{k,l\}$.

3.3. A Cartan subspace. Consider regular $\mathbf{Z}_5$-graded subalgebras of type $\mathbf{A}_4$ in $\mathfrak{g}$ such that their degree 0 component is a Cartan subalgebra, and simple roots and the lowest root have degree 1. They are classified by embeddings of the extended system of simple roots $\mathbf{A}_4^{(1)}$ in Δ_1 preserving the inner product. It is easy to see that there is a unique such subalgebra up to W_0-conjugacy, say, the algebra $\mathfrak{A}_1$ generated by root vectors ${}_{12}e_4,{}_{34}e_1,{}_{51}e_3,{}_{23}e_5,{}_{45}e_2$. The algebra $\mathfrak{A}_2$ generated by ${}_{13}e_2,{}_{52}e_1,{}_{41}e_5,{}_{35}e_4,{}_{24}e_3$ is the unique subalgebra of the same kind commuting with $\mathfrak{A}_1$. It follows that the elements $u_1={}_{12}e_4+{}_{34}e_1+{}_{51}e_3+{}_{23}e_5+{}_{45}e_2$ and $u_2={}_{13}e_2+{}_{52}e_1+{}_{41}e_5+{}_{35}e_4+{}_{24}e_3$ are semisimple and commute with each other. Hence $\mathfrak{c}=\langle u_1,u_2\rangle$ is an abelian subspace in $\mathfrak{g}_1$ consisting of semisimple elements. It can be included in a $\mathbf{Z}_5$-graded Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ constructed as follows. The subspace $\mathfrak{h}_k$ is generated by $u_1^{(k)},u_2^{(k)}$, where $u_i^{(k)}$ is the sum of root vectors of degree k in $\mathfrak{A}_i$. An easy argument from [13, 3.1] shows that $\mathfrak{c}$ is a Cartan subspace. (Otherwise there exists a Cartan subspace of dimension ≥ 3, whence the minimal algebraic torus containing it has dimension $\geq 3 \cdot 4 = 12$, whereas $\operatorname{rk} \mathfrak{g} = 8$.)

3.4. The Weyl group. Now we want to determine the Weyl group W associated with the chosen Cartan subspace $\mathfrak{c}$. By definition, W is the group of linear transformations of $\mathfrak{c}$ generated by elements of G_0 normalizing $\mathfrak{c}$. First note that if a scalar transformation $\alpha \cdot \mathbf{1}$ belongs to W, then $\alpha^{10} = 1$. Indeed, we may regard u_i as operators $\bigwedge^2 U^* \to V$ and consider $u_1 \oplus u_2 : \bigwedge^2 U^* \to V \oplus V$. An element $g \in N_0(\mathfrak{c})$ acting on $\mathfrak{c}$ as $\alpha \cdot \mathbf{1}$ multiplies $u_1 \oplus u_2$ by α, hence $\bigwedge^{10}(u_1 \oplus u_2) \neq 0$ by α^{10}; but recall that $g \in SL(U) \times SL(V)$ acts on a one-dimensional space $\operatorname{Hom}(\bigwedge^{10}(\bigwedge^2 U^*), \bigwedge^{10}(V \oplus V))$ trivially. It is easy to pick elements from G_0, diagonal in the bases ${}_je, e_j$, that multiply u_k by arbitrary and independent 5-th roots of unity. Also note that a pair of monomial transformations of U and V given by the cyclic permutation $(1\,2\,4\,3)$ of basic vectors ${}_je, e_j$ combined with the multiplication by -1 yields a transformation $r \in W$, whose matrix in the basis u_1, u_2 is $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. It already follows that W is irreducible and its center is isomorphic to $\mathbf{Z}_{10}$.

Consider the (unique) $\mathbf{Z}_5$-graded Cartan subalgebra $\mathfrak{h} \supset \mathfrak{c}$ constructed in 3.3. The automorphism θ acts naturally on $\mathfrak{h}^*$ and permutes the roots of $\mathfrak{g}$ relative to $\mathfrak{h}$. It is represented by an integral matrix in a base of simple roots and has eigenvalues $\omega^{\pm 1}, \omega^{\pm 2}$ of multiplicity 2, where $\omega = e^{\frac{2\pi i}{5}}$. The minimal polynomial $\mu_\theta(t) = t^4 + t^3 + t^2 + t + 1$ is irreducible over $\mathbb{Q}$. The restriction of θ to each invariant subspace of $\mathfrak{h}^*$ generated by roots has an integral matrix as well, whence its eigenvalues are $\omega^{\pm 1}, \omega^{\pm 2}$ (of equal multiplicities). It follows that for each root α, $\operatorname{rk}\{\alpha, \theta\alpha, \theta^2\alpha, \theta^3\alpha, \theta^4\alpha\} = 4$ and $\sum_{k=0}^4 \theta^k \alpha = 0$. We conclude that $\{\alpha, \theta\alpha, \theta^2\alpha, \theta^3\alpha, \theta^4\alpha\}$ is the extended system of simple roots of type $\mathbf{A}_4^{(1)}$ of a certain $\mathfrak{h}$-regular θ-invariant simple subalgebra $\mathfrak{A}(\alpha)$, whose 20 roots are $\theta^k\alpha + \ldots + \theta^{k+l}\alpha$ $(k, l \geq 0)$.

The 240 roots of $\mathfrak{g}$ relative to $\mathfrak{h}$ fall into 12 such subsystems of type $\mathbf{A}_4$, hence we have 12 subalgebras $\mathfrak{A}(\alpha)$ of type $\mathbf{A}_4$, $\mathfrak{A}_1$ and $\mathfrak{A}_2$ being among them. We have a decomposition $\mathfrak{h} = \mathfrak{h}(\alpha) \oplus \mathfrak{h}^\alpha$, where $\mathfrak{h}(\alpha)$ is a Cartan subalgebra in $\mathfrak{A}(\alpha)$ and $\mathfrak{h}^\alpha$ is the centralizer of $\mathfrak{A}(\alpha)$ in $\mathfrak{h}$. Since there are no outer automorphisms of $\mathfrak{A}(\alpha)$ of order 5, the restriction $\theta|_{\mathfrak{A}(\alpha)}$ is induced by an inner automorphism $\operatorname{Ad} g_\alpha$, where g_α is an element of the subgroup $A(\alpha)$ of G with Lie algebra $\mathfrak{A}(\alpha)$. Clearly, $\theta(g_\alpha) = g_\alpha$, whence $g_\alpha \in G_0$. We have $\operatorname{Ad} g_\alpha|_{\mathfrak{h}} = \theta|_{\mathfrak{h}(\alpha)} \oplus \mathbf{1}|_{\mathfrak{h}^\alpha}$, and $w_\alpha = \operatorname{Ad} g_\alpha|_{\mathfrak{c}} \in W$ has eigenspaces $\mathfrak{h}(\alpha)_1 = \mathfrak{c} \cap \mathfrak{h}(\alpha)$ and $\mathfrak{h}_1^\alpha = \mathfrak{c} \cap \mathfrak{h}^\alpha$ and is represented by the matrix $\begin{pmatrix} \omega & 0 \\ 0 & 1 \end{pmatrix}$ in the eigenbasis.

Thus W contains 48 (complex) reflections w_α^k $(k = 1, \ldots, 4)$ of order 5, and its center is isomorphic to $\mathbf{Z}_{10}$. Among irreducible finite reflection groups acting on a 2-dimensional space, only one has these properties, namely the group No. 16 in the list of Shephard–Todd [10]. Its order is 600, and it can be represented as $W = \mathbf{Z}_5 \times \widetilde{I}$, where $\mathbf{Z}_5$ is generated by $z = \omega \cdot \mathbf{1}$ and $\widetilde{I}$ is the binary icosahedral group. The above 12 reflections w_α are all conjugated, and there are no other reflections in W but w_α^k.

The binary icosahedral group is examined in detail in [7]. In the basis u_1, u_2, $\widetilde{I}$ is generated by the matrices (see remark 3.1 below)

$$c = \begin{pmatrix} \omega & 0 \\ 0 & \omega^{-1} \end{pmatrix}, \quad r = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \text{and } s = \frac{1}{\sqrt{5}} \begin{pmatrix} \omega - \omega^{-1} & \omega^2 - \omega^{-2} \\ \omega^2 - \omega^{-2} & \omega^{-1} - \omega \end{pmatrix}.$$

Each element $w \in W$ can be expressed uniquely as $w = \pm z^a c^b r^i s^j c^k r^l$, where $a, b, k = 0, \dots, 4$; $i, j, l = 0, 1$; and $i = 1 \implies j = 0 \implies k = l = 0$.

Let z_1, z_2 be the coordinates on $\mathfrak{c}$ in the basis u_1, u_2. Then the basic invariants of $\widetilde{I}$ are

$$\begin{aligned} f &= z_1 z_2(-z_1^{10} + 11 z_1^5 z_2^5 + z_2^{10}) \\ g &= \frac{H(f)}{121} = -z_1^{20} - z_2^{20} - 228 z_1^{15} z_2^5 + 228 z_1^5 z_2^{15} - 494 z_1^{10} z_2^{10} \\ h &= \frac{J(f,g)}{20} = z_1^{30} + z_2^{30} - 522 z_1^{25} z_2^5 + 522 z_1^5 z_2^{25} - 10005 z_1^{20} z_2^{10} - 10005 z_1^{10} z_2^{20} \end{aligned}$$

where $H(\cdot)$ denotes the Hessian and $J(\cdot,\cdot)$ the Jacobian, and the basic syzygy is $1728 f^5 - g^3 - h^2 = 0$. The polynomial f is an eigenfunction for z of eigenvalue ω^{-2}, and g, h are z-invariant. It follows that g, h are (algebraically independent) basic invariants for $W : \mathfrak{c}$, which separate W-orbits.

3.5. Mixed elements. The centralizer $\mathfrak{z}(u)$ of $u \in \mathfrak{c}$ is an $\mathfrak{h}$-regular subalgebra spanned by $\mathfrak{h}$ and by the root subspaces $\mathfrak{g}_\alpha$ such that $\alpha(u) = 0$, or equivalently, $w_\alpha u = u$. The set of $u \in \mathfrak{c}$ such that $\mathfrak{z}(u)' \neq 0$ is a union of 12 lines in $\mathfrak{c}$ composing one projective W-orbit given by the equation $f = 0$ [7]. By 2.6, the semisimple part u of a mixed vector $x = u + e$ is equivalent to a nonzero vector from one of these lines. Hence the canonical form for u is $u = \lambda u_1$, where λ is determined up to multiplication by a 10-th root of unity.

For such u, $\mathfrak{z}(u)' = \mathfrak{A}_2$, $\mathfrak{z}(u)'_0$ is its Cartan subalgebra, and $\mathfrak{z}(u)'_1$ is spanned by 5 root vectors corresponding to the extended system of simple roots (of degree 1) of $\mathfrak{A}_2$ relative to $\mathfrak{z}(u)'_0$. These root vectors are just the summands in the expression for u_2 (3.3). Consider the coordinates on $\mathfrak{z}(u)'_1$ w. r. t. this basis of root vectors. Nilpotent orbits for the action $Z_0(u)^0 : \mathfrak{z}(u)'_1$ are given by the condition that coordinates from a certain subset are zero and other coordinates are nonzero.

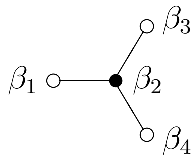
A pair of monomial transformations of U, V given by the same permutation $(1\,2\,3\,4\,5)$ fixes u_i and permutes basic vectors, their roots, and coordinates cyclically. Hence certain $Z_0(u)^0$-orbits are glued together under the action of $Z_0(u)$. No other orbits are glued together, since the respective sets of roots cannot be transformed into each other via an automorphism of $\mathfrak{z}(u)'$. (Just look at their inner products!)

Remark 3.1. In fact, the discussion in 3.4 determines W only up to conjugation by transposition of u_1, u_2. To make the correct choice, one should determine the set of $u = \lambda_1 u_1 + \lambda_2 u_2 \in \mathfrak{c}$ such that $\mathfrak{z}(u)_0 \neq 0$ explicitly. Consider the metabelian Lie algebra $L = U^* \oplus V$ with the structure tensor u. Then $\mathfrak{z}(u)_0$ is identified with the subalgebra of $\operatorname{Der} L \cap \mathfrak{sl}(L)$ leaving U^* stable. These derivations are determined by their restriction to U^*, and the defining relations of L impose 25 linear equations on 25 matrix entries of a derivation restricted to U^*. This linear system splits into five 5×5 subsystems. Four of them have determinant $-\lambda_1^{10} + 11\lambda_1^5\lambda_2^5 + \lambda_2^{10}$, and the 5-th one (involving diagonal entries) has corank 5 iff $\lambda_1\lambda_2 = 0$, and has corank 1 (and no nonzero traceless solutions) otherwise. Thence $\mathfrak{z}(u)_0 \neq 0$ iff $\lambda_1\lambda_2(-\lambda_1^{10} + 11\lambda_1^5\lambda_2^5 + \lambda_2^{10}) = 0$, and our choice of W in 3.4 is correct.

3.6. Nilpotent elements. Consider the covering loop algebra $\mathfrak{G}$ of $\mathfrak{g}$. The mapping $\alpha \mapsto (\bar{\alpha}, \deg \alpha)$ is an embedding $Q \hookrightarrow \mathbb{Z}\Delta \times \mathbb{Z}$. Under this identification, the extended system of simple roots of $\mathfrak{g}$ gives rise to a base $({}_i\varepsilon - {}_{i+1}\varepsilon, 0)$, $(\varepsilon_i - \varepsilon_{i+1}, 0)$ $(i = 1, \dots, 4)$, $({}_4\varepsilon + {}_5\varepsilon + \varepsilon_5, 1)$ of Q. As explained in 2.5, each semisimple $\mathbb{Z}$-graded T-regular subalgebra $\mathfrak{s} \subset \mathfrak{g}$ (given by its system of simple roots $\Pi(\mathfrak{s}) \subset \Delta$) is canonically embedded in $\mathfrak{G}$ (so that $\Pi(\mathfrak{s})_k \hookrightarrow \Delta_k \times \{k\}$), and $\mathfrak{s}$ is complete iff its root lattice $Q(\mathfrak{s}) = \mathbb{Z}\Pi(\mathfrak{s}) \subset Q$ is saturated.

There are 61 types of complete subalgebras of $\mathfrak{G}$ (given by types of various subsystems of Π). For some of these types, there are several (but no more than 11) locally flat algebras, and some of these algebras admit several (but no more than 4) non-conjugated embeddings in $\mathfrak{G}$. Summing up, we obtain 138 locally flat subalgebras of $\mathfrak{G}$. Some of them are not complete. Excluding them, we obtain, up to conjugacy, 105 supports of nilpotent elements, hence 106 nilpotent orbits of G_0 in $\mathfrak{g}_1$ (including 0).

Let us give an example of computation. Consider the locally flat algebra of type $\mathbf{D}_4(a_1)$. Its Dynkin diagram is given by the following picture, where the simple roots of degree 1 are indicated by white nodes, and the simple root of degree 0 is blackened:



An embedding of its system of simple roots in Δ is given by 3 pairwise orthogonal roots of degree 1, and one root of degree 0, whose inner product with these three equals 1. Up to a permutation of indices, there exists a unique such subset in Δ, namely $\Pi(\mathfrak{s}) = \{\, \beta_1 = {}_1\varepsilon + {}_5\varepsilon + \varepsilon_3,\ \beta_2 = {}_2\varepsilon - {}_1\varepsilon,\ \beta_3 = {}_1\varepsilon + {}_4\varepsilon + \varepsilon_2,\ \beta_4 = {}_1\varepsilon + {}_3\varepsilon + \varepsilon_1 \,\}$. The coordinates of β_i, considered as elements of Q, in the base Π are given by the rows of the following matrix:

$$\begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

The invariant factors of this matrix are equal to 1. Hence the sublattice $Q(\mathfrak{s}) = \mathbb{Z}\Pi(\mathfrak{s}) \subset Q$ is saturated and the respective regular locally flat subalgebra $\mathfrak{s} \subset \mathfrak{g}$ is complete.

The sum of root vectors corresponding to the roots β_1, $\beta_2 + \beta_3$, $\beta_2 + \beta_4$, β_4 is a nilpotent element from the dense orbit of the θ-representation for the algebra of type $\mathbf{D}_4(a_1)$. Hence $\mathfrak{s}$ is a support of $e = (\,153\;242\;231\;131\,)$ (in the notation of 1.5).

The characteristic $h = (\mathrm{diag}(x_1, \ldots, x_5), \mathrm{diag}(y_1, \ldots, y_5))$ of e is given by the solution of the linear system

$$\begin{cases} x_2 - x_1 &= 0 \\ x_1 + x_5 + y_3 &= 2 \\ x_1 + x_4 + y_2 &= 2 \\ x_1 + x_3 + y_1 &= 2 \\ x_1 + \ldots + x_5 &= 0 \\ y_1 + \ldots + y_5 &= 0 \end{cases}$$

orthogonal to all solutions of the corresponding homogeneous system. One finds $x_1 = x_2 = \frac{6}{5}$, $x_3 = x_4 = x_5 = -\frac{4}{5}$, $y_1 = y_2 = y_3 = \frac{8}{5}$, $y_4 = y_5 = -\frac{12}{5}$. Hence h lies in the positive Weyl chamber, and its indices are $(\,0\,2\,0\,0\quad 0\,0\,4\,0\,)$. (If $x_1, \ldots, x_5$ and/or $y_1, \ldots, y_5$ were not in decreasing order, then we would apply an appropriate pair of permutations of $1, \ldots, 5$ to $\Pi(\mathfrak{s})$, e, and h, so as to put them in order.)

Finally, we obtain information on the stabilizer of e. All eigenspaces $\mathfrak{g}_{i,k}(h)$ $(i = 0, 1;\; k \geq 0)$ are zero, except the following cases:

$$\begin{aligned}
\mathfrak{g}_{0,0}(h) &= \mathfrak{z}_0(h) = \left\{ \left(\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}, \begin{pmatrix} C & 0 \\ 0 & D \end{pmatrix} \right) \;\middle|\; A, D \in M_2(\mathbb{C}); \right. \\
&\qquad\qquad \left. B, C \in M_3(\mathbb{C});\ \operatorname{tr} A + \operatorname{tr} B = \operatorname{tr} C + \operatorname{tr} D = 0 \right\} \\
\mathfrak{g}_{0,2}(h) &= \left\{ \left(\begin{pmatrix} 0 & X \\ 0 & 0 \end{pmatrix}, 0 \right) \;\middle|\; X \in M_{2,3}(\mathbb{C}) \right\} \\
\mathfrak{g}_{0,4}(h) &= \left\{ \left(0, \begin{pmatrix} 0 & Y \\ 0 & 0 \end{pmatrix} \right) \;\middle|\; Y \in M_{3,2}(\mathbb{C}) \right\} \\
\mathfrak{g}_{1,2}(h) &= \langle\, {}_{ij}e_k \mid i = 1, 2;\; j = 3, 4, 5;\; k = 1, 2, 3 \,\rangle \\
\mathfrak{g}_{1,4}(h) &= \langle\, {}_{12}e_k \mid k = 1, 2, 3 \,\rangle
\end{aligned}$$

Hence the dimension formula (3) yields

$$\begin{aligned}
\dim \mathfrak{z}_0(e) &= \dim \mathfrak{g}_{0,0}(h) - \dim \mathfrak{g}_{1,2}(h) + \dim \mathfrak{g}_{0,2}(h) - \dim \mathfrak{g}_{1,4}(h) + \dim \mathfrak{g}_{0,4}(h) \\
&= 24 - 18 + 6 - 3 + 6 = 15
\end{aligned}$$

The Levi part of $\mathfrak{z}_0(e)$ is the stabilizer subalgebra of general position for the representation of $\mathfrak{g}_0(h) = \mathfrak{g}_{0,0}(h)$ in $\mathfrak{g}_1(h) = \mathfrak{g}_{1,2}(h)$. This is the natural representation of $\mathfrak{sl}_2(\mathbb{C}) \oplus \mathfrak{sl}_3(\mathbb{C}) \oplus \mathfrak{sl}_3(\mathbb{C}) \oplus \mathfrak{sl}_2(\mathbb{C}) \oplus \mathbb{C}^2$ in $\mathbb{C}^2 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3$, where the last $\mathfrak{sl}_2(\mathbb{C})$ acts trivially and the torus $\mathbb{C}^2$ acts by homotheties via the weight $(5, 2)$. The stabilizer of general position of the semisimple part of $\mathfrak{g}_0(h)$ has the type $\mathbf{A}_1 + \mathbf{T}_2$ and is smaller than the projective stabilizer. Hence the whole stabilizer has the type $\mathbf{A}_1 + \mathbf{T}_2 + \mathbf{T}_1 = \mathbf{A}_1 + \mathbf{T}_3$.

4. Metabelian algebras of signature $(6, 3)$

In this section, we classify the orbits of the θ-group $\bigwedge^2 SL_6(\mathbb{C}) \otimes SL_3(\mathbb{C})$ or, equivalently, metabelian Lie algebras of signature (m, n), $m \leq 6$, $n \leq 3$.

4.1. Formulation of results. We retain the notation of 1.5 with $m = 6,\ n = 3$. The classification of tensors in $\bigwedge^2 U \otimes V$ under the action of $SL(U) \times SL(V)$ is performed along the same lines as in 3.1. Here all tensors fall into 7 families according to the type of the stabilizer of their semisimple parts. The canonical forms for semisimple parts are given below, and the canonical forms for nilpotent parts are listed in tables 3–8. The notation in tables 3–8 is similar to that of tables 1–2, with two additional remarks. First, the canonical form for nilpotent parts in the 5-th family is represented in two ways, see 4.6.4 for details. Secondly, the characteristics of nilpotent parts in the 6-th family are indicated by 3 numerical indices (i. e., values of simple roots of $\mathfrak{z}(u)'_0$) and values of 3 linearly dependent weights generating the dual to the center of $\mathfrak{z}(u)'_0$, see 4.6.5 for details.

Here is the classification.

Family 1. This family contains only semisimple tensors. The canonical form is

$$u = \lambda_1 u_1 + \lambda_2 u_2 + \lambda_3 u_3, \qquad \begin{array}{l}(\lambda_1^3 - \lambda_2^3)(\lambda_1^3 - \lambda_3^3)(\lambda_2^3 - \lambda_3^3) \neq 0,\\ \lambda_1\lambda_2\lambda_3((\lambda_1^3 + \lambda_2^3 + \lambda_3^3)^3 - (3\lambda_1\lambda_2\lambda_3)^3) \neq 0,\end{array}$$

where

$$\begin{aligned} u_1 &= {}_{12}e_1 + {}_{34}e_2 + {}_{56}e_3, \\ u_2 &= {}_{54}e_1 + {}_{16}e_2 + {}_{32}e_3, \\ u_3 &= {}_{36}e_1 + {}_{52}e_2 + {}_{14}e_3. \end{aligned}$$

The coefficients λ_i are determined up to the action of a certain finite group W described in 4.4. Two canonical forms are equivalent iff they have the same values of

$$\begin{gathered} \lambda_1^6 + \lambda_2^6 + \lambda_3^6 - 10(\lambda_1^3\lambda_2^3 + \lambda_1^3\lambda_3^3 + \lambda_1^3\lambda_3^3), \\ (\lambda_1^3 + \lambda_2^3 + \lambda_3^3)((\lambda_1^3 + \lambda_2^3 + \lambda_3^3)^3 + (6\lambda_1\lambda_2\lambda_3)^3), \quad \text{and} \\ (\lambda_1^3 - \lambda_2^3)^2(\lambda_1^3 - \lambda_3^3)^2(\lambda_2^3 - \lambda_3^3)^2. \end{gathered}$$

The stabilizer subalgebra of u is a one-dimensional torus.

Family 2. The canonical form of a semisimple part is

$$u = \lambda u_2 + \mu u_3, \qquad \lambda, \mu, \lambda^3 \pm \mu^3 \neq 0,$$

where λ, μ are determined up a permutation, simultaneous sign change and multiplication by two independent cubic roots of unity.

Family 3. The canonical form of a semisimple part is

$$u = \lambda u_1 + \mu(u_2 + u_3), \qquad \lambda, \mu, \lambda^3 - \mu^3, \lambda^3 + 8\mu^3 \neq 0,$$

where λ, μ are determined up the action of a certain group of order 72 (group No. 5 in the list of Shephard–Todd [10]). Two canonical forms are equivalent iff they have the same values of

$$\begin{gathered} \lambda^6 - 20\lambda^3\mu^3 - 8\mu^6 \qquad \text{and} \\ (\lambda^3 + 2\mu^3)((\lambda^3 + 2\mu^3)^3 + 108\lambda^3\mu^6). \end{gathered}$$

Table 3: Nilpotent parts of elements of family 2

No.	Canonical form	Support	Characteristic	Stabilizer	
				dim	Type
1	121 342	$\mathbf{A}_2$	2 2	1	$\mathbf{T}_1$
2	121	$\mathbf{A}_1$	2 0	2	$\mathbf{T}_2$
3	0			3	$\mathbf{T}_3$

Table 4: Nilpotent parts of elements of family 3

No.	Canonical form	Support	Characteristic	Stabilizer	
				dim	Type
1	531 152 313	$\mathbf{A}_1$	2	1	0
2	0			3	$\mathbf{A}_1$

Family 4. The canonical form of a semisimple part is

$$u = \lambda(u_2 + u_3), \qquad \lambda \neq 0,$$

where λ is determined up to multiplication by a 6-th root of unity.

Table 5: Nilpotent parts of elements of family 4

No.	Canonical form	Support	Characteristic	Stabilizer	
				dim	Type
1	121 342 531 152 313	$\mathbf{A}_2 + \mathbf{A}_1$	2 2 2	1	0
2	121 531 152 313	$2\mathbf{A}_1$	2 0 2	2	$\mathbf{T}_1$
3	531 152 313	$\mathbf{A}_1$	0 0 2	3	$\mathbf{T}_2$
4	121 342	$\mathbf{A}_2$	2 2 0	3	$\mathbf{A}_1$
5	121	$\mathbf{A}_1$	2 0 0	4	$\mathbf{A}_1 + \mathbf{T}_1$
6	0			5	$\mathbf{A}_1 + \mathbf{T}_2$

Family 5. The canonical form of a semisimple part is

$$u = \lambda(u_3 - u_2), \qquad \lambda \neq 0,$$

where λ is determined up to multiplication by a 6-th root of unity.

Family 6. The canonical form of a semisimple part is

$$u = \lambda u_1, \qquad \lambda \neq 0,$$

where λ is determined up to multiplication by a 6-th root of unity.

Family 7. This family consists of nilpotent elements.

Table 6: Nilpotent parts of elements of family 5

No.	Canonical form		Support	Characteristic	Stabilizer	
	in S^3Q	in $\bigwedge^2 U \otimes V$			dim	Type
1	$\frac{1}{2}x^2z + \frac{1}{6}y^3$	123 161 521 342	$\mathbf{G}_2$	2 4	1	$\mathbf{T}_1$
2	$\frac{1}{2}x^2z + \frac{1}{2}xy^2$	123 161 521 142 322 341	$\mathbf{A}_2$	2 2	2	$\mathbf{T}_1$
3	$\frac{1}{6}x^3 + \frac{1}{6}y^3$	121 342	$\mathbf{G}_2(f)$	0 2	3	$\mathbf{T}_1$
4	$\frac{1}{2}x^2y$	122 141 321	$\mathbf{A}_1$	1 1	4	$\mathbf{T}_2$
5	$\frac{1}{6}x^3$	121	$\mathbf{A}_1$	1 0	6	$\mathbf{A}_1 + \mathbf{T}_1$
6	0	0			9	$\mathbf{A}_2 + \mathbf{T}_1$

Table 7: Nilpotent parts of elements of family 6

No.	Canonical form	Support	Characteristic	Stabilizer	
				dim	Type
1	143 162 233 252 351	$\mathbf{A}_5$	6 6 6 8 −4 −4	1	0
2	143 152 233 361	$\mathbf{A}_4$	6 6 0 2 2 −4	2	$\mathbf{T}_1$
3	143 152 361 451	$\mathbf{A}_3 + \mathbf{A}_1$	6 0 0 −4 2 2	2	$\mathbf{T}_1$
4	133 152 361 451	$2\mathbf{A}_2$	4 2 2 −4 2 2	3	$\mathbf{T}_1$
5	133 162 243 252	$2\mathbf{A}_2$	0 0 0 8 −4 −4	3	$\mathbf{A}_1$
6	143 162 351	$\mathbf{A}_3$	6 1 1 −2 1 1	3	$\mathbf{T}_2$
7	133 152 361	$\mathbf{A}_2 + \mathbf{A}_1$	3 3 0 −1 −1 2	4	$\mathbf{T}_2$
8	133 361 451	$\mathbf{A}_2 + \mathbf{A}_1$	2 1 1 −4 5 −1	4	$\mathbf{T}_2$
9	152 361 451	$\mathbf{A}_2 + \mathbf{A}_1$	2 1 1 −4 −1 5	4	$\mathbf{T}_2$
10	133 152 351	$3\mathbf{A}_1$	2 2 2 0 0 0	5	$\mathbf{T}_2$
11	152 361	$\mathbf{A}_2$	2 2 0 −2 −2 4	5	$\mathbf{T}_3$
12	133 152	$2\mathbf{A}_1$	2 1 1 2 −1 −1	6	$\mathbf{T}_3$
13	133 243	$2\mathbf{A}_1$	0 0 0 2 2 −4	7	$2\mathbf{A}_1 + \mathbf{T}_1$
14	133	$\mathbf{A}_1$	1 1 0 1 1 −2	8	$\mathbf{A}_1 + \mathbf{T}_3$
15	0			11	$3\mathbf{A}_1 + \mathbf{T}_2$

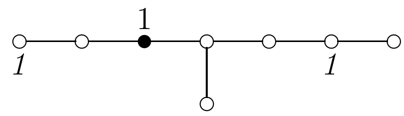
Table 8: Elements of family 7

No.	Canonical form	Signature	Support	Characteristic	Stabilizer	
					dim	Type
1	143 162 233 252 261 342 351	$(6,3)$	$\mathbf{E}_7$	6 6 6 6 6 6 12	1	0
2	153 162 233 242 252 261 341	$(6,3)$	$\mathbf{E}_7(a_1)$	6 6 6 0 6 6 6	2	0
3	133 152 161 243 252 342 351	$(6,3)$	$\mathbf{E}_7(a_2)$	0 6 0 6 6 6 6	2	0
4	143 161 233 242 251 341 352	$(6,3)$	$\mathbf{E}_7(b)$	6 0 6 0 6 0 6	3	0
5	143 152 233 261 342 351	$(6,3)$	$\mathbf{E}_6$	2 4 2 6 4 6 6	3	$\mathbf{T}_1$
6	123 162 252 342 361 451	$(6,3)$	$\mathbf{E}_6$	0 6 0 6 0 6 12	3	$\mathbf{A}_1$
7	133 142 153 161 243 252 341	$(6,3)$	$\mathbf{E}_7(c_1)$	0 6 0 0 6 6 0	4	0
8	143 151 233 262 341 342	$(6,3)$	$\mathbf{E}_6(a_1)$	0 6 0 6 0 0 6	4	$\mathbf{T}_1$
9	133 152 242 261 342 451	$(6,3)$	$\mathbf{E}_6(a_1)$	4 0 2 4 2 6 6	4	$\mathbf{T}_1$
10	153 162 233 242 251 341	$(6,3)$	$\mathbf{D}_6$	6 1 5 1 5 1 5	4	$\mathbf{T}_1$
11	123 141 152 242 261 351 362	$(6,3)$	$\mathbf{E}_7(c_2)$	0 0 6 0 0 0 6	5	0
12	133 152 242 261 351 451	$(6,3)$	$\mathbf{D}_6(a_1)$	1 5 0 1 5 6 1	5	$\mathbf{T}_1$
13	143 152 161 233 242 351	$(6,3)$	$\mathbf{A}_6$	2 2 2 2 4 4 2	5	$\mathbf{T}_1$
14	143 152 233 242 261 341	$(6,3)$	$\mathbf{D}_5 + \mathbf{A}_1$	1 5 1 1 4 5 1	5	$\mathbf{T}_1$

Table 8: (Continued)

No.	Canonical form	Signature	Support	Characteristic	Stabilizer	
					dim	Type
15	143 152 161 233 242 251 341	$(6,3)$	$\mathbf{A}_5+\mathbf{A}_2$	2 2 2 2 2 2 2	6	0
16	123 161 241 252 342 351	$(6,3)$	$\mathbf{E}_6(b)$	2 0 4 0 2 0 6	6	$\mathbf{T}_1$
17	133 152 233 242 261 341	$(6,3)$	$\mathbf{D}_6(a_2)$	0 1 5 0 1 1 5	6	$\mathbf{T}_1$
18	142 153 232 261 341	$(6,3)$	$\mathbf{D}_5$	2 4 2 0 4 6 0	6	$\mathbf{T}_2$
19	133 152 161 242 251 351	$(6,3)$	$\mathbf{D}_5(a_1)+\mathbf{A}_1$	2 0 2 2 2 2 4	7	$\mathbf{T}_1$
20	152 161 233 242 251 341	$(6,3)$	$\mathbf{A}_5+\mathbf{A}_1$	1 1 4 1 1 1 4	7	$\mathbf{T}_1$
21	122 133 151 243 261 342	$(6,3)$	$\mathbf{E}_6(b)$	0 0 0 6 0 6 0	7	$\mathbf{A}_1$
22	133 162 242 251 341	$(6,3)$	$\mathbf{A}_5$	2 1 3 1 1 1 5	7	$\mathbf{T}_2$
23	123 132 242 341 561	$(6,3)$	$\mathbf{D}_5$	0 6 0 0 0 6 6	7	$2\mathbf{A}_1$
24	143 152 161 232 251 341	$(6,3)$	$\mathbf{A}_4+\mathbf{A}_2$	2 2 0 2 2 2 2	8	$\mathbf{T}_1$
25	143 152 161 233 241 342	$(6,3)$	$\mathbf{A}_5+\mathbf{A}_1$	3 0 3 3 0 0 3	8	$\mathbf{A}_1$
26	131 153 162 233 242 261	$(6,3)$	$\mathbf{A}_4+\mathbf{A}_2$	0 6 0 0 0 0 0	8	$\mathbf{A}_1$
27	121 153 233 252 342 451	$(5,3)$	$\mathbf{A}_3+\mathbf{A}_2+\mathbf{A}_1$	0 0 0 0 6 0 0	8	$\mathbf{A}_1$
28	133 161 242 251 351	$(6,3)$	$\mathbf{D}_5(a_1)$	1 0 2 3 1 3 3	8	$\mathbf{T}_2$
29	123 161 162 252 341	$(6,3)$	$\mathbf{D}_5(a_1)$	0 3 0 3 0 0 6	8	$\mathbf{A}_1+\mathbf{T}_1$
30	123 132 242 361 451	$(6,3)$	$\mathbf{A}_5$	0 1 0 5 0 6 1	8	$\mathbf{A}_1+\mathbf{T}_1$
31	123 152 232 261 341	$(6,3)$	$\mathbf{A}_4+\mathbf{A}_1$	1 2 1 0 2 3 3	9	$\mathbf{T}_2$
32	143 152 161 232 251	$(6,3)$	$\mathbf{A}_4+\mathbf{A}_1$	1 4 0 1 1 1 1	9	$\mathbf{T}_2$
33	143 151 232 261 341	$(6,3)$	$\mathbf{A}_4+\mathbf{A}_1$	0 3 0 3 0 3 0	9	$\mathbf{T}_2$
34	141 152 233 242 351	$(5,3)$	$\mathbf{A}_3+\mathbf{A}_2$	0 0 1 0 5 0 1	9	$\mathbf{T}_2$
35	152 161 233 241 342	$(6,3)$	$\mathbf{A}_5$	2 0 4 2 0 0 4	9	$\mathbf{A}_1+\mathbf{T}_1$
36	123 152 161 251 342	$(6,3)$	$\mathbf{D}_4+\mathbf{A}_1$	1 2 0 3 1 1 5	9	$\mathbf{A}_1+\mathbf{T}_1$
37	123 152 161 232 251 341	$(6,3)$	$\mathbf{A}_3+\mathbf{A}_2+\mathbf{A}_1$	2 0 2 0 2 2 2	10	$\mathbf{T}_1$
38	143 151 232 243 341	$(5,3)$	$\mathbf{D}_4(a_1)+\mathbf{A}_1$	0 1 0 1 4 1 0	10	$\mathbf{T}_2$
39	143 151 232 261	$(6,3)$	$\mathbf{A}_4$	0 4 0 2 0 2 0	10	$\mathbf{T}_3$
40	133 152 161 232 241	$(6,3)$	$\mathbf{A}_3+\mathbf{A}_2$	2 1 1 1 1 1 2	11	$\mathbf{T}_2$
41	123 152 232 251 341	$(5,3)$	$\mathbf{A}_3+2\mathbf{A}_1$	1 0 1 0 4 1 1	11	$\mathbf{T}_2$
42	131 153 242 253	$(5,3)$	$\mathbf{D}_4(a_1)$	0 2 0 0 4 0 0	11	$\mathbf{T}_3$
43	123 152 261 341	$(6,3)$	$\mathbf{A}_4$	2 2 0 0 2 4 2	11	$\mathbf{A}_1+\mathbf{T}_2$
44	123 151 261 342	$(6,3)$	$\mathbf{D}_4$	0 2 0 4 0 2 4	11	$2\mathbf{A}_1+\mathbf{T}_1$
45	121 342 561 562	$(6,2)$	$\mathbf{D}_4(a_1)$	0 0 0 0 0 0 6	11	$3\mathbf{A}_1$
46	123 142 151 232 261 341	$(6,3)$	$3\mathbf{A}_2$	0 2 0 2 0 2 2	12	$\mathbf{A}_1$
47	133 152 232 241	$(5,3)$	$\mathbf{A}_3+\mathbf{A}_1$	1 1 0 1 3 0 1	12	$\mathbf{T}_3$
48	142 151 232 461	$(6,2)$	$\mathbf{A}_3+\mathbf{A}_1$	0 0 0 1 0 1 5	12	$2\mathbf{A}_1+\mathbf{T}_1$
49	133 142 151 232 241	$(5,3)$	$2\mathbf{A}_2+\mathbf{A}_1$	1 0 1 1 2 1 1	13	$\mathbf{T}_2$
50	142 151 232 261 341	$(6,2)$	$2\mathbf{A}_2+\mathbf{A}_1$	0 1 0 1 0 1 4	13	$\mathbf{A}_1+\mathbf{T}_1$
51	122 153 161 251 341	$(6,3)$	$\mathbf{A}_3+2\mathbf{A}_1$	3 0 0 0 3 3 0	13	$2\mathbf{A}_1$
52	123 132 161 241 351	$(6,3)$	$2\mathbf{A}_2+\mathbf{A}_1$	1 0 2 0 1 3 0	14	$\mathbf{A}_1+\mathbf{T}_1$
53	123 131 162 242 251	$(6,3)$	$2\mathbf{A}_2+\mathbf{A}_1$	0 3 0 0 0 0 3	14	$2\mathbf{A}_1$
54	142 153 161 231	$(6,3)$	$\mathbf{A}_3+\mathbf{A}_1$	3 0 1 0 2 2 0	14	$2\mathbf{A}_1+\mathbf{T}_1$
55	122 153 251 341	$(5,3)$	$\mathbf{A}_3+\mathbf{A}_1$	2 0 0 0 4 2 0	14	$2\mathbf{A}_1+\mathbf{T}_1$
56	123 131 152 242 251	$(5,3)$	$\mathbf{A}_2+3\mathbf{A}_1$	0 2 0 0 2 0 2	15	$\mathbf{A}_1+\mathbf{T}_1$
57	123 141 232 351	$(5,3)$	$2\mathbf{A}_2$	0 0 2 0 2 2 0	15	$\mathbf{A}_1+\mathbf{T}_2$
58	131 162 242 251	$(6,2)$	$2\mathbf{A}_2$	0 2 0 0 0 0 4	15	$2\mathbf{A}_1+\mathbf{T}_1$
59	142 153 231	$(5,3)$	$\mathbf{A}_3$	2 0 1 0 3 1 0	15	$2\mathbf{A}_1+\mathbf{T}_2$
60	123 142 151 231	$(5,3)$	$\mathbf{A}_2+2\mathbf{A}_1$	1 1 0 1 1 1 1	16	$\mathbf{T}_3$
61	131 152 242 251	$(5,2)$	$\mathbf{A}_2+2\mathbf{A}_1$	0 1 0 0 2 0 3	16	$\mathbf{A}_1+\mathbf{T}_2$
62	122 341 561	$(6,2)$	$\mathbf{A}_3$	0 1 0 0 0 2 4	16	$\mathbf{C}_2+\mathbf{A}_1+\mathbf{T}_1$
63	122 161 251 341	$(6,2)$	$\mathbf{A}_2+2\mathbf{A}_1$	0 1 0 1 0 2 2	17	$2\mathbf{A}_1+\mathbf{T}_1$
64	121 133 243 342	$(4,3)$	$\mathbf{A}_2+2\mathbf{A}_1$	0 0 0 3 0 0 0	17	$3\mathbf{A}_1$
65	142 151 231	$(5,2)$	$\mathbf{A}_2+\mathbf{A}_1$	1 0 0 1 1 1 2	18	$\mathbf{A}_1+\mathbf{T}_3$
66	123 131 242	$(4,3)$	$\mathbf{A}_2+\mathbf{A}_1$	0 1 0 2 0 0 1	18	$\mathbf{A}_1+\mathbf{T}_3$
67	122 133 141 231	$(4,3)$	$4\mathbf{A}_1$	1 0 1 1 0 1 0	20	$2\mathbf{A}_1+\mathbf{T}_1$
68	121 342	$(4,2)$	$\mathbf{A}_2$	0 0 0 2 0 0 2	21	$3\mathbf{A}_1+\mathbf{T}_2$
69	122 131 241	$(4,2)$	$3\mathbf{A}_1$	0 1 0 1 0 1 1	22	$2\mathbf{A}_1+\mathbf{T}_2$
70	121 132 143	$(4,3)$	$3\mathbf{A}_1$	2 0 0 1 0 0 0	23	$\mathbf{A}_2+\mathbf{A}_1+\mathbf{T}_1$
71	121 132 233	$(3,3)$	$3\mathbf{A}_1$	0 0 2 0 0 0 0	25	$2\mathbf{A}_2$
72	121 341 561	$(6,1)$	$3\mathbf{A}_1$	0 0 0 0 0 3 0	26	$\mathbf{C}_3+\mathbf{A}_1$
73	121 132	$(3,2)$	$2\mathbf{A}_1$	1 0 1 0 0 0 1	26	$\mathbf{A}_2+\mathbf{A}_1+\mathbf{T}_2$
74	121 341	$(4,1)$	$2\mathbf{A}_1$	0 0 0 1 0 2 0	27	$\mathbf{C}_2+2\mathbf{A}_1+\mathbf{T}_1$
75	121	$(2,1)$	$\mathbf{A}_1$	0 1 0 0 0 1 0	32	$\mathbf{A}_3+2\mathbf{A}_1+\mathbf{T}_1$
76	0				43	$\mathbf{A}_5+\mathbf{A}_2$

4.2. The $\mathbf{Z}_3$-graded algebra $\mathbf{E}_7$. Let $\mathfrak{g}$ be a simple Lie algebra of the type $\mathbf{E}_7$. We define its $\mathbf{Z}_3$-grading and the respective θ-representation by the following labelling of the affine Dynkin diagram $\mathbf{E}_7^{(1)}$ (cf. 2.4):



(We omit zero labels and blacken the unique simple root of positive degree. The upright digit indicates the degree of this root, and italic digits indicate numerical labels of the highest weight of the θ-representation.)

The precise description of the $\mathbf{Z}_3$-graded algebra $\mathfrak{g}$ may be obtained as follows. We consider a direct sum

$$\mathfrak{g} = (\textstyle\bigwedge^2 U^* \otimes V^*) \oplus (\mathfrak{sl}(U) \oplus \mathfrak{sl}(V)) \oplus (\textstyle\bigwedge^2 U \otimes V)$$

and introduce a Lie algebra structure on it so that the above direct sum is a $\mathbf{Z}_3$-grading

$$\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1.$$

The commutator maps $\mathfrak{g}_i \times \mathfrak{g}_j \to \mathfrak{g}_{i+j}$ are determined as in 3.2. Here are the precise formulas:

$$\begin{aligned}
[\alpha^* \otimes v^*, \alpha \otimes v] &= (\langle \alpha^*, \alpha \rangle /6 \cdot \mathbf{1} - \mathrm{C}(\alpha^* \otimes \alpha)) \cdot \langle v^*, v \rangle \\
&\qquad \oplus \langle \alpha^*, \alpha \rangle /2 \cdot (\langle v^*, v \rangle /3 \cdot \mathbf{1} - v^* \otimes v) \\
[u_1 \wedge u_2 \otimes v_1, u_3 \wedge u_4 \otimes v_2] &= 2\iota(u_1 \wedge u_2 \wedge u_3 \wedge u_4) \otimes \iota(v_1 \wedge v_2) \\
[u_1^* \wedge u_2^* \otimes v_1^*, u_3^* \wedge u_4^* \otimes v_2^*] &= -2\iota(u_1^* \wedge u_2^* \wedge u_3^* \wedge u_4^*) \otimes \iota(v_1^* \wedge v_2^*)
\end{aligned}$$

(where $\alpha \in \bigwedge^2 U$; $\alpha^* \in \bigwedge^2 U^*$; $u_i \in U$; $v, v_i \in V$; $u^*, u_i^* \in U^*$; $v^*, v_i^* \in V^*$).

Let us describe the root system Δ of $\mathfrak{g}$ relative to $\mathfrak{t}$. Denote by Δ_k the set of nonzero weights of $\mathfrak{t}$ in $\mathfrak{g}_k$. We have $\Delta_0 = \{{}_i\varepsilon - {}_j\varepsilon, \varepsilon_i - \varepsilon_j\}$, $\Delta_1 = \{{}_i\varepsilon + {}_j\varepsilon + \varepsilon_k\}$, $\Delta_{-1} = \{-{}_i\varepsilon - {}_j\varepsilon - \varepsilon_k\}$ $(i \neq j)$.

The Weyl group W_0 permutes ${}_i\varepsilon$ and ε_i arbitrarily. We choose ${}_i\varepsilon - {}_{i+1}\varepsilon$ $(i = 1, \dots, 5)$, $\varepsilon_j - \varepsilon_{j+1}$ $(j = 1, 2)$ as simple roots of $\mathfrak{g}_0$. Together with ${}_5\varepsilon + {}_6\varepsilon + \varepsilon_3$, they compose an extended system of simple roots of $\mathfrak{g}$.

An invariant inner product on $\mathfrak{t}^*$ can be determined by the formulas $({}_i\varepsilon, {}_j\varepsilon) = -\frac{1}{6}$, $({}_i\varepsilon, {}_i\varepsilon) = \frac{5}{6}$, $(\varepsilon_s, \varepsilon_t) = -\frac{1}{3}$, $(\varepsilon_s, \varepsilon_s) = \frac{2}{3}$, $({}_i\varepsilon, \varepsilon_s) = 0$ $(i \neq j,\ s \neq t)$. In particular, we have $(\sum {}_i x {}_i\varepsilon + \sum x_s \varepsilon_s, \sum {}_i y {}_i\varepsilon + \sum y_s \varepsilon_s) = \sum {}_i x {}_i y + \sum x_s y_s$ whenever $\sum {}_i x = \sum x_s = 0$. The inner products of roots of degree 1 can be computed by formula (4).

4.3. A Cartan subspace. Consider regular $\mathbf{Z}_3$-graded subalgebras of type $\mathbf{A}_2$ in $\mathfrak{g}$ such that their degree 0 component is a Cartan subalgebra, and simple roots and the lowest root have degree 1. They are classified by embeddings of the extended system of simple roots $\mathbf{A}_2^{(1)}$ in Δ_1 preserving the inner product. Up to W_0-conjugacy, there is a unique such subalgebra, say, the algebra $\mathfrak{A}_1$ generated by root vectors ${}_{12}e_1, {}_{34}e_2, {}_{56}e_3$. Consider also an algebra $\mathfrak{A}_2$ generated by ${}_{54}e_1, {}_{16}e_2, {}_{32}e_3$, and $\mathfrak{A}_3$ generated by ${}_{36}e_1, {}_{52}e_2, {}_{14}e_3$. Making use of (4), it is easy to see that $(\mathfrak{A}_1, \mathfrak{A}_2, \mathfrak{A}_3)$ is the unique, up to W_0-conjugacy, triple of subalgebras of the above type commuting with each other. It follows that the elements $u_1 = {}_{12}e_1 + {}_{34}e_2 + {}_{56}e_3$, $u_2 = {}_{54}e_1 + {}_{16}e_2 + {}_{32}e_3$, $u_3 = {}_{36}e_1 + {}_{52}e_2 + {}_{14}e_3$ are semisimple and commute with each other. Hence $\mathfrak{c} = \langle u_1, u_2, u_3\rangle$ is an abelian subspace in $\mathfrak{g}_1$ consisting of semisimple elements. It can be included in a $\mathbf{Z}_3$-graded Cartan subalgebra $\mathfrak{h} = \mathfrak{c}^* \oplus \langle c\rangle \oplus \mathfrak{c}$ of $\mathfrak{g}$, where $c = (\operatorname{diag}(1,-1,1,-1,1,-1),0) \in \mathfrak{g}_0$, and $\mathfrak{c}^* = \langle u^1, u^2, u^3\rangle \subset \mathfrak{g}_{-1}$ is generated by semisimple elements $u^i \in \mathfrak{A}_i$ obtained from u_i by lifting the indices. (The u^i are sums of root vectors of degree -1 in $\mathfrak{A}_i$.) The same argument as in 3.3 shows that $\mathfrak{c}$ is a Cartan subspace.

4.4. The Weyl group. In order to find the Weyl group W associated with $\mathfrak{c}$, we first collect sufficiently many linear transformations of $\mathfrak{c}$ induced by elements of G_0 normalizing $\mathfrak{c}$. It is easy to pick elements from G_0, diagonal in the bases ${}_je, e_j$, that multiply u_k by arbitrary and independent cubic roots of unity. The permutation $(1\,2)(3\,4)(5\,6)$ of basic vectors ${}_je$ combined with the multiplication by i induces a linear transformation of $\mathfrak{c}$ transposing u_2, u_3 and leaving u_1 fixed. Other transpositions of u_k can be obtained in the same way. It follows that W contains a subgroup $G(3,1,3)$ of monomial transformations in the basis u_k, whose nonzero matrix entries are cubic roots of unity. In particular, W is irreducible. Also note that a pair of monomial transformations of U and V given by permutations $(1\,2)(5\,4)(3\,6)$ and $(2\,3)$ of basic vectors combined with the multiplication by i and -1, respectively, yields $-\mathbf{1} \in W$.

Consider the root system of $\mathfrak{g}$ relative to $\mathfrak{h}$. The automorphism θ acts naturally on $\mathfrak{h}^*$ and permutes the roots. It is represented by an integral matrix in a base of simple roots and has eigenvalues $1, \omega, \omega^{-1}$, where $\omega = e^{\frac{2\pi i}{3}}$. Moreover, the restriction of θ to each invariant subspace of $\mathfrak{h}^*$ generated by roots has an integral matrix, too, whence eigenvalues ω, ω^{-1} occur with equal multiplicities.

The root system is divided in two subclasses. The first one consists of roots α such that $\alpha(c) = 0$. Since $\dim \mathfrak{z}(c) = 79$, there are 72 such roots. The second subclass consists of remaining 54 roots β, which satisfy $\beta(c) \neq 0$. Each root α in the first subclass has nonzero projections on θ-eigenspaces of eigenvalues $\omega^{\pm 1}$, and $\alpha + \theta\alpha + \theta^2\alpha = 0$. It follows that $\operatorname{rk}\{\alpha, \theta\alpha, \theta^2\alpha\} = 2$ and $\{\alpha, \theta\alpha, \theta^2\alpha\}$ is the extended system of simple roots of type $\mathbf{A}_2^{(1)}$ of a certain $\mathfrak{h}$-regular θ-invariant simple subalgebra $\mathfrak{A}(\alpha)$, whose 6 roots are $\pm\theta^k\alpha$ $(k = 0,1,2)$.

The 72 roots of the first subclass fall into 12 such subsystems of type $\mathbf{A}_2$, hence we obtain 12 subalgebras $\mathfrak{A}(\alpha)$ of type $\mathbf{A}_2$, including $\mathfrak{A}_1, \mathfrak{A}_2, \mathfrak{A}_3$. We have a decomposition $\mathfrak{h} = \mathfrak{h}(\alpha) \oplus \mathfrak{h}^\alpha$, where $\mathfrak{h}(\alpha)$ is a Cartan subalgebra in $\mathfrak{A}(\alpha)$ and $\mathfrak{h}^\alpha$ is the centralizer of $\mathfrak{A}(\alpha)$ in $\mathfrak{h}$. (Note that $c \in \mathfrak{h}^\alpha$.) The same reasoning as in 3.4 shows that there exists an element $g_\alpha \in G_0$ such that $\operatorname{Ad} g_\alpha|_{\mathfrak{h}} = \theta|_{\mathfrak{h}(\alpha)} \oplus \mathbf{1}|_{\mathfrak{h}^\alpha}$, and $w_\alpha = \operatorname{Ad} g_\alpha|_{\mathfrak{c}} \in W$ has eigenspaces $\mathfrak{h}(\alpha)_1 = \mathfrak{c} \cap \mathfrak{h}(\alpha)$ and $\mathfrak{h}^\alpha_1 = \mathfrak{c} \cap \mathfrak{h}^\alpha = \operatorname{Ker} \alpha|_{\mathfrak{c}}$ and is represented by the matrix

$$\begin{pmatrix} \omega & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

in the eigenbasis.

Thus W contains 24 reflections $w_\alpha^{\pm 1}$ of order 3, and $-\mathbf{1}$. Among irreducible finite reflection groups acting on a 3-dimensional space, only one has these properties, namely the group No. 26 in the list of Shephard–Todd. The order of W

is 1296, and it is generated by $-\mathbf{1}$, the subgroup $G(3,1,3)$, and a 3-fold reflection

$$s = \frac{-i}{\sqrt{3}} \begin{pmatrix} \omega & \omega^2 & \omega^2 \\ \omega^2 & \omega & \omega^2 \\ \omega^2 & \omega^2 & \omega \end{pmatrix}$$

(in the basis u_k), see [10]. The above 12 reflections w_α are conjugated, and there are no 3-fold reflections in W other than $w_\alpha^{\pm 1}$.

However, W contains in addition 9 reflections of order 2. They can be described as follows. Let β be a root from the second subclass. Suppose that $\beta|_{\mathfrak{c}^*\oplus\mathfrak{c}} \neq 0$; then β has nonzero projection on each eigenspace of θ. It follows that $\operatorname{rk}\{\beta, \theta\beta, \theta^2\beta\} = 3$. We claim that $\theta^k\beta$ are pairwise orthogonal. Indeed, if their inner products are equal to -1, then $\operatorname{rk}\{\beta, \theta\beta, \theta^2\beta\} = 2$. If their inner products are equal to 1, then $\{\, \theta^k\beta - \theta^{k+l}\beta \mid l = 1, 2 \,\}$ is a root subsystem of type $\mathbf{A}_2$ in the first subclass. Since all such root subsystems are conjugated (by elements of G_0 normalizing $\mathfrak{c}$), each one of them originates from a subsystem $\{\pm\beta, \pm\theta\beta, \pm\theta^2\beta\}$ of the second subclass, as above. Hence 12 subsystems of type $\mathbf{A}_2$ in the first subclass give rise to at least $12 \cdot 6 = 72$ roots in the second subclass, whereas it contains only 54 roots. Our claim is proved. It follows that $\{\pm\theta^k\beta\}$ is a root system of type $3\mathbf{A}_1$, where θ acts by a cyclic permutation of components.

The 54 roots of the second subclass fall into 9 such subsystems. (If there is a root β such that $\beta|_{\mathfrak{c}^*\oplus\mathfrak{c}} = 0$, then there are exactly two such roots. But this cannot happen, because the number of roots in the second subclass not belonging to these subsystems is divisible by 6.) Hence we obtain 9 subalgebras $\mathfrak{B}(\beta)$ of type $3\mathbf{A}_1$, and θ acts on each $\mathfrak{B}(\beta)$ by a cyclic permutation of its components. As above, we have a decomposition $\mathfrak{h} = \mathfrak{h}(\beta) \oplus \mathfrak{h}^\beta$, where $\mathfrak{h}(\beta)$ is a Cartan subalgebra in $\mathfrak{B}(\beta)$ and $\mathfrak{h}^\beta$ is the centralizer of $\mathfrak{B}(\beta)$ in $\mathfrak{h}$. (Note that $c \in \mathfrak{h}(\beta)$.) Let $B(\beta)$ be the subgroup of G corresponding to $\mathfrak{B}(\beta)$. Take a θ-invariant element g_β from the normalizer of $\mathfrak{h}(\beta)$ in $B(\beta)$ such that $\operatorname{Ad} g_\beta|_{\mathfrak{h}(\beta)} = -\mathbf{1}$, $\operatorname{Ad} g_\beta|_{\mathfrak{h}^\beta} = \mathbf{1}$. We have $g_\beta \in G_0$, and $r_\beta = \operatorname{Ad} g_\beta|_{\mathfrak{c}} \in W$ is a reflection of order two. These 9 reflections r_β are conjugated and all 2-fold reflections in W are obtained in this way.

Among $\mathfrak{B}(\beta)$, there is a subalgebra $\mathfrak{B}_0$ constructed as follows. Let E_{ij} denote (the linear operator on U given in the basis ${}_ie$ by) the (i,j)-th matrix unit. Put

$$\begin{aligned} e_0 &= E_{12} + E_{34} + E_{56}, & f_0 &= E_{21} + E_{43} + E_{65}, & h_0 &= c, \\ e_1 &= {}_{53}e_1 + {}_{15}e_2 + {}_{31}e_3, & f_1 &= {}_{46}e_1 + {}_{62}e_2 + {}_{24}e_3, & h_1 &= u_3 - u_2, \\ e_{-1} &= {}^{46}e^1 + {}^{62}e^2 + {}^{24}e^3, & f_{-1} &= {}^{53}e^1 + {}^{15}e^2 + {}^{31}e^3, & h_{-1} &= u^3 - u^2. \end{aligned}$$

Then $\mathfrak{B}_0 = \langle\, e_k, f_k, h_k \mid k = 0, \pm 1 \,\rangle$ is a graded $\mathfrak{h}$-regular semisimple subalgebra of type $3\mathbf{A}_1$, $\deg e_k = \deg f_k = \deg h_k = k \in \mathbf{Z}_3$, and the representations $(\mathfrak{B}_0)_0 : (\mathfrak{B}_0)_{\pm 1}$ are isomorphic to the adjoint representation of $(\mathfrak{B}_0)_0 \cong \mathfrak{sl}_2(\mathbb{C})$ under the identifications $e_0 \mapsto e_{\pm 1}$, $f_0 \mapsto f_{\pm 1}$, $h_0 \mapsto h_{\pm 1}$.

Let z_1, z_2, z_3 be the coordinates on $\mathfrak{c}$ in the basis u_1, u_2, u_3. The basic projective invariants of W are

$$f = z_1^6 + z_2^6 + z_3^6 - 10(z_1^3 z_2^3 + z_1^3 z_3^3 + z_2^3 z_3^3)$$

$$
\begin{aligned}
g &= (z_1^3+z_2^3+z_3^3)(z_1^3+z_2^3+z_3^3+6z_1z_2z_3) \\
&\qquad \times (z_1^3+z_2^3+z_3^3+6\omega z_1z_2z_3)(z_1^3+z_2^3+z_3^3+6\omega^2 z_1z_2z_3) \\
p &= (z_1^3-z_2^3)(z_1^3-z_3^3)(z_2^3-z_3^3) \\
q &= z_1z_2z_3(z_1+z_2+z_3)(z_1+\omega z_2+\omega^2 z_3)(z_1+\omega^2 z_2+\omega z_3) \\
&\quad \times (z_1+z_2+\omega z_3)(z_1+\omega z_2+z_3)(\omega z_1+z_2+z_3) \\
&\quad \times (z_1+z_2+\omega^2 z_3)(z_1+\omega^2 z_2+z_3)(\omega^2 z_1+z_2+z_3)
\end{aligned}
$$

(see [10]) connected with the basic syzygy

$$
(5) \qquad 6912q^3-4g^3+186624p^4+f^6-6f^4g-864f^3p^2+9f^2g^2+2592fp^2g=0.
$$

The polynomials p, q are products of mirror functionals of all 2-fold and 3-fold reflections, respectively. It is easily shown that p is invariant under all w_α, and each r_β multiplies it by -1. Similarly, q is invariant under all r_β, and each w_α multiplies it by ω^{-1}. It follows that $f, g, h = p^2$ are (algebraically independent) basic invariants for $W : \mathfrak{c}$, which separate W-orbits.

4.5. Nilpotent elements. Our next task is to classify nilpotent elements. We proceed as in 3.6 using the methods of 2.5. The root lattice Q of the covering loop algebra $\mathfrak{G}$ is embedded in $\mathbb{Z}\Delta \times \mathbb{Z}$, so that the extended system of simple roots of $\mathfrak{g}$ gives rise to a base $\Pi = \{\, ({}_i\varepsilon - {}_{i+1}\varepsilon, 0), (\varepsilon_j - \varepsilon_{j+1}, 0), ({}_5\varepsilon + {}_6\varepsilon + \varepsilon_3, 1) \mid i = 1, \dots, 5;\ j = 1, 2 \,\}$ of Q. There are 37 types of complete subalgebras of $\mathfrak{G}$ (given by types of various subsystems of Π). For some of these types, there are several (but no more than 6) locally flat algebras, and some of these algebras admit several (but no more than 4) non-conjugated embeddings in $\mathfrak{G}$. Summing up, we obtain 93 locally flat subalgebras of $\mathfrak{G}$. Some of them are not complete. Excluding them, we obtain, up to conjugacy, 75 supports of nilpotent elements, hence 76 nilpotent orbits of G_0 in $\mathfrak{g}_1$ (including 0).

4.6. Mixed elements. The centralizer $\mathfrak{z}(u)$ of $u \in \mathfrak{c}$ is an $\mathfrak{h}$-regular subalgebra spanned by $\mathfrak{h}$ and the root subspaces $\mathfrak{g}_\alpha$ such that $\alpha(u) = 0$. The latter equality is equivalent to $w_\alpha u = u$ or $r_\alpha u = u$ if α belongs to the first or second subclass of the root system, respectively. Therefore u is a semisimple part of a mixed element iff u lies on mirror hyperplanes of some reflections from W.

In this subsection, we use the following shorthand terminology. The mirror hyperplane of an s-fold reflection will be called an *s-mirror*. Linear functionals defining 2- or 3-mirrors in $\mathfrak{c}$ (*mirror functionals*) are easily obtained by decomposing p and q into linear factors.

The collection of mirrors containing u may belong to one of the following types.

4.6.1. *One 3-mirror.* The set of such u is determined by the conditions $p(u) \neq 0,\ q(u) = 0,\ g(u) \neq 0$. Passing to a conjugate, we may assume that the equation of the mirror is $z_1 = 0 \implies \mathfrak{z}(u)' = \mathfrak{A}_1$, $\mathfrak{z}(u)'_0$ is its Cartan subalgebra, and $\mathfrak{z}(u)'_1$ is spanned by three summands of u_1, which are the root vectors corresponding to the extended system of simple roots (of degree 1) of $\mathfrak{A}_1$ relative to $\mathfrak{z}(u)'_0$. These roots and root vectors are permuted cyclically by a pair of monomial transformations of U, V given by permutations $(1\,3\,5)(2\,4\,6)$, $(1\,2\,3)$. The respective element of G_0 fixes u and glues together some nilpotent orbits of the action $Z_0(u)^0 : \mathfrak{z}(u)'_1$. Finally, nilpotent orbits for $Z_0(u) : \mathfrak{z}(u)'_1$ are determined by the condition that a certain number of coordinates (in the basis of root vectors) vanish and other coordinates are nonzero.

4.6.2. *One* 2*-mirror.* The respective subset of $u \in \mathfrak{c}$ is determined by the conditions $p(u) = 0,\ q(u) \neq 0$. We may assume that the equation of the mirror is $z_2 = z_3 \implies \mathfrak{z}(u)' = \mathfrak{B}_0$, $\mathfrak{z}(u)'_0 = \langle e_0, f_0, h_0 \rangle \cong \mathfrak{sl}_2(\mathbb{C})$, and $\mathfrak{z}(u)'_1 = \langle e_1, f_1, h_1 \rangle$ is isomorphic to the adjoint representation of $\mathfrak{sl}_2(\mathbb{C})$. Hence there is a unique nilpotent orbit for the action $Z_0(u) : \mathfrak{z}(u)'_1$—the orbit of e_1.

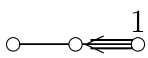
4.6.3. *One* 3*-mirror and one* 2*-mirror.* The respective subset in $\mathfrak{c}$ is determined by the equations $p(u) = 0,\ q(u) = 0,\ f^2(u) = 4g(u)$. We may assume that the equations of the mirrors are $z_1 = 0,\ z_2 = z_3 \implies \mathfrak{z}(u)' = \mathfrak{A}_1 \oplus \mathfrak{B}_0$, and nilpotent orbits for the action $Z_0(u) : \mathfrak{z}(u)'_1$ are direct products (or sums, is a reader prefers) of nilpotent orbits of the above two types.

Remark 4.1. If $p = q = 0$, then the syzygy (5) transforms into $(f^2 - g)^2(f^2 - 4g) = 0$, so that either $f^2 = 4g$ or $f^2 = g$. The first possibility has been just considered, and the second one is considered below (4.6.5).

4.6.4. *Four* 3*-mirrors.* The respective subset of $\mathfrak{c}$ is determined by the conditions $p(u) \neq 0,\ q(u) = 0,\ g(u) = 0$. We may assume that the equations of the mirrors are $z_1 = 0,\ z_1 + z_2 + z_3 = 0,\ \omega z_1 + z_2 + z_3 = 0,\ \omega^2 z_1 + z_2 + z_3 = 0 \implies u = \lambda(u_3 - u_2)$. The centralizer $\mathfrak{g}_u = \mathfrak{z}(u)$ is a graded reductive subalgebra of type $\mathbf{D}_4 + \mathbf{T}_3$ described as follows. First, its center is $\mathfrak{z}(\mathfrak{g}_u) = \langle h_{-1}, c, h_1 \rangle$. Further, consider a 3-dimensional space $Q = \langle x, y, z \rangle$ and let $\iota_1, \iota_2, \iota_3$ be three embeddings of Q in $U \oplus V$ mapping the triple (x, y, z) to $({}_1e, {}_3e, {}_5e)$, $({}_2e, {}_4e, {}_6e)$, and (e_1, e_2, e_3), respectively. Then $\mathfrak{sl}(Q)$ is embedded diagonally in $\mathfrak{sl}(U) \oplus \mathfrak{sl}(V)$ via $\iota_1, \iota_2, \iota_3$, and S^3Q is embedded in $\bigwedge^2 U \otimes V$ by mapping, for $q_1, q_2, q_3 \in Q$,

$$q_1 q_2 q_3 \mapsto \sum_{(i_1, i_2, i_3)} \iota_1(q_{i_1}) \wedge \iota_2(q_{i_2}) \otimes \iota_3(q_{i_3}),$$

where the sum is taken over all permutations of $(1, 2, 3)$. Similarly, S^3Q^* is embedded in $\bigwedge^2 U^* \otimes V^*$. One verifies that $\mathfrak{z}(u)' = S^3Q^* \oplus \mathfrak{sl}(Q) \oplus S^3Q$ is a graded subalgebra of type $\mathbf{D}_4$, whose grading is determined by the following labelling of the affine Dynkin diagram $\mathbf{D}_4^{(3)}$:



It is known (and easy to verify using the methods of 2.5, see [15, 5.3], [14, §5]) that there are 5 nonzero nilpotent orbits for the action $(Z(u)')^0_0 : \mathfrak{z}(u)'_1$, whose characteristics cannot be transformed into each other via an automorphism of $\mathfrak{z}(u)'_0$, see table 6. Hence these orbits are not glued together under the action of $Z_0(u)$ and are exactly nilpotent orbits for $Z_0(u) : \mathfrak{z}(u)'_1$.

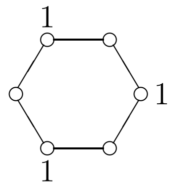
4.6.5. *Two 3-mirrors and three 2-mirrors.* The respective subset of $\mathfrak{c}$ is determined by the equations $p(u) = 0$, $q(u) = 0$, $f^2(u) = g(u)$. We may assume that the equations of the mirrors are $z_2 = 0$, $z_3 = 0$, $z_2 - z_3 = 0$, $z_2 - \omega z_3 = 0$, $z_2 - \omega^2 z_3 = 0 \implies u = \lambda u_1$. The centralizer $\mathfrak{g}_u = \mathfrak{z}(u)$ is a graded reductive subalgebra of type $\mathbf{A}_5 + \mathbf{T}_2$ described as follows. For its center, we have $\mathfrak{z}(\mathfrak{g}_u) = \langle u^1, u_1 \rangle$. Let $U_1 = \langle {}_1e, {}_2e \rangle$, $U_2 = \langle {}_3e, {}_4e \rangle$, $U_3 = \langle {}_5e, {}_6e \rangle$. We identify V with $\bigwedge^2 U_1^* \oplus \bigwedge^2 U_2^* \oplus \bigwedge^2 U_3^*$ by mapping $e_1 \mapsto {}^{12}e$, $e_2 \mapsto {}^{34}e$, $e_3 \mapsto {}^{56}e$ and thus embed $\mathfrak{gl}(U_1) \oplus \mathfrak{gl}(U_2) \oplus \mathfrak{gl}(U_3)$ in $\mathfrak{gl}(U) \oplus \mathfrak{gl}(V)$. There are also natural embeddings

$$\bigoplus_{(i,j,k)} U_i \otimes U_j \otimes \textstyle\bigwedge^2 U_k^* \hookrightarrow \bigwedge^2 U \otimes V \quad \text{and} \quad \displaystyle\bigoplus_{(i,j,k)} U_i^* \otimes U_j^* \otimes \textstyle\bigwedge^2 U_k \hookrightarrow \bigwedge^2 U^* \otimes V^*$$

(the direct sums are taken over all cyclic permutations of $(1,2,3)$). One verifies that

$$\mathfrak{z}(u)' = \bigoplus_{(i,j,k)} U_i^* \otimes U_j^* \otimes \textstyle\bigwedge^2 U_k \oplus \Big((\mathfrak{sl}(U) \oplus \mathfrak{sl}(V)) \cap \displaystyle\bigoplus \mathfrak{gl}(U_i) \Big) \oplus \bigoplus_{(i,j,k)} U_i \otimes U_j \otimes \textstyle\bigwedge^2 U_k^*$$

is a $\mathfrak{t}$-regular subalgebra of type $\mathbf{A}_5$, whose grading is determined by the following labelling of the affine Dynkin diagram $\mathbf{A}_5^{(1)}$:



We may assume that characteristics of nilpotent orbits for $Z_0(u)^0 : \mathfrak{z}(u)'_1$ lie in the positive Weyl chamber of $\mathfrak{t}$, and we indicate them by 6 indices, where the first triple is given by the values of the simple roots ${}_1\varepsilon - {}_2\varepsilon$, ${}_3\varepsilon - {}_4\varepsilon$, ${}_5\varepsilon - {}_6\varepsilon$ of $\mathfrak{z}(u)'_0$ and the second triple is composed of the values of the weights ${}_1\varepsilon + {}_2\varepsilon - 2\varepsilon_1$, ${}_3\varepsilon + {}_4\varepsilon - 2\varepsilon_2$, ${}_5\varepsilon + {}_6\varepsilon - 2\varepsilon_3$, which generate the dual to the 2-dimensional center of $\mathfrak{z}(u)'_0$.

A pair of monomial transformations of U, V determined by permutations $(1\,3\,5)(2\,4\,6)$, $(1\,2\,3)$ fixes u and permutes simple components $\mathfrak{sl}(U_i)$ of $\mathfrak{z}(u)'_0$ and both triples in each characteristic cyclically. On the contrary, an outer automorphism transposing two simple components of $\mathfrak{z}(u)'_0$ cannot be induced by an element of $Z_0(u)$. (That is why the elements No. 8 and 9 in table 7 are not equivalent.) Indeed, suppose for example that a pair $(a, b) \in SL(U) \times SL(V)$ fixes u and permutes U_1 and U_2 leaving U_3 stable. Then we must have

$$\begin{aligned} a({}_{12}e) &= \lambda_{34}e, & a({}_{34}e) &= \mu_{12}e, & a({}_{56}e) &= \nu_{56}e, \\ b(e_1) &= \lambda^{-1}e_2, & b(e_2) &= \mu^{-1}e_1, & b(e_3) &= \nu^{-1}e_3 \end{aligned}$$

$\implies \det a = \lambda\mu\nu = 1$, $\det b = -(\lambda\mu\nu)^{-1} = 1$. A contradiction! It is also clear that an element of $Z_0(u)$ normalizing each $\mathfrak{sl}(U_i)$ leaves each U_i stable, whence belongs to $Z_0(u)^0$. Therefore two nilpotent orbits for the action $Z_0(u)^0 : \mathfrak{z}(u)'_1$ are glued together under the action of $Z_0(u)$ iff their characteristics are obtained from each other by a cyclic permutation of both triples. This remark allows to complete the classification of nilpotent orbits for the action $Z_0(u) : \mathfrak{z}(u)'_1$.

References

[1] Elashvili, A. G., *Canonical forms and stationary subalgebras of points in general position for simple linear Lie groups*, Funk. Anal. i Prilozhen. **6** (1972), 51–62 (Russian), English translation: Funct. Anal. Appl. **6** (1972), 44–53.

[2] —, *Stationary subalgebras of points in general position for irreducible linear Lie groups*, Funk. Anal. i Prilozhen. **6** (1972) 65–78 (Russian), English translation: Funct. Anal. Appl. **6** (1972), 139–148.

[3] Gantmacher, F. R., "The theory of matrices," 3-rd edn., Nauka, Moscow, 1967 (Russian), English translation of the 1-st edn.: Chelsea, New York, 1959.

[4] Gauger, M., *On the classification of metabelian Lie algebras*, Trans. Amer. Math. Soc. **179** (1973), 293–329.

[5] Kac, V., *Some remarks on nilpotent orbits*, J. Algebra **64** (1980), 190–213.

[6] —, "Infinite dimensional Lie algebras," 3-rd edn., Cambridge Univ. Press, Cambridge, 1990.

[7] Klein, F., „Vorlesungen über das Ikosaeder und die Auflösung der Gleichungen vom fünften Grade", Teubner, Leipzig, 1884.

[8] Maltsev, A. I., *On semisimple subgroups of Lie groups*, Izvestiya AN SSSR, ser. matem. **8** (1944), 143–174 (Russian).

[9] —, *On solvable Lie algebras*, Izvestiya AN SSSR, ser. matem. **9** (1945), No. 5, 329–356 (Russian), English translation: Amer. Math. Soc. Transl. (1) **9** (1962), 228–262.

[10] Shephard, G. C., and J. A. Todd, *Finite unitary reflection groups*, Canad. J. Math. **6** (1954), 274–304.

[11] Vinberg, E. B., *On the linear groups associated to periodic automorphisms of semisimple algebraic groups*, Dokl. AN SSSR **221** (1975), 767–770 (Russian), English translation: Soviet Math. Dokl. **16** (1975), 406–409.

[12] —, *On the classification of the nilpotent elements of graded Lie algebras*, Dokl. AN SSSR **225** (1975), 745–748 (Russian), English translation: Soviet Math. Dokl. **16** (1975), 1517–1520.

[13] —, *The Weyl group of a graded Lie algebra*, Izvestiya AN SSSR, ser. matem. **40** (1976), 488–526 (Russian), English translation: Math. USSR Izvestiya **10** (1976), 463–495.

[14] —, *Classification of homogeneous nilpotent elements of a semisimple graded Lie algebra*, Trudy Sem. Vektor. Tenzor. Anal. **19** (1979), 155–177 (Russian), English translation: Selecta Math. Sovietica **6** (1987), 15–35.

[15] Vinberg, E. B., and A. G. Elashvili, *Classification of trivectors of a 9-dimensional space*, Trudy Sem. Vektor. Tenzor. Anal. **18** (1978), 197–233 (Russian), English translation: Selecta Math. Sovietica **7** (1988), 63–98.

Philippinenhöfer Weg, 26
Kassel, 34127, Deutschland

Chair of Algebra
Department of Mathematics
Moscow State University
Moscow, 119899, Russia
timashev@mech.math.msu.su

Received January 28, 1998