

Almost Transitive Actions on Spaces with the Rational Homotopy of Sphere Products

Oliver Bletz-Siebert

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Abstract. We determine the structure of transitive actions of compact Lie groups on spaces which have the dimension and the (rational) homotopy groups of a product $\mathbb{S}^1 \times \mathbb{S}^m$ of spheres.

These homogeneous spaces arise in several geometric contexts and may be considered as \mathbb{S}^1 -bundles over certain spaces, e.g. over lens spaces and over certain quotients of Stiefel manifolds.

Furthermore, we show that if a non-compact simply connected Lie group acts transitively on such a space, then the orbits of the maximal compact subgroups are all simply connected rational cohomology spheres of codimension one and hence classified. We obtain this by giving a short proof of the existence and structure of the natural bundle of Gorbatsevich under these much less general assumptions. In this special case the proof gets considerably shorter by the use of the homotopy properties of the spaces in question and a theorem of Mostert on the structure of orbit spaces of compact Lie groups on manifolds.

Introduction

By a well-known theorem of Montgomery [14], the maximal compact subgroups of a Lie group acting transitively on a compact simply connected manifold still act transitively on the manifold. This is not true if the manifold is not simply connected, as already the example of the action of the real numbers on the 1-sphere \mathbb{S}^1 shows. Even if the simply connected Lie group acts effectively, the maximal compact subgroups need not act transitively, see Example 5.

We show that sometimes the maximal compact subgroups of a transitive Lie group are not too far from being transitive in the sense that their orbits are of codimension 1. The structure of manifolds admitting an action of a compact connected Lie group with an orbit of codimension 1 was determined by Mostert. To be precise, we prove the following result, where $\pi_{\bullet}(X) \otimes \mathbb{Q}$ denotes the graded group $\bigoplus_{k \in \mathbb{N}} \pi_k(X) \otimes \mathbb{Q}$.

Theorem 1. *Let X be an $(m+1)$ -dimensional compact homogeneous space of a simply connected Lie group G with $\pi_1(X) \cong \mathbb{Z}$ and*

$$\pi_{\bullet}(X) \otimes \mathbb{Q} \cong \pi_{\bullet}(\mathbb{S}^1 \times \mathbb{S}^m) \otimes \mathbb{Q}.$$

Then the orbits of all maximal compact subgroups K of G are simply connected rational cohomology m -spheres and $\pi_{\bullet}(\mathbb{S}^1 \times (K \cdot x)) \cong \pi_{\bullet}(X)$ for every $x \in X$. All these K -orbits are equivalent and of dimension m . In particular, K is not transitive and G is not compact.

From this theorem we deduce the structure of a homogeneous space of a compact Lie group if the homogeneous space meets the other assumptions of Theorem 1.

Theorem 2. *Assume A is the commutator group of a compact connected Lie group K and H is a closed subgroup of K . Set $X = K/H$. If*

$$\dim X = m + 1, \quad \pi_1(X) \cong \mathbb{Z} \text{ and } \pi_{\bullet}(X) \otimes \mathbb{Q} \cong \pi_{\bullet}(\mathbb{S}^1 \times \mathbb{S}^m) \otimes \mathbb{Q}, \quad (1)$$

then there is a complementary one-dimensional torus group $T \cong \text{SO}(2)$ for A such that $T \cdot A$ is transitive on X and $A/A \cap H$ is a simply connected rational cohomology m -sphere. Conversely, if the centre of K is one-dimensional, if $H^1 \subset A$ and if $A/A \cap H$ is a simply connected rational cohomology m -sphere, then also (1) holds.

In both cases all the A -orbits on X are equivalent; furthermore, $A \cap H$ is connected, H/H^1 is cyclic, $\pi_{\bullet}(A/A \cap H) \otimes \mathbb{Q} \cong \pi_{\bullet}(\mathbb{S}^m) \otimes \mathbb{Q}$ and $\mathbb{S}^1 \times (A/A \cap H)$ is a covering space of X .

In Theorem 2 one might restrict the action of A to a minimal normal subgroup acting transitively on the orbits of A ; if there is no such proper normal subgroup, then (the action of) A is called *irreducible*.

The compact Lie groups acting transitively on spheres and even on spaces with the rational cohomology of spheres were determined by various authors, see the explanations before Theorem 10; there, for sake of completeness, we cite the classification result for such homogeneous spaces and give tables of the transitive groups. In particular, the irreducible groups A of Theorem 2 (and their actions) are explicitly known. Furthermore, by Theorem 2 the homotopy type of X determines the homotopy type of $A/A \cap H$; a look on Theorem 10 shows that the space $A/A \cap H$ is determined by $\pi_{\bullet}(X)$, and if $A/A \cap H$ is not a sphere, then even the irreducible action on the A -orbits is determined. Of course, a similar observation holds for Theorem 1. (Note that by the last column of table 2 in Theorem 10 one can distinguish the spheres by their homotopy from the rational cohomology spheres which are not genuine spheres, and the spaces of the last sort may be distinguished from each other by their homotopy since the Stiefel manifolds $\text{SO}(2k+1)/\text{SO}(2k-1)$ are $(2k-2)$ -connected).

In [2] we show that a quotient of a compact connected Lie group and a closed connected subgroup has the rational cohomology of $\mathbb{S}^1 \times \mathbb{S}^m$ if it meets the assumptions in Theorem 2; the conclusion on the cohomology of a manifold even holds more generally under the assumptions in Corollary 4.

Such a homogeneous space is certainly not simply connected and transitive groups are not semi-simple. This contrasts with the case where $\mathbb{S}^1 \times \mathbb{S}^m$ is replaced by $\mathbb{S}^k \times \mathbb{S}^m$ with $k > 1$.

Kramer [13] classified the simply connected homogeneous spaces with the rational cohomology of $\mathbb{S}^k \times \mathbb{S}^m$ for $3 \leq k \leq m$ and m odd, and Wolfram [23] classified these simply connected homogeneous spaces in the case $2 = k < m$ and

m odd. It is implicit in [13, 3.C] that these homogeneous spaces with the rational cohomology of $\mathbb{S}^k \times \mathbb{S}^m$ have also the rational *homotopy* of $\mathbb{S}^k \times \mathbb{S}^m$ if $k > 1$. As mentioned above, in the case $k = 1$ the reverse implication holds: a homogeneous spaces with the rational homotopy of $\mathbb{S}^1 \times \mathbb{S}^m$ as in Theorem 1 or 2 has the rational cohomology of $\mathbb{S}^1 \times \mathbb{S}^m$.

Spaces with the rational cohomology of sphere products $\mathbb{S}^k \times \mathbb{S}^m$ occur for example as focal manifolds of isoparametric hypersurfaces with four principal curvatures in spheres. These focal manifolds may be seen as special cases of point and line spaces of generalized quadrangles, i.e. of buildings of type C_2 in the sense of Tits. And in this more general situation it still holds that the homogeneous (point or line) spaces have the rational cohomology of sphere products (Kramer [12]). In all these examples the spaces in question are simply connected for $k > 1$, and as stated above it follows that they also have the rational homotopy of $\mathbb{S}^k \times \mathbb{S}^m$. For $k = 1$ it can be shown ([2]) that in this geometrical context (as point or line space of a homogeneous generalized quadrangle with parameters $(1, m)$) these spaces have the rational homotopy of $\mathbb{S}^1 \times \mathbb{S}^m$.

\mathbb{S}^1 -fibre bundles over lens spaces and over quotients of Stiefel manifolds

In this section we give another interpretation of the homogeneous spaces appearing in Theorem 2.

First we recall the exact sequence of homotopy groups for a fibre bundle, see Steenrod [22, 17.4]. If $E \rightarrow B$ is a fibre bundle projection with typical fibre F , then there is a long exact sequence of groups

$$\cdots \rightarrow \pi_n(F) \rightarrow \pi_n(E) \rightarrow \pi_n(B) \rightarrow \pi_{n-1}(F) \rightarrow \cdots \rightarrow \pi_1(B).$$

We will use this exact sequence in the case of fibre bundles $G/H \rightarrow G/U$, where G is a Lie group and H and U are closed subgroups of G with $H \subseteq U$; these fibre bundles have typical fibres U/H , see Steenrod [22, 9.6]. Furthermore, if G is connected and H trivial, then the exact homotopy sequence may be extended to end in

$$\cdots \rightarrow \pi_1(U) \rightarrow \pi_1(G) \rightarrow \pi_1(G/U) \rightarrow \pi_o(U) \rightarrow 1,$$

where $\pi_o(U) = U/U^1$ is the group of connected components of U .

These fibre bundles arising from Lie groups are used to give the following interpretation of the spaces appearing in Theorem 2. Assume A/H is one of the homogeneous spaces of Theorem 10 below, i.e. A/H is a simply connected rational cohomology m -sphere, in particular we have $\pi_\bullet(A/H) \otimes \mathbb{Q} \cong \pi_\bullet(\mathbb{S}^m) \otimes \mathbb{Q}$, see Kramer [13, 2.4]. Let Z be a finite cyclic group in the normalizer of H in $\text{SO}(2) \times A$; note that H is connected. Then the quotient $(\text{SO}(2) \times A)/(Z \cdot H)$ is a space of the type considered in Theorem 2 (for $m \geq 2$). For simplicity we assume that Z intersects $\text{SO}(2)$ and H trivially. (It would be enough to ask that Z intersects A in H .)

We denote the projection of $\text{SO}(2) \times A/H$ to the second factor A/H by pr_2 . The bundle

$$\frac{(\text{SO}(2) \times A)}{(Z \cdot H)} \rightarrow \frac{\text{SO}(2) \times A}{\text{SO}(2) \times \text{pr}_2(Z) \cdot H} \cong \frac{A}{\text{pr}_2(Z) \cdot H}$$

has typical fibre $\frac{\mathrm{SO}(2) \times_{\mathrm{pr}_2(Z)} H}{Z \cdot H} \cong \frac{\mathrm{SO}(2) \times_{\mathrm{pr}_2(Z)} Z}{Z} \cong \mathrm{SO}(2)$. Another interpretation of this \mathbb{S}^1 -fibre bundle projection is the following. On the one hand the finite cyclic group Z acts on $\mathrm{SO}(2) \times A/H$, on the other hand it acts as $\mathrm{pr}_2(Z)$ on A/H ; the induced map between the Z -orbit spaces $(\mathrm{SO}(2) \times A)/(Z \cdot H)$ and $A/(\mathrm{pr}_2(Z) \cdot H)$ is the mentioned bundle projection.

For example, for $k \geq 2$ let $A = \mathrm{SO}(2k+1)$ and $H = \mathrm{SO}(2k-1)$; furthermore, let Z be a finite cyclic subgroup diagonally embedded into the product $\mathrm{SO}(2) \times \mathrm{SO}(2)$, where the second factor is interpreted as the (connected component of the) centralizer of H in A . Then $(\mathrm{SO}(2) \times \mathrm{SO}(2k+1))/(Z \cdot \mathrm{SO}(2k-1))$ is a \mathbb{S}^1 -fibre bundle over the cyclic quotient $\mathrm{SO}(2k+1)/(\mathrm{pr}_2(Z) \cdot \mathrm{SO}(2k-1))$ of the Stiefel manifold

$$\mathrm{SO}(2k+1)/\mathrm{SO}(2k-1)$$

of pairs of orthonormal vectors in \mathbb{R}^{2k+1} .

As another example consider the lens space L obtained as the orbit space of $\mathbb{S}^{2n+1} \subset \mathbb{C}^{n+1}$ under the action of the finite cyclic group generated by the diagonal matrix with entries $\xi^{q_1}, \xi^{q_2}, \dots, \xi^{q_{n+1}}$, where $\xi = \exp(\frac{2\pi i}{p})$ for a prime number p and where q_1, q_2, \dots, q_{n+1} are prime to p with $q_1 + q_2 + \dots + q_{n+1} = 0 \pmod{p}$, see e.g. Spanier [21, 2.6.10]. If we identify $\mathbb{S}^{2n+1} = \mathrm{SU}(n+1)/\mathrm{SU}(n)$ and if we denote by Z a cyclic subgroup of $\mathrm{SO}(2) \times \mathrm{SU}(n+1)$ that projects onto the group generated by $\mathrm{diag}(\xi^{q_1}, \xi^{q_2}, \dots, \xi^{q_{n+1}}) \in \mathrm{SU}(n+1)$, then the orbit space $(\mathrm{SO}(2) \times \mathrm{SU}(n+1))/(Z \cdot \mathrm{SU}(n))$ is a \mathbb{S}^1 -fibre bundle over the lens space L .

Proofs of the theorems

Suppose X is a homogeneous space as in Theorem 1. First, we will determine the cohomology (over the rationals) of the universal covering space of X .

Lemma 3. *Let X be a compact connected topological $(m+1)$ -manifold with infinite fundamental group such that $\pi_k(X)$ is finite for $1 < k < m$. If s is the free rank of $\pi_m(X)$, then the rational cohomology groups of the universal covering space \tilde{X} are given by*

$$H^k(\tilde{X}; \mathbb{Q}) \cong \begin{cases} \mathbb{Q} & \text{if } k = 0, \\ \mathbb{Q}^s & \text{if } k = m > 1, \\ 0 & \text{else.} \end{cases}$$

Proof. The cohomology groups of the manifold \tilde{X} are trivial in dimensions greater than $m+1 = \dim \tilde{X}$. As a covering space with infinite fibres, \tilde{X} is a non-compact manifold, and it follows that $H^{m+1}(\tilde{X}) = 0$, see Bredon [7, VI.7.12 and 14].

Since \tilde{X} is the universal covering space of X , we have

$$\pi_k(\tilde{X}) \otimes \mathbb{Q} \cong \pi_k(X) \otimes \mathbb{Q} = 0 \quad \text{for } 1 < k < m$$

and $\pi_m(\tilde{X}) \otimes \mathbb{Q} \cong \pi_m(X) \otimes \mathbb{Q} \cong \mathbb{Q}^s$ (if $m > 1$). There is a rational version of the Hurewicz theorem, see e.g. Spanier [21, 9.6.15], which now implies that the rational cohomology groups in the 'middle' dimensions vanish, i.e. $H^k(\tilde{X}; \mathbb{Q}) = 0$ for $0 < k < m$, and that $H^m(\tilde{X}; \mathbb{Q}) \cong \mathbb{Q}^s$ (if $m > 1$). ■

The assumptions on the homotopy groups in the last lemma are in particular fulfilled if the fundamental group is abelian of free rank 1 and if the only non-vanishing higher homotopy groups are up to torsion the same as those of \mathbb{S}^m .

Corollary 4. *Let X be a compact connected topological $(m+1)$ -manifold with an abelian fundamental group such that $\pi_\bullet(X) \otimes \mathbb{Q} \cong \pi_\bullet(\mathbb{S}^1 \times \mathbb{S}^m) \otimes \mathbb{Q}$ and $m \geq 2$. Then the universal covering space \tilde{X} of X has the rational cohomology of an m -sphere, i.e.*

$$H^\bullet(\tilde{X}; \mathbb{Q}) \cong H^\bullet(\mathbb{S}^m; \mathbb{Q}).$$

We will consider spaces X which are homogeneous spaces of Lie groups, and we will determine their structure. First, we consider the case where the Lie group is simply connected. For simply connected compact homogeneous spaces the maximal compact subgroups still act transitively on the homogeneous space by a result of Montgomery [14]. Montgomery generalized this to homogeneous spaces with finite fundamental group; see also Salzmänn et al. [20, 96.19] for a proof, which inspired the proof of Theorem 1. For infinite fundamental groups this is false in general as the following example shows.

Example 5. The image of $\mathbb{R} \times \mathrm{SU}(n)$ in $\mathrm{GL}_{n+1}(\mathbb{C})$ under the injection

$$(t, A) \mapsto \begin{pmatrix} e^{it\sqrt{2}} & \\ & e^{it}A \end{pmatrix}$$

acts effectively and transitively on $\mathbb{S}^1 \times \mathbb{S}^{2n-1} \subset \mathbb{C}^{n+1}$ and has no transitive compact subgroup, because the orbits of $\mathrm{SU}(n)$ are $(2n-1)$ -spheres.

We are now going to describe the orbits of compact Lie groups acting transitively on spaces with the homotopy considered above. We start with the case of maximal compact subgroups of simply connected Lie groups.

Proof of Theorem 1. First note that the assumptions imply $m \geq 2$. We begin the proof as in Salzmänn et al. [20, 96.19]. We choose a point $x \in X$ such that K_x^1 is maximal compact in G_x^1 : for some $y \in X$ take a maximal compact subgroup L of G containing a maximal compact subgroup of G_y^1 , then L_y^1 is maximal compact in G_y^1 and because of $(gLg^{-1})_{g \cdot y} = gL_y g^{-1}$ every $x = g \cdot y$ with $K = gLg^{-1}$ is as claimed; note that every compact subgroup lies in a maximal one and that these are connected and all conjugate. (In the end we will see that all K -orbits are of the same type. Therefore, the choice of x is actually arbitrary.)

Consider the commutative diagram

$$\begin{array}{ccc} K/K_x^1 & \xrightarrow{j} & G/G_x^1 \\ \downarrow p & & \downarrow q \\ K/K_x & \xrightarrow{\alpha} & G/G_x \end{array}$$

where j and α are the natural mappings and indeed embeddings, see e.g. [20, 96.9]. Being inclusions of maximal compact subgroups, $K \hookrightarrow G$ and $K_x^1 \hookrightarrow G_x^1$ are

homotopy equivalences, and so they induce isomorphisms in homotopy. Therefore, the map j of the diagram also induces an isomorphism in homotopy by the five-lemma. But then j induces also an isomorphism in homology, see the Whitehead theorem [21, 7.5.9]. Since p and q are covering maps, α induces isomorphisms $\alpha_{\#} : \pi_n(K/K_x) \rightarrow \pi_n(G/G_x)$ for all $n \geq 2$.

In the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \pi_1(K/K_x^1) & \longrightarrow & \pi_1(K/K_x) & \longrightarrow & K_x/K_x^1 & \longrightarrow & 0 \\ & & \downarrow \cong & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \pi_1(G/G_x^1) & \longrightarrow & \pi_1(G/G_x) & \longrightarrow & G_x/G_x^1 & \longrightarrow & 0 \end{array}$$

we have $\pi_1(G/G_x) \cong \pi_1(X) \cong \mathbb{Z}$ by assumption.

As G/G_x^1 is simply connected and as j induces isomorphisms in homotopy, K/K_x^1 is simply connected. Furthermore, G_x/G_x^1 is infinite. This means that G/G_x^1 is the universal covering space of X and that it is not compact and not homeomorphic to the compact space K/K_x^1 , the universal covering space of K/K_x . Therefore, K is not transitive on X . In particular, the group G is not compact.

As j induces isomorphisms in homotopy and (co)homology, we have

$$\pi_{\bullet}(K/K_x^1) \otimes \mathbb{Q} \cong \pi_{\bullet}(G/G_x^1) \otimes \mathbb{Q} \cong \pi_{\bullet}(\mathbb{S}^m) \otimes \mathbb{Q}$$

and by Corollary 4 also $H^{\bullet}(K/K_x^1; \mathbb{Q}) \cong H^{\bullet}(G/G_x^1; \mathbb{Q}) \cong H^{\bullet}(\mathbb{S}^m; \mathbb{Q})$.

Therefore K/K_x^1 is a simply connected rational cohomology m -sphere. As K/K_x^1 is compact, orientable, and has the rational cohomology of \mathbb{S}^m , one has $\dim K \cdot x = \dim K/K_x = \dim K/K_x^1 = m$.

Up to now we only used that the fundamental group of X is infinite; the fact that it is torsion free will be exploited in the sequel. We now apply Mostert's theorem [16] for orbits of codimension 1 in manifolds (compare also [17] and Hofmann and Mostert [11]). There are just the two possibilities $X/K \approx \mathbb{S}^1$ or $X/K \approx [0, 1]$. We will show that the second case does not occur.

As K is compact and simply connected, K -orbits have finite fundamental groups. If $X/K \approx [0, 1]$, then X may be described as a double mapping cylinder of orbit projections of a principal orbit onto two exceptional orbits of the action of K , see Mostert's theorem [16]. The injections of the two exceptional orbits into X induce mappings of their fundamental groups into the fundamental group of X . By the Seifert-Van Kampen theorem these images generate the fundamental group of X . But as $\pi_1(X) \cong \mathbb{Z}$ by assumption, the images of finite groups in $\pi_1(X)$ are trivial and cannot generate $\pi_1(X)$. Hence, $X/K \approx [0, 1]$ is not possible.

Therefore, $X/K \approx \mathbb{S}^1$. But then all orbits are of the same type K/K_x , and there is a fibre bundle $K/K_x \rightarrow X \rightarrow X/K \approx \mathbb{S}^1$, whose homotopy sequence shows that $\pi_1(K/K_x) \cong 0$, i.e. $K_x = K_x^1$ is connected. Hence the orbits of K are simply connected rational cohomology m -spheres, and the homotopy sequence of $K/K_x \rightarrow X \rightarrow \mathbb{S}^1$ shows that $\pi_{\bullet}(X) \cong \pi_{\bullet}(\mathbb{S}^1 \times (K \cdot x))$. ■

Remark 6. Note that the simply connected homogeneous rational cohomology spheres, which appear in Theorem 1 and in Theorem 2, are classified; for the convenience of the reader we list them at the end, see Theorem 10.

Remark 7. The last proof shows that if $\pi_1(X)$ in Theorem 1 is infinite abelian but fails to be torsion free, then there still is for every $x \in X$ a maximal compact subgroup L of G such that L/L_x^1 is a simply connected rational cohomology m -sphere, and all the L -orbits are equivalent. Furthermore, for every maximal compact subgroup there is such a point $x \in X$.

Remark 8. The fibre bundle $K/K_x \rightarrow X \rightarrow X/K$ is in the case treated here the natural bundle of Gorbatsevich. In [9] he proved that for a compact homogeneous space $X = G/H$ of a simply connected Lie group G there is a subgroup H' of finite index in H such that $K/(K \cap H') \rightarrow G/H' \rightarrow X/K$ is a fibre bundle whose fibre $K/(K \cap H')$ has a finite fundamental group (and where K is a maximal compact subgroup of G). Furthermore, he showed that if additionally the fundamental group of X is torsion free, then one may take $H' = H$ and then the fibre $K/(K \cap H')$ is simply connected.

We gave a different direct proof (not relying on the results of Gorbatsevich) that $K/K_x \rightarrow X \rightarrow X/K$ is a fibre bundle under the additional assumptions on the homotopy, since we need these assumptions anyway to describe the structure of the homogeneous spaces we are interested in; furthermore, the assumptions on the homotopy considerably simplify the proof.

If the transitive Lie group is compact (and hence not simply connected), its action may also be seen in the light of Theorem 1, applied to the simply connected covering group. We stress this by deducing Theorem 2 from Theorem 1. It is not really necessary to invoke Theorem 1, see the remark after the proof.

Proof of Theorem 2. First assume that (1) holds, i.e. that X is an $(m+1)$ -dimensional homogeneous space of K with infinite cyclic fundamental group and that X has the same rational homotopy as $\mathbb{S}^1 \times \mathbb{S}^m$. Note that the assumptions imply $m \geq 2$. The homotopy sequence of $H \rightarrow K \rightarrow X$ shows that the centre of K has positive dimension, since this dimension is the free rank of $\pi_1(K)$. The connected component of the centre still acts transitively on the orbit space X/A of the commutator group A of K . Up to a finite covering, A is the maximal compact subgroup of the universal covering group of K ; hence A is connected. By Theorem 1 the orbits of A in X are simply connected rational cohomology spheres, and these orbits are all equivalent; furthermore, $X/A \approx \mathbb{S}^1$. It follows that there is a complementary torus group $T \cong \text{SO}(2)$ in K acting transitively on X/A ; in particular $T \cap H$ is finite, and $T \cdot A$ is transitive on X .

It also holds that $A \cap H$ is connected since $A/A \cap H$ is simply connected. A simply connected space with the rational cohomology of an m -sphere has also the rational homotopy of an m -sphere (see Cartan-Serre [8, Prop. 3] and Kramer [13, 2.4]); therefore $\pi_\bullet(A/A \cap H) \otimes \mathbb{Q} \cong \pi_\bullet(\mathbb{S}^m) \otimes \mathbb{Q}$.

The fact that H/H^1 is cyclic follows from the homotopy sequence of the fibre bundle $H \rightarrow K \rightarrow X$. Finally, $K/H^1 \approx \mathbb{S}^1 \times A/A \cap H$ is a covering space of X . This also shows that the additional conclusions of Theorem 2 hold if we assume (1).

Now assume that the centre of K is one-dimensional (i.e. $K = \text{SO}(2) \cdot A$) and that the quotient $A/A \cap H$ is a simply connected rational cohomology m -sphere for some subgroup H of K with $H^1 \subset A$. Note that the conclusions of the second paragraph only require that $A/A \cap H$ be a simply connected rational cohomology m -sphere. Hence, these conclusions hold under the assumptions of

this paragraph. The cohomology structure of the orientable manifold $A/A \cap H$ implies that $\dim A/A \cap H = m$, see e.g. Bredon [7, VI.7.14], and it follows that $\dim K/H = \dim K/H^1 = 1 + \dim A/H^1 = m + 1$. We may assume that A is simply connected and that $K = \mathrm{SO}(2) \times A$. We consider the projection

$$p : \mathrm{SO}(2) \times A \rightarrow \mathrm{SO}(2)$$

from K to the first factor of K . Because of $H^1 \subset A$ the image $p(H)$ of H in $\mathrm{SO}(2)$ is finite, hence $\mathrm{SO}(2)/p(H) \cong \mathrm{SO}(2)$. Furthermore, $A/A \cap H \approx (p(H) \times A)/H$, and there is the fibre bundle $A/A \cap H \rightarrow (\mathrm{SO}(2) \times A)/H \rightarrow \mathrm{SO}(2)/p(H)$ and an exact sequence

$$\pi_1(A/A \cap H) \rightarrow \pi_1(K/H) \rightarrow \pi_1(\mathrm{SO}(2)),$$

which shows that $\pi_1(K/H)$ is infinite cyclic since $A/A \cap H$ is simply connected and K/H is finitely covered by $\mathrm{SO}(2) \times A/A \cap H$. Furthermore,

$$\pi_\bullet(K/H) \otimes \mathbb{Q} \cong \pi_\bullet(\mathrm{SO}(2) \times A/A \cap H) \otimes \mathbb{Q} \cong \pi_\bullet(\mathbb{S}^1 \times \mathbb{S}^m) \otimes \mathbb{Q}.$$

Hence, (1) holds. ■

Remark 9. One may obtain the first part of Theorem 2 also as follows. Up to a finite covering we may assume that $K = Z \times A$, where Z is the connected component of the centre of K and A is the commutator group. The projection p from K onto Z induces a fibre bundle $A/(A \cap H) \rightarrow K/H \rightarrow Z/p(H)$ whose homotopy sequence shows that $\pi_1(Z/p(H)) \cong \mathbb{Z}$, since $\pi_1(A/(A \cap H))$ is finite. It follows that $Z/p(H) \approx \mathbb{S}^1$ and that $A/(A \cap H)$ is simply connected. Hence, there is a torus group T in Z complementary to $p(H)$ such that $T \times A$ is transitive on $X = K/H$. Furthermore, $A/(A \cap H)$ is a rational (homotopy and by the rational Hurewicz theorem) cohomology m -sphere. The other claims follow as above.

With the notation of the last theorem, $\mathbb{R} \times A/A \cap H$ is the universal covering space of X ; this explains Corollary 4.

The rational cohomology of X and of the bundle $H^1 \rightarrow K \rightarrow K/H^1$ is determined in [2]. In particular, it is shown there that K/H^1 has the (rational) cohomology of the sphere product $\mathbb{S}^1 \times \mathbb{S}^m$.

If in the last theorem K is assumed to act irreducibly, then it follows that K is locally isomorphic to $\mathrm{SO}(2) \times A$, where A is one of the almost simple Lie groups appearing in the tables of the following theorem, in which for the sake of completeness and for the convenience of the reader we give the table of homogeneous cohomology spheres. This classification was obtained by various authors. We give a short overview over the classification. Montgomery and Samelson [15] determined the structure of compact connected Lie groups that act effectively and transitively on spheres. Borel ([3] and [4]) gave an explicit list of these almost simple Lie groups. He also proved that if a homogeneous integral (co)homology sphere, i.e. a homogeneous space of a compact connected Lie group with the same integral (co)homology as a sphere, is additionally simply connected, then it is indeed an ordinary sphere.

Poncet [19] proved that these transitive actions on spheres are all equivalent to actions of subgroups of the respective orthogonal group and hence linear. Finally, Bredon [6, 1.1, 1.2] showed that a homogeneous cohomology sphere is either

a 1-sphere or simply connected (and hence a sphere) or the Poincaré homology 3-sphere, i.e. the quotient space $\mathrm{SO}(3)/I$ of $\mathrm{SO}(3)$ and the group I of rotational symmetries of an icosahedron, see Bredon [7, VI.8.10]. This gives the classification of homogeneous integral cohomology spheres. For tables similar to those of Theorem 10, see also Salzmann et al. [20, 95.4], Grundhöfer-Knarr-Kramer [10, 1.1] and Biller [1, 3.1.1]. There all the effective actions on spheres are given, whereas we only give the irreducible ones in Theorem 10.

To consider the simply connected homogeneous spaces with the *rational* cohomology of an n -sphere, first assume that n is even; then the homogeneous space has Euler characteristic 2 and is indeed a sphere by Borel-De Siebenthal [5] and Borel [3]. A compact connected one-dimensional manifold is a 1-sphere. For $n \geq 3$ odd assume additionally that the transitive Lie group G acts irreducibly; recall that this means that there is no proper transitive normal subgroup in G . Then G is almost simple by Kramer [13, 3.7] and Onishchik [18, §18, Prop. 2(i)]. For the classification of the transitive actions of almost simple Lie groups on rational cohomology spheres of odd dimension $n \geq 3$ see Kramer [13, 6.A] from where we took table 2 of Theorem 10; cf. also Onishchik [18, Ch. 5, §18, Table 10], where $\mathrm{Sp}(2)$ appears as $\mathrm{SO}(5)$.

Theorem 10. *Let G be a compact connected Lie group and H a closed subgroup such that G acts effectively and irreducibly on G/H and such that*

$$H^\bullet(G/H; \mathbb{Q}) \cong H^\bullet(\mathbb{S}^n; \mathbb{Q})$$

for some $n \in \mathbb{N}$. Assume that G/H is simply connected, if $n > 1$. Then either G/H is a sphere and (G, H, n) is one of the triples of irreducible homogeneous spheres in table 1, or n is odd and (G, H, n) is one of the triples in table 2.

| G | H | n |
|--------------------|--------------------|--------------------|
| $\mathrm{SO}(n+1)$ | $\mathrm{SO}(n)$ | $n \in \mathbb{N}$ |
| $\mathrm{SU}(k)$ | $\mathrm{SU}(k-1)$ | $2k-1, k \geq 2$ |
| $\mathrm{Sp}(k)$ | $\mathrm{Sp}(k-1)$ | $4k-1, k \geq 2$ |
| G_2 | $\mathrm{SU}(3)$ | 6 |
| $\mathrm{Spin}(7)$ | G_2 | 7 |
| $\mathrm{Spin}(9)$ | $\mathrm{Spin}(7)$ | 15 |

Table 1: Homogeneous cohomology spheres

| G | H | $\mathrm{Cen}_G(H)^1$ | n | |
|---------------------|--|-----------------------|--------|--|
| $\mathrm{SU}(3)$ | $\mathrm{SO}(3)$ | 1 | 5 | $\pi_3(G/H) = \mathbb{Z}_4$ |
| $\mathrm{SO}(2k+1)$ | $\mathrm{SO}(2k-1)$ | $\mathrm{SO}(2)$ | $4k-1$ | $\pi_{2k-1}(G/H) = \mathbb{Z}_2, k \geq 2$ |
| $\mathrm{Sp}(2)$ | ${}^{\mathbb{H}}\rho_{3\lambda_1}$ | 1 | 7 | $\pi_3(G/H) = \mathbb{Z}_{10}$ |
| G_2 | $\mathrm{SU}(2)$ | $\mathrm{SU}(2)$ | 11 | $\pi_3(G/H) = \mathbb{Z}_2$ |
| G_2 | ${}^{\mathbb{R}}\rho_{\lambda_1} + {}^{\mathbb{R}}\rho_{2\lambda_1}$ | $\mathrm{SU}(2)$ | 11 | $\pi_3(G/H) = \mathbb{Z}_3$ |
| G_2 | $2{}^{\mathbb{R}}\rho_{2\lambda_1}$ | 1 | 11 | $\pi_3(G/H) = \mathbb{Z}_4$ |
| G_2 | ${}^{\mathbb{R}}\rho_{6\lambda_1}$ | 1 | 11 | $\pi_3(G/H) = \mathbb{Z}_{28}$ |

Table 2: Homogeneous rational cohomology spheres

(The symbols ${}^{\mathbb{H}}\rho_{3\lambda_1}$, ${}^{\mathbb{R}}\rho_{\lambda_1} + {}^{\mathbb{R}}\rho_{2\lambda_1}$, $2{}^{\mathbb{R}}\rho_{2\lambda_1}$ and ${}^{\mathbb{R}}\rho_{6\lambda_1}$ appearing in table 2 stand for the images of certain representations of Lie groups locally isomorphic to $SU(2)$, cf. Kramer [13, Ch. 4 and 6.A].)

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Oliver Bletz-Siebert
Mathematisches Institut
Universität Würzburg
Am Hubland
97074 Würzburg, Germany
bletz-siebert@mathematik.uni-wuerzburg.de

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