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A Combinatorial Construction of $\mathbf{G}_2$

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Abstract. We show how to construct the simple exceptional Lie algebra of type G_2 by explicitly constructing its 7 dimensional representation. Technically no knowledge of Lie theory is required. The structure constants have a combinatorial meaning involving convex subsets of a partially ordered multiset of six elements. These arise from playing the Numbers and Mutation games on a certain directed multigraph.

1. Introduction

The classification of finite dimensional simple Lie algebras over the complex numbers goes back to W. Killing [12] and E. Cartan [3]. Four families of 'classical' Lie algebras of type A, B, C, D correspond to the special linear, odd orthogonal, symplectic and even orthogonal algebras, with five exceptional Lie algebras of types G_2, F_4, E_6, E_7 and E_8 of dimensions $14, 56, 78, 133$ and 248 respectively. The construction of these exceptional Lie algebras has been treated by several authors beginning with Cartan [4]. In principle the Serre relations allow us to write down structure constants for any simple Lie algebra from the Dynkin diagram, but for many applications one desires to know explicit structure constants. A general construction valid for all simple Lie algebras was given by Harish-Chandra [7], while for the exceptional Lie algebras Tits [16] has given a uniform treatment. Both of these are perhaps rather abstract.

Jacobson [10] and [11] constructs the Lie algebra $\mathfrak{g}_2$ of type G_2 as the derivations of a certain non-associative Cayley algebra. Adams [1] shows that the corresponding Lie group G_2 is the subgroup of $Spin(7)$ fixing a point in S^7. Humphreys [9] shows how to explicitly write down basis elements of $\mathfrak{g}_2$ in a somewhat ad hoc fashion, from knowledge of the simple Lie algebra of type B_3. Fulton and Harris [6] devote most of a chapter to a construction using detailed facts from the representation theory of sl_2 and sl_3. Harvey [8] and Baez [2] relate G_2 to the octonions.

Our goal here is to give a direct combinatorial construction of $\mathfrak{g}_2$, requiring no knowledge of Lie theory. Indeed the entire construction is given in this

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Introduction. The rest of the paper contains relatively simple mnemonics for remembering the relevant pictures, a discussion of connections with Lie theory, a sketch of the proof of the main Theorem, and an explicit table of commutation relations.

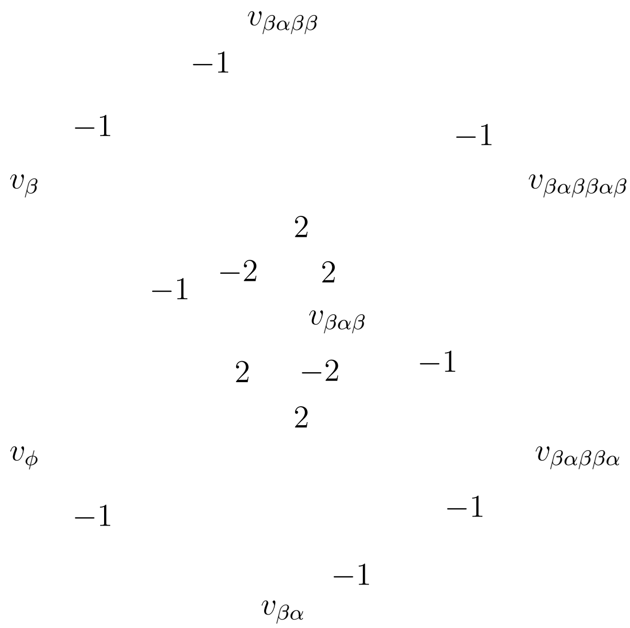

Figure 1

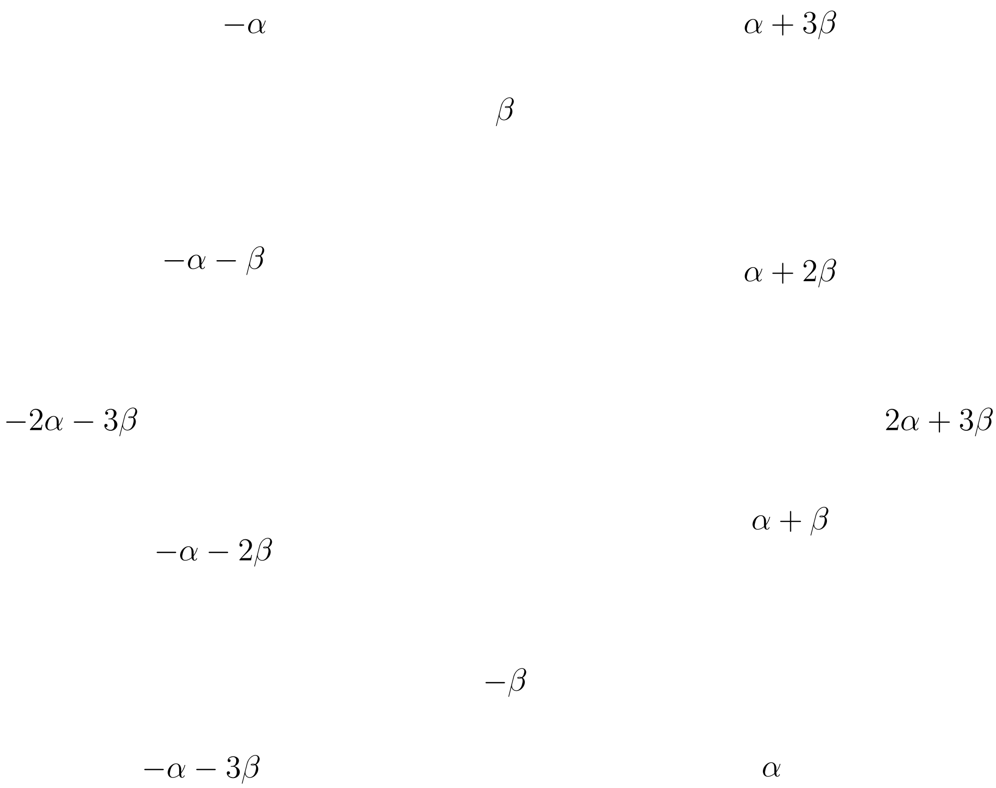

Figure 2

Consider the directed graph of Figure 1, consisting of vertices forming a hexagon, together with its center, and certain directed edges. We adopt the convention that an edge with two opposite arrows on it signifies two directed edges of opposite orientation. This directed graph will be called the G_2 *hexagon*, and note that in Figure 1 its directed edges are labelled by integers, adjacent to the arrows, which we call *weights*, where the default weight is one. Each edge of the G_2 hexagon has a direction given by one of the twelve vectors in Figure 2, called *roots*.

Note that each root is an integral linear combination of the basis vectors α and β, also called simple roots, either with all non-negative coefficients, or with all non-positive coefficients. It will be useful to consider both Figures in the same plane and sharing a common origin.

To each root γ we associate an operator X_γ on the complex span V of the vertices of the G_2 hexagon, defined by the rule that X_γ takes a vertex v to n times a vertex w precisely when there is an edge of the G_2 hexagon from v to w of direction γ and weight n. If we use the labelling of the vertices shown, then for example the operator X_α takes v_β to $v_{\beta\alpha}$ and takes $v_{\beta\alpha\beta\beta}$ to $v_{\beta\alpha\beta\beta\alpha}$, while it sends all other vertices to 0, while the operator $X_{\alpha+2\beta}$ takes v_ϕ to $2v_{\beta\alpha\beta}$, v_β to $-v_{\beta\alpha\beta\beta}$, $v_{\beta\alpha}$ to $-v_{\beta\alpha\beta\beta\alpha}$, and $v_{\beta\alpha\beta}$ to $v_{\beta\alpha\beta\beta\alpha\beta}$, while it sends the other vertices to 0.

The bracket $[X,Y]$ of two operators X and Y is given by the rule

$$[X,Y]v = XYv - YXv.$$

We now define additional operators H_γ, one for each root γ, by the commutation relation

$$H_\gamma = [X_\gamma, X_{-\gamma}].$$

It is not hard to verify that each of these H_γ is a *scalar* operator; it multiplies each vertex by a scalar, which in fact is the restriction of a linear function to the G_2 hexagon, assumed to be centered at the origin.

Theorem 1.1. *The span of the operators $\{X_\gamma, H_\gamma \mid \gamma$ is a root$\}$ is closed under brackets and forms a 14-dimensional Lie algebra $\mathfrak{g} \simeq \mathfrak{g}_2$. A basis of $\mathfrak{g}_2$ consists of $\{X_\gamma \mid \gamma$ is a root$\} \cup \{H_\alpha, H_\beta\}$.*

With respect to the basis of vertices of the G_2 hexagon, each of the operators X_γ, H_γ can be seen to have integer matrix entries in the set $\{-2,-1,0,1,2\}$, with each row and column containing at most one nonzero entry. This implies that the structure equations of $\mathfrak{g}$ can be read off easily from the G_2 hexagon by simply observing how the operators compose. For example, $[X_\alpha, X_\beta] = X_{\alpha+\beta}$, as the reader may quickly verify by checking the actions on a suitable vertex, such as ϕ.

The remainder of the paper shows that Figures 1 and 2 are not arbitrary, but can be systematically developed from the combinatorics of convex subsets of a certain partially ordered multiset, which in turn arises by playing the 'mutation and numbers games' on a particular graph. Ultimately all the information is contained in the chain $\beta < \alpha < \beta < \beta < \alpha < \beta$. The numbers game was defined by Mozes [13], and studied by Proctor [14] and Eriksson [5], while the mutation game is in a precise sense dual to the numbers game and has its origins in the theory of reflection groups and root systems (see [19]). These games are of considerable independent interest, leading quickly to deep aspects of Lie theory.

The construction presented here complements the development of [17], in which we combinatorially constructed the minuscule representations of the simply laced Lie algebras, also without explicit use of Lie theory. In that situation we showed how to construct the minuscule posets defined by Proctor from combinatorics associated to the mutation game, and how to realize the explicit action of a basis of the Lie algebra by raising and lowering operators on the space of ideals of minuscule posets.

Minuscule representations have the property that all weights are conjugate under the Weyl group, so that the geometry and order structure of this orbit of weights naturally determines much about the representation. As a consequence, the minuscule situation does not involve chains of weight vectors of length greater than one, so the weights involved are all ± 1, although the calculation of the parities of the weights is necessarily more involved than for G_2. So a key new ingredient in the present paper is dealing with weight chains of length more than one.

The present work is also related to a recent construction of the irreducible representations of $sl(3)$ (different from Gelfand Tsetlin) utilizing the combinatorial geometry of three dimensional polytopes which we call *diamonds* [18]. This model shares the pleasant feature that matrix coefficients are always integers. Another common aspect of all the situations described is that we represent an entire linear basis of the Lie algebra $\mathfrak{g}$, not just a choice of Chevalley generators.

To agree with standard usage, we split the set of operators $\{X_\gamma \mid \gamma \text{ is a root}\}$ into operators X_γ, Y_γ for γ a positive root. These are then raising and lowering operators in the familiar terminology of the physics literature. The paper concludes with a list of all commutation relations.

2. The mutation and numbers games on a graph

Consider the directed graph G_2 consisting of two vertices α and β with three edges from α to β and one edge from β to α. We may encapsulate this information as follows.

Figure 3

α β

We will now show that both Figure 1 and Figure 2 arise by playing two remarkable graph-theoretical games on the graph G_2 of Figure 3.

Given any directed graph Z, we may associate to it a distinguished collection $R = R(Z)$ of integer valued functions on the vertices of Z by playing the *mutation game*. The elements of R are called *roots*. A root is any function obtained by starting with a delta function, or *simple root*, which has value one at some vertex and is zero elsewhere, and performing an arbitrary sequence of *mutations*, defined as follows.

A mutation at a vertex z applied to a function f has the effect of leaving the values of f at all vertices except z unchanged, while the new value at z is obtained by negating the current value at z and adding a copy of the function values at each neighbour w of z, once for every directed edge from w to z. Since a mutation at z applied to the delta function at z results in a sign change, the set of roots is symmetric under change of sign.

With the convention that we identify the vertices α and β with the delta functions on them, one may now check that the set $R(G_2)$ consists of the following *positive* roots together with their negatives.

$$R^+(G_2) = \{\ \alpha, \beta, \alpha+\beta, \alpha+2\beta, \alpha+3\beta, 2\alpha+3\beta\}. \tag{1}$$

Indeed as functions on G_2, these are obtained from the delta functions α and β by mutations in the orders

$$\alpha \to \alpha+3\beta \to 2\alpha+3\beta$$

and

$$\beta \to \alpha+\beta \to \alpha+2\beta.$$

Now associate α and β to two vectors in the plane whose lengths are in the ratio $\sqrt{3}:1$ respectively, making an angle of $5\pi/6$ as in Figure 2. The other roots, which are linear combinations of α and β, then correspond to the other vectors in the same Figure, and the mutations at the vertices α and β act on these vectors by reflections in the lines perpendicular to α and β respectively. Thus $R(G_2)$ is indeed a *root system* in the classical sense.

The *numbers game* is also played with integer valued functions on the vertices of a directed graph. Given such a function g and a vertex z, we will say that a *dual mutation*, or *firing*, at z replaces the value of g at z with its negative, and simultaneously adds a copy of $g(z)$ to each neighbour w of z, once for each directed edge from z to w, and leaves all other vertices unchanged.

Of particular interest is playing the numbers game under the condition that dual mutations only can occur at vertices where the current value of the function is positive. The systematic study of sequences of such positive firings yields many remarkable combinatorial structures. The minuscule posets studied by Proctor and others in the context of Bruhat orders in Coxeter groups occur when we begin with a Dynkin diagram and a minuscule vertex (for which the associated fundamental representation is minuscule). However, many more fascinating objects are created by arranging sequences starting with more general configurations into partially ordered multisets by comparing only firings at adjacent vertices (see [19]).

The only example relevant for us here starts with a delta function at β, here denoted by recording the function values in the ordered pair $(0,1)$. Proceeding with positive dual mutations, we get the sequence

$$(0,1), (1,-1), (-1,2), (1,-2), (-1,1), (0,-1).$$

If we record, using multiplicative notation, those positive function values at each stage in this dual mutation sequence, we get what we call the 'change sequence' $\beta\alpha\beta^2\alpha\beta$, which we write as $\beta\alpha\beta\beta\alpha\beta$.

Form a partially ordered multiset (*pomset*) P by linearly ordering the terms

$$\beta < \alpha < \beta < \beta < \alpha < \beta.$$

This pomset is then a chain. Recall that a subset I of a poset is called an *ideal* if $x \in I$ and $y \leq x$ implies $y \in I$. Extending the notion here in the obvious way, we will consider the seven ideals of P to be the seven initial chains of P, including the empty chain. We associate to each of these ideals a vertex of a graph in the plane whose edges have 'directions' corresponding to roots. More specifically we will connect an ideal v to an ideal u by an edge of *direction* γ precisely when

$$\sum_{x \in u} x - \sum_{x \in v} x = \gamma.$$

This yields the G_2 hexagon. For example there is an edge of direction α from $\beta\alpha\beta\beta$ to $\beta\alpha\beta\beta\alpha$, and an edge of direction $-\alpha - 2\beta$ from $\beta\alpha\beta\beta$ to β.

It remains to prescribe the *weight* $w(e)$ of a directed edge e. A submultiset $L \subseteq P$ is called a *layer* if it is convex in the usual sense. This means here only that L is a chain of some consecutive elements of P, such as for example $\beta\beta\alpha\beta$ or $\alpha\beta\beta$. Such a layer L will be called a γ*–layer*, for some positive root $\gamma \in R^+$, if

$$\sum_{x \in L} x = \gamma.$$

We let $\mathcal{L}_\gamma$ denote the collection of γ-layers, which itself is partially ordered by setting $L_1 \leq L_2$ if $I(L_1) \subseteq I(L_2)$ where $I(S)$ is the ideal generated by a submultiset $S \subseteq P$, that is,

$$I(S) = \{z |\ z \leq w \text{ for some } w \in S\}.$$

For example if $\gamma = \alpha + 2\beta$, then $\mathcal{L}_\gamma$ contains four γ-layers, namely $L_0 = \beta\alpha\beta < L_1 = \alpha\beta\beta < L_2 = \beta\beta\alpha < L_3 = \beta\alpha\beta$, as shown in Figure 4.

L_1 L_3

$\beta \quad \alpha \quad \beta \quad \beta \quad \alpha \quad \beta$ Figure 4

L_0 L_2

If $\gamma' = \alpha + \beta$, then $\mathcal{L}_{\gamma'}$ also contains four γ'-layers, namely $L'_0 = \beta\alpha < L'_1 = \alpha\beta < L'_2 = \beta\alpha < L'_3 = \alpha\beta$, as shown in Figure 5.

L'_1 L'_3

$\beta \quad \alpha \quad \beta \quad \beta \quad \alpha \quad \beta$ Figure 5

L'_0 L'_2

Note that there is some possibility for confusion here as we regard L_0 and L_3 to be distinct layers, despite the fact that as sequences they appear the same. The same holds for the pair L'_0 and L'_2 as well as the pair L'_1 and L'_3.

For each positive root γ there is a unique minimal γ-layer which we will denote by L_m, and a unique maximal γ-layer which we will denote by L_M.

Given a γ-layer L, define the *parity* $\epsilon(L)$ (respectively *dual parity* $\tilde{\epsilon}(L)$) of L, to be $(-1)^n$ where n is the number of $\alpha - \beta$ interchanges required in passing from L to the minimal γ-layer L_m (respectively maximal γ-layer L_M) by switching adjacent α and β's. This is well-defined.

Thus continuing the examples above, $\epsilon(L_0) = 1$, $\epsilon(L_1) = -1$, $\epsilon(L_2) = -1$, and $\epsilon(L_3) = 1$, while the dual parities agree with the parities since L_m and L_M have the same form. On the other hand $\epsilon(L'_0) = 1$, $\epsilon(L'_1) = -1$, $\epsilon(L'_2) = 1$, and $\epsilon(L'_3) = -1$, while $\tilde{\epsilon}(L'_0) = -1$, $\tilde{\epsilon}(L'_1) = 1$, $\tilde{\epsilon}(L'_2) = 1$, and $\tilde{\epsilon}(L'_3) = -1$, since in this case the minimal and maximal γ'-layers have opposite parities.

A *ladder* of γ-layers will be a sequence $L_1 < L_2 < \ldots < L_k$ of disjoint γ-layers such that $L_1 \cup \ldots \cup L_i$ is a layer for all $i = 1, \ldots, k$. Such a ladder has *size* k, and will be said to *start* at L_1 and *finish* at L_k. For a γ-layer L let $s(L)$ be the maximum size of a ladder starting at L, and let $f(L)$ be the maximum size of a ladder finishing at L.

Thus in the first example above $s(L_0) = 2$, $s(L_1) = s(L_2) = s(L_3) = 1$, while $f(L_0) = f(L_1) = f(L_2) = 1$, $f(L_3) = 2$. In the second example we find $s(L'_0) = 1$, $s(L'_1) = 2, s(L'_2) = s(L'_3) = 1$, while $f(L'_0) = f(L'_1) = 1$, $f(L'_2) = 2$, $f(L'_3) = 1$.

For a positive root γ and a γ-layer L, define the *weight* $w(L)$ and the *dual weight* $\tilde{w}(L)$ by

$$w(L) = \epsilon(L)s(L) \quad \tilde{w}(L) = \tilde{\epsilon}(L)f(L). \tag{2}$$

Ideals in P correspond to vertices in the G_2 hexagon, and for positive roots γ, γ-layers correspond to γ-edges, so we label the γ-edge e with the weight $w(L)$ of the corresponding γ-layer L. A $(-\gamma)$-edge of direction the negative of a positive root γ will be labelled with the dual weight $\tilde{w}(L)$ of the corresponding γ-layer. The G_2 hexagon is symmetrical about the origin, and this definition produces a pattern of weights on it which is still symmetrical about the origin.

3. Structure equations for semisimple Lie algebras

We now remind the reader about some elementary facts about the structure of a semisimple Lie algebra $\mathfrak{g}$ over the complex numbers. This is intended to motivate what follows, so is not strictly necessary for the construction itself. If R is the root system of $\mathfrak{g}$, then there exists a decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \sum_{\gamma \in R} \mathfrak{g}_\gamma$$

where $\mathfrak{h}$ is a *Cartan subalgebra* (maximal commutative subalgebra of semisimple elements), and each $\mathfrak{g}_\gamma$ is a one dimensional *root space* with the property that $[H, X_\gamma]$ is always a multiple of X_γ for $H \in \mathfrak{h}$ and $X_\gamma \in \mathfrak{g}_\gamma$.

Suppose one chooses, perhaps arbitrarily, a nonzero element X_γ from each root space $\mathfrak{g}_\gamma$. Then from general considerations (see for example Humphreys [9]) one knows that for $\gamma, \gamma' \in R$, $[X_\gamma, X_{\gamma'}] = 0$ unless $\gamma + \gamma'$ is also a root, in which case $[X_\gamma, X_{\gamma'}]$ is some multiple of $X_{\gamma+\gamma'}$, or unless $\gamma = -\gamma'$, in which case $[X_\gamma, X_{-\gamma}]$ is a known element of the Cartan subalgebra $\mathfrak{h}$. So the problem involved in describing $\mathfrak{g}$ completely essentially involves two steps: to choose explicit basis elements X_γ in each root space $\mathfrak{g}_\gamma$ and to determine the corresponding constants

$n_{\gamma,\gamma'}$ in the structural equations

$$[X_\gamma, X_{\gamma'}] = n_{\gamma,\gamma'} X_{\gamma+\gamma'},$$

one for each pair of roots (γ, γ') whose sum is also a root. Then all other structure constants of the Lie algebra will be given. Somewhat surprisingly, the difficulty in this task lies not so much in finding the absolute values of the $n_{\gamma,\gamma'}$, rather it is the signs which are difficult to determine. See Samelson [15], Tits [16] for a discussion of this point.

4. The 7-dimensional representation of G_2

The Lie algebra $\mathfrak{g}_2$ will be realized by operators on the vector space spanned by the vertices of the G_2 hexagon, which we have seen are labelled by the ideals in the pomset P arising from playing the numbers game on a particular initial function. To abide by the usual conventions, we will actually associate to each positive root γ two operators, X_γ and Y_γ, called raising and lowering operators in the physics literature. Together with two scalar operators H_α and H_β, we get a total of 14 operators on a seven dimensional space. All of these operators are entirely encoded in Figure 1 and we will then check that they form a Lie algebra of operators. This will be $\mathfrak{g}_2$. We also observe that if we augment our set of operators to a larger set of 18, still in the span of the original 14, then we obtain what we call a *bracket set*; the bracket of any two of them is a multiple of one in the set.

Define a complex vector space

$$V = \operatorname{span}\{v_I | \ I \text{ is an ideal of } P\}.$$

For any layer $L \subseteq P$ define operators X_L, Y_L on V by

$$X_L(v_I) = \begin{cases} v_{I\cup L} & \text{if } I \cup L \text{ is an ideal and } I \cap L = \emptyset \\ 0 & \text{otherwise.} \end{cases}$$

$$Y_L(v_I) = \begin{cases} v_{I\setminus L} & \text{if } L \subseteq I \text{ and } I \setminus L \text{ is an ideal} \\ 0 & \text{otherwise.} \end{cases}$$

For $\gamma \in R^+$ define operators X_γ, Y_γ and H_γ on V by the following rules.

$$X_\gamma = \sum_{L \in \mathcal{L}_\gamma} w(L) X_L$$

$$Y_\gamma = \sum_{L \in \mathcal{L}_\gamma} \tilde{w}(L) Y_L$$

$$H_\gamma = [X_\gamma, Y_\gamma].$$

So the X_γ and Y_γ operators correspond exactly to the edges of the G_2 hexagon that are in the γ and $-\gamma$ directions respectively. In particular X_γ sends a vertex u of the hexagon to n times a vertex v precisely when there is a γ-edge from u to v of weight n, and similarly Y_γ sends a vertex u of the hexagon to n times a vertex v precisely when there is a $(-\gamma)$-edge from u to v of weight n.

A set $\{Z_\gamma \mid \gamma \in \Gamma\}$ of operators on a vector space will be defined to be a *bracket set* if for all $\gamma, \gamma' \in \Gamma$, $[Z_\gamma, Z_{\gamma'}]$ is a multiple of some $Z_{\gamma''}$, $\gamma'' \in \Gamma$. If $\{Z_\gamma \mid \gamma \in \Gamma\}$ is a bracket set, clearly $\mathfrak{g} = \operatorname{span}\{Z_\gamma \mid \gamma \in \Gamma\}$ is a Lie algebra of operators. Our main result is the following:

Theorem 4.1. $\{X_\gamma, Y_\gamma, H_\gamma \mid \gamma \in R^+\}$ *is a bracket set inside* $End(V)$ *spanning a 14-dimensional Lie algebra* $\mathfrak{g} \simeq \mathfrak{g}_2$. *The set* $\{X_\gamma, Y_\gamma \mid \gamma \in R^+\} \cup \{H_\alpha, H_\beta\}$ *is a basis for* $\mathfrak{g}$.

Proof. The proof will require some careful checking involving the G_2 hexagon. Recall first that the symmetry of the edges and weights of the G_2 hexagon ensures that for any commutation relation, there is a corresponding relation with all the X operators replaced by Y operators and all the scalar operators H_γ replaced by their negatives. This reduces the number of relations we need check by a factor of two. We proceed in steps.

Step 1. Let us say that $\{\gamma, \gamma'; \gamma''\}$ is a *positive root triple* if all three are positive roots and $\gamma + \gamma' = \gamma''$. The positive root triples are $\{\alpha, \beta; \alpha+\beta\}, \{\beta, \alpha+\beta; \alpha+2\beta\}, \{\beta, \alpha+2\beta; \alpha+3\beta\}, \{\alpha, \alpha+3\beta; 2\alpha+3\beta\}$, and $\{\alpha+\beta, \alpha+2\beta; 2\alpha+3\beta\}$. For every such triple, there are a number of triangles in the G_2 hexagon with sides having these directions. Each of these triples leads to six commutation relations of which we need only check three from the above remark. Checking one of these relations involves examining all the triangles associated to that triple and checking consistency of weights between them.

Let us illustrate this for the triple $\{\alpha, \beta; \alpha+\beta\}$. We need check that we can find constants A, B, C such that $[X_\alpha, X_\beta] = AX_{\alpha+\beta}$, $[X_{\alpha+\beta}, Y_\beta] = BX_\alpha$, and $[X_{\alpha+\beta}, Y_\alpha] = CX_\beta$. There are four triangles associated to this triple- they are $(\phi, \beta, \beta\alpha), (\beta, \beta\alpha, \beta\alpha\beta), (\beta\alpha\beta, \beta\alpha\beta\beta, \beta\alpha\beta\beta\alpha)$, and $(\beta\alpha\beta\beta, \beta\alpha\beta\beta\alpha, \beta\alpha\beta\beta\alpha\beta)$. For one of the triangles, say the first, we observe that the product of the weights of the α and β edges is the weight of the $\alpha+\beta$ edge, and the latter edge is the vector sum of *first* the β edge and *second* the α edge, yielding $[X_\alpha, X_\beta] = X_\alpha X_\beta - X_\beta X_\alpha = X_{\alpha+\beta}$ when applied to v_ϕ.

Now we check that in the other three triangles, exactly the same commutation relation holds. Then we check the same works for the other two relations associated to this triple. Finally we check that the same procedure holds for all positive root triples.

Step 2. There are some potential commutation relations corresponding to the three rectangles in the G_2 hexagon. These relations involve pairs of roots which are at right angles to each other, namely $\{\beta, 2\alpha+3\beta\}, \{\alpha+\beta, \alpha+3\beta\}, \{\alpha, \alpha+2\beta\}$ and pairs obtained from them by negating one or both of the entries. Checking that the commutation relations between any X or Y operators involving such a pair is zero amounts to noting that for any one of the rectangles in the G_2 hexagon, products of weights along adjacent edges is constant.

Step 3. For each of the six positive roots γ, one checks that $H_\gamma = [X_\gamma, Y_\gamma]$ is a scalar operator which multiplies each vertex by a number which is the restriction to the hexagon of a *linear* function f_γ in the plane. In particular, H_α corresponds to the linear function f_α which has value -1 on the vertices v_β and $v_{\beta\alpha\beta\beta}$, value 1 on the vertices $v_{\beta\alpha}$ and $v_{\beta\alpha\beta\beta\alpha}$, and value zero on the rest, while H_β corresponds to the linear function f_β which has value 2 at the vertex $v_{\beta\alpha\beta\beta}$, value 1 on the vertices v_β and $v_{\beta\alpha\beta\beta\alpha\beta}$, value zero on $v_{\beta\alpha\beta}$ etc. In particular these two linear functions are linearly independent and so any of the linear functions f_γ is a linear combination of them. ¿From this it follows that $[H_\gamma, X_\gamma]$ is always a multiple of X_γ for any root γ.

Finally the Jacobi relation (valid since we are working with operators) and the results from Step 1 show that $[X_{\gamma'}, H_\gamma] = [X_{\gamma'}, [X_\gamma, Y_\gamma]] = -[X_\gamma, [Y_\gamma, X_{\gamma'}]] - [Y_\gamma, [X_{\gamma'}, X_\gamma]$ is a multiple of $X_{\gamma'}$. For example $[H_\beta, X_\gamma] = f_\beta(\gamma)X_\gamma$, where we consider γ as a vector in the plane. That is, we regard Figures 1 and 2 to share a common origin in the plane, allowing linear functionals to be evaluated at roots.

This shows that altogether the set of operators $\{X_\gamma, Y_\gamma, H_\gamma \mid \gamma \in R^+\}$ is a bracket set of operators, so in particular spans a finite dimensional Lie algebra. It is easy to see that all the $\{X_\gamma, Y_\gamma \mid \gamma \in R^+\}$ are linearly independent, so we get a 14 dimensional Lie algebra. Clearly the span of H_α, H_β is a Cartan subalgebra and the roots with respect to it are exactly $R(G_2)$, in other words, this Lie algebra is $\mathfrak{g}_2$.

Here is the list of all the non-zero commutation relations, expressed in the basis described in the Theorem.

$$
\begin{array}{lll}
[X_\alpha, X_\beta] = X_{\alpha+\beta} & [X_\beta, X_{\alpha+\beta}] = 2X_{\alpha+2\beta} & [X_\beta, X_{\alpha+2\beta}] = 3X_{\alpha+3\beta} \\
[X_\alpha, X_{\alpha+3\beta}] = X_{2\alpha+3\beta} & [X_{\alpha+\beta}, X_{\alpha+2\beta}] = 3X_{2\alpha+3\beta} & \\
[Y_\alpha, Y_\beta] = Y_{\alpha+\beta} & [Y_\beta, Y_{\alpha+\beta}] = 2Y_{\alpha+2\beta} & [Y_\beta, Y_{\alpha+2\beta}] = 3Y_{\alpha+3\beta} \\
[Y_\alpha, Y_{\alpha+3\beta}] = Y_{2\alpha+3\beta} & [Y_{\alpha+\beta}, Y_{\alpha+2\beta}] = 3Y_{2\alpha+3\beta} &
\end{array}
$$

$$
\begin{array}{lll}
[X_\alpha, Y_{\alpha+\beta}] = Y_\beta & [X_\alpha, Y_{2\alpha+3\beta}] = Y_{\alpha+3\beta} & [X_\beta, Y_{\alpha+3\beta}] = Y_{\alpha+2\beta} \\
[X_\beta, Y_{\alpha+2\beta}] = 2Y_{\alpha+\beta} & [X_\beta, Y_{\alpha+\beta}] = -3Y_\alpha & [X_{\alpha+\beta}, Y_\alpha] = -X_\beta \\
[X_{\alpha+\beta}, Y_\beta] = 3X_\alpha & [X_{\alpha+\beta}, Y_{\alpha+2\beta}] = 2Y_\beta & [X_{\alpha+2\beta}, Y_\beta] = -2X_{\alpha+\beta}
\end{array}
$$

$$
\begin{array}{lll}
[X_{\alpha+2\beta}, Y_{\alpha+\beta}] = -2X_\beta & [X_{\alpha+2\beta}, Y_{\alpha+3\beta}] = -Y_\alpha & [X_{\alpha+2\beta}, Y_{2\alpha+3\beta}] = -Y_{\alpha+\beta} \\
[X_{\alpha+3\beta}, Y_{2\alpha+3\beta}] = Y_\alpha & [X_{2\alpha+3\beta}, Y_\alpha] = -X_{\alpha+3\beta} & [X_{2\alpha+3\beta}, Y_{\alpha+\beta}] = X_{\alpha+2\beta} \\
[X_{2\alpha+3\beta}, Y_{\alpha+2\beta}] = X_{\alpha+\beta} & [X_{2\alpha+3\beta}, Y_{\alpha+3\beta}] = -X_\alpha &
\end{array}
$$

$$
\begin{array}{ll}
[X_\alpha, Y_\alpha] = H_\alpha & [X_\beta, Y_\beta] = H_\beta \\
[X_{\alpha+\beta}, Y_{\alpha+\beta}] = H_{\alpha+\beta} = 3H_\alpha + H_\beta & [X_{\alpha+2\beta}, Y_{\alpha+2\beta}] = H_{\alpha+2\beta} = 2H_\alpha + 3H_\beta \\
[X_{\alpha+3\beta}, Y_{\alpha+3\beta}] = H_{\alpha+3\beta} = H_\alpha + H_\beta & [X_{2\alpha+3\beta}, Y_{2\alpha+3\beta}] = H_{2\alpha+3\beta} = 2H_\alpha + H_\beta
\end{array}
$$

$$
\begin{array}{ll}
[H_\alpha, X_\gamma] = f_\alpha(\gamma)X_\gamma & [H_\alpha, Y_\gamma] = -f_\alpha(\gamma)Y_\gamma. \\
[H_\beta, X_\gamma] = f_\beta(\gamma)X_\gamma & [H_\beta, Y_\gamma] = -f_\beta(\gamma)Y_\gamma
\end{array}
$$

■

If $\mathfrak{g}$ is a finite dimensional Lie algebra, it seems interesting to inquire about the existence of spanning bracket sets inside $\mathfrak{g}$; do finite ones necessarily exist for example? If so, what is the minimum number of elements in a spanning bracket set? This is perhaps a useful invariant for Lie algebras.

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