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Harmonic analysis on SU(n,n)/SL(n,C) x $\mathbf{R}_+^*$

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Abstract. We find an explicit expression for the spherical functions on the ordered symmetric space $\mathcal{M}=SU(n,n)/SL(n,\mathbb{C})\times\mathbb{R}_+^*$, we formulate and prove a Paley-Wiener theorem for the spherical Laplace transform on $\mathcal{M}$ and we find an inversion formula for the Abel transform on $\mathcal{M}$.

0. Introduction

Let $\mathcal{M} = SU(n,n)/SL(n,\mathbb{C}) \times \mathbb{R}_+^*$, let $\mathfrak{a}^-$ be the negative Weyl chamber of a certain Cartan subspace $\mathfrak{a}$ for $\mathcal{M}$, let $\lambda \in \mathfrak{a}_\mathbb{C}^*$, the complex dual of $\mathfrak{a}$, and let $A^- = \exp \mathfrak{a}^-$. Let Φ_λ denote the Harish-Chandra series on the Riemannian dual $\mathcal{M}^d = SU(n,n)/S(U(n) \times U(n))$ of $\mathcal{M}$. G. Ólafsson proved in [9], §5 an expansion formula (for general ordered symmetric spaces):

$$\varphi_\lambda(a) = \sum_{w\in W_0} c(w\lambda)\Phi_{w\lambda}(a), \quad a \in A^-,$$

for the spherical functions φ_λ on $\mathcal{M}$ (see §3 for a precise definition and construction of φ_λ), where $c(\lambda)$ is the c-function for $\mathcal{M}$ and W_0 is some Weyl group.

The Berezin-Karpelevič formula for the spherical functions ψ_λ^d on $\mathcal{M}^d$ was proved by B. Hoogenboom, see [6], using the Harish-Chandra expansion of ψ_λ^d and an explicit expression for Φ_λ. We use the expansion formula above to prove a similar (explicit) formula for the spherical functions φ_λ on $\mathcal{M}$.

The spherical Laplace transform $\mathcal{L}$ on $\mathcal{M}$ is defined in terms of integrating against the spherical functions. We use the explicit formulae for the spherical functions on $\mathcal{M}$ and $\mathcal{M}^d$ to prove a Paley-Wiener Theorem for the spherical Laplace transform, generalizing results in the rank 1 case obtained by G. Ólafsson and the first author, see [1].

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The Abel transform on $\mathcal{M}$ is related to the spherical Laplace transform $\mathcal{L}$ by the classical Laplace transform on the cone $c_{\max} \subset \mathfrak{a}$. We find an inversion formula for the Abel transform, using an approach similar to the method used by C. Meaney for the inversion formula for the Abel transform on $\mathcal{M}^d$, see [8].

We have tried to keep notations and proofs to a minimum in order to make the presentation as clear as possible, we refer to [3], [5] and [9] for more details on spherical functions and the spherical Laplace and Abel transforms defined on ordered symmetric spaces.

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1. Notation and preliminaries

Let n be a positive integer and let $G^c = SU(n,n)$ denote the connected group of matrices with determinant 1 preserving the hermitian form

$$(x,y) = x_1\bar{y}_1 + \cdots + x_n\bar{y}_n - x_{n+1}\bar{y}_{n+1} - \cdots - x_{2n}\bar{y}_{2n},\ x,y \in \mathbb{C}^{2n}.$$

The Lie algebra $\mathfrak{g}^c = \mathfrak{su}(n,n)$ is given by $2n \times 2n$-matrices of the form

$$\mathfrak{g}^c = \left\{ \begin{pmatrix} a & c \\ c^* & b \end{pmatrix} \middle| \, a = -a^*,\, b = -b^*,\, \operatorname{tr}(a+b) = 0 \right\},$$

where a, b and c are $n \times n$-matrices. It is isomorphic (by c-duality) to

$$\mathfrak{g} = \left\{ \begin{pmatrix} \alpha & \beta \\ \gamma & -\alpha^* \end{pmatrix} \middle| \, \beta = \beta^*,\, \gamma = \gamma^*,\, \Im \operatorname{tr} \alpha = 0 \right\}.$$

We embed $\mathfrak{h} = \mathfrak{sl}(n,\mathbb{C}) \oplus \mathbb{R} \cong \{\alpha \in \mathfrak{gl}(n,\mathbb{C}) \mid \Im \operatorname{tr} \alpha = 0\}$ in the diagonal as follows:

$$\alpha \mapsto \begin{pmatrix} \alpha & \\ & -\alpha^* \end{pmatrix}.$$

Let G and H denote the analytic subgroups of $GL(2n,\mathbb{C})$ with Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$ respectively. The involution σ on $\mathfrak{g}$ given by

$$\sigma \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} \alpha & -\beta \\ -\gamma & \delta \end{pmatrix},$$

fixes $\mathfrak{h}$. The -1 eigenspace $\mathfrak{q}$ of σ is given by:

$$\mathfrak{q} = \left\{ \begin{pmatrix} & \beta \\ \gamma & \end{pmatrix} \middle| \, \beta = \beta^*,\, \gamma = \gamma^* \right\}.$$

Let $\mathcal{M} = G/H \cong SU(n,n)/SL(n,\mathbb{C}) \times \mathbb{R}_+^*$, then G/H is an ordered symmetric space of Cayley type, see [5] or [9], §1.

Let θ be the classical Cartan involution on $\mathfrak{g}$, i.e. $\theta(X) = -X^*$, $X \in \mathfrak{g}$, and let $\mathfrak{k}$ and $\mathfrak{p}$ denote the ± 1-eigenspaces of θ. Let $K \cong S(U(n) \times U(n))$

denote the maximal compact subgroup of G with Lie algebra $\mathfrak{k}$. Then G/K is isometric to the Riemannian dual $\mathcal{M}^d$ of $\mathcal{M}$, see [5] and [9], §1 for details.

We choose a Cartan subspace $\mathfrak{a} \subset \mathfrak{p} \cap \mathfrak{q}$ for $\mathcal{M}$ as follows:

$$\mathfrak{a} = \left\{ X_T = \begin{pmatrix} 0 & T \\ T & 0 \end{pmatrix} \middle| T = \operatorname{diag}(t_1/2, \ldots, t_n/2),\, t_1, \ldots, t_n \in \mathbb{R} \right\}.$$

We note that $\mathfrak{a}$ also is a Cartan subspace of $\mathfrak{p}$. We identify $\mathfrak{a}$ and $\mathbb{R}^n$ via the map $\mathbb{R}^n \ni t = (t_1, \ldots, t_n) \mapsto T = \operatorname{diag}(t_1/2, \ldots, t_n/2)$. Let $\gamma_i \in \mathfrak{a}^*$ be defined by: $\gamma_i(t) = -t_i$ for $i = 1, \ldots, n$. We identify the complexified dual $\mathfrak{a}^*_{\mathbb{C}}$ and $\mathbb{C}^n$ by the map:

$$\mathbb{C}^n \ni \lambda = (\lambda_1, \ldots, \lambda_n) \mapsto -\sum_j \lambda_j \gamma_j.$$

The root system $\Delta = \Delta(\mathfrak{g}, \mathfrak{a})$ is given by $\Delta = \{\pm\gamma_i\} \cup \left\{ \frac{\gamma_j \pm \gamma_i}{2} \right\}$, with multiplicity $m_\alpha = 2$ for the short roots $\alpha = \frac{\gamma_j \pm \gamma_i}{2}$ and $m_\alpha = 1$ for the long roots $\alpha = \pm\gamma_i$. Let $\Delta^+ = \{\gamma_i\} \cup \left\{ \frac{\gamma_j \pm \gamma_i}{2}, i < j \right\}$ be a set of positive roots. Let furthermore Δ_0 denote the root system $\Delta_0 = \left\{ \frac{\gamma_j - \gamma_i}{2} \right\}$ with positive roots $\Delta_0^+ = \left\{ \frac{\gamma_j - \gamma_i}{2}, i < j \right\}$. The negative Weyl chamber $\mathfrak{a}^-$ is given by:

$$\mathfrak{a}^- = \{ t \in \mathbb{R}^n \mid 0 < t_1 < t_2 < \cdots < t_{n-1} < t_n \}.$$

Let $W \cong \{\pm 1\}^n \times \mathfrak{S}_n$ and $W_0 \cong \mathfrak{S}_n$ (the permutation group of n elements) denote the Weyl groups of the root systems Δ and Δ_0 respectively. Let finally $\mathfrak{n} = \sum_{\alpha \in \Delta^+} \mathfrak{g}_\alpha$, $\bar{\mathfrak{n}} = \sum_{\alpha \in \Delta^+} \mathfrak{g}_{-\alpha}$, $A = \exp \mathfrak{a}$, $A^- = \exp \mathfrak{a}^-$, $N = \exp \mathfrak{n}$ and $\bar{N} = \exp \bar{\mathfrak{n}}$, where $\exp$ is the exponential mapping from $\mathfrak{g}$ to G.

Let $x \in \mathbb{R}^n$ and $r \in \mathbb{R}$. We will use the notation $x \geq r$ $(x > r)$ if $x_j \geq r$ $(x_j > r)$ for all j. Let $C_{\max}$ be the (unique) closed H-invariant cone in $\mathfrak{q}$ defined by $C_{\max} \cap \mathfrak{a} := c_{\max} = \{ t \in \mathbb{R}^n | t \geq 0 \}$. Let $S = \exp(C_{\max})H$ be the associated semigroup in G, and let S^o denote the interior of S. Let finally $S^o_A := S^o \cap A = \exp c^o_{\max}$.

Let $\eta : \mathbb{D}(\mathcal{M}) \to \mathbb{D}(\mathcal{M}^d)$ denote the Flensted-Jensen isomorphism between the commutative algebras of invariant differential operators on $\mathcal{M}$ and $\mathcal{M}^d$ respectively (mapping the Laplace-Beltrami operator Δ on $\mathcal{M}$ onto the Laplace-Beltrami operator $\Delta^d = \eta(\Delta)$ on $\mathcal{M}^d$). Let $\Pi(D)$ and $\Pi^d(D^d)$ denote the radial part (on A^-) of $D \in \mathbb{D}(\mathcal{M})$ and $D^d \in \mathbb{D}(\mathcal{M}^d)$ respectively. There exists a unique map $C_c^\infty(H \backslash S^o / H) \ni f \mapsto f^d \in C_c^\infty(K \backslash G / K)$ such that $f_{|A^-} = f^d_{|A^-}$ and $\Pi(D) f_{|A^-} = \Pi^d(\eta(D)) f^d_{|A^-}$, see [5] or [9], §4 for more details.

Let P_λ and Q_λ denote Legendre functions of the first and second kind. We note that

$$P_{\lambda - \frac{1}{2}}(\cosh t) = \varphi_{i\lambda}^{(0, -\frac{1}{2})}(t) = \varphi_{2i\lambda}^{(0,0)}(t/2),$$

and

$$\frac{\Gamma(\lambda + 1)}{\Gamma\left(\frac{1}{2}\right) \Gamma\left(\lambda + \frac{1}{2}\right)} Q_{\lambda - \frac{1}{2}}(\cosh t) = \Phi_{-i\lambda}^{(0, -\frac{1}{2})}(t) = \Phi_{-2i\lambda}^{(0,0)}(t),$$

where $\varphi_\lambda^{(\alpha,\beta)}$ and $\Phi_\lambda^{(\alpha,\beta)}$ denote Jacobi functions of the first and second kind. We can furthermore view $P_{\lambda - \frac{1}{2}}(\cosh t)$ and $Q_{\lambda - \frac{1}{2}}(\cosh t)$ as spherical functions on the Riemannian symmetric space $SO_o(1,2)/SO(2)$, respectively on the ordered symmetric space $SO_o(1,2)/SO_o(1,1)$, of rank 1. From e.g. [7], §2, we get the following estimates on $P_{\lambda-\frac{1}{2}}(\cosh t)$ and $Q_{\lambda-\frac{1}{2}}(\cosh t)$:

$$\left|P_{\lambda-\frac{1}{2}}(\cosh t)\right| \leq ce^{(|\Re\lambda|-\frac{1}{2})|t|},$$

for all $t \in \mathbb{R}$ and $\lambda \in \mathbb{C}$, for some constant c; and, for any $r > 0$:

$$\left|\frac{\Gamma(\lambda+1)}{\Gamma(\lambda+\frac{1}{2})}\right| \left|Q_{\lambda-\frac{1}{2}}(\cosh t)\right| \leq c_r e^{-(\Re\lambda+\frac{1}{2})t},$$

for $\Re\lambda \geq 0$ and $t \geq r > 0$, where c_r is a constant only depending on r.

2. The spherical Fourier transform on $\mathcal{M}^d = SU(n,n)/S(U(n)\times U(n))$

In this section we recall some well-known definitions and results for the spherical Fourier transform on $\mathcal{M}^d$, see e.g. [4], Chapter 4.

Let $\lambda \in \mathbb{C}^n$. The Poisson kernel for $\mathcal{M}^d$ is defined by:

$$NAK \ni nak = x \mapsto a^{\lambda+\rho} =: p_\lambda^d(x),$$

where $\rho = \sum_{\alpha\in\Delta^+} m_\alpha \alpha$. The spherical functions on $\mathcal{M}^d$ can be written as:

$$\psi_\lambda^d(x) = \int_K p_\lambda^d(kx)dk,$$

for $x \in G$. The spherical functions are bi-K-invariant, $\psi_\lambda^d(\exp 0) = 1$ and $D\psi_\lambda^d = \gamma(D)(\lambda)\psi_\lambda^d$ for all $D \in D(\mathcal{M}^d)$ and all $\lambda \in \mathbb{C}^n$, where γ is the Harish-Chandra isomorphism. They are furthermore invariant under the action of the Weyl group W, i.e. $\psi_{w\lambda}^d = \psi_\lambda^d$ for all $w \in W$.

Let Λ denote the simple roots in Δ^+. The Harish-Chandra series:

$$\Phi_\lambda(a) = a^{\rho-\lambda} \sum_{\mu\in(\mathbb{N}\cup\{0\})\Lambda} a^\mu \Gamma_\mu(\lambda), \quad a \in A^-,$$

is a solution of the differential equation $\Delta^d\Phi_\lambda(a) = (\lambda^2-\rho^2)\Phi_\lambda(a)$ for $a \in A^-$, where $\Gamma_0(\lambda) \equiv 1$ and $\Gamma_\mu(\lambda)$, $\mu \in \mathbb{N}\Lambda$ is determined by recursion. The Harish-Chandra expansion formula states that:

$$\psi_{-\lambda}^d(a) = \psi_\lambda^d(a^{-1}) = \sum_{w\in W} c^d(w\lambda)\Phi_{w\lambda}(a), \quad a \in A^-,$$

where the Harish-Chandra c-function c^d for $\mathcal{M}^d$ is given by (modulo constants):

$$c^d(\lambda) := \int_{\bar{N}} p_\lambda^d(\bar{n})d\bar{n} = \prod_j \frac{\Gamma(-\lambda_j)}{\Gamma(-\lambda_j+\frac{1}{2})} \prod_{i<j}(\lambda_i^2-\lambda_j^2)^{-1}.$$

The Harish-Chandra series on $\mathcal{M}^d$ is given by:

$$\Phi_\lambda(\exp t) = \pi^{-n/2} \prod_i \frac{\Gamma(\lambda_i+1)}{\Gamma(\lambda_i+\frac{1}{2})} \frac{\prod_i Q_{\lambda_i-\frac{1}{2}}(\cosh t_i)}{\delta_1(t)}, \tag{1}$$

for $t>0$, where

$$\delta_1(t) = \prod_{\alpha=\frac{\gamma_j\pm\gamma_i}{2}, i<j} \sinh\langle -\alpha, t\rangle = 2^{n(n-1)/2} \prod_{i<j} (\cosh t_j - \cosh t_i),$$

see [6], Theorem 2. Using the Harish-Chandra expansion formula, this yields the Berezin-Karpelevič formula for the spherical functions on $\mathcal{M}^d$:

$$\psi_\lambda^d(\exp t) = \frac{c}{\prod_{i<j}(\lambda_j^2-\lambda_i^2)} \frac{\det\left(P_{\lambda_i-\frac{1}{2}}(\cosh t_j)\right)}{\delta_1(t)},$$

for all $t\in\mathbb{R}^n$, where c is a constant, see [6] for more details.

The spherical Fourier transform $\mathcal{F}$ on $\mathcal{M}^d$ is defined for any function $f\in C_c^\infty(K\backslash G/K)$ as:

$$\mathcal{F}(f)(\lambda) = \int_G f(x)\psi_{-\lambda}^d(x)dx = \int_{A^-} f(a)\psi_{-\lambda}^d(a)\delta(a)da,$$

where $\delta(\exp t) = \prod_{\alpha\in\Delta^+}\sinh^{m_\alpha}\langle -\alpha, t\rangle = \delta_1(t)^2\prod_j \sinh t_j$. The inversion formula for $\mathcal{F}$ reads (after normalizing $d\lambda$ suitably):

$$f(x) = \int_{i\mathbb{R}^n} \mathcal{F}(f)(\lambda)\psi_\lambda^d(x)|c^d(\lambda)|^{-2}d\lambda,$$

for all $f\in C_c^\infty(K\backslash G/K)$ and $x\in G$.

Let $R>0$. Let $C_R^\infty(K\backslash G/K) := \{f\in C_c^\infty(K\backslash G/K) | \mathrm{supp} f\subset \exp B_R\}$, where $B_R := \{t\in\mathbb{R}^n | \, |t|\le R\}$. Define the Paley-Wiener space $\mathcal{H}_R(\mathbb{C}^n)$ as the space of W-invariant holomorphic functions g on $\mathbb{C}^n$ of exponential type R, i.e. satisfying the estimate:

$$\sup_{\lambda\in\mathbb{C}^n} e^{-R|\Re\lambda|}(1+|\lambda|)^N|g(\lambda)| < \infty,$$

for all $N\in\mathbb{N}$. Furthermore denote by $\mathcal{H}(\mathbb{C}^n)$ the union of the spaces $\mathcal{H}_R(\mathbb{C}^n)$ for all $R>0$.

Theorem 1 (The Paley-Wiener Theorem). *The Fourier transform is a bijection of $C_c^\infty(K\backslash G/K)$ onto $\mathcal{H}(\mathbb{C}^n)$. More precisely it is a bijection of $C_R^\infty(K\backslash G/K)$ onto $\mathcal{H}_R(\mathbb{C}^n)$ for all $R>0$.*

3. Spherical functions on $\mathcal{M} = SU(n,n)/SL(n,\mathbb{C})\times\mathbb{R}_+^*$

We define spherical functions on $\mathcal{M}$ according to [9], Definition 4.1:

Definition 2. An H-biinvariant continuous function $\varphi : S^o \to \mathbb{C}$ is called a spherical function if there exists a character χ of $\mathbb{D}(\mathcal{M})$ such that (in the sense of distributions) $D\varphi = \chi(D)\varphi$ for all $D \in \mathbb{D}(\mathcal{M})$.

Define the Poisson kernel for $\mathcal{M}$ (and the open orbit NAH) by:

$$NAH \ni nah = x \mapsto a^{\rho-\lambda} =: p_\lambda(x),$$

and $p_\lambda \equiv 0$ on $G\backslash NAH$. We note that $hx \in S \subset NAH$ for all $h \in H$ and $x \in S$, see [3], Theorem 4.2. We can construct spherical functions φ_λ as follows:

$$\varphi_\lambda(x) := \int_H p_\lambda(hx)dh,$$

for $x \in S^o$, and $D\varphi_\lambda = \gamma(D)(\lambda)\varphi_\lambda$ for all $D \in \mathbb{D}(\mathcal{M})$, whenever the integral exists, see [3], §5 and [9], Theorem 4.10.

The asymptotic behavior of φ_λ as $t \to \infty$, $t \in \mathfrak{a}^-$ is given by:

$$\lim_{t\to\infty} e^{(\lambda-\rho)t}\varphi_\lambda(\exp t) = c(\lambda) = c_0(\lambda)c_\Omega(\lambda),$$

see [3], §6 for details, where c is the c-function for $\mathcal{M}$ given by:

$$c(\lambda) := \int_{\bar{N}\cap NAH} p_{-\lambda}(\bar{n})d\bar{n},$$

the function c_Ω is given by (modulo constants):

$$c_\Omega(\lambda) := \int_{K\cap NAH} p_{-\lambda}(k)dk = \prod_j \frac{\Gamma(\lambda_j + \frac{1}{2})}{\Gamma(\lambda_j + 1)} \prod_{i<j}(\lambda_i + \lambda_j)^{-1},$$

see [2], Corollaire 5.2, and c_0 is the c-function for a Riemannian symmetric space with root system Δ_0, given by (modulo constants):

$$c_0(\lambda) = \prod_{i<j}(\lambda_j - \lambda_i)^{-1}.$$

We note that c_Ω is W_0-invariant, i.e. $c_\Omega(w\lambda) = c_\Omega(\lambda)$ for $w \in W_0$.

Considering asymptotics of the spherical functions and the correspondence between (the radial parts of) invariant differential operators on $\mathcal{M}$, respectively on $\mathcal{M}^d$, we obtain the following expansion formula for φ_λ:

$$\varphi_\lambda(a) = c_\Omega(\lambda) \sum_{w\in W_0} c_0(w\lambda)\Phi_{w\lambda}(a), \quad a \in A^-, \tag{2}$$

for λ in a dense open subset of $\mathbb{C}^n$, see [9], Theorem 5.7. We use this expansion formula to find an explicit expression for φ_λ:

Theorem 3. *The spherical functions on $\mathcal{M}$ are given by:*

$$\varphi_\lambda(\exp t) = \frac{c}{\prod_{i<j}(\lambda_j^2 - \lambda_i^2)} \frac{\det\left(Q_{\lambda_i - \frac{1}{2}}(\cosh t_j)\right)}{\delta_1(t)},$$

for $\lambda \geq 0$ and $t > 0$, where c is a constant. The map $\lambda \to \varphi_\lambda(\exp t)$ extends (for fixed $t > 0$) to a meromorphic function with simple poles for $\lambda_i \in -\mathbb{N} + \frac{1}{2}$, $(i = 1, \dots n)$ and $\lambda_i = -\lambda_j$ $(i \neq j)$.

Proof. The expansion formula (2) yields:

$$\varphi_\lambda(\exp t) = c_\Omega(\lambda) c_0(\lambda) \sum_{w \in W_0} \varepsilon(w) \Phi_{w\lambda}(\exp t) = c(\lambda) \sum_{w \in W_0} \varepsilon(w) \Phi_{w\lambda}(\exp t),$$

since $c_0(w\lambda) = \varepsilon(w) c_0(\lambda)$ for all $w \in W_0 = \mathfrak{S}_n$, where $\varepsilon(w)$ denotes the sign of the permutation $w \in \mathfrak{S}_n$. Inserting the explicit expression (1) of the Harish-Chandra series Φ_λ gives the result by definition of the determinant. ∎

We easily get the following estimates of the spherical functions on $\mathcal{M}$:

Lemma 4. *Let $r > 0$. There exists a constant c_r such that*

$$|\delta_1(t) \varphi_\lambda(\exp t)/c(\lambda)| \leq c_r e^{-\min_{w \in W_0} \langle w \Re\lambda, t \rangle} \leq c_r e^{-\langle \Re\lambda, r t_o \rangle},$$

for $\Re\lambda \geq 0$ and $t \geq r$, where $t_o = (1, \dots, 1)$.

Proof. Let $r > 0$, then:

$$|\delta_1(t) \varphi_\lambda(\exp t)/c(\lambda)| = c \left| \frac{\Gamma(\lambda_j + 1)}{\Gamma(\lambda_j + \frac{1}{2})} \det\left(Q_{\lambda_i - \frac{1}{2}}(\cosh t_j)\right) \right|$$
$$\leq c e^{-\min_{w \in W_0} \langle w \Re\lambda, t \rangle},$$

for $\lambda \geq 0$ and $t \geq r$, for some constants c. ∎

From the two expansion formulae for the spherical functions we finally obtain the following correspondence between the spherical functions on $\mathcal{M}^d$ and $\mathcal{M}$:

$$\psi_\lambda^d(a^{-1}) = \psi_{-\lambda}^d(a) = \sum_{w \in W_0 \backslash W} \frac{c^d(w\lambda)}{c(w\lambda)} \varphi_{w\lambda}(a), \quad a \in A^-,$$

see also [9], Theorem 5.9. We note that the fraction $\frac{c^d(\lambda)}{c(\lambda)}$ is W_0-invariant.

4. The spherical Laplace transform on $\mathcal{M}$

We define the normalized spherical Laplace transform $\mathcal{L}^o$ on $\mathcal{M}$ as (cf. [3], §8):

$$\mathcal{L}^o(f)(\lambda) = c_\Omega(\lambda)^{-1} \int_{A^-} f(a) \varphi_\lambda(a) \delta(a) da,$$

for any $f \in C_c^\infty(H\backslash S^o/H) \cong C_c^\infty(S_A^o)^{W_0}$ (the left-W_0-invariant functions in $C_c^\infty(S_A^o)$), whenever the integral converges. From the explicit expression for φ_λ, we see that the function $\lambda \mapsto \mathcal{L}^o(f)(\lambda)$ extends to a meromorphic function on $\mathbb{C}^n$ with at most simple poles for $\lambda_i \in -\mathbb{N}\,(i = 1, \dots, n)$.

Let $f \in C_c^\infty(S_A^o)^{W_0}$. We see that $\mathcal{L}^o f$ satisfies the following functional equation:

$$\mathcal{F}(f^d)(\lambda) = \sum_{w \in W_o\backslash W} c_1(w\lambda)\mathcal{L}^o(f)(w\lambda), \tag{3}$$

almost everywhere (and the right hand side extends to an analytic function), where

$$c_1(\lambda) := c^d(\lambda)/c_0(\lambda) = \prod_j \frac{\Gamma(-\lambda_j)}{\Gamma(-\lambda_j + \frac{1}{2})} \prod_{i<j}(-\lambda_i - \lambda_j)^{-1}.$$

The inversion formula for the normalized spherical Laplace transform is an easy consequence of (3) and the inversion formula for the spherical Fourier transform, see also [9], Theorem 6.13:

Theorem 5 (The Inversion Formula). *Let $f \in C_c^\infty(S_A^o)^{W_0}$. Then*

$$f(a) = \frac{|W|}{|W_0|} \int_{i\mathbb{R}^n} \mathcal{L}^o(f)(\lambda)\psi_\lambda^d(a)\frac{d\lambda}{c_0(\lambda)c^d(-\lambda)},$$

for all $a \in S_A^o$.

Let $R > r > 0$ and define $C_{r,R}^\infty(S_A^o)^{W_0} := \{f \in C_c^\infty(S_A^o)^{W_0} | \operatorname{supp} f \subset \exp(C_r \cap B_R)\}$, where $C_r := \{t \in \mathbb{R}^n | t \geq r\}$. Lemma 4 and (3) suggest the following definition of the Paley-Wiener space, the supposed image space of the normalized spherical Laplace transform acting on $C_c^\infty(S_A^o)^{W_0}$ (or on the subspaces $C_{r,R}^\infty(S_A^o)^{W_0}$):

Definition 6. Let $R > r > 0$. We define the Paley-Wiener space $PW_{r,R}(\mathbb{C}^n)$ as the space of W_0-invariant meromorphic functions g on $\mathbb{C}^n$, with at most simple poles for $\lambda_i \in -\mathbb{N}\,(i = 1, \dots, n)$, such that (i)

$$\sup_{\Re\lambda \geq 0} e^{\Re\langle\lambda, rt_o\rangle}(1 + |\lambda|)^N |g(\lambda)/c_0(\lambda)| < \infty,$$

for all $N \in \mathbb{N}$, and (ii) the c_1-weighted average

$$\mathrm{P}^{\mathrm{av}} g(\lambda) = \sum_{w \in W_0\backslash W} c_1(w\lambda)g(w\lambda)$$

extends to a function in $\mathcal{H}_R(\mathbb{C}^n)$. Furthermore denote by $PW(\mathbb{C}^n)$ the union of the spaces $PW_{r,R}(\mathbb{C}^n)$ over all $R > r > 0$.

It is easily seen that $\mathcal{L}^o$ maps $C_{r,R}^\infty(S_A^o)^{W_0}$ into $PW_{r,R}(\mathbb{C}^n)$ for all $R > r > 0$ (since $\mathcal{L}^o(\Delta f)(\lambda) = (\lambda^2 - \rho^2)\mathcal{L}^o f(\lambda)$ for all $f \in C_c^\infty(S_A^o)^{W_0}$). We remark that $\mathrm{P}^{\mathrm{av}}\mathcal{L}^o$ acts injectively on $C_c^\infty(S_A^o)^{W_0}$, since $\mathrm{P}^{\mathrm{av}}\mathcal{L}^o(f) = \mathcal{F}(f^d) = 0$ implies $f = f^d = 0$ on A^- for any $f \in C_c^\infty(S_A^o)^{W_0}$ by injectivity of the spherical Fourier transform. The following lemma, due to H. Schlichtkrull in the rank 1 case, see [1], Lemma 7, shows that P^{av} is injective on $PW(\mathbb{C}^n)$:

Lemma 7. *Let g be meromorphic function on $\mathbb{C}^n$ that satisfies item (i) of Definition 6 (for some $r > 0$). Assume that $\mathrm{P}^{\mathrm{av}} g = 0$. Then $g = 0$.*

Proof. Let $g_1(\lambda) = g(\lambda)/c^d(-\lambda)c_0(\lambda)$ and let $W_1 := \{\pm 1\}^n \cong W_0 \backslash W$. Then $\mathrm{P}^{\mathrm{av}} g(\lambda) = |W_1| c^d(\lambda) c^d(-\lambda) \mathrm{av} g_1(\lambda)$, where

$$\mathrm{av} g_1(\lambda) := \frac{1}{|W_1|} \sum_{w \in W_1} g_1(w\lambda)$$

is the average of g_1 over W_1. It follows from the assumption $\mathrm{P}^{\mathrm{av}} g = 0$ that $\mathrm{av} g_1 = 0$. The function g_1 also satisfies item (i) of Definition 6, in particular, $g_1(i\,\cdot\,) \in L^1(\mathbb{R}^n)$. Let

$$\gamma(s) = \int_{\mathbb{R}^n} g_1(i\lambda) e^{i\langle s, \lambda \rangle}\, d\lambda, \qquad s \in \mathbb{R}^n,$$

denote the Euclidean Fourier transform of $g_1(i\,\cdot\,)$. The condition (i) implies that g_1 is holomorphic in an open set containing $\{z \in \mathbb{C}^n \,|\, \Re z \geq 0\}$, and the standard argument with Cauchy's theorem gives that γ is supported on C_r. On the other hand, the average $\mathrm{av}\gamma$ of γ is the Fourier transform of $\mathrm{av} g_1(i\cdot)$, which vanishes, hence $\mathrm{av}\gamma$ vanishes as well. Hence $\gamma = 0$ by the support condition. Since the Euclidean Fourier transform is injective on $L^1(\mathbb{R}^n)$, we conclude that g_1, and hence also g, vanishes. ■

Theorem 8 (The Paley-Wiener Theorem). *The normalized spherical Laplace transform $\mathcal{L}^o$ is a bijection of $C_c^\infty(S_A^o)^{W_0}$ onto $PW(\mathbb{C}^n)$. More precisely it is a bijection of $C_{r,R}^\infty(S_A^o)^{W_0}$ onto $PW_{r,R}(\mathbb{C}^n)$ for all $R > r > 0$.*

Proof. It only remains to show that the normalized spherical Laplace transform maps $C_{r,R}^\infty(S_A^o)^{W_0}$ onto $PW_{r,R}(\mathbb{C})$ for all $R > r > 0$.

We define an auxiliary function Ξ_λ^d by:

$$\begin{aligned}\Xi_\lambda^d(\exp t) &= \sum_{w \in W_1} c^d(w\lambda) \Phi_{w\lambda}(\exp t) = \frac{c}{\prod_{i<j}(\lambda_j^2 - \lambda_i^2)} \frac{\prod_j P_{\lambda_j - \frac{1}{2}}(\cosh t_j)}{\delta_1(t)} \\ &= c^d(-\lambda) c \prod_j \frac{\Gamma(\lambda_j + \frac{1}{2})}{\Gamma(\lambda_j)} \frac{\prod_j P_{\lambda_j - \frac{1}{2}}(\cosh t_j)}{\delta_1(t)},\end{aligned}$$

for $\lambda_i \neq \pm\lambda_j$ $(i \neq j)$ and $t_i \neq t_j$ $(i \neq j)$. Hence $\psi_\lambda^d = \sum_{w \in W_0} \Xi_{w\lambda}^d$, and we can rewrite the inversion formula as:

$$\begin{aligned}f(a) &= \frac{|W|}{|W_0|} \int_{i\mathbb{R}^n} \mathcal{L}^o f(\lambda) \psi_\lambda^d(a) \frac{d\lambda}{c_0(\lambda) c^d(-\lambda)} \\ &= |W| \int_{i\mathbb{R}^n} \mathcal{L}^o f(\lambda) \Xi_\lambda^d(a) \frac{d\lambda}{c_0(\lambda) c^d(-\lambda)},\end{aligned}$$

for all $a \in A^-$, by W_0-invariance of the measure $d\lambda$.

Consider the wave packet $\mathcal{I}g \in C^\infty(S_A^o)^{W_0}$ of $g \in PW_{r,R}(\mathfrak{a}_{\mathbb{C}}^*)$ defined by the inversion formula(e) (for $a \in A^-$):

$$\begin{aligned}\mathcal{I}g(a) &= \frac{|W|}{|W_0|}\int_{i\mathbb{R}^n} g(\lambda)\psi_\lambda^d(a)\frac{d\lambda}{c_0(\lambda)c^d(-\lambda)}\\ &= |W|\int_{i\mathbb{R}^n} g(\lambda)\Xi_\lambda^d(a)\frac{d\lambda}{c_0(\lambda)c^d(-\lambda)}.\end{aligned}$$

Fix $r > 0$ and assume that $t \notin C_r$. There exists $\lambda_o > 0$ such that $\langle \lambda_o, t - rt_o\rangle = -\varepsilon < 0$ ($t_o = (1, \ldots, 1)$). This yields the following estimate:

$$\left|\Xi_{\lambda+\mu\lambda_o}^d(\exp t)/c^d(-\lambda-\mu\lambda_o)\right| \leq c(1+|\lambda+\mu\lambda_0|)^{n/2}e^{\mu\langle\lambda_o, rt_o\rangle}e^{-\mu\varepsilon},$$

for $\mu \geq 0$ and $\lambda \in i\mathbb{R}^n$, for some constants c not depending on λ. By Cauchy's theorem and a contour shift we get:

$$\begin{aligned}\mathcal{I}g(\exp t) &= |W|\int_{i\mathbb{R}^n}\frac{g(\lambda)}{c_0(\lambda)}\frac{\Xi_\lambda^d(\exp t)}{c^d(-\lambda)}d\lambda\\ &= |W|\int_{i\mathbb{R}^n}\frac{g(\lambda+\mu\lambda_o)}{c_0(\lambda+\mu\lambda_o)}\frac{\Xi_{\lambda+\mu\lambda_o}^d(\exp t)}{c^d(-\lambda-\mu\lambda_o)}d\lambda\\ &\to 0 \quad \text{for} \quad \mu\to\infty.\end{aligned}$$

By continuity and W_0-invariance this shows that $\mathcal{I}g$ is identically zero on $S_A^o \backslash \exp C_r$.

An easy calculation shows that (for $a \in A^-$):

$$\begin{aligned}\mathcal{I}g(a) &= \frac{|W|}{|W_o|}\int_{i\mathbb{R}^n} g(\lambda)\psi_\lambda^d(a)\frac{d\lambda}{c_0(\lambda)c^d(-\lambda)}\\ &= \int_{i\mathbb{R}^n} \mathrm{P}^{\mathrm{av}}g(\lambda)\psi_\lambda^d(a)\left|c^d(\lambda)\right|^{-2}d\lambda,\end{aligned}$$

which we recognize as the inverse Fourier transform of $\mathrm{P}^{\mathrm{av}}g \in \mathcal{H}_R(\mathbb{C})$, whence $\mathcal{I}g(a) = 0$ for $a \in S_A^o\backslash \exp B_R$ by the Paley-Wiener theorem for the spherical Fourier transform on $\mathcal{M}^d$.

Since $\mathrm{P}^{\mathrm{av}}\mathcal{L}^o f = \mathcal{F}f^d$ for all $f \in C_c^\infty(S_A^o)^{W_0}$, the above also yields:

$$\mathrm{P}^{\mathrm{av}}\mathcal{L}^o\mathcal{I}g = \mathcal{F}(\mathcal{I}g)^d = \mathrm{P}^{\mathrm{av}}g,$$

for all $g \in PW(\mathbb{C}^n)$, hence Lemma 7 implies that $\mathcal{L}^o\mathcal{I}g = g$ for all $g \in PW(\mathbb{C}^n)$ and we conclude that $\mathcal{L}^o$ maps $C_{r,R}^\infty(S_A^o)^{W_0}$ onto $PW_{r,R}(\mathbb{C}^n)$ for all $R > r > 0$.■

5. The Abel transform on $\mathcal{M} = SU(n,n)/SL(n,\mathbb{C}) \times \mathbb{R}_+^*$

The Abel transform $\mathcal{A}$ of an H-invariant function f on the semigroup S is defined as (cf. [3], §8):

$$\mathcal{A}f(a) = a^{-\rho}\int_N f(na)dn,$$

for $a \in A$, whenever this integral exists (we put $f(x) \equiv 0$ for $x \in NAH \backslash S$). It has the following connection to the spherical Laplace transform (for $\lambda \gg 0$ and otherwise by analytic continuation):

$$\mathcal{L}f(\lambda) = \int_{\exp c_{\max}} a^{-\lambda} \mathcal{A}f(a)da = \mathcal{L}_A(\mathcal{A}f)(\lambda),$$

where $\mathcal{L}_A$ is the Euclidean Laplace transform on A with respect to the cone $c_{\max}$, see [3], Proposition 8.5.

Using the explicit expression of the spherical functions from Theorem 3, we get (modulo constants):

$$\prod_{i<j}(\lambda_j^2 - \lambda_i^2)\mathcal{L}(f)(\lambda) = \int_{t_n>t_{n-1}>\ldots t_2>t_1>0} f(\exp t) \det\left(Q_{\lambda_i - \frac{1}{2}}(\cosh t_j)\right) \frac{\delta(t)}{\delta_1(t)} dt$$

$$= \int_{t_n>t_{n-1}>\ldots t_2>t_1>0} f(\exp t) \det\left(Q_{\lambda_i - \frac{1}{2}}(\cosh t_j)\right) \delta_1(t) \prod_j \sinh t_j dt$$

$$= \sum_{w\in W_0} \int_{t_n>t_{n-1}>\ldots t_2>t_1>0} f(\exp t)\epsilon(w) \prod_j Q_{\lambda_j - \frac{1}{2}}(\cosh wt_j)\delta_1(t) \prod_j \sinh t_j dt$$

$$= \sum_{w\in W_0} \int_{t_n>t_{n-1}>\ldots t_2>t_1>0} f(\exp t) \prod_j Q_{\lambda_j - \frac{1}{2}}(\cosh wt_j)\delta_1(wt) \prod_j \sinh t_j dt$$

$$= \int_{c_{\max}} f(\exp t)\delta_1(t) \left\{ \prod_j Q_{\lambda_j - \frac{1}{2}}(\cosh t_j) \sinh t_j \right\} dt$$

$$= \mathcal{L}_1^{\otimes}(f(\exp \cdot) \cdot \delta_1)(\lambda) = \mathcal{L}_A \mathcal{A}_1^{\otimes}(f(\exp \cdot) \cdot \delta_1)(\lambda),$$

where $\mathcal{L}_1^{\otimes}$ is the n-fold tensor product of the Laplace transform $\mathcal{L}_1$ on the ordered symmetric space $SO_o(1,2)/SO_o(1,1)$ of rank 1:

$$\mathcal{L}_1 f(\lambda) = \int_0^\infty f(t) Q_{\lambda - \frac{1}{2}}(\cosh t) \sinh t dt,$$

for $f \in C_c^\infty(\mathbb{R}_+)$, and $\mathcal{A}_1^{\otimes}$ is the n-fold tensor product of the Abel transform $\mathcal{A}_1$ on $SO_o(1,2)/SO_o(1,1)$:

$$\mathcal{A}_1 f(t) = \int_0^t f(\tau)(2\cosh t - 2\cosh \tau)^{-1/2} \sinh \tau d\tau,$$

for $f \in C_c^\infty(\mathbb{R}_+)$, see [3], §10 for details (we have identified A^- in the rank 1 case with $\mathbb{R}_+$ via the map $a_t \mapsto t$).

We furthermore have:

$$\prod_{i<j} (\lambda_j^2 - \lambda_i^2)\, \mathcal{L}(f) = \mathcal{L}_A \left(\prod_{i<j} (\partial_j^2 - \partial_i^2) \mathcal{A}(f) \right)(\lambda),$$

which implies that:

$$\left(\prod_{i<j} (\partial_j^2 - \partial_i^2)\, \mathcal{A}(f) \right) = \mathcal{A}_1^{\otimes}(f(\exp \cdot) \cdot \delta_1),$$

by injectivity of the Laplace transform $\mathcal{L}_A$. Finally, inverting one coordinate at a time, we get by [3], §10:

Theorem 9. *Let* $f \in C_c^\infty(S_A^o)^{W_0}$. *Then:*

$$f(\exp t) = c\delta_1(t)^{-1} \prod_j \left(\frac{1}{\sinh t_j} \frac{d}{dt_j} \right) \int_0^{t_n} \cdots \int_0^{t_1} \left(\prod_{k<l} (\partial_l^2 - \partial_k^2) \mathcal{A} f \right) (\exp \tau)$$
$$\times \prod_j \left((\cosh t_j - \cosh \tau_j)^{-1/2} \sinh \tau_j \right) d\tau_1 \dots d\tau_n,$$

for $t \in \mathfrak{a}^-$, *for some constant* c.

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