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Lie quasi-bialgebras with quasi-triangular decomposition

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Abstract. A class of Lie bialgebras and Lie quasi-bialgebras related to a triangular decomposition of the underlying Lie algebras is discussed. New examples are presented.

1. Introduction

A *Lie bialgebra* is a vector space which is simultaneously a Lie algebra and a Lie coalgebra, both structures connected by a cocycle condition. This fundamental concept was introduced by Drinfeld [5] as the infinitesimal counterpart of the notion of *Poisson-Lie group*: a Lie group which is a Poisson manifold, both structures related by imposing the multiplication to be a Poisson manifold mapping. Poisson-Lie groups appear naturally in deformation-quantization theory. Their quantizations are the quantum groups. The subsidiary notion of *Lie quasi-bialgebra* was again introduced by Drinfeld in his approach to the quantization of classical solutions of the quantum Yang-Baxter equations [7]. Being more flexible that Lie bialgebras, the context of Lie quasi-bialgebras allows to use twistings, a technical tool that became very useful.

In this article, we present a unified way to endow Lie algebras with additional data (a so-called "triangular decomposition" or "quasi-triangular decomposition", see Definition 3.1), a Lie bialgebra or Lie quasi-bialgebra cobracket. Then we provide a systematic iterative way of constructing Lie algebras with quasi-triangular decomposition, analogous to a construction of Witt [14].

The paper is organized as follows: in §2, we recall the necessary definitions and results, mostly due to Drinfeld. In §3, we introduce the notion of Lie algebra with quasi-triangular decomposition and show that a Lie algebra with (quasi)-triangular decomposition is a factorizable Lie (quasi)-bialgebra. Examples of Lie algebras with triangular decomposition are given in §4: some of them were known, as Kac-Moody Lie algebras [6], extended Heisenberg algebras [4]; some of them are new, e. g. motion Lie algebras with respect to the adjoint representation.

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As a byproduct, we provide new examples of classical r-matrices. In Section §5, we discuss more examples arising form the analogue of Witt's construction; in particular, we endow many generalized Heisenberg algebras with Lie bialgebra structures. These examples are also new, see however [12].

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2. Preliminaries

For simplicity of the exposition, we shall work over $\mathbb{C}$. We collect in this section the necessary definitions and theorems, due mostly to Drinfeld [7], [5]; see [3] for further properties of Lie quasi-bialgebras. By abuse of notation, ad will mean a representation which is tensor product of copies of the adjoint representation.

Definition 2.1. A *Lie quasi-bialgebra* is a triple $(\mathfrak{g}, \delta, \phi)$, where $\mathfrak{g}$ is a Lie algebra, $\delta : \mathfrak{g} \to \wedge^2\mathfrak{g} \subset \mathfrak{g} \otimes \mathfrak{g}$ is a 1-cocycle and $\phi \in \wedge^3\mathfrak{g} \subset \mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g}$ such that the following axioms hold:

$$\operatorname{Alt}(\delta \otimes \mathrm{id})\delta(x) = \operatorname{ad} x(\phi), \quad x \in \mathfrak{g}; \tag{1}$$

$$\operatorname{Alt}(\delta \otimes \mathrm{id} \otimes \mathrm{id})(\phi) = 0, \tag{2}$$

where Alt is the alternation map and "1-cocycle" means that δ is linear and $\delta([x, y]) = \operatorname{ad} x(\delta(y)) - \operatorname{ad} y(\delta(x))$. Examples of 1-cocycles are the 1-coboundaries: if $r \in \mathfrak{g} \otimes \mathfrak{g}$, then $\partial r : \mathfrak{g} \to \mathfrak{g} \otimes \mathfrak{g}$, the map given by $\partial r(x) := \operatorname{ad} x(r) - r$, is called the coboundary of r. Furthermore, if $\phi = 0$ we say that $(\mathfrak{g}, \delta)$ is a *Lie bialgebra.* So that equation (1) becomes $\operatorname{Alt}(\delta \otimes \mathrm{id})\delta(x) = 0$ and equation (2) is identically satisfied. The equality $\operatorname{Alt}(\delta \otimes \mathrm{id})\delta(x) = 0$ is called the co-Jacobi identity.

Definition 2.2. Let $(\mathfrak{g}, \delta)$ be a Lie bialgebra. A Lie subalgebra $\mathfrak{h} \subset \mathfrak{g}$ is a *Lie subbialgebra* if $\delta(\mathfrak{h}) \subset \mathfrak{h} \otimes \mathfrak{h}$.

Definition 2.3. A *Manin pair* is a data $(\mathfrak{p}, \mathfrak{p}_1, \mathfrak{p}_2)$, where $\mathfrak{p}$ is a Lie algebra provided with a $\mathfrak{p}$-invariant, symmetric, non degenerate bilinear form $< \mid >$: $\mathfrak{p} \times \mathfrak{p} \to \mathbb{C}$, $\mathfrak{p}_1$ is an isotropic Lie subalgebra of $\mathfrak{p}$ and $\mathfrak{p}_2$ is an isotropic subspace of $\mathfrak{p}$ complementary to $\mathfrak{p}_1$. That is, $\mathfrak{p}_1 \oplus \mathfrak{p}_2 = \mathfrak{p}$, $< \mathfrak{p}_i | \mathfrak{p}_i >= 0$, $i = 1, 2$. If $\mathfrak{p}_2$ is a Lie subalgebra of $\mathfrak{p}$, we say that $(\mathfrak{p}, \mathfrak{p}_1, \mathfrak{p}_2)$ is a *Manin triple.*

The terminology "Manin pair" is justified as follows: to a Manin pair $(\mathfrak{p}, \mathfrak{p}_1, \mathfrak{p}_2)$ corresponds a Lie quasi-bialgebra structure on $\mathfrak{p}_1$; and changing the complementary subspace $\mathfrak{p}_2$ amounts only to a twisting of this structure, so that up to twisting only the pair $(\mathfrak{p}, \mathfrak{p}_1)$ counts. See below for details. We remark that Drinfeld does not fixes the isotropic complement $\mathfrak{p}_2$ since he is interested in the notion of Lie quasi-bialgebra up to twisting.

We now recall the relation between Manin pairs and Lie quasi-bialgebras. Let $(\mathfrak{p}, \mathfrak{p}_1, \mathfrak{p}_2)$ be a Manin pair such that $\mathfrak{p}$ is finite dimensional. Then there is a Lie quasi-bialgebra structure on $\mathfrak{p}_1$. Indeed, the restriction of the bracket to $\mathfrak{p}_2 \otimes \mathfrak{p}_2 \to \mathfrak{p} = \mathfrak{p}_1 \oplus \mathfrak{p}_2$ has two components: $[\,,\,]_2 : \mathfrak{p}_2 \otimes \mathfrak{p}_2 \to \mathfrak{p}_2$ and

$[\,,\,]_1 : \mathfrak{p}_2 \otimes \mathfrak{p}_2 \to \mathfrak{p}_1$. Since $\mathfrak{p}_1 \simeq \mathfrak{p}_2^*$ and $\mathfrak{p}_1 \otimes \mathfrak{p}_1 \simeq (\mathfrak{p}_2 \otimes \mathfrak{p}_2)^*$, the first defines by transposition a cobracket $\delta : \mathfrak{p}_1 \to \mathfrak{p}_1 \otimes \mathfrak{p}_1$; that is, $< \delta(x)|u \otimes v >=< x|[u,v] >$, $x \in \mathfrak{p}_1$, $u, v \in \mathfrak{p}_2$. Similarly, the second defines an element $\psi \in \mathfrak{p}_1 \otimes \mathfrak{p}_1 \otimes \mathfrak{p}_1$. Put $\phi = -\psi$. Then $(\mathfrak{p}_1, \delta, \phi)$ is a Lie quasi-bialgebra.

Conversely, let $(\mathfrak{g}, \delta, \phi)$ be a finite dimensional Lie quasi-bialgebra. Put $\mathfrak{p} = \mathfrak{g} \oplus \mathfrak{g}^*$, $\mathfrak{p}_1 = \mathfrak{g}$, $\mathfrak{p}_2 = \mathfrak{g}^*$ and endow $\mathfrak{p}$ with the canonical scalar product. Let $\delta^* : \mathfrak{g}^* \otimes \mathfrak{g}^* \to \mathfrak{g}^*$ be the transpose of the bracket and let $\theta : \mathfrak{g}^* \otimes \mathfrak{g}^* \to \mathfrak{g}$ be induced by ϕ. Take $\delta^* - \theta$ as the commutator in $\mathfrak{g}^*$; it takes values in $\mathfrak{g} \oplus \mathfrak{g}^*$. Then $[x, l]$ can be uniquely defined for $x \in \mathfrak{g}$, $l \in \mathfrak{g}^*$ so that $\mathfrak{p}$ is a Lie algebra and the scalar product in $\mathfrak{p}$ is invariant. Explicitly, if $\{x_i\}$ is a basis of $\mathfrak{g}$, $\{x^i\}$ is the dual basis in $\mathfrak{g}^*$ and

$$[x_i, x_j] = c_{ij}^k x_k, \qquad \delta(x_i) = d_i^{jk} x_j \otimes x_k, \qquad \text{and} \qquad \phi = \phi^{ijl} x_i \otimes x_j \otimes x_l \tag{3}$$

(here and below, summation is assumed for repeated indices), then $\{x_i\} \cup \{x^i\}$ is a basis of $\mathfrak{p}$ and

$$\begin{aligned} &< x_i|x^j > = \delta_i^j, && < x_i|x_j >= 0, && < x^i|x^j > = 0, \\ &[x^i, x^j] = d_k^{ij} x^k - \phi^{ijl} x_l, && [x_i, x^j] = d_i^{jk} x_k + c_{li}^j x^l. && \end{aligned} \tag{4}$$

It is clear from the preceding discussion that $\mathfrak{p}_2$ is a Lie subalgebra if and only if $\phi = 0$.

Example 2.4. We recall that, for a Lie algebra $\mathfrak{g}$ and a $\mathfrak{g}$-module V, the motion Lie algebra $\mathfrak{g} \ltimes V$ is the vector space $\mathfrak{g} \oplus V$ with the bracket

$$[(x,u),(y,v)] = ([x,y], x.v - y.u), \qquad x, y \in \mathfrak{g}, u, v \in V.$$

If $(\mathfrak{g}, 0, 0)$ is the trivial Lie bialgebra with underlying Lie algebra $\mathfrak{g}$ then $\mathfrak{p} = \mathfrak{g} \oplus \mathfrak{g}^*$ is the motion Lie algebra with respect to the coadjoint representation.

Remark 2.5. If $(\mathfrak{g}, \delta)$ is a finite dimensional Lie bialgebra and $(\mathfrak{p}, \mathfrak{p}_1, \mathfrak{p}_2)$ is the corresponding Manin triple, there is a one-to-one correspondence between subbialgebras $\mathfrak{q} \subset \mathfrak{g}$ and subalgebras $\mathfrak{q}$ of $\mathfrak{p}_1 = \mathfrak{g}$ such that $\mathfrak{q}^\perp \cap \mathfrak{p}_2$ is an ideal of $\mathfrak{p}_2$.

Let $(\mathfrak{g}, \delta)$ be a finite dimensional Lie bialgebra and let $(\mathfrak{p}, \mathfrak{p}_1, \mathfrak{p}_2)$ be the corresponding Manin triple. The double of $\mathfrak{g}$ is the Lie bialgebra $\mathfrak{d}(\mathfrak{g})$ whose underlying Lie algebra is $\mathfrak{p}$ and whose Lie cobracket is ∂r, where r is the image of the canonical element of $\mathfrak{g} \otimes \mathfrak{g}^*$ under the embedding $\mathfrak{g} \otimes \mathfrak{g}^* \hookrightarrow \mathfrak{d}(\mathfrak{g}) \otimes \mathfrak{d}(\mathfrak{g})$ (the canonical element is $e_i \otimes e^i$, where e_i is a basis of $\mathfrak{g}$ and e^i is the dual basis in $\mathfrak{g}^*$). Let $(\mathfrak{q}, \mathfrak{q}_1, \mathfrak{q}_2)$ be the Manin triple corresponding to the Lie bialgebra $\mathfrak{d}(\mathfrak{g})$ and identify $\mathfrak{q}_2$ with $\mathfrak{p}$ by means of the bilinear form $\langle\,|\,\rangle$; the Lie bracket in $\mathfrak{q}_2$, denoted $[\,,\,]_*$, is

$$[u,v]_* = [v_1, u_1] + [u_2, v_2],$$

where u_i, v_i belongs to $\mathfrak{p}_i$ and the bracket in the right hand side is that of $\mathfrak{p}$. Indeed, $\langle \delta(x)|u \otimes v\rangle = \sum_i (\langle [x, e_i]|u\rangle\langle e^i|v\rangle + \langle e_i|u\rangle\langle [x, e^i]|v\rangle) = \langle [x, \sum_i \langle e^i|v\rangle e_i]|u\rangle + \langle [x, \sum_i \langle e_i|u\rangle e^i]|v\rangle = \langle [x, v_1]|u\rangle + \langle [x, u_2]|v\rangle = \langle x|[v_1, u]\rangle + \langle x|[u_2, v]\rangle$. Here, $x \in \mathfrak{q}_1$, $u, v \in \mathfrak{q}_2$.

Remark 2.6. Let now $(\mathfrak{p}, \mathfrak{p}_1, \mathfrak{p}_2)$ be a Manin pair with $\mathfrak{p}$ not necessarily finite dimensional. We decompose again $[x,y] = [x,y]_1 + [x,y]_2$ for $x, y \in \mathfrak{p}_2$ and let $\delta : (\mathfrak{p}_2)^* \to (\mathfrak{p}_2 \otimes \mathfrak{p}_2)^*$ be the transpose of $[\,,\,]_2$. Identifying $\mathfrak{p}_1$ with a subspace of $(\mathfrak{p}_2)^*$, the space of those $x \in \mathfrak{p}_1$ such that $\delta(x) \in \mathfrak{p}_1 \otimes \mathfrak{p}_1$ is a Lie subalgebra of $\mathfrak{p}_1$. Let $\{x_i : i \in I\}$ be a basis of $\mathfrak{p}_1$ and assume that there exists a family $\{x^i : i \in I\}$ in $\mathfrak{p}_2$ such that $\langle x_i | x^j \rangle = \delta_i^j$. If the support of the family $(d_k^{ij})_{i,j \in I}$ is finite for each k, then $\delta(\mathfrak{p}_1) \subset \mathfrak{p}_1 \otimes \mathfrak{p}_1$.

Let ϕ^{ijl} be given by $[x^i, x^j]_1 = -\phi^{ijl} x_l$. If the support of the family $(\phi^{ijl})_{i,j,l \in I}$ is finite then it defines $\phi \in \mathfrak{p}_1 \otimes \mathfrak{p}_1 \otimes \mathfrak{p}_1$ and $(\mathfrak{p}_1, \delta, \phi)$ is a Lie quasi-bialgebra. In fact, a weak version would be that $\operatorname{ad} x(\phi^{ijl}) \in \mathfrak{p}_1 \otimes \mathfrak{p}_1 \otimes \mathfrak{p}_1$ for any $x \in \mathfrak{p}_1$.

We now recall the notion of twisting of Lie quasi-bialgebras [7]. If r is an element of $\mathfrak{g} \otimes \mathfrak{g}$, then set

$$\widehat{r} := [r^{12}, r^{13}] + [r^{12}, r^{23}] + [r^{13}, r^{23}]; \tag{5}$$

the identity $\widehat{r} = 0$ is the classical Yang-Baxter equation (CYBE). If $r = r^{ij} x_i \otimes x_j$, then $\widehat{r} = \widehat{r}^{ijk} x_i \otimes x_j \otimes x_k$, where, keeping the notation from (3),

$$\widehat{r}^{ijk} = r^{sj} r^{tk} c_{st}^i + r^{is} r^{tk} c_{st}^j + r^{is} r^{jt} c_{st}^k. \tag{6}$$

Let $(\mathfrak{g}, \delta, \phi)$ be a Lie quasi-bialgebra and let $r \in \wedge^2 \mathfrak{g}$. Put

$$\delta_r = \delta(x) + \operatorname{ad} x r, \qquad \phi_r = \phi + \operatorname{Alt}(\delta \otimes \operatorname{id}) r - \widehat{r}. \tag{7}$$

Then $(\mathfrak{g}, \delta_r, \phi_r)$ is also a Lie quasi-bialgebra; we shall say that it is obtained from $(\mathfrak{g}, \delta, \phi)$ by *twisting via* r. If $(\mathfrak{g}, \delta)$ is a Lie bialgebra, $\widehat{r} = 0$ and $\operatorname{Alt}(\delta \otimes \operatorname{id}) r = 0$ then $(\mathfrak{g}, \delta_r)$ is a Lie bialgebra. These hypotheses hold if $r \in \wedge^2 \mathfrak{g}_0$ where $\mathfrak{g}_0$ is an abelian subalgebra of $\mathfrak{g}$ such that $\delta(\mathfrak{g}_0) = 0$.

Lemma 2.8 below is stated in [7] without proof; we include one for completeness. We need the following elementary linear algebra facts:

Remark 2.7. (a). Let W be a vector subspace of a vector space V. Fix a complement U of W in V, i. e. $V = W \oplus U$. There is a bijection between the set of all complements of W in V and $\hom(U, W)$. Explicitly, if Z is such a complement and $x \in U$ then write $x = x_W + x_Z$, with $x_W \in W$, $x_Z \in Z$ and define $\varphi_Z(x) := x_W$. Conversely, if $\varphi \in \hom(U, W)$ then $Z :=$ the image of Φ, where $\Phi(x) := x - \varphi(x)$, is a complement of W.

(b). If in addition V is provided with a non-degenerate symmetric bilinear form $(\,,\,)$, W is isotropic and admits an isotropic complement U then there is a bijection between the set of all isotropic complements of W in V and $\{\varphi \in \hom(U, W) : (\varphi(x), y) = -(x, \varphi(y))\}$.

(c). If in addition V is finite dimensional, there is a bijection between the set of all isotropic complements of W in V and $\wedge^2 W$. Explicitly, let (x_i) be a basis of W and let $x^j \in U$ such that $(x_i, x^j) = \delta_i^j$. If $r = r^{ij} x_i \otimes x_j \in \wedge^2 W$ then the subspace $V_r := \langle x^i + r^{ji} x_j \rangle$ of V is an isotropic complement of W.

Lemma 2.8. *Let $(\mathfrak{g}, \delta, \phi)$ be a finite dimensional Lie quasi-bialgebra and let $(\mathfrak{p}, \mathfrak{p}_1, \mathfrak{p}_2)$ be the corresponding Manin pair. Then changing the isotropic complement of $\mathfrak{p}_1$ amounts to twisting the corresponding Lie quasi-bialgebra. Explicitly,* *if $r \in \wedge^2\mathfrak{g}$ and $\mathfrak{u} = \mathfrak{p}_{-r}$ corresponds to $-r$ as described in Remark 2.7 (c), then the Manin pair $(\mathfrak{p}, \mathfrak{p}_1, \mathfrak{u})$ corresponds to the Lie quasi-bialgebra $(\mathfrak{g}, \delta_r, \phi_r)$.*

Proof. Let $x_1, \dots, x_n$ be a basis of $\mathfrak{p}_1$ and $x^1, \dots, x^n$ be the dual basis in $\mathfrak{p}_2$; so equations (3) and (4) hold. Let $u^1, \dots, u^n$ be the basis in $\mathfrak{u}$ given by $u^i := x^i + r^{ij}x_j$. Let $\widehat{r}$ have the same meaning as in (6). Then

$$\begin{aligned}
[u^i, u^j] &= [r^{il}x_l + x^i, r^{jk} + x_k x^j] \\
&= [x^i, x^j] + r^{jk}[x^i, x_k] + r^{il}[x_l, x^j] + r^{il}r^{jk}[x_l, x_k] \\
&= d_s^{ij}x^s - \phi^{ijs}x_s + r^{kj}d_k^{is}x_s + r^{kj}c_{sk}^i x^s \\
&\quad + r^{il}d_l^{js}x_s + r^{il}c_{sl}^j x^s + r^{il}r^{jk}c_{lk}^s x_s \\
&= d_s^{ij}u^s + r^{kj}c_{sk}^i u^s + r^{il}c_{sl}^j u^s - \phi^{ijk}x_k \\
&\quad - < u^i \otimes u^j \otimes \mathrm{id} \,|\, \mathrm{Alt}(\delta \otimes \mathrm{id})r > + \widehat{r}^{ijk}x_k
\end{aligned} \tag{8}$$

It follows that $\phi_r = \phi + \mathrm{Alt}(\delta \otimes \mathrm{id})r - \widehat{r}$ and

$$\begin{aligned}
\delta_r(x_t) &= d_t^{ij}x_i \otimes x_j + r^{kj}c_{tk}^i x_i \otimes x_j + r^{il}c_{tl}^j x_i \otimes x_j \\
&= \delta(x_t) + \mathrm{ad}\, x_t(r^{kj}x_k) \otimes x_j + r^{ik}x_i \otimes \mathrm{ad}\, x_t(x_k) = \delta(x_t) + \mathrm{ad}\, x_t(r).
\end{aligned}$$

■

Definition 2.9. A Lie quasi-bialgebra $(\mathfrak{g}, \delta, \phi)$ is *quasitriangular* if there exists $r \in \mathfrak{g} \otimes \mathfrak{g}$, such that:

1. the coboundary of r is the cobracket of $\mathfrak{g}$, i.e. $\partial r = \delta$, and
2. $\widehat{r} = \phi$. (The definition of $\widehat{r}$ is given in formula (5).)

So that if $\mathfrak{g}$ is Lie bialgebra (i.e. $\phi = 0$), then it is quasitriangular if and only if $\partial r = \delta$ and r satisfies the classical Yang-Baxter equation. The following result is also stated in [7] without proof; a proof appears in [3].

Lemma 2.10. *Let $(\mathfrak{g}, \delta, \phi)$ be a finite dimensional Lie quasi-bialgebra and let $(\mathfrak{p}, \mathfrak{p}_1, \mathfrak{p}_2)$ be the corresponding Manin pair. Let $\mathfrak{u}$ be a subspace of $\mathfrak{p}$ such that $\mathfrak{p} = \mathfrak{p}_1 \oplus \mathfrak{u}$, and let $r \in \mathfrak{g} \otimes \mathfrak{g}$ be the tensor associated to $\mathfrak{u}$ (i.e. $\mathfrak{p}_r = \mathfrak{u}$). Then*

(a) $[\mathfrak{p}_1, \mathfrak{u}] \subset \mathfrak{u}$ if and only if the coboundary of r is the cobracket of $\mathfrak{g}$.

(b) $[\mathfrak{p}, \mathfrak{u}] \subset \mathfrak{u}$ if and only if the coboundary of r is the cobracket of $\mathfrak{g}$ and $\widehat{r} = \phi$. In other words, $(\mathfrak{g}, \delta, \phi)$ is quasitriangular if and only if $\mathfrak{p}_1$ admits a complementary ideal in $\mathfrak{p}$.

Proof. (a). Let $x_1, \dots, x_n$ be a basis of $\mathfrak{g} = \mathfrak{p}_1$ and $x^1, \dots, x^n$ be a dual basis in $\mathfrak{g}^* = \mathfrak{p}_2$. Let $u^i = r^{ji}x_j + x^i$; then $[x_l, u^i] = (r^{ti}c_{lt}^k + d_l^{ik})x_k + c_{sl}^i x^s$. Thus $[x_l, u^i] \in \mathfrak{u}$ if and only if $(r^{ti}c_{lt}^k + d_l^{ik})x_k + c_{sl}^i x^s = \alpha_{sl}^i(r^{ks}x_k + x^s)$, for some scalars α_{sl}^i. This happens if and only if $c_{sl}^i = \alpha_{sl}^i$ and $r^{ti}c_{lt}^k + d_l^{ik} = c_{sl}^i r^{ks}$, or $d_l^{ik} = r^{ti}c_{tl}^k + r^{kt}c_{tl}^i$, for all j, k, l. On the other hand,

$$\partial r(x_l) = (r^{ti}c_{lt}^k + r^{kt}c_{lt}^i)x_k \otimes x_i = -(r^{kt}c_{tl}^i + r^{ti}c_{tl}^k)x_k \otimes x_i. \tag{9}$$

That is, $[\mathfrak{p}_1, \mathfrak{u}] \subset \mathfrak{u}$ if and only if $d_l^{ik} = r^{ti}c_{tl}^k + r^{kt}c_{tl}^i$ for all j, k, l if and only if $\partial r(x_l) = d^{ki}x_k \otimes x_i = \delta(x_l)$.

For (b), we can assume that $\partial r = \delta$ by (a). We have $[x^l, u^k] = (r^{sk} d^{tl}_s - \phi^{lkt})x_t + (r^{sk} c^l_{sp} + d^{lk}_p)x^p$. Thus $[x^l, u^k] \in \mathfrak{u}$ if and only if $[x^l, u^k] = (r^{sk} c^l_{sp} + d^{lk}_p)(r^{tp} x_t + x^p)$, i.e. $r^{sk} d^{tl}_s - \phi^{lkt} = r^{sk} c^l_{sw} r^{tw} + d^{lk}_s r^{ts}$. Thus

$$r^{sk} r^{wl} c^t_{sw} + r^{sk} r^{tw} c^l_{sw} + r^{sk} r^{tw} c^l_{ws} + r^{wl} r^{ts} c^k_{sw} + r^{kw} r^{ts} c^l_{sw} = \phi^{lkt}.$$

The second and the third term cancel because of the antisymmetry of the bracket. Performing some permutations in the others terms, we have

$$r^{sk} r^{wl} c^t_{sw} + r^{ts} r^{wl} c^k_{sw} + r^{ts} r^{kw} c^l_{sw} = \phi^{lkt}.$$

That is, $\mathfrak{u}$ is an ideal if and only if $\widehat{r} = \phi$ and $\partial r = \delta$. ■

Definition 2.11. [13]. A quasitriangular Lie quasi-bialgebra $(\mathfrak{g}, r)$ is *factorizable* if the map $\mathfrak{g}^* \to \mathfrak{g}$, $\alpha \mapsto \langle \alpha \otimes \mathrm{id}, r + \tau(r) \rangle$, is a bijection, where τ is the usual transposition.

3. Lie algebras with quasi-triangular decomposition

In this section we introduce the notions of "Lie algebra with quasi-triangular decomposition" and "Lie algebra with triangular decomposition"; these definitions are inspired by [6, Ex. 3.2] and are related to but not the same as the notion discussed in [11]. We show that such a Lie algebra has a canonical structure of quasitriangular Lie quasi-bialgebra. We give two proofs of this fact; the second one uses the double and suggests a method of constructing Lie algebras with triangular decomposition.

Definition 3.1. Let $\mathfrak{g}$ be a Lie algebra. We shall say that the collection $(\mathfrak{g}_0, \mathfrak{g}_+, \mathfrak{g}_-, (\,|\,))$ is a *quasi-triangular decomposition* (QTD) of $\mathfrak{g}$ if $\mathfrak{g}_0$ is a subalgebra of $\mathfrak{g}$, $\mathfrak{g}_-, \mathfrak{g}_+$ are subspaces of $\mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_+$, and $(\,|\,) : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ is a $\mathfrak{g}$-invariant, non degenerate, symmetric, bilinear form such that

$$0 = (\mathfrak{g}_+ | \mathfrak{g}_+) = (\mathfrak{g}_- | \mathfrak{g}_-) = (\mathfrak{g}_+ | \mathfrak{g}_0) = (\mathfrak{g}_0 | \mathfrak{g}_-).$$

Furthermore, we shall say that $(\mathfrak{g}_0, \mathfrak{g}_+, \mathfrak{g}_-, (\,|\,))$ is a *triangular decomposition* (TD) if $\mathfrak{g}_0$ is abelian, $\mathfrak{g}_-, \mathfrak{g}_+$ are subalgebras of $\mathfrak{g}$ and $[\mathfrak{g}_\pm, \mathfrak{g}_0] \subset \mathfrak{g}_\pm$.

In what follows, we shall simply say "$\mathfrak{g}$ is a Lie algebra with quasi-triangular decomposition or triangular decomposition", without mentioning the data defining it. We shall use the notation $x = x_+ + x_0 + x_-$ for $x \in \mathfrak{g}$, if $x_j \in \mathfrak{g}_j$, $j \in \{+, 0, -\}$.

Theorem 3.1. *Let $\mathfrak{g}$ be a finite dimensional Lie algebra with QTD (respectively TD). Then $\mathfrak{g}$ admits a canonical structure of Lie quasi-bialgebra (resp. Lie bialgebra), which is quasitriangular. If $\mathfrak{g}$ has another structure of QTD with the same non-degenerate invariant form then the corresponding structures of Lie quasi-bialgebra are related by a twist.*

Proof. Let $\mathfrak{p} = \mathfrak{g} \times \mathfrak{g}$ with the product Lie algebra structure, $\mathfrak{p}_1 = \{(a, a) : a \in \mathfrak{g}\}$ and $\mathfrak{p}_2 = \{(a_- + a_0, a_+ - a_0) : a_- \in \mathfrak{g}_-,\ a_+ \in \mathfrak{g}_+,\ a_0 \in \mathfrak{g}_0\}$. Then $\mathfrak{p}_1$ is a Lie subalgebra of $\mathfrak{p}$ and $\mathfrak{p}_2 \subset \mathfrak{p}$ is a subspace complementary to $\mathfrak{p}_1$. Let $<\ |\ >: \mathfrak{p} \times \mathfrak{p} \to$

$\mathbb{C}$ be the bilinear form defined by $<(x,y)|(u,v)>=(x|u)-(y|v)$. Then $<\,|\,>$ is $\mathfrak{p}$-invariant, non degenerate and $<\mathfrak{p}_1|\mathfrak{p}_1>=0$. If $(x,y)=(x_-+x_0,x_+-x_0)$ and $(u,v)=(u_-+u_0,u_+-u_0)$ belong to $\mathfrak{p}_2$, then

$$<(x,y)|(u,v)>=(x,u)-(y|v)=(x_0|u_0)-(-x_0|-u_0)=0;$$

that is $<\mathfrak{p}_2|\mathfrak{p}_2>=0$. Hence $(\mathfrak{p},\mathfrak{p}_1,\mathfrak{p}_2)$ is a Manin pair and $\mathfrak{p}_1\simeq\mathfrak{g}$ has a structure of Lie quasi-bialgebra. If $\mathfrak{g}$ has a TD, then $\mathfrak{p}_2$ is a Lie subalgebra of $\mathfrak{p}$ and $\mathfrak{g}$ has a structure of Lie bialgebra. Since $\mathfrak{u}=\{(0,x):x\in\mathfrak{g}\}$ is an ideal complementary to $\mathfrak{p}_1$, $\mathfrak{g}$ is quasitriangular by Lemma 2.10.

Let $(\mathfrak{g}'_0,\mathfrak{g}'_+,\mathfrak{g}'_-,(\,|\,))$ be another QTD of $\mathfrak{g}$; its Manin pair is $(\mathfrak{p},\mathfrak{p}_1,\mathfrak{p}'_2)$, where $\mathfrak{p}'_2=\{(a_-+a_0,a_+-a_0):a_-\in\mathfrak{g}'_-,a_+\in\mathfrak{g}'_+,a'_0\in\mathfrak{g}'_0\}$. The corresponding bilinear form is again $<(x,y)|(u,v)>=(x|u)-(y|v)$; we have two Manin pairs that only differ in the complement of $\mathfrak{p}_1$, so Lemma 2.8 applies. ■

Remark 3.2. $\delta=0$ if and only if $[\mathfrak{g}_+,\mathfrak{g}_+]=[\mathfrak{g}_-,\mathfrak{g}_-]=[\mathfrak{g},\mathfrak{g}_0]=0$.

Let $\mathfrak{g}=\mathfrak{g}_+\oplus\mathfrak{g}_0\oplus\mathfrak{g}_-$ be a finite dimensional Lie algebra with TD, and consider on $\mathfrak{g}$ the structure of Lie bialgebra provided by Theorem 3.1.

Lemma 3.3. *(a) $\mathfrak{b}_+=\mathfrak{g}_0\oplus\mathfrak{g}_+$ and $\mathfrak{b}_-=\mathfrak{g}_0\oplus\mathfrak{g}_-$ are Lie subbialgebras of $\mathfrak{g}$. As Lie algebras, $\mathfrak{b}_\pm^*\cong\mathfrak{b}_\mp$.*

(b) $\mathfrak{d}(\mathfrak{b}_+)$ is isomorphic as a Lie algebra to the direct product $\mathfrak{g}\times\mathfrak{g}_0$.

Proof. (a). Keep the notation of the proof of Theorem 3.1. The subspace orthogonal to $\mathfrak{b}_+$ (resp., $\mathfrak{b}_-$) in $\mathfrak{p}_2$ is $0\times\mathfrak{g}_+$ (resp., $\mathfrak{g}_-\times 0$) which is clearly an ideal of $\mathfrak{p}_2$, and clearly $\mathfrak{p}_2/(0\times\mathfrak{g}_+)$ (resp., $\mathfrak{p}_2/\mathfrak{g}_-\times 0)$) is isomorphic to $\mathfrak{b}_-$ (resp., $\mathfrak{b}_+$), as Lie algebras. Notice that the pairing $\langle\,,\,\rangle$ between $\mathfrak{b}_+$ and $\mathfrak{b}_-$ is

$$\langle x,y\rangle=(x_0|y_0)+(x|y). \tag{10}$$

(b). Let $\Upsilon:\mathfrak{d}(\mathfrak{b}_+)\to\mathfrak{g}\times\mathfrak{g}_0$ be the linear isomorphism $\Upsilon(x_++x_0,y_0+y_-)=(x_++x_0+y_0+y_-,x_0-y_0)$. We want to show that $\Upsilon([u,v])=[\Upsilon(u)\Upsilon(v)]$; it suffices to consider $u=x\in\mathfrak{b}_+$, $v=y\in\mathfrak{b}_-$. Let us write $[x,y]=[x,y]_1+[x,y]_2$, where $[x,y]_1\in\mathfrak{b}_+$, $[x,y]_2\in\mathfrak{b}_-$. We deduce easily from (10) that

$$[x,y]_1=[x,y]_++\frac{1}{2}[x,y]_0,\quad [x,y]_2=[x,y]_-+\frac{1}{2}[x,y]_0. \tag{11}$$

Indeed, if $u\in\mathfrak{b}_-$, $\langle[x,y]_1,u\rangle=\langle x,[y,u]\rangle=(x|[y,u])=\langle[x,y]_++\frac{1}{2}[x,y]_0,u\rangle$. Now (11) implies our claim. ■

Let $\{x_j:j\in J\}$ be a basis of $\mathfrak{g}_+$, $\{y_j\}$ be its dual basis in $\mathfrak{g}_-$, $\{h_i:i\in I\}$ is an orthonormal basis of $\mathfrak{g}_0$. Then the dual basis of $B=\{x_j\}\cup\{y_j\}\cup\{h_i\}$ in $\mathfrak{p}_2$ is constituted by the vectors

$$x_j^*=(y_j,0),\quad y_j^*=(0,-x_j)\quad\text{and}\quad h_i^*=\frac{1}{2}(h_i,-h_i),\qquad j\in J,i\in I. \tag{12}$$

Corollary 3.4. *Let $\mathfrak{g}$ be a finite dimensional Lie algebra with TD. Then the Lie cobracket on $\mathfrak{g}$ provided by Theorem 3.1 is ∂r_0, where, in the notation above,*

$$r_0=\sum_{j\in J}x_j\otimes y_j+\frac{1}{2}\sum_{i\in I}h_i\otimes h_i. \tag{13}$$

This gives a new proof of the quasitriangularity of $\mathfrak{g}$.

Proof. Preserve the notation of the preceding proof. The orthogonal of the ideal $\Upsilon^{-1}(0 \times \mathfrak{g}_0)$ is $\{(u,v) \in \mathfrak{d}(\mathfrak{b}_+) : u_0 = v_0\}$, clearly a Lie subalgebra of the dual of $\mathfrak{d}(\mathfrak{b}_+)$. Then $\mathfrak{d}(\mathfrak{b}_+)/\Upsilon^{-1}(0 \times \mathfrak{g}_0) \simeq \mathfrak{g}$ inherits a Lie bialgebra structure and the canonical projection is a morphism of Lie bialgebras. We claim that this Lie bialgebra structure coincides with the structure defined in Theorem 3.1. Let $\{(u,v) \in \mathfrak{d}(\mathfrak{b}_+) : u_0 = v_0\} \to \mathfrak{p}_2$ be the application $(u,v) \mapsto (v,-u)$; it is easy to check that it is an isomorphism of Lie algebras. Since the introduced isomorphisms preserve the corresponding dualities, the claim follows. Let r be the canonical element of $\mathfrak{d}(\mathfrak{b}_+)$. It is easy to see that the image of r under the above projection is r_0; the latter satisfies CYBE because the former does. ■

Corollary 3.5. $\delta(\mathfrak{g}_0) = 0$.

Proof. If $H \in \mathfrak{g}_0$, then write $[H, x_j] = \sum_i \varphi_{ji}(H) x_i$. It follows from the invariance of the bilinear form that $[H, y_j] = -\sum_i \varphi_{ij}(H) y_i$. Hence $\delta(H) = \sum_{j\in J}[H, x_j] \otimes y_j + \sum_{j\in J} x_j \otimes [H, y_j] = 0$. ■

Corollary 3.6. *A finite dimensional Lie bialgebra with TD is factorizable.* ■

Lemma 3.3 and Corollary 3.4 suggest the following method of constructing Lie algebras with TD.

Theorem 3.2. *Let $\mathfrak{b}$ be a finite dimensional Lie bialgebra. Consider $\mathfrak{b} \subset \mathfrak{d}(\mathfrak{b})$ the double of $\mathfrak{b}$. Assume that*

(a) there exists an abelian subalgebra $\mathfrak{h}$ such that, as vector spaces, $\mathfrak{b} = \mathfrak{h} \oplus [\mathfrak{b}, \mathfrak{b}]$;

(b) $\mathfrak{h}^\perp = [\mathfrak{b}^*, \mathfrak{b}^*]$; there exists an abelian subalgebra $\widetilde{\mathfrak{h}}$ such that, as vector spaces, $\mathfrak{b}^* = \widetilde{\mathfrak{h}} \oplus [\mathfrak{b}^*, \mathfrak{b}^*]$, and $\widetilde{\mathfrak{h}}^\perp = [\mathfrak{b}, \mathfrak{b}]$;

(c) for any $x \in \mathfrak{h}$, there exists a unique $\widetilde{x} \in \widetilde{\mathfrak{h}}$ such that $\operatorname{ad}\widetilde{x}$ coincides with $\operatorname{ad} x$ on $\mathfrak{b}^* \subset \mathfrak{d}(\mathfrak{b})$.

Given h in $\mathfrak{h}$, let $\widehat{h}$ be the unique element of $\widetilde{\mathfrak{h}}$ such that $\langle x|\widehat{h}\rangle = \langle \widetilde{x}|h\rangle$, for all x in $\mathfrak{h}$.

Let $\mathfrak{g}_+ = [\mathfrak{b}, \mathfrak{b}]$, $\mathfrak{g}_- = [\mathfrak{b}^*, \mathfrak{b}^*]$, $\mathfrak{g}_0 = \{h + \widehat{h} : h \in \mathfrak{h}\}$. Then

$$\mathfrak{g} =: \mathfrak{g}_- \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_+$$

is a Lie subalgebra of $\mathfrak{d}(\mathfrak{b})$ with TD. The non-degenerate invariant bilinear form is the one inherited from $\mathfrak{d}(\mathfrak{b})$.

Proof. First, we remark that if $x \in \mathfrak{h}$, $u \in \mathfrak{b}$ and $w \in \mathfrak{b}^*$, then

$$< \operatorname{ad}(x)u|w > = < [x,u]|w > = - < u|[x,w] > = - < u|\operatorname{ad}(x)w > .$$

By (c) we know that $\operatorname{ad}(x)w = \operatorname{ad}(\widetilde{x})w$, hence

$$- < u|\operatorname{ad}(x)w > = - < u|\operatorname{ad}(\widetilde{x})w > = - < u|[\widetilde{x},w] > = < \operatorname{ad}(\widetilde{x})u|w > .$$

Thus, $\operatorname{ad}(\widetilde{x})$ coincides with $\operatorname{ad}(x)$ on $\mathfrak{b}$.

Let $\mathfrak{r} = \{x - \widetilde{x} : x \in \mathfrak{h}\}$, then $\mathfrak{r}$ is an ideal of $\mathfrak{d}(\mathfrak{b})$. In fact if $u \in \mathfrak{b}$, then $[u, x - \widetilde{x}] = [u, x] - [u, \widetilde{x}] = 0$ (see above). In an analogous way, for (c), we obtain that if $w \in \mathfrak{b}^*$, then $[w, x - \widetilde{x}] = 0$. So, $[\mathfrak{r}, \mathfrak{d}(\mathfrak{b})] = 0$ and clearly $\mathfrak{r}$ is an ideal.

Let $z \in \mathfrak{r}^\perp$, so $z = u+w$, $u \in \mathfrak{b}$, $w \in \mathfrak{b}^*$ and $< u|x-\widetilde{x} > + < w|x-\widetilde{x} >= 0$ for all $x \in \mathfrak{h}$. For (a) and (b) we have $u = h_1 + u_1$ and $w = s_1 + w_1$, with $h_1 \in \mathfrak{h}$, $u_1 \in [\mathfrak{b}, \mathfrak{b}]$, $s_1 \in \widetilde{\mathfrak{h}}$ and $w_1 \in [\mathfrak{b}^*, \mathfrak{b}^*]$. So, $0 =< z|x - \widetilde{x} >=< h_1 + u_1| - \widetilde{x} > + < s_1 + w_1|x >=< h_1| - \widetilde{x} > + < s_1|x >$. Thus, $< h_1|\widetilde{x} >=< s_1|x >$ for all $x \in \mathfrak{h}$, then by hypothesis $s_1 = \widehat{h_1}$. This implies that

$$\mathfrak{r}^\perp = \mathfrak{g}_- \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_+ = \mathfrak{g}.$$

As $\mathfrak{r}$ is an ideal, $\mathfrak{g}$ is also an ideal. In particular, $\mathfrak{g}$ is a subalgebra of $\mathfrak{d}(\mathfrak{b})$.

■

4. Examples.

Example 4.1. A Lie algebra $\mathfrak{g}$ with TD such that $\mathfrak{g}_0 = 0$ is equivalent to a Manin triple $(\mathfrak{g}, \mathfrak{g}_+, \mathfrak{g}_-)$.

Example 4.2. Let $\mathfrak{g}$ be a Lie bialgebra with TD. If we twist via $r \in \wedge^2\mathfrak{g}_0$ then $(\mathfrak{g}, \delta_r)$ is a Lie bialgebra (use Corollary 3.5).

Example 4.3. Let $\mathfrak{g}$ be a Lie algebra and let $< \mid >$ be a non degenerate invariant bilinear form on $\mathfrak{g}$. Then $(\mathfrak{g}, 0, 0, < \mid >)$ is a QTD. It is a TD if and only if $\mathfrak{g}$ is abelian.

Example 4.4. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the complexification of a decomposition of a real simple Lie algebra. It is known that the representation of $\mathfrak{k}$ on $\mathfrak{p}$ is either irreducible or a direct sum of two irreducible components; in the latter case, the corresponding symmetric space is hermitian. See [9]. Assume we are in the hermitian case, i. e. that $\mathfrak{p} = \mathfrak{p}_1 \oplus \mathfrak{p}_2$ as $\mathfrak{k}$-modules, with $\mathfrak{p}_1$ and $\mathfrak{p}_2$ irreducible. Then $(\mathfrak{k}, \mathfrak{p}_1, \mathfrak{p}_2, (\,|\,))$, where $(\,|\,)$ is the Killing form, is a QTD. It is seldom a TD; only when $\mathfrak{g} = sl(2, \mathbb{C})$.

Example 4.5. Let A be a symmetrizable complex matrix of size $n\times$ and let $D = (d_1, \dots d_n)$ be an invertible diagonal matrix such that $DA = A^tD$. Let $\tilde{\mathfrak{g}} = \tilde{\mathfrak{g}}(A)$ be the Lie algebra defined in [10, §1.2] and let $\mathfrak{g} = \mathfrak{g}(A)$ be the corresponding contragradient Lie algebra [10, Ch. 1]. We preserve the notation $\mathfrak{h}$, $\tilde{\mathfrak{n}}^\pm$, e_i, f_i, $\alpha_i^\vee$, etc. from *loc cit.* Let $h_i = d_i\alpha_i^\vee$. Let $\mathfrak{r}$ be the unique maximal ideal among the ideals intersecting $\mathfrak{h}$ trivially; then $\mathfrak{g} \simeq \tilde{\mathfrak{g}}/\mathfrak{r}$. Then $\mathfrak{g}$ has a well-known triangular decomposition, cf [10, 1.2, 2.2], which gives rise to a Lie bialgebra structure by the method of Proposition 3.1. It is well-known [6] that the corresponding cobracket is given by

$$\delta(h_i) = 0, \quad \delta(e_i) = \frac{1}{2}(e_i \otimes h_i - h_i \otimes e_i) \quad \text{and} \quad \delta(f_i) = \frac{1}{2}(f_i \otimes h_i - h_i \otimes f_i). \tag{14}$$

Alternatively, it is not difficult to see that formula (14) determines a Lie bialgebra structure on $\tilde{\mathfrak{g}}$ or $\mathfrak{g}$. Indeed, [2, Ch. II §2 Prop. 8] allows to define the 1-cocycle on $\tilde{\mathfrak{g}}$ or $\mathfrak{g}$. The co-Jacobi identity is also easy to check; it suffices to verify it on generators.

Example 4.6. Let $\mathfrak{g}$ be a Lie algebra and $(\mathfrak{h}, \mathfrak{n}_+, \mathfrak{n}_-, (\,|\,))$ a triangular decomposition of $\mathfrak{g}$. Let $\mathfrak{l} = \mathfrak{g} \times \mathfrak{g}$ be the motion Lie algebra with respect to the adjoint representation, cf. Example 2.4. Take

$$\mathfrak{l}_0 = \mathfrak{h} \times \mathfrak{h}, \quad \mathfrak{l}_+ = \mathfrak{n}_+ \times \mathfrak{n}_+, \quad \text{and} \quad \mathfrak{l}_- = \mathfrak{n}_- \times \mathfrak{n}_-.$$

Thus $\mathfrak{l} = \mathfrak{l}_+ \oplus \mathfrak{l}_0 \oplus \mathfrak{l}_-$. Let $k_{\mathfrak{l}}(\,|\,) : \mathfrak{l} \times \mathfrak{l} \to \mathbb{C}$ be defined by

$$k_{\mathfrak{l}}((x, y)|(u, v)) = (x|u) + (y|u) + (x|v).$$

Then $(\mathfrak{l}_0, \mathfrak{l}_+, \mathfrak{l}_-, k_{\mathfrak{l}}(|))$ is a TD of $\mathfrak{l}$.

Now, we assume that $\mathfrak{g}$ is a simple Lie algebra and $\mathfrak{g} = \mathfrak{n}_+ \oplus \mathfrak{h} \oplus \mathfrak{n}_-$ is the usual decomposition, where $\mathfrak{h}$ is a Cartan subalgebra, $\mathfrak{n}_\pm$ is the span of the positive, resp. negative, root vectors. In this context, $(|)$ will be the Killing form. Let A be the Cartan matrix of $\mathfrak{g}$, Φ be the root system of $\mathfrak{g}$, Φ^+ be the set of positive roots and $\Pi = \{\alpha_1, \dots, \alpha_n\}$ be the set of simple roots. We choose $a_\alpha \in \mathfrak{g}_\alpha - \{0\}$, $\alpha \in \Phi$, ($\mathfrak{g}_\alpha$ is the root space) and $H_i, m_i \in \mathfrak{h}$ such that:

$$(H_i|H) = \alpha_i(H), \ \forall H \in \mathfrak{h}, \quad [a_{\alpha_i}, a_{-\alpha_i}] = H_i, \quad (a_\alpha|a_{-\alpha}) = 1, \quad (H_i|m_j) = \delta_{ij}.$$

Let us consider the following elements of $\mathfrak{l}$:

$$\begin{array}{lllll} x_\alpha = (a_\alpha, 0), & y_\alpha = (0, a_{-\alpha}), & u_\alpha = (0, a_\alpha), & v_\alpha = (a_{-\alpha}, -a_{-\alpha}), & \alpha \in \Phi, \\ h_i = (H_i, 0), & l_i = (m_i, 0), & r_i = (0, H_i), & s_i = (m_i, -m_i) & i = 1, \dots, n. \end{array}$$

Then it is clear that $\{x_\alpha, u_\alpha\}_{\alpha \in \Phi^+}$ (resp. $\{h_i, r_i\}_{1 \le i \le n}$) is a basis of $\mathfrak{l}_+$ (resp. $\mathfrak{l}_0$), whose dual basis is $\{y_\alpha, v_\alpha\}_{\alpha \in \Phi^+}$ (resp. $\{l_i, s_i\}_{1 \le i \le n}$).

Applying (4) and (12), we obtain the cobracket δ:

$$\delta(x_{\pm\alpha_i}) = \frac{1}{2} x_{\pm\alpha_i} \wedge r_i + \frac{1}{2} u_{\pm\alpha_i} \wedge (h_i - r_i) \quad \delta(u_{\pm\alpha_i}) = \frac{1}{2} u_{\pm\alpha_i} \wedge r_i \quad \delta(h_i) = \delta(r_i) = 0.$$

The corresponding r-matrix is given by

$$r_0 = \sum_{\alpha \in \Phi^+} x_\alpha \otimes y_\alpha + u_\alpha \otimes v_\alpha + \frac{1}{2} \sum_{i,t} (m_t|m_i)(h_i \otimes h_t - r_i \otimes r_t) \tag{15}$$

Note that (15) is a new example of a classical r-matrix.

Remark 4.7. Let V be a $\mathfrak{g}$-module and consider the motion Lie algebra $\mathfrak{g} \oplus V$, i. e. with the Lie bracket given by $[(x, y), (u, v)] = ([x, u], xv - uy)$. Suppose that $\mathfrak{g} \oplus V$ admits a non degenerate invariant bilinear form $(\,|\,)$. Then $(V|\mathfrak{g}V) = 0$. If V is irreducible and non trivial, $(V|V) = 0$ and we obtain a monomorphism of $\mathfrak{g}$-modules $V \hookrightarrow \mathfrak{g}^*$. Assume that $\mathfrak{g}$ is simple: then $V \simeq \mathfrak{g}^*$. If in addition $\mathfrak{g}$ is finite dimensional, identify $\mathfrak{g}^*$ with $\mathfrak{g}$ via the Killing form. Then any invariant non-degenerate bilinear form on $\mathfrak{g} \oplus V$ is $a(x|u) + b(y|u) + b(x|v)$, for some scalars a, b. Let c be a scalar and let T_c be the Lie algebra automorphism of $\mathfrak{g} \oplus V$, $T_c((x, v)) = (x, cv)$. By using an appropriate T_c, we may assume that an invariant non-degenerate bilinear form on $\mathfrak{g} \oplus V$ is a multiple of the one considered in Example 4.6.

Example 4.8. The extended Heisenberg algebras have a quasitriangular Lie bialgebra structure considered in [4] as well as their quantizations. It is easy to see that the Lie bialgebra structure arises from a TD; see Example 5.6 below.

Example 4.9. Let $\mathcal{L} = \mathbb{C}[t, t^{-1}]$ be the algebra of Laurent polynomials in t. Recall that the residue of a Laurent polynomial P is defined by $\operatorname{Res} P =$ the coefficient of P at degree -1. Let $\phi : \mathcal{L} \times \mathcal{L} \to \mathbb{C}$ be defined by $\phi(P, Q) = \operatorname{Res} \frac{dP}{dt} Q$. Then

$$\phi(P, Q) = -\phi(Q, P), \tag{16}$$

$$\phi(PQ, R) + \phi(QR, P) + \phi(RP, Q) = 0 \quad (P, Q, R \in \mathcal{L}). \tag{17}$$

Let $\mathfrak{g}$ be a Lie algebra with QTD. As in [10], consider the *loop algebra* $\mathcal{L}(\mathfrak{g}) := \mathcal{L} \otimes \mathfrak{g}$, with the bracket $[\,,\,]_0$ given by $[P \otimes x, Q \otimes y]_0 = PQ \otimes [x, y]$, $P, Q \in \mathcal{L}$, $x, y \in \mathfrak{g}$. Let $\psi : \mathcal{L}(\mathfrak{g}) \times \mathcal{L}(\mathfrak{g}) \to \mathbb{C}$, $\psi(P \otimes x, Q \otimes y) = (x|y)\phi(P, Q)$. It is easy to check, using (16), (17) and the symmetry and invariance of $(\,|\,))$, that ψ is a 2-cocycle on $\mathcal{L}(\mathfrak{g})$:

$$\psi(a, b) = -\psi(b, a),$$
$$\psi([a, b]_0, c) + \psi([b, c]_0, a) + \psi([c, a]_0, b) = 0, \qquad a, b, c \in \mathcal{L}(\mathfrak{g}).$$

Denote by $\widetilde{\mathcal{L}}(\mathfrak{g})$ the extension of the Lie algebra $\mathcal{L}(\mathfrak{g})$ by a 1-dimensional center, associated to the cocycle ψ. Explicity, $\widetilde{\mathcal{L}}(\mathfrak{g}) = \mathcal{L}(\mathfrak{g}) \oplus \mathbb{C}K$ and the bracket is given by

$$[a + \lambda_1 K, b + \lambda_2 K]_1 = [a, b]_0 + \psi(a, b)K, \qquad a, b \in \mathcal{L}(\mathfrak{g});\ \lambda_1, \lambda_2 \in \mathbb{C}.$$

The derivation $t\frac{d}{dt} : \mathcal{L} \to \mathcal{L}$ extends to a derivation of $\mathcal{L}(\mathfrak{g})$ by $t\frac{d}{dt}(x \otimes P) = x \otimes t\frac{d}{dt}P$. Let $\hat{\mathcal{L}}(\mathfrak{g})$ be the Lie algebra obtained by adjoining to $\widetilde{\mathcal{L}}(\mathfrak{g})$ a derivation D which acts on $\mathcal{L}(\mathfrak{g})$ as $t\frac{d}{dt}$ and which kills K. In other words, $\hat{\mathcal{L}}(\mathfrak{g})$ is $\hat{\mathcal{L}}(\mathfrak{g}) = \mathcal{L}(\mathfrak{g}) \oplus \mathbb{C}K \oplus \mathbb{C}D$ with the bracket

$$[a + \lambda_1 K + \mu_1 D, b + \lambda_2 K + \mu_2 D] = [a, b]_0 + \psi(a, b)K + \mu_1 D(b) - \mu_2 D(a),$$

$a, b \in \mathcal{L}(\mathfrak{g}), \lambda_1, \lambda_2, \mu_1, \mu_2 \in \mathbb{C}$. Even more explicity

$$[x \otimes t^m + \lambda_1 K + \mu_1 D, y \otimes t^n + \lambda_2 K + \mu_2 D] =$$
$$[x, y] \otimes t^{m+n} + m\delta_{m,-n}(x|y)K + \mu_1 n y \otimes t^n - \mu_2 m x \otimes t^m,$$

$x, y \in \mathfrak{g}$, $m, n \in \mathbb{Z}$, $\lambda_1, \lambda_2, \mu_1, \mu_2 \in \mathbb{C}$. We extend the form $(\,|\,)$ to a form $(\,|\,)_t$ on $\mathcal{L}(\mathfrak{g})$ by:

$$(P \otimes x | Q \otimes y)_t = \operatorname{Res}(t^{-1}PQ)(x|y).$$

Then we extend further $(\,|\,)_t$ to a bilinear symmetric form $(\,|\,)$ on $\hat{\mathcal{L}}(\mathfrak{g})$ imposing $(K|D) = 1$, $(K|\mathcal{L}(\mathfrak{g}) \oplus K) = 0$ and $(D|\mathcal{L}(\mathfrak{g}) \oplus D) = 0$. It is easy to see that $(\,|\,)$ is non degenerate and $\hat{\mathcal{L}}(\mathfrak{g})$-invariant (see [10, p. 102]).

We see, with all these coventions, that here are *two* QTD of $\hat{\mathcal{L}}(\mathfrak{g})$, namely $(G_0, G_+, G_-, (\,|\,))$ and $(G_0, L_+, L_-, (\,|\,))$, where

$$\begin{aligned} G_+ &= \mathfrak{g}_+ \otimes \mathbb{C}[t, t^{-1}] \oplus \mathfrak{g}_0 \otimes t\mathbb{C}[t], \\ G_- &= \mathfrak{g}_- \otimes \mathbb{C}[t, t^{-1}] \oplus \mathfrak{g}_0 \otimes t^{-1}\mathbb{C}[t^{-1}], \\ G_0 &= \mathfrak{g}_0 \otimes 1 \oplus \mathbb{C}K \oplus \mathbb{C}D, \\ L_+ &= (\mathfrak{g}_- + \mathfrak{g}_0) \otimes t\mathbb{C}[t] \oplus \mathfrak{g}_+ \otimes \mathbb{C}[t], \\ L_- &= (\mathfrak{g}_+ + \mathfrak{g}_0) \otimes t^{-1}\mathbb{C}[t^{-1}] \oplus \mathfrak{g}_- \otimes \mathbb{C}[t^{-1}]. \end{aligned}$$

If $\mathfrak{g}$ is a Lie algebra with TD then these are TD of $\hat{\mathcal{L}}(\mathfrak{g})$. It can be shown that these two QTD give rise to Lie quasi-bialgebra structures which are topological twistings of each other.

5. A variation of Witt's construction

We now discuss a family of examples arising from a construction due to Witt [14], see also [8]. Let $\mathfrak{g}$ be a Lie algebra, let V be a vector space and V^* be the dual vector space of V. Let $\pi : \mathfrak{g} \to \mathrm{End}(V)$ and $\rho : \mathfrak{g} \to \mathrm{End}(V^*)$ be representations of $\mathfrak{g}$ and denote $\widetilde{\mathfrak{g}} = V^* \oplus \mathfrak{g} \oplus V$.

Lemma 5.1. *Let $\beta : V \times V^* \to \mathfrak{g}$ be a bilinear form. Then the bracket*

$$\begin{aligned} &[(\lambda, x, v), (\lambda', x', v')] \\ &\qquad = (\rho(x)\lambda' - \rho(x')\lambda, [x, x]' + \beta(v, \lambda') - \beta(v', \lambda), \pi(x)v' - \pi(x')v), \end{aligned} \tag{18}$$

$\lambda, \lambda' \in V^*$, $x, x' \in \mathfrak{g}$ and $v, v' \in V$, defines a Lie algebra structure on $\widetilde{\mathfrak{g}}$ if and only if for all $x \in \mathfrak{g}$, $v, v' \in V$ and $\lambda, \lambda' \in V^*$

$$[x, \beta(v, \lambda)] = \beta(\pi(x)v, \lambda) + \beta(v, \rho(x)\lambda), \tag{19}$$

$$\pi(\beta(v, \lambda))v' = \pi(\beta(v', \lambda))v \quad \text{and} \quad \rho(\beta(v, \lambda))\lambda' = \rho(\beta(v, \lambda'))\lambda. \tag{20}$$

Proof. The antisymmetry of the bracket (18) is evident. A straightforward computation shows that the Jacobi identity is equivalent to (19), (20). ∎

Lemma 5.2. *Suppose that $\mathfrak{g}$ is provided with a $\mathfrak{g}$-invariant nondegenerate symmetric bilinear form $<\,|\,>$. Let $<\,|\,> : V \times V^* \to \mathbb{C}$ be the canonical bilinear form. Extend these forms to $\widetilde{\mathfrak{g}}$ in the following way*

$$< \lambda + x + v | \lambda' + x' + v' > = < v' | \lambda > + < x | x' > + < v | \lambda' >, \tag{21}$$

$x, x' \in \mathfrak{g}$, $v, v' \in V$, $\lambda, \lambda' \in V^*$.

Suppose that $\rho = \pi^*$ with respect to the form on $V \times V^*$ and define $\beta : V \times V^* \to \mathfrak{g}$ by

$$< \beta(v, \lambda) | x > = < \pi(x)v | \lambda >, \quad (x \in \mathfrak{g}, v \in V, \lambda \in V^*). \tag{22}$$

Then the bracket (18) defines a Lie algebra structure on $\widetilde{\mathfrak{g}}$ if and only if the equations (20) hold. In such case, the form $<\,|\,>$ on $\widetilde{\mathfrak{g}}$ is $\widetilde{\mathfrak{g}}$-invariant.

Proof. It is clear that if $v \in V$ and $\lambda \in V^*$, then $[v, \lambda] \in \mathfrak{g}$ is the unique element such that $< x|[v, \lambda] >=< \pi(x)v|\lambda >$ for all $x \in \mathfrak{g}$. Hence, (22) implies (19). Thus the bracket (18) define a Lie algebra structure if and only if (20) hold. Let us check that the $\widetilde{\mathfrak{g}}$-invariance of (21): let $x \in \mathfrak{g}$, $v \in V$ and $\lambda \in V^*$, then

$$< x|[v, \lambda] >=< \beta(v, \lambda)|x >=< \pi(x)v|\lambda >=< [x, v]|\lambda >$$

and

$$< v|[x, \lambda] >=< v|\rho(x)\lambda >= - < \beta(v, \lambda)|x >= - < \pi(x)v|\lambda >=< [v, x]|\lambda > .$$

We can deduce the other cases from the definition of $< \mid >$ and the invariance of the form on $\mathfrak{g}$.

■

Remark 5.3. Instead of defining β by (22), we could define π by the formula (22); then we should check that π is a representation of $\mathfrak{g}$.

Corollary 5.4. *Let $\mathfrak{g}$ be a Lie algebra with QTD (respectively, with TD). Let $\pi : \mathfrak{g} \to V$ be a representation and let ρ, β be as in Lemma 5.2. Then $(\mathfrak{g}_0, \mathfrak{g}_+ \oplus V^*, \mathfrak{g}_- \oplus V, < \mid >)$ is a QTD (respectively, TD) of $\widetilde{\mathfrak{g}}$ if and only if the equations (20) hold. In such case, the motion Lie algebra $\mathfrak{g} \oplus V$ is a subbialgebra of $\widetilde{\mathfrak{g}}$.*

Proof. We leave the first part to the reader. Let $(\mathfrak{p}, \mathfrak{p}_1, \mathfrak{p}_2)$ be the Manin triple associated to $(\widetilde{\mathfrak{g}}, \delta)$ as in Theorem 3.1. Clearly, $\mathfrak{q} = \{(t, t) : t \in \mathfrak{g} \oplus V\}$ is a subalgebra of $\mathfrak{p}_1$ and $\mathfrak{q}^\perp \cap \mathfrak{p}_2 = \{(v, 0) : v \in V\}$ is an ideal of $\mathfrak{p}_2$, so $\mathfrak{g} \oplus V$ is a subbialgebra of $\widetilde{\mathfrak{g}}$. ■

Example 5.5. We preserve the notation above. We assume that $\mathfrak{g}$ is a finite dimensional semisimple Lie algebra, the invariant bilinear form is the Killing form and V a finite dimensional representation of $\mathfrak{g}$. Let $C_\mathfrak{g}$ be the value of the action of the Casimir element on the adjoint representation and assume that the action of the Casimir element on V has a single eigenvalue C_V. Then equations (20) hold whenever

$$\frac{2 \dim V}{\dim \mathfrak{g}} + \frac{C_\mathfrak{g}}{C_V} = 2.$$

Indeed, let $M = V \oplus V^*$ and let ψ be the symmetric bilinear form on M which restricted to $V \times V^*$ is the usual evaluation and such that V and V^* are isotropic. It is clearly $\mathfrak{g}$-invariant. On the other hand, it is clear that the Casimir element acts on V^* and *a fortiori* on M with a single eigenvalue C_V. The claim then follows from [8, Th. 12.1].

Example 5.6. We now consider the opposite situation to the example above. Let $\mathfrak{g}$ be a finite dimensional Lie algebra and let $\pi : \mathfrak{g} \to \mathrm{End}(V)$ be a finite dimensional representation. Let $\mathfrak{l} = \mathfrak{g} \oplus \mathfrak{g}^*$ be the motion Lie algebra corresponding to the coadjoint representation. We extend π to a representation of $\mathfrak{l}$ of the same name by letting $\mathfrak{g}^*$ act by 0. The bilinear form on $\mathfrak{l}$ given by evaluation between $\mathfrak{g}$ and $\mathfrak{g}^*$, and such that $\mathfrak{g}$ and $\mathfrak{g}^*$ are isotropic, is invariant (e. g. by Example 2.4). Let $\beta : V \times V^* \to \mathfrak{g}^*$ be the bilinear map given by $< \beta(v, \lambda)|x >=< \pi(x)v|\lambda >$, $v \in V$, $\lambda \in V^*$, $x \in \mathfrak{g}$. Then equations (20) hold because $\mathfrak{g}^* \subset \ker \pi$; therefore $\widetilde{\mathfrak{l}} := V^* \oplus \mathfrak{l} \oplus V$ has a Lie algebra structure by Lemma 5.2. Furthermore, if $\mathfrak{g}$ has a TD then $\mathfrak{l}$ also does by Example 4.6 (note that the bilinear form considered in Example 4.6 is not the same as the one coming from Example 2.4; however (22) holds for both). By Corollary 5.4, $\widetilde{\mathfrak{l}}$ also has a TD.

Lemma 5.7. *The Lie subalgebra* $\mathfrak{h} = V^* \oplus \mathfrak{g}^* \oplus V$*; it is a of* $\widetilde{\mathfrak{l}}$ *is a Lie subbialgebra.*

Proof. Let $(\mathfrak{p}, \mathfrak{p}_1, \mathfrak{p}_2)$ be the Manin triple corresponding to $\widetilde{\mathfrak{l}}$ as constructed in Theorem 3.1. By Remark 2.5, it is enough to show that $\mathfrak{h} \cap \mathfrak{p}_2$ is an ideal of $\mathfrak{p}_2$. This is not difficult to see using the definitions. ■

Notice that $\mathfrak{h}$ is a two-step nilpotent Lie algebra, or Heisenberg-type Lie algebra since Heisenberg Lie algebras correspond to the case $\dim \mathfrak{g} = 1$. Hence the procedure just described allows to obtain many new Lie bialgebras with underlying Lie algebra of Heisenberg-type and to provide many new examples of factorizable Lie bialgebras.

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