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# An Equivalence between Categories of Modules for Generalized Kac-Moody Lie Algebras

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**Abstract.** An equivalence between categories of modules for a generalized Kac-Moody algebra and modules for an appropriate parabolic subalgebra is shown. In particular, properties such as the irreducibility and complete reducibility of a module whose weights satisfy certain conditions can be determined by restriction to a subalgebra.

## 1. Introduction

Generalized Kac-Moody algebras are Lie algebras determined by certain symmetrizable matrices as in [11], and [1]. Let $\mathfrak{g}$ be a generalized Kac-Moody algebra. In [10] some examples of irreducible highest weight modules for $\mathfrak{g}$ which are generalized Verma modules are given. These generalized Verma modules are modules that are induced from an irreducible highest weight module for a "parabolic" subalgebra $\mathfrak{p} = (\mathfrak{g}_S + \mathfrak{h}) \oplus \mathfrak{u}^+$, where $\mathfrak{g}_S$ is a generalized Kac-Moody subalgebra of $\mathfrak{g}$ (often $\mathfrak{g}_S$ can be chosen to be a Kac-Moody or semi-simple Lie algebra). In this paper, we show that the converse is also true, that in fact any irreducible $\mathfrak{g}$ module whose weights satisfy the appropriate conditions is a generalized Verma module for a parabolic subalgebra. More generally, it is shown that a subcategory of the category $\mathcal{O}$ of $\mathfrak{g}$-modules is equivalent to a subcategory of the category $\mathcal{O}$ of the Lie algebra $\mathfrak{r} = \mathfrak{g}_S + \mathfrak{h}$.

This result is most useful in cases where the subalgebra $\mathfrak{g}_S$ can be chosen to be a semi-simple or Kac-Moody Lie algebra, so that some of the representation theory for the generalized Kac-Moody algebra can be reduced to that of the semi-simple or Kac-Moody subalgebra.

In §2 of this paper, we review the definition of generalized Kac-Moody algebra, introduce notation and define the category of $\mathfrak{g}$ modules that we use. Section 3 contains the statement and proof of the main theorem. The equivalence of categories is proven by using the functor determined by inducing a module for the parabolic subalgebra $\mathfrak{p}$ to $\mathfrak{g}$, and the functor determined by considering the set of elements of a $\mathfrak{g}$-module $V$ annihilated by $\mathfrak{u}^+$. The method of proof is similar to that in [4] (see also [5] and [12]). Section 4 contains some applications.

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## 2. The Setting

We recall the definition of a generalized Kac-Moody algebra. The definition given here is similar to Borcherds' original definition [1] and agrees with the definition in [9]. A variety of other definitions exist, for example [2], [11], [13]. We refer the reader to [11] and [13] for general definitions involving Lie algebras defined by symmetrizable matrices, of which generalized Kac-Moody algebras are an example. Definitions and proofs regarding the basic structure of generalized Kac-Moody algebras are presented in [1],[8], [9] and [10]. For the convenience of the reader, we include the definitions that differ from the definitions for the more familiar Kac-Moody algebras.

We define a generalized Kac-Moody algebra over $\mathbb{C}$ by specifying generators and relations. Let $\mathbb{Z}_+$ denote the nonnegative integers. Let $I$ be a countable index set, $A = (a_{ij})_{i,j\in I}$ a matrix with entries in $\mathbb{R}$ satisfying the conditions:

1. $A$ is symmetrizable.
2. If $a_{ii} > 0$, then $a_{ii} = 2$.
3. For all $k \neq j$ $a_{jk} \leq 0$ and for all $i \in I$ such that $a_{ii} > 0$, $a_{ij} \in -\mathbb{Z}_+$ for all $j \in I$.

Let $\mathfrak{g}(A)'$ be the Lie algebra with generators $e_i, f_i, h_i, i \in I$ and defining relations: For all $i, j \in I$

$$[h_i, h_j] = 0, [e_i, f_j] = \delta_{ij} h_i$$
$$[h_i, e_j] = a_{ij} e_j, [h_i, f_j] = -a_{ij} f_j.$$

For all $i \in I$ such that $a_{ii} > 0$

$$(\operatorname{ad} e_i)^{-a_{ij}+1} e_j = 0, (\operatorname{ad} f_i)^{-a_{ij}+1} f_j = 0.$$

Finally, for all $i, j \in I$ such that $a_{ij} = a_{ji} = 0$

$$[e_i, e_j] = 0, [f_i, f_j] = 0.$$

Define degree derivations $\partial_i$, $i \in I$, on $\mathfrak{g}(A)'$ by taking $\partial_i(e_j) = \delta_{ij}$, $\partial_i(f_j) = -\delta_{ij}$ and $\partial_i(h_j) = 0$. Let $\mathfrak{d}$ be the abelian Lie algebra generated by the $\partial_i$, $i \in I$.

**Definition 1.** The generalized Kac-Moody algebra $\mathfrak{g} = \mathfrak{g}(A)$ is defined to be the semidirect product $\mathfrak{g}(A)' \ltimes \mathfrak{d}$.

We also call any Lie algebra of the form $\mathfrak{g}(A)/\mathfrak{c}$ where $\mathfrak{c}$ is a central ideal a generalized Kac-Moody algebra. Borcherds' Monster Lie algebra is an example of this kind.

The Lie algebra $\mathfrak{g}$ has a triangular decomposition $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$, where $\mathfrak{n}^\pm$ is the subalgebra generated by the $e_i$ (resp. the $f_i$), and the Cartan subalgebra $\mathfrak{h}$ is the abelian subalgebra spanned by the $h_i$ and the $\partial_i$. There is a symmetric invariant bilinear form on $\mathfrak{g}$, and on $\mathfrak{h}^*$, which will be denoted by $(\cdot,\cdot)$ in both cases.

The *roots* of $\mathfrak{g}$ are the nonzero elements $\varphi$ of $\mathfrak{h}^*$ such that $\mathfrak{g}^\varphi = \{x \in \mathfrak{g} | [h,x] = \varphi(h)x \text{ for all } h \in \mathfrak{h}\}$ is nonzero. Denote the set of roots of $\mathfrak{g}$ by $\Delta$. There is a system of positive roots $\Delta_+$ and $\Delta = \Delta_+ \cup -\Delta_+$. Roots with positive square norm are called real, and roots with non-positive square norm are called imaginary. Denote the set of real roots by $\Delta_R$. The set of simple roots, denoted $\{\alpha_i\}_{i\in I} \subset \mathfrak{h}^*$, are defined by the conditions

$$[h, e_i] = \alpha_i(h)e_i \text{ for all } h \in \mathfrak{h}.$$

Note that $\alpha_j(h_i) = a_{ij}$, and $\alpha_j(\partial_i) = \delta_{ij}$ for all $i,j \in I$ and that the $\alpha_i$ for $i \in I$ are linearly independent. The bilinear form is chosen so that $(\alpha_i,\alpha_j) = a_{ij}$ for all $i,j \in I$. Note that for $i \in I$ $(\alpha_i,\alpha_i) = a_{ii}$ may be non-positive, so that simple roots may be imaginary.

Given an element $\mu \in \mathfrak{h}^*$ of the form $\lambda - \sum_{i\in I} n_i\alpha_i$, $n_i \in \mathbb{Z}_+$, we define the *depth* $d_\lambda(\mu) = \sum_{i\in I} n_i$.

Given a $\mathfrak{g}$-module and $\lambda \in \mathfrak{h}^*$ let

$$M^\lambda = \{v \in M | h\cdot v = \lambda(h)v, \text{for all } h \in \mathfrak{h}\}.$$

The elements $\lambda \in \mathfrak{h}^*$ for which $M^\lambda \neq 0$ are called the *weights* of $M$. Given a $\mathfrak{g}$-module $M$ let $P(M)$ denote the set of weights of $M$. A weight $\lambda \in (\mathfrak{h})^*$ is *dominant* if $(\lambda,\alpha_i) \in \mathbb{R}$ and $(\lambda,\alpha_i) \geq 0$ for all $i \in I$. A weight $\lambda$ is called *integral* if $\lambda(h_i) \in \mathbb{Z}_+$ for all $i \in I$ such that $a_{ii} > 0$. Denote by $P_+$ the set of dominant integral weights.

**Definition 2.** A $\mathfrak{g}$-module $M$ is a *standard* module if $M$ is a highest weight module with highest weight $\mu \in P_+$ and highest weight vector $v$ such that:

1. for $i \in I$, if $(\mu,\alpha_i) = 0$ then $f_i \cdot v = 0$;
2. if $\alpha_i$ $(i \in I)$ is real then $f_i^{n_i+1} \cdot v = 0$, where $n_i = 2(\mu,\alpha_i)/(\alpha_i,\alpha_i)$ (necessarily a nonnegative integer).

Given any $\mu \in P_+$ there is a unique (up to isomorphism) standard irreducible highest weight $\mathfrak{g}$-module of highest weight $\mu$, denote this module by $L_\mathfrak{g}(\mu)$ [9].

Fix $S$ a distinguished subset of $I$ containing $\{i \in I | \alpha_i \in \Delta_R\}$. Denote by $\mathfrak{g}_S$ the generalized Kac-Moody subalgebra of $\mathfrak{g}$ associated to the matrix $(a_{ij})_{i,j\in S}$. Assume that $S$ is chosen so that the resulting matrix $(a_{ij})_{i,j\in S}$ is indecomposible. Note that if $S = \{i \in I | \alpha_i \in \Delta_R\}$ then $\mathfrak{g}_S$ is a Kac-Moody algebra. In the case of the monster Lie algebra we can choose $\mathfrak{g}_S = \mathfrak{sl}_2$; (see §4 below).

We make the following definitions: $\Delta^S = \Delta \cap \coprod_{i\in S} \mathbb{Z}\alpha_i$, $\Delta^S_+ = \Delta_+ \cap \Delta^S$ and $\Delta^S_- = \Delta_- \cap \Delta^S$. Denote by $\mathfrak{h}_S$ the span of the $h_i$, $i \in S$. There is a root space decomposition

$$\mathfrak{g}_S = \coprod_{\varphi\in\Delta^S_+} \mathfrak{g}^\varphi \oplus \mathfrak{h}_S \oplus \coprod_{\varphi\in\Delta^S_-} \mathfrak{g}^\varphi.$$

Define the following subalgebras of $\mathfrak{g}$:

$$\mathfrak{u}^+ = \coprod_{\varphi\in\Delta_+\setminus\Delta_+^S} \mathfrak{g}^\varphi; \quad \mathfrak{u}^- = \coprod_{\varphi\in\Delta_-\setminus\Delta_-^S} \mathfrak{g}^\varphi; \quad \mathfrak{r} = \mathfrak{g}_S + \mathfrak{h}.$$

Let $\mathfrak{p} = \mathfrak{r} \oplus \mathfrak{u}^+$, the "parabolic" subalgebra of $\mathfrak{g}$ determined by $S$. Then

$$\mathfrak{g} = \mathfrak{u}^- \oplus \mathfrak{r} \oplus \mathfrak{u}^+ = \mathfrak{u}^- \oplus \mathfrak{p}.$$

If $M$ is a $\mathfrak{r}$-module we define a $\mathfrak{p}$-module which we also call $M$ by letting $\mathfrak{u}^+$ act as zero on $M$. Note that if $M$ is irreducible, or completely reducible as a $\mathfrak{g}_S$ or $\mathfrak{p}$-module, then the $\mathfrak{p}$-module $M$ is irreducible, or completely reducible as well.

We recall the definition of generalized Verma module of [6]. Let $\lambda \in P_+$, and consider the standard (irreducible) highest weight $\mathfrak{r}$-module $L(\lambda)$ associated to $\lambda$ (this is a standard module for $\mathfrak{g}_S$). The highest weight space of $L(\lambda)$ (as a $\mathfrak{g}_S$-module) is a weight space for $\mathfrak{h}$, with weight $\lambda$. Let $\mathfrak{u}^+$ act trivially on $L(\lambda)$; this gives $L(\lambda)$ the structure of an irreducible $\mathfrak{p}$-module. Define the generalized Verma module $V^{L(\lambda)}$ to be the induced module $U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} L(\lambda)$.

**Definition 3.** Let $\mathcal{O}_\mathfrak{g}$ denote the category of $\mathfrak{g}$-modules that are weight modules whose weight spaces are finite dimensional, and whose set of weights lies in a finite union of sets of the form

$$D(\lambda) = \{\lambda - \sum_{i\in I} n_i\alpha_i | \lambda \in \mathfrak{h}^*\}.$$

The morphisms are $\mathfrak{g}$-module homomorphisms.

**Definition 4.** Let $\mathcal{O}_\mathfrak{g}^S$ be the full subcategory of $\mathcal{O}_\mathfrak{g}$ whose objects are $\mathfrak{g}$-modules $V \in \mathcal{O}_\mathfrak{g}$ such that every weight $\mu \in P(V)$ satisfies $(\mu, \alpha_i) > 0$ for all $i \in I\setminus S$.

Let $M \in \mathcal{O}_\mathfrak{g}$. Recall that a highest weight series for $M$ is an increasing filtration

$$(0) \subset M_1 \subset M_2 \subset \cdots$$

of submodules of $M$ satisfying $\bigcup_{i=0}^\infty M_i = M$. For $M_{i+1} \neq M_i$ the module $M_{i+1}/M_i$ is a highest weight module. We extend a well known result of [6] to modules in $\mathcal{O}_\mathfrak{g}$, where the index set $I$ may be countably infinite.

**Proposition 2.1.** *Let $M \in \mathcal{O}_\mathfrak{g}$. Then $M$ has a highest weight series $\{M_i\}_{i\in\mathbb{Z}}$.*

**Proof.** Let $\lambda_1, \lambda_2, \ldots \lambda_r$ be a set of weights of such that

$$P(M) \subset D(\lambda_1) \cup D(\lambda_2) \ldots \cup D(\lambda_r).$$

With out loss of generality we may assume that the sets $D(\lambda_i)$ are distinct. Then any $\mu \in P(M)$ is of the form $\mu = \lambda_j - \beta$ for some $j$ $1 \leq j \leq r$, with

$\beta = \sum n_i\alpha_i$ where $n_i \in \mathbb{Z}_+$. Define $d(\mu) = \sum in_i$. Since $d(\mu) = N$ implies $n_i = 0$ for sufficiently large $i$ the set $\{\mu \in P(M) | d(\mu) = N\}$ is finite for all $N \in \mathbb{Z}_+$. Now a filtration of $M$ can be constructed using the same argument as in the case where $I$ is finite [6]: Let $\mu_1$ be a weight in $P(M)$ with minimal $d(\mu_1)$. Choose a nonzero weight vector $x_1 \in M^\mu$. By the minimality of $d(\mu_1)$ the vector $x_1$ is a highest weight vector. Denote by $M_1$ the highest weight module generated by $x_1$. To construct a module $M_2$ with $(0) \subset M_1 \subset M_2$ and $M_2/M_1$ a highest weight module repeat the above argument for the module $M/M_1$. The series $d(\mu_k)$ is increasing, and can only remain at one value for finitely many $k$. Thus eventually $M^\mu = M^\mu_n = M^\mu_{n+1} \dots$ for every $\mu \in P(M)$. This construction gives the desired highest weight series. ∎

In order to prove the main theorem we will use the following result from category theory [7].

**Lemma 2.2.** *Let $\mathcal{A}$ and $\mathcal{B}$ be categories. Let $F$ be a functor from $\mathcal{A}$ to $\mathcal{B}$. The categories are equivalent if and only if*

1. *$F$ is full, that is, for all objects $V_1, V_2$ of $\mathcal{A}$ the mapping*

$$Hom_{\mathcal{A}}(V_1, V_2) \to Hom_{\mathcal{B}}(F(V_1), F(V_2))$$

   *described by $f \mapsto F(f)$ is surjective.*
2. *$F$ is faithful, that is, the above mapping is injective.*
3. *For every object $M$ of $\mathcal{B}$ there is an object $V$ of $\mathcal{A}$ such that $M$ is isomorphic to $F(V)$.* ∎

## 3. The equivalence of categories

We will show that the category $\mathcal{O}^S_{\mathfrak{g}}$ is equivalent to the category $\mathcal{O}^S_{\mathfrak{r}}$.

If $V \in \mathcal{O}^S_{\mathfrak{g}}$ let $V^{\mathfrak{u}^+}$ denote the the set of elements of $V$ annihilated by the action of $\mathfrak{u}^+$. The space $V^{\mathfrak{u}^+}$ is an $\mathfrak{g}_s$-module since $U(\mathfrak{g}_s)V^{\mathfrak{u}^+} \subset V^{\mathfrak{u}^+}$. Likewise $V^{\mathfrak{u}^+}$ is an $\mathfrak{r}$-module and a $\mathfrak{p}$-module. Note that the subspace $V^{\mathfrak{u}^+}$ considered as an $\mathfrak{r}$-module is in $\mathcal{O}^S_{\mathfrak{r}}$.

The functors between our categories $\mathcal{O}^S_{\mathfrak{g}}$ and $\mathcal{O}^S_{\mathfrak{r}}$ are simply restriction and induction.

**Definition 5.** The functor $R : \mathcal{O}^S_{\mathfrak{g}} \to \mathcal{O}^S_{\mathfrak{r}}$ is defined as follows: $R(V) = V^{\mathfrak{u}^+}$ for $V \in \mathcal{O}^S_{\mathfrak{g}}$ and given a pair $V, U$ of modules in $\mathcal{O}^S_{\mathfrak{g}}$ and a module homomorphism $f : V \to U$ define $R(f) = \bar{f}$ to be the restriction of the homomorphism $f$ to $V^{\mathfrak{u}^+}$.

**Definition 6.** The functor $\mathrm{Ind}^{\mathfrak{g}}_{\mathfrak{p}} : \mathcal{O}^S_{\mathfrak{r}} \to \mathcal{O}^S_{\mathfrak{g}}$ is defined by $\mathrm{Ind}^{\mathfrak{g}}_{\mathfrak{p}}(N) = U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} N$ for $N$ an object of $\mathcal{O}^S_{\mathfrak{r}}$, considered as a $\mathfrak{p}$-module with $\mathfrak{u}^+$ acting as multiplication by $0$, and $\mathrm{Ind}^{\mathfrak{g}}_{\mathfrak{p}}(f) = 1 \otimes f$ for $f$ a morphism in $\mathcal{O}^S_{\mathfrak{r}}$, considered as a morphism in $\mathcal{O}^S_{\mathfrak{p}}$.

**Lemma 3.1.** *Let $M \in \mathcal{O}_{\mathfrak{r}}^S$. Then $(Ind_{\mathfrak{p}}^{\mathfrak{g}} M)^{\mathfrak{u}^+} = 1 \otimes M$.*

**Proof.** By definition $M$ is annihilated by $\mathfrak{u}^+$ as a $\mathfrak{p}$-module, so $1 \otimes M \subset (\mathrm{Ind}_{\mathfrak{p}}^{\mathfrak{g}} M)^{\mathfrak{u}^+}$.

Using the Poincare-Birkhoff-Witt theorem, we have an isomorphism $\mathrm{Ind}_{\mathfrak{p}}^{\mathfrak{g}} M \simeq U(\mathfrak{u}^-) \otimes_{\mathbb{C}} M$. Consider $U(\mathfrak{u}^-) \otimes_{\mathbb{C}} M = (U(\mathfrak{u}^-)\mathfrak{u}^- \otimes_{\mathbb{C}} M) \oplus (1 \otimes M)$.

Let $w \in U(\mathfrak{u}^-)\mathfrak{u}^- \otimes_{\mathbb{C}} M$ be nonzero. Since $M$ is a weight module, we may assume $w$ is a linear combination of vectors of the form $f_{i_1} f_{i_2} \ldots f_{i_j} \otimes v$, where $v \in M$ is a weight vector of weight $\lambda$, and the $f_{i_1} f_{i_2} \cdots f_{i_j}$, $i_1 \le i_2 \le \cdots \le i_j$ is an element of the Poincare-Birkhoff-Witt basis of $U(\mathfrak{u}^-) \subset U(\mathfrak{n}^-)$. We will consider the elements $f_{i_1} f_{i_2} \cdots f_{i_j} \otimes v$, $i_1 \le i_2 \le \cdots \le i_j$ to be basis vectors of the vector space $U(\mathfrak{u}^-)\mathfrak{u}^- \otimes v$. Let $w = f_{i_1} f_{i_2} \ldots f_{i_j} \otimes v$. Not all of the indices $i_1, i_2, \ldots i_j$ are in $S$, by the definition of $\mathfrak{u}^-$. Assume that $i_k \in I \backslash S$, we will show that $e_{i_k} w \ne 0$. The vector $w$ can be rewritten as $w = f_{i_1} f_{i_2} \cdots f_{i_{k-1}} f_{i_k}^n f_{i_{k+1}} \cdots f_{i_j} \otimes v$ where $n \ge 1$, $i_i \le i_2 \le \cdots \le i_{k-1} < i_k < i_{k+1} \le \cdots \le i_j$, renaming the indices if necessary. Using the commutation relations of the Lie algebra,

$$\begin{aligned} e_{i_k} w &= f_{i_1} \cdots f_{i_{k-1}} \left( e_{i_k} f_{i_k}^n \right) \cdots f_{i_j} \otimes v \\ &= f_{i_1} \cdots f_{i_{k-1}} \left( f_{i_k}^n e_{i_k} + n f_{i_k}^{n-1} h_{i_k} + -a_{i_k i_k} \frac{(n-1)n}{2} f_{i_k}^{n-1} \right) f_{i_{k+1}} \cdots f_{i_j} \otimes v \\ &= f_{i_1} \cdots f_{i_k}^n e_{i_k} f_{i_{k+1}} \cdots f_{i_j} \otimes v + n f_{i_1} \cdots f_{i_k}^{n-1} h_{i_k} f_{i_{k+1}} \cdots f_{i_j} \otimes v \\ &\quad + -a_{i_k i_k} \frac{(n-1)n}{2} f_{i_1} \cdots f_{i_k}^{n-1} f_{i_{k+1}} \cdots f_{i_j} \otimes v. \end{aligned}$$

Since $v$ is in $M$ it is annihilated by $e_{i_k} \in \mathfrak{u}^+$, and one has

$$f_{i_1} \cdots f_{i_k}^n e_{i_k} f_{i_{k+1}} \cdots f_{i_j} \otimes v = f_{i_1} \cdots f_{i_k}^n f_{i_{k+1}} \cdots f_{i_j} e_{i_k} \otimes v = 0.$$

It is also true that

$$h_{i_k} f_{i_{k+1}} \cdots f_{i_j} \otimes v = f_{i_{k+1}} \cdots f_{i_j} h_{i_k} \otimes v + \sum_{s=1}^{j-k} -a_{i_k i_{k+s}} f_{i_{k+1}} \cdots f_{i_j} \otimes v.$$

Thus

$$\begin{aligned} & e_{i_k} w \\ &= n f_{i_1} \cdots f_{i_k}^{n-1} f_{i_{k+1}} \cdots f_{i_j} h_{i_k} \otimes v + n \sum_{s=1}^{j-k} -a_{i_k i_{k+s}} f_{i_1} \cdots f_{i_k}^{n-1} f_{i_{k+1}} \cdots f_{i_j} \otimes v \\ &\quad - a_{i_k i_k} \frac{(n-1)n}{2} f_{i_1} \cdots f_{i_k}^{n-1} f_{i_{k+1}} \cdots f_{i_j} \otimes v \\ &= n\lambda(h_{i_k}) f_{i_1} \cdots f_{i_k}^{n-1} f_{i_{k+1}} \cdots f_{i_j} \otimes v - n \sum_{s=1}^{j-k} a_{i_k i_{k+s}} f_{i_1} \cdots f_{i_k}^{n-1} f_{i_{k+1}} \cdots f_{i_j} \otimes v \\ &\quad - a_{i_k i_k} \frac{(n-1)n}{2} f_{i_1} \cdots f_{i_k}^{n-1} f_{i_{k+1}} \cdots f_{i_j} \otimes v \\ &= \left[ n\lambda(h_{i_k}) + n \sum_{s=1}^{j-k} -a_{i_k i_{k+s}} + -a_{i_k i_k} \frac{(n-1)n}{2} \right] f_{i_1} \cdots f_{i_k}^{n-1} f_{i_{k+1}} \cdots f_{i_j} \otimes v. \\ &\ne 0. \end{aligned}$$

The above expression is nonzero because $\lambda(h_{i_k}) > 0$, $a_{i_k i_{k+s}} \leq 0$ for all $1 \leq s \leq j-k$ and $a_{i_k i_k} \leq 0$ for $i_k \in I \backslash S$. Let $w' = f_{i'_1} \cdots f_{i'_l} \otimes v$, $i'_1 \leq i'_2 \leq \cdots i'_l$ be a basis vector not equal to $w$. If $f_{i_k}$ does not occur in $w'$ we have $e_{i_k} w' = 0$, and if $f_{i_k}$ does occur in $w'$ then the lists $i_1 \leq i_2 \leq \cdots \leq i_j$ and $i'_1 \leq i'_2 \leq \cdots \leq i'_l$ do not agree, so by the above computation $e_{i_k} w$ is not in the linear span of $e_{i_k} w'$. Now consider an element $w \in U(\mathfrak{u}^-)\mathfrak{u}^- \otimes v$ that is a linear combination of the basis vectors $f_{i_1} f_{i_2} \ldots f_{i_j} \otimes v$, $i_1 \leq i_2 \cdots \leq i_j$, choose an index $i_k \in I \backslash S$ that appears in the first term of $w$. Then $e_{i_k} w$ is a linear combination of the basis vectors, with nonzero first term, so is nonzero. We conclude that $w$ is not annihilated by $\mathfrak{u}^+$, that $(\mathrm{Ind}_{\mathfrak{p}}^{\mathfrak{g}} M)^{\mathfrak{u}^+} \subset 1 \otimes M$, and finally that $(\mathrm{Ind}_{\mathfrak{p}}^{\mathfrak{g}} M)^{\mathfrak{u}^+} = 1 \otimes M$. ■

**Proposition 3.2.** *Let $V \in \mathcal{O}_{\mathfrak{g}}^S$ be a highest weight module. Then $V = U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} V^{\mathfrak{u}^+}$.*

**Proof.** Let $\lambda$ be the highest weight of $V$, with highest weight vector $v_0$. The induced module $U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} V^{\mathfrak{u}^+}$ is also a highest weight module for $\mathfrak{g}$. There is a natural map $\mu : U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} V^{\mathfrak{u}^+} \to U(\mathfrak{g}) V^{\mathfrak{u}^+} = V$, which we will show is an isomorphism. The map $\mu$ is surjective because $V^{\mathfrak{u}^+}$ generates the module $V$, as $V^{\mathfrak{u}^+}$ contains the highest weight vector $v_0$. Let $W = \mathrm{Ker}\,\mu$. If $W$ is nontrivial then $W$ contains a weight vector $w \neq 0$ whose weight is of minimal depth. Applying an element of $\mathfrak{u}^+$ to $w \in W$ reduces the depth of $w$, so that $w$ is annihilated by $\mathfrak{u}^+$. By Lemma 3.1 $w \in 1 \otimes V^{\mathfrak{u}^+}$, which contradicts the fact that $\mu$ is injective when restricted to $1 \otimes V^{\mathfrak{u}^+}$. ■

**Proposition 3.3.** *Let $V \in \mathcal{O}_{\mathfrak{g}}^S$, then $V = U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} N$ for some $N \in \mathcal{O}_{\mathfrak{r}}^S$.*

**Proof.** Consider the highest weight series

$$(0) = V_0 \subset V_1 \subset V_2 \subset \cdots$$

where $\bigcup_{i=0}^{\infty} V_i = V$. Note $\bigcup_{i=0}^{\infty} V_i^{\mathfrak{u}^+} = (\bigcup_{i=0}^{\infty} V_i)^{\mathfrak{u}^+} = V^{\mathfrak{u}^+}$.

We will prove the existence of an isomorphism by induction on $i \in \mathbb{Z}_+$. The module $V_1$ is by assumption a highest weight module so Proposition 3.2 shows the proposition holds in this case. Similarly, the result holds for the highest weight module $V_{i+1}/V_i$ any $i \in \mathbb{Z}_+$. Assume that $V_i = U(\mathfrak{g}) \otimes V_i^{\mathfrak{u}^+}$. Applying the five lemma to the diagram

$$\begin{array}{ccccccccc} 0 & \to & U(\mathfrak{g}) \otimes V_i^{\mathfrak{u}^+} & \longrightarrow & U(\mathfrak{g}) \otimes V_{i+1}^{\mathfrak{u}^+} & \longrightarrow & U(\mathfrak{g}) \otimes (V_{i+1}/V_i)^{\mathfrak{u}^+} & \to & 0 \\ & & \approx \downarrow & & \downarrow & & \approx \downarrow & & \\ 0 & \to & V_i & \longrightarrow & V_{i+1} & \longrightarrow & V_{i+1}/V_i & \to & 0 \end{array}$$

implies $V_{i+1} = U(\mathfrak{g}) \otimes V_{i+1}^{\mathfrak{u}^+}$, so by induction $V_i = U(\mathfrak{g}) \otimes V_i^{\mathfrak{u}^+}$ for all $i \in \mathbb{Z}_+$. Thus

$$V = \bigcup U(\mathfrak{g}) \otimes V_i^{\mathfrak{u}^+} = U(\mathfrak{g}) \otimes \bigcup V_i^{\mathfrak{u}^+} = U(\mathfrak{g}) \otimes V^{\mathfrak{u}^+}.$$ ■

**Theorem 3.4.** *The category $\mathcal{O}_{\mathfrak{g}}^S$ is equivalent to the category $\mathcal{O}_{\mathfrak{r}}^S$.*

**Proof.** First we show that the functor $\mathrm{Ind}_{\mathfrak{p}}^{\mathfrak{g}} : \mathcal{O}_{\mathfrak{r}}^S \to \mathcal{O}_{\mathfrak{g}}^S$ is full. All objects $V_1, V_2$ of $\mathcal{O}_{\mathfrak{g}}^S$ are, by Proposition 3.2, of the form $V_i = U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} V_i^{\mathfrak{u}^+}$ and

$V_2 = U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} V_2^{\mathfrak{u}^+}$. Then any $\mathfrak{g}$-homomorphism $g : V_1 \to V_2$ is determined by its action on $V_1^{\mathfrak{u}^+}$

$$g(U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} V_1^{\mathfrak{u}^+}) = U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} \bar{g}(V_1^{\mathfrak{u}^+}).$$

Hence $g = 1 \otimes \bar{g}$.

Since $1 \otimes f = 1 \otimes g$ if and only if $f = g$ the functor $\mathrm{Ind}_{\mathfrak{p}}^{\mathfrak{g}}$ is faithful.

Finally, Proposition 3.3 says that for every object $V$ of $\mathcal{O}_{\mathfrak{g}}^S$ there is an $N \in \mathcal{O}_{\mathfrak{r}}^S$ with $\mathrm{Ind}_{\mathfrak{p}}^{\mathfrak{g}}(N) = V$. The conditions of Lemma 2.2 are satisfied and the categories are equivalent. ■

## 4. Applications

Our first corollary includes Proposition 4.2 of [10].

**Corollary 4.1.** *Fix $S \subset I$. For $\mu \in P_+$ satisfying $(\mu, \alpha_i) > 0$ for $i \in I \backslash S$ the irreducible highest weight module $L_{\mathfrak{g}}(\mu)$ is a generalized Verma module:*

$$L_{\mathfrak{g}}(\mu) = Ind_{\mathfrak{p}}^{\mathfrak{g}} L(\mu),$$

*here $L(\mu)$ is the irreducible highest weight $\mathfrak{p}$-module of highest weight $\mu$, which is a standard irreducible module for the generalized Kac-Moody algebra $\mathfrak{g}_S$. Conversely, any generalized Verma module $V^{L(\mu)} = Ind_{\mathfrak{p}}^{\mathfrak{g}} L(\mu)$ is an irreducible highest weight $\mathfrak{g}$-module.* ■

The following complete reducibility result appears in [14](where he uses the additional assumption that $I$ is finite). For a Kac-Moody algebra $\mathfrak{l}$, $\mu \in \mathfrak{h}^*$, and an $\mathfrak{l}$-module $M$ in $\mathcal{O}_{\mathfrak{l}}$ define the multiplicity $[M : L(\mu)]$ of the irreducible $\mathfrak{l}$-module $L(\mu)$ in $M$ to be the number of proper factors of type $L(\mu)$ in any local composition series of $M$ at $\mu$ see [13].

**Corollary 4.2.** *Let $\mathfrak{g}$ be a generalized Kac-Moody algebra. Take $S = \{i \in I | a_{ii} > 0\}$, so that $\mathfrak{g}_S$ is a Kac-Moody algebra. Let $V$ be a module in the category $\mathcal{O}_{\mathfrak{g}}^S$. Then $V$ is completely reducible and*

$$V = \bigoplus_{\mu \in P_+} [V^{\mathfrak{u}^+}, L(\mu)] L_{\mathfrak{g}}(\mu).$$

**Proof.** Since $\mathfrak{g}_S$ is a Kac-Moody algebra and $V^{\mathfrak{u}^+}$ is an integrable $\mathfrak{g}_S$-module the complete reducibility of integrable modules of a Kac-Moody algebra implies

$$V^{\mathfrak{u}^+} = \oplus_{\mu \in P_+} [V^{\mathfrak{u}^+}, L_{\mathfrak{r}}(\mu)] L_{\mathfrak{r}}(\mu).$$

Applying the main theorem gives

$$\mathrm{Ind}_{\mathfrak{p}}^{\mathfrak{g}} V^{\mathfrak{u}^+} = \oplus_{\mu \in P_+} [V^{\mathfrak{u}^+}, L_S(\mu)] \mathrm{Ind}_{\mathfrak{p}}^{\mathfrak{g}} L_{\mathfrak{r}}(\mu)$$

So by Corollary 4.1 and Proposition 3.2

$$V = \oplus_{\mu \in P_+} [V^{\mathfrak{u}^+}, L_{\mathfrak{r}}(\mu)] L_{\mathfrak{g}}(\mu). \qquad \blacksquare$$

For a module $V \in \mathcal{O}_{\mathfrak{g}}$ recall the formal character is defined as

$$\mathrm{ch}(V) = \sum_{\lambda \in \mathfrak{h}^*} \mathrm{dim} V_\lambda e^\lambda.$$

Using Theorem 3.4, we can compute the character of a module $V \in \mathcal{O}_{\mathfrak{g}}^S$ by computing the product of the $\mathfrak{h}$ weight modules $U(\mathfrak{u}^-)$ and $V^{\mathfrak{u}^+}$.

**Corollary 4.3.** *Let $V \in \mathcal{O}_{\mathfrak{g}}^S$. Then*

$$ch(V) = ch(U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} V^{\mathfrak{u}^+}) = ch(U(\mathfrak{u}^-)) \cdot ch(V^{\mathfrak{u}^+}) \qquad \blacksquare$$

Depending upon the structure of the Lie algebra, $\mathrm{ch}(U(u^-))$ and $V^{\mathfrak{u}^+}$ may be computed more readily than $\mathrm{ch}(V)$. For example, choose $S$ so that $a_{ij} \neq 0$ for all $i, j \in I \backslash S$. The subalgebra $\mathfrak{g}_S$ of $\mathfrak{g}$ and hence its universal enveloping algebra $\mathcal{U}(\mathfrak{g}_S)$ act via the adjoint action on $\mathfrak{g}$. Identify the root $-\alpha_j \in -\Delta_+$ of $\mathfrak{g}$ with the weight of the highest weight vector $f_j$ of the standard $\mathfrak{g}_S$-module $L_{\mathfrak{g}_S}(-\alpha_i) = \mathcal{U}(\mathfrak{g}_S) \cdot f_j \subset \mathfrak{g}$. By Theorem 5.1 in [9] $\mathfrak{g} = \mathfrak{u}^+ \oplus (\mathfrak{g}_S + \mathfrak{h}) \oplus \mathfrak{u}^-$, where $\mathfrak{u}^-$ is the free Lie algebra over the vector space $\sum_{j \in I \backslash S} L(-\alpha_i)$.

Then

$$\mathrm{ch}(U(\mathfrak{u}^-)) = 1 - \sum_{j \in I \backslash S} \mathrm{ch}(L_S(-\alpha_j)) = \prod_{\varphi \in \Delta_+ \backslash \Delta_+^S} (1 - e^{-\varphi})^{\dim F(V)^{-\varphi}}$$

and

$$\mathrm{ch}(V) = \prod_{\varphi \in \Delta_+ \backslash \Delta_+^S} (1 - e^{-\varphi})^{\dim F(V)^{-\varphi}} \mathrm{ch}(V^{\mathfrak{u}^+}). \tag{1}$$

In the case of the monster Lie algebra, one can choose $S$ so that $\mathfrak{g}_S = \mathfrak{sl}_2$. Then each $L(-\alpha_i)$ is a finite dimensional $\mathfrak{sl}_2$-module of highest weight $-\alpha_1$. For this case equation (1) specializes to Borcherds' product formula for the modular function $j(\tau)$ [2]. Formula (1) appears in [8],[9] and is used in [10] to further elucidate a portion of the monstrous moonshine phenomenon already studied in [2].

Computing the homology $H_n(\mathfrak{g}, V)$ for $V \in \mathcal{O}_{\mathfrak{g}}^S$ can also be reduced to computing the homology groups for modules for the smaller Lie algebra $\mathfrak{g}_S$.

**Corollary 4.4.** *For a $g$-module $V$ in $\mathcal{O}_{\mathfrak{g}}^S$*

$$H_n(\mathfrak{g}, V^t) \simeq H_n(\mathfrak{p}, (V^{u^+})^t).$$

**Proof.** This corollary follows from Theorem 3.4 and Proposition 4.2 of [3]. $\blacksquare$

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