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On the Riemann-Lie Algebras and Riemann-Poisson Lie Groups

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Abstract. A Riemann-Lie algebra is a Lie algebra $\mathcal{G}$ such that its dual $\mathcal{G}^*$ carries a Riemannian metric compatible (in the sense introduced by the author in C. R. Acad. Sci. Paris, **t. 333**, Série I, (2001) 763–768) with the canonical linear Poisson structure of $\mathcal{G}^*$. The notion of Riemann-Lie algebra has its origins in the study, by the author, of Riemann-Poisson manifolds (see Differential Geometry and its Applications, **Vol. 20, Issue 3**(2004), 279–291).
In this paper, we show that, for a Lie group G, its Lie algebra $\mathcal{G}$ carries a structure of Riemann-Lie algebra iff G carries a flat left-invariant Riemannian metric. We use this characterization to construct examples of Riemann-Poisson Lie groups (a Riemann-Poisson Lie group is a Poisson Lie group endowed with a left-invariant Riemannian metric compatible with the Poisson structure).

1. Introduction

Riemann-Lie algebras first arose in the study by the author of Riemann-Poisson manifolds (see [2]). A Riemann-Poisson manifold is a Poisson manifold (P,π) endowed with a Riemannian metric $\langle,\rangle$ such that the couple $(\pi,\langle,\rangle)$ is compatible in the sense introduced by the author in [1]. The notion of Riemann-Lie algebra appeared when we looked for the Riemannian metrics compatible with the canonical Poisson structure on the dual of a Lie algebra. We pointed out (see [2]) that the dual space $\mathcal{G}^*$ of a Lie algebra $\mathcal{G}$ carries a Riemannian metric compatible with the linear Poisson structure iff the Lie algebra $\mathcal{G}$ carries a structure which we called Riemann-Lie algebra. Moreover, the isotropy Lie algebra at a point on a Riemann-Poisson manifold is a Riemann-Lie algebra. In particular, the dual Lie algebra of a Riemann-Poisson Lie group is a Riemann-Lie algebra (a Riemann-Poisson Lie group is a Poisson Lie group endowed with a left-invariant Riemannian metric compatible with the Poisson structure). In this paper, we will show that a Lie algebra $\mathcal{G}$ carries a structure of Riemann-Lie algebra iff $\mathcal{G}$ is a semi-direct

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product of two abelian Lie algebras. Hence, according to a well-known result of Milnor [5], we deduce that, for a Lie group G, its Lie algebra carries a structure of Riemann-Lie algebra iff G carries a flat left-invariant Riemannian metric. We apply this geometrical characterization to construct examples of Riemann-Poisson Lie groups. In particular, we give many examples of bialgebras $(\mathcal{G},[\ ,\ ],\mathcal{G}^*,[\ ,\ ]^*)$ such that both $(\mathcal{G},[\ ,\ ])$ and $(\mathcal{G}^*,[\ ,\ ]^*)$ are Riemann-Lie algebras.

2. Definitions and main results

Notations. Let G be a connected Lie group and $(\mathcal{G},[\ ,\ ])$ its Lie algebra. For any $u\in\mathcal{G}$, we denote by u^l (resp. u^r) the left-invariant (resp. right-invariant) vector field of G corresponding to u. We denote by θ the right-invariant Maurer-Cartan form on G given by

$$\theta(u^r)=-u,\quad u\in\mathcal{G}.\tag{1}$$

Let $\langle;\rangle$ be a scalar product on $\mathcal{G}$ i.e. a bilinear, symmetric, non-degenerate and positive definite form on $\mathcal{G}$.

Let us enumerate some mathematical objets which are naturally associated with $\langle,\rangle$:

1. an isomorphism $\#:\mathcal{G}^*\longrightarrow\mathcal{G}$;
2. a scalar product $\langle,\rangle^*$ on the dual space $\mathcal{G}^*$ by
$$\langle\alpha,\beta\rangle^*=\langle\#(\alpha),\#(\beta)\rangle\qquad\alpha,\beta\in\mathcal{G}^*;\tag{2}$$
3. a left-invariant Riemannian metric $\langle,\rangle^l$ on G by
$$\langle u^l,v^l\rangle^l=\langle u,v\rangle\qquad u,v\in\mathcal{G};\tag{3}$$
4. a left-invariant contravariant Riemannian metric $\langle,\rangle^{*l}$ on G by
$$\langle\alpha,\beta\rangle^{*l}_g=\langle T_e^*L_g(\alpha),T_e^*L_g(\beta)\rangle^*\tag{4}$$

where $\alpha,\beta\in\Omega^1(G)$ and L_g is the left translation of G by g.

The infinitesimal Levi-Civita connection associated with $(\mathcal{G},[\ ,\ ],\langle,\rangle)$ is the bilinear map $A:\mathcal{G}\times\mathcal{G}\longrightarrow\mathcal{G}$ given by

$$2\langle A_uv,w\rangle=\langle[u,v],w\rangle+\langle[w,u],v\rangle+\langle[w,v],u\rangle,\quad u,v,w\in\mathcal{G}.\tag{5}$$

Note that A is the unique bilinear map from $\mathcal{G}\times\mathcal{G}$ to $\mathcal{G}$ which verifies:

1. $A_uv-A_vu=[u,v]$;
2. for any $u\in\mathcal{G}$, $A_u:\mathcal{G}\longrightarrow\mathcal{G}$ is skew-adjoint i.e.
$$\langle A_uv,w\rangle+\langle v,A_uw\rangle=0,\quad v,w\in\mathcal{G}.$$

The terminology used here is motivated by the fact that the Levi-Civita connection ∇ associated with $(G,\langle,\rangle^l)$ is given by

$$\nabla_{u^l}v^l=(A_uv)^l\qquad u,v\in\mathcal{G}.\tag{6}$$

We will introduce now a Lie subalgebra of $\mathcal{G}$ which will play a crucial role in this paper.

For any $u \in \mathcal{G}$, we denote by $ad_u : \mathcal{G} \longrightarrow \mathcal{G}$ the endomorphism given by $ad_u(v) = [u, v]$, and by $ad_u^t : \mathcal{G} \longrightarrow \mathcal{G}$ its adjoint given by

$$\langle ad_u^t(v), w \rangle = \langle v, ad_u(w) \rangle \qquad v, w \in \mathcal{G}.$$

The space

$$S_{\langle,\rangle} = \{u \in \mathcal{G}; ad_u + ad_u^t = 0\} \tag{7}$$

is obviously a subalgebra of $\mathcal{G}$. We call $S_{\langle,\rangle}$ *the orthogonal subalgebra associated with* $(\mathcal{G}, [\ ,\ ], \langle,\rangle)$.

Remark 2.1. The scalar product $\langle,\rangle$ is bi-invariant if and only if $\mathcal{G} = S_{\langle,\rangle}$. In this case $\mathcal{G}$ is the product of an abelian Lie algebra and a semi-simple and compact Lie algebra (see [5]). The general case where $\langle,\rangle$ is not positive definite has been studied by A. Medina and P. Revoy in [4] and they called a Lie algebra $\mathcal{G}$ with an inner product $\langle,\rangle$ such that $\mathcal{G} = S_{\langle,\rangle}$ an orthogonal Lie algebra which justifies the terminology used here.

Let $(\mathcal{G}, [\ ,\ ], \langle,\rangle)$ be a Lie algebra endowed with a scalar product.

The triple $(\mathcal{G}, [\ ,\ ], \langle,\rangle)$ is called *a Riemann-Lie algebra* if

$$[A_u v, w] + [u, A_w v] = 0 \tag{8}$$

for all $u, v, w \in \mathcal{G}$ and where A is the infinitesimal Levi-Civita connection associated to $(\mathcal{G}, [\ ,\ ], \langle,\rangle)$.

From the relation $A_u v - A_v u = [u, v]$ and the Jacobi identity, we deduce that (8) is equivalent to

$$[u, [v, w]] = [A_u v, w] + [v, A_u w] \tag{9}$$

for any $u, v, w \in \mathcal{G}$. We refer the reader to [2] for the origins of this definition.

Briefly, if $(\mathcal{G}, [\ ,\ ], \langle,\rangle)$ is a Lie algebra endowed with a scalar product. The scalar product $\langle,\rangle$ defines naturally a contravariant Riemannian metric on the Poisson manifold $\mathcal{G}^*$ which we denote also by $\langle,\rangle$. If we denote by π^l the canonical Poisson tensor on $\mathcal{G}^*$, $(\mathcal{G}^*, \pi^l, \langle,\rangle)$ is a Riemann-Poisson manifold iff the triple $(\mathcal{G}, [\ ,\ ], \langle,\rangle)$ is a Riemann-Lie algebra.

Characterization of Riemann-Lie algebras. With the notations and the definitions above, we can state now the main result of this paper.

Theorem 2.2. *Let G be a Lie group, $(\mathcal{G}, [\ ,\ ])$ its Lie algebra and $\langle,\rangle$ a scalar product on $\mathcal{G}$. Then, the following assertions are equivalent:*

1) $(\mathcal{G}, [\ ,\ ], \langle,\rangle)$ is a Riemann-Lie algebra.

2) $(\mathcal{G}^*, \pi^l, \langle,\rangle)$ is a Riemann-Poisson manifold (π^l is the canonical Poisson tensor on $\mathcal{G}^*$ and $\langle,\rangle$ is considered as a contravariant metric on $\mathcal{G}^*$).

3) The 2-form $d\theta \in \Omega^2(G, \mathcal{G})$ is parallel with respect the Levi-Civita connection ∇ i.e. $\nabla d\theta = 0$.

4) $(G, \langle,\rangle^l)$ is a flat Riemannian manifold.

5) The orthogonal subalgebra $S_{\langle,\rangle}$ of $(\mathcal{G}, [\ ,\ ], \langle,\rangle)$ is abelian and $\mathcal{G}$ split as an orthogonal direct sum $S_{\langle,\rangle} \oplus \mathcal{U}$ where $\mathcal{U}$ is a commutative ideal.

The equivalence "1) ⇔ 2)" of this theorem was proven in [2] and the equivalence "4) ⇔ 5)" was proven by Milnor in [5]. We will prove the equivalence "1) ⇔ 3)" and the equivalence "1) ⇔ 5)".

The equivalence "1) ⇔ 3)" is a direct consequence of the following formula which it is easy to verify:

$$\nabla d\theta(u^l, v^l, w^l)_g = Ad_g\left([u,[v,w]] - [A_u v, w] - [v, A_u w]\right), \quad u,v,w \in \mathcal{G}, g \in G. \tag{10}$$

If G is compact and connected, the condition $\nabla d\theta = 0$ implies that $d\theta$ is harmonic and, according to the Hodge Theorem must vanishes since it is exact. Now, the vanishing of $d\theta$ is equivalent to G being abelian and hence we get the following lemma which will be used in the proof of the equivalence "1) ⇔ 5)" in Section 3.

Lemma 2.3. *Let G be a compact, connected and non abelian Lie group. Then the Lie algebra of G does not admit any structure of Riemann-Lie algebra.*

A proof of the equivalence "1) ⇔ 5)" will be given in Section 3.

Examples of Riemann-Poisson Lie groups. This subsection is devoted to the construction, using Theorem 2.2, of some examples of Riemann-Poisson Lie groups. A Riemann-Poisson Lie group is a Poisson Lie group with a left-invariant Riemannian metric compatible with the Poisson structure (see [2]).

We refer the reader to [6] for background material on Poisson Lie groups.

Let G be a Poisson Lie group with Lie algebra $\mathcal{G}$ and π the Poisson tensor on G. Pulling π back to the identity element e of G by left translations, we get a map $\pi_l : G \longrightarrow \mathcal{G} \wedge \mathcal{G}$ defined by $\pi_l(g) = (L_{g^{-1}})_* \pi(g)$ where $(L_g)_*$ denotes the tangent map of the left translation of G by g. Let

$$d_e\pi : \mathcal{G} \longrightarrow \mathcal{G} \wedge \mathcal{G}$$

be the intrinsic derivative of π at e given by

$$v \mapsto L_X \pi(e),$$

where X can be any vector field on G with $X(e) = v$.

The dual map of $d_e\pi$

$$[\ ,\ ]_e : \mathcal{G}^* \wedge \mathcal{G}^* \longrightarrow \mathcal{G}^*$$

is exactly the Lie bracket on $\mathcal{G}^*$ obtained by linearizing the Poisson structure at e. The Lie algebra $(\mathcal{G}^*, [\ ,\ ]_e)$ is called the dual Lie algebra associated with the Poisson Lie group (G, π).

We consider now a scalar product $\langle , \rangle^*$ on $\mathcal{G}^*$. We denote by $\langle , \rangle^{*l}$ the left-invariant contravariant Riemannian metric on G given by (4).

We have shown (cf. [2] Lemma 5.1) that $(G, \pi, \langle , \rangle^{*l})$ is a Riemann-Poisson Lie group iff, for any $\alpha, \beta, \gamma \in \mathcal{G}^*$ and for any $g \in G$,

$$[Ad_g^*\left(A_\alpha^*\gamma + ad_{\pi_l(g)(\alpha)}^*\gamma\right), Ad_g^*(\beta)]_e + [Ad_g^*(\alpha), Ad_g^*\left(A_\beta^*\gamma + ad_{\pi_l(g)(\beta)}^*\gamma\right)]_e = 0, \tag{11}$$

where $A^* : \mathcal{G}^* \times \mathcal{G}^* \longrightarrow \mathcal{G}^*$ is the infinitesimal Levi-Civita connection associated to $(\mathcal{G}^*, [\ ,\ ]_e, \langle , \rangle^*)$ and where $\pi_l(g)$ also denotes the linear map from $\mathcal{G}^*$ to $\mathcal{G}$ induced by $\pi_l(g) \in \mathcal{G} \wedge \mathcal{G}$.

This complicated equation can be simplified enormously in the case where the Poisson tensor arises from a solution of the classical Yang-Baxter equation. However, we need to work more in order to give this simplification.

Let G be a Lie group and let $r \in \mathcal{G} \wedge \mathcal{G}$. We will also denote by $r : \mathcal{G}^* \longrightarrow \mathcal{G}$ the linear map induced by r. Define a bivector π on G by

$$\pi(g) = (L_g)_* r - (R_g)_* r \qquad g \in G.$$

(G, π) is a Poisson Lie group if and only if the element $[r, r] \in \mathcal{G} \wedge \mathcal{G} \wedge \mathcal{G}$ defined by

$$[r, r](\alpha, \beta, \gamma) = \alpha([r(\beta), r(\gamma)]) + \beta([r(\gamma), r(\alpha)]) + \gamma([r(\alpha), r(\beta)]) \tag{12}$$

is ad-invariant. In particular, when r satisfies the Yang-Baxter equation

$$[r, r] = 0, \tag{$Y-B$}$$

it defines a Poisson Lie group structure on G and, in this case, the bracket of the dual Lie algebra $\mathcal{G}^*$ is given by

$$[\alpha, \beta]_e = ad^*_{r(\beta)}\alpha - ad^*_{r(\alpha)}\beta, \quad \alpha, \beta \in \mathcal{G}^*. \tag{13}$$

We will denote by $[\ \ ]_r$ this bracket.

We will give now another description of the solutions of the Yang-Baxter equation which will be useful latter.

We observe that to give $r \in \mathcal{G} \wedge \mathcal{G}$ is equivalent to give a vectorial subspace $S_r \subset \mathcal{G}$ and a non-degenerate 2-form $\omega_r \in \wedge^2 S_r^*$.

Indeed, for $r \in \mathcal{G} \wedge \mathcal{G}$, we put $S_r = Im\, r$ and $\omega_r(u, v) = r(r^{-1}(u), r^{-1}(v))$ where $u, v \in S_r$ and $r^{-1}(u)$ is any antecedent of u by r.

Conversely, let (S, ω) be a vectorial subspace of $\mathcal{G}$ with a non-degenerate 2-form. The 2-form ω defines an isomorphism $\omega^b : S \longrightarrow S^*$ by $\omega^b(u) = \omega(u, .)$, we denote by $\# : S^* \longrightarrow S$ its inverse and we put

$$r = \# \circ i^*$$

where $i^* : \mathcal{G}^* \longrightarrow S^*$ is the dual of the inclusion $i : S \hookrightarrow \mathcal{G}$.

With this observation in mind, the following proposition gives another description of the solutions of the Yang-Baxter equation.

Proposition 2.4. *Let $r \in \mathcal{G} \wedge \mathcal{G}$ and (S_r, ω_r) its associated subspace. The following assertions are equivalent:*

1) $[r, r] = 0$.

2) $r([\alpha, \beta]_r) = [r(\alpha), r(\beta)]$. ($[\ ,\ ]_r$ *is the bracket given by* (13)).

3) S_r *is a subalgebra of* $\mathcal{G}$ *and*

$$\delta\omega_r(u, v, w) := \omega_r(u, [v, w]) + \omega_r(v, [w, u]) + \omega_r(w, [u, v]) = 0$$

for any $u, v, w \in S_r$.

Proof. The proposition follows from the following formulas:

$$\gamma(r([\alpha,\beta]_r) - [r(\alpha), r(\beta)]) = -[r,r](\alpha,\beta,\gamma), \qquad \alpha,\beta,\gamma \in \mathcal{G}^* \tag{14}$$

and, if S is a subalgebra,

$$[r,r](\alpha,\beta,\gamma) = -\delta\omega_r(r(\alpha), r(\beta), r(\gamma)). \tag{15}$$

■

This proposition shows that to give a solution of the Yang-Baxter equation is equivalent to give a symplectic subalgebra of $\mathcal{G}$. We recall that a symplectic algebra (see [3]) is a Lie algebra S endowed with a non-degenerate 2-form ω such that $\delta\omega = 0$.

Remark 2.5. Let G be a Lie group, $\mathcal{G}$ its Lie algebra and S an even dimensional abelian subalgebra of $\mathcal{G}$. Any non-degenerate 2-form ω on S verifies the assertion 3) in Proposition 2.1 and hence (S,ω) defines a solution of the Yang-Baxter equation and then a structure of Poisson Lie group on G.

The following proposition will be crucial in the simplification of the equation (11).

Proposition 2.6. *Let $(\mathcal{G}, [\ ,\ ], \langle,\rangle)$ be a Lie algebra endowed with a scalar product, $r \in \mathcal{G}\wedge\mathcal{G}$ a solution of $(Y-B)$ and (S_r, ω_r) its associated symplectic Lie algebra. Then, $S_r \subset S_{\langle,\rangle}$ iff the infinitesimal Levi-Civita connection A^* associated with $(\mathcal{G}^*, [\ ,\ ]_r, \langle,\rangle^*)$ is given by*

$$A^*_\alpha\beta = -ad^*_{r(\alpha)}\beta, \qquad \alpha,\beta \in \mathcal{G}^*. \tag{16}$$

Moreover, if $S_r \subset S_{\langle,\rangle}$, the curvature of A^* vanishes and hence $(\mathcal{G}^*, [\ ,\ ]_r, \langle,\rangle^*)$ is a Riemann-Lie algebra.

Proof. A^* is the unique bilinear map from $\mathcal{G}^* \times \mathcal{G}^*$ to $\mathcal{G}^*$ such that:

1) $A^*_\alpha\beta - A^*_\beta\alpha = [\alpha,\beta]_r$ for any $\alpha,\beta \in \mathcal{G}^*$;

2) the endomorphism $A^*_\alpha : \mathcal{G}^* \longrightarrow \mathcal{G}^*$ is skew-adjoint with respect to $\langle,\rangle^*$.

The bilinear map $(\alpha,\beta) \mapsto -ad^*_{r(\alpha)}\beta$ verifies 1) obviously and verifies 2) iff $S_r \subset S_{\langle,\rangle}$.

If $A^*_\alpha\beta = -ad^*_{r(\alpha)}\beta$, the curvature of A^* is given by

$$R(\alpha,\beta)\gamma = A^*_{[\alpha,\beta]_r}\gamma - \left(A^*_\alpha A^*_\beta\gamma - A^*_\beta A^*_\alpha\gamma\right) = ad^*_{r([\alpha,\beta]_r)-[r(\alpha),r(\beta)]}\gamma = 0$$

from Proposition 2.4 2). We conclude by Theorem 2.2. ■

Proposition 2.7. *Let $(\mathcal{G}, [\ ,\ ], \langle,\rangle)$ be a Lie algebra with a scalar product. Let $r \in \mathcal{G}\wedge\mathcal{G}$ be a solution of $(Y-B)$ such that S_r is a subalgebra of the orthogonal subalgebra $S_{\langle,\rangle}$. Then S_r is abelian.*

Proof. S_r is unimodular and symplectic and then solvable (see [3]). Also S_r carries a bi-invariant scalar product so S_r must be abelian (see [5]). ∎

We can now simplify the equation (11) and give the construction of Riemann-Poisson Lie groups announced before.

Let G be a Lie group, $(\mathcal{G}, [\ ,\ ])$ its Lie algebra and $\langle , \rangle$ a scalar product on $\mathcal{G}$. We assume that the orthogonal subalgebra $S_{\langle , \rangle}$ contains an abelian even dimensional subalgebra S endowed with a non-degenerate 2-form ω.

As in Remark 2.5, (S, ω) defines a solution r of $(Y-B)$ and then a Poisson Lie tensor π on G. It is easy to see that, for any $g \in G$,

$$\pi^l(g) = r - Ad_g(r).$$

This implies that (11) can be rewritten

$$[Ad_g^*\left(A_\alpha^*\gamma + ad_{r(\alpha)}^*\gamma\right), Ad_g^*(\beta)]_r + [Ad_g^*(\alpha), Ad_g^*\left(A_\beta^*\gamma + ad_{r(\beta)}^*\gamma\right)]_r = [Ad_g^*\left(ad_{Ad_g(r)(\alpha)}^*\gamma\right), Ad_g^*(\beta)]_r + [Ad_g^*(\alpha), Ad_g^*\left(ad_{Ad_g(r)(\beta)}^*\gamma\right)]_r.$$

Now, since $S \subset S_{\langle , \rangle}$, we have by Proposition 2.6

$$A_\alpha^*\gamma + ad_{r(\alpha)}^*\gamma = 0$$

for any $\alpha, \gamma \in \mathcal{G}^*$. On other hand, it is easy to get the formula

$$Ad_g^*[ad_{r(\alpha)}^*\beta] = ad_{(Ad_{g^{-1}}r)(Ad_g^*\alpha)}^*(Ad_g^*\beta), \qquad g \in G, \alpha, \beta \in \mathcal{G}^*.$$

Finally, $(G, \pi, \langle , \rangle^{*l})$ is a Riemann-Poisson Lie group iff

$$[ad_{r(\alpha)}^*\gamma, \beta]_r + [\alpha, ad_{r(\beta)}^*\gamma]_r = 0, \qquad \alpha, \beta, \gamma \in \mathcal{G}^*.$$

But, also since $A_\alpha^*\gamma + ad_{r(\alpha)}^*\gamma = 0$, this condition is equivalent to $(\mathcal{G}^*, [\ ]_r, \langle , \rangle^*)$ is a Riemann-Lie algebra which is true by Proposition 2.6. So, we have shown:

Theorem 2.8. *Let G be a Lie group, $(\mathcal{G}, [\ ,\ ])$ its Lie algebra and $\langle , \rangle$ a scalar product on $\mathcal{G}$. Let S be an even dimensional abelian subalgebra of the orthogonal subalgebra $S_{\langle , \rangle}$ and ω a non-degenerate 2-form on S. Then, the solution of the Yang-Baxter equation associated with (S, ω) defines a structure of Poisson Lie group (G, π) and $(G, \pi, \langle , \rangle^{*l})$ is a Riemann-Poisson Lie group.*

Let us enumerate some important cases where this theorem can be used.

1) Let G be a compact Lie group and $\mathcal{G}$ its Lie algebra. For any bi-invariant scalar product $\langle , \rangle$ on the Lie algebra $\mathcal{G}$, $S_{\langle , \rangle} = \mathcal{G}$. By Theorem 2.8, we can associate to any even dimensional abelian subalgebra of $\mathcal{G}$ a Riemann-Poisson Lie group structure on G.

2) Let $(\mathcal{G}, [\ ,\ ], \langle , \rangle)$ be a Riemann-Lie algebra. By Theorem 2.2, the orthogonal subalgebra $S_{\langle , \rangle}$ is abelian and any even dimensional subalgebra of $S_{\langle , \rangle}$ gives rise to a structure of a Riemann-Poisson Lie group on any Lie group whose the Lie algebra is $\mathcal{G}$. Moreover, we get a structure of bialgebra $(\mathcal{G}, [\ ,\ ], \mathcal{G}^*, [\ ,\ ]_r)$ where both $\mathcal{G}$ and $\mathcal{G}^*$ are Riemann-Lie algebras.

Finally, we observe that the Riemann-Lie groups constructed above inherit the properties of Riemann-Poisson manifolds (see [2]). Namely, the symplectic leaves of these Poisson Lie groups are Kählerian and their Poisson structures are unimodular.

3. Proof of the equivalence "1) ⇔ 5)" in Theorem 2.2

In this section we will give a proof of the equivalence "1) ⇔ 5)" in Theorem 2.2. The proof is a sequence of lemmas. Namely, we will show that, for a Riemann-Lie algebra $(\mathcal{G}, [\ ,\ ], \langle , \rangle)$, the orthogonal subalgebra $S_{\langle , \rangle}$ is abelian. Moreover, $S_{\langle , \rangle}$ is the $\langle , \rangle$-orthogonal of the ideal $[\mathcal{G}, \mathcal{G}]$. This result will be the key of the proof.

We begin by a characterization of Riemann-Lie subalgebras.

Proposition 3.1. *Let $(\mathcal{G}, [\ ,\ ], \langle , \rangle)$ be a Riemann-Lie algebra and $\mathcal{H}$ a subalgebra of $\mathcal{G}$. For any $u, v \in \mathcal{H}$, we put $A_u v = A_u^0 v + A_u^1 v$, where $A_u^0 v \in \mathcal{H}$ and $A_u^1 v \in \mathcal{H}^\perp$. Then, $(\mathcal{H}, [\ ,\ ], \langle , \rangle)$ is a Riemann-Lie algebra if and only if, for any $u, v, w \in \mathcal{H}$, $[A_u^1 v, w] + [v, A_u^1 w] \in \mathcal{H}^\perp$.*

Proof. We have, from (9), that for any $u, v, w \in \mathcal{H}$

$$[u, [v, w]] = [A_u^0 v, w] + [v, A_u^0 w] + [A_u^1 v, w] + [v, A_u^1 w].$$

Now $A^0 : \mathcal{H} \times \mathcal{H} \longrightarrow \mathcal{H}$ is the infinitesimal Levi-Civita connection associated with the restriction of $\langle , \rangle$ to $\mathcal{H}$ and the proposition follows. ■

We will introduce now some objects which will be useful latter.

Let $(\mathcal{G}, [\ ,\ ], \langle , \rangle)$ a Lie algebra endowed with a scalar product.

From (5), we deduce that the infinitesimal Levi-Civita connection A associated to $\langle , \rangle$ is given by

$$A_u v = \frac{1}{2}[u, v] - \frac{1}{2}\left(ad_u^t v + ad_v^t u\right) \quad u, v \in \mathcal{G}. \tag{17}$$

On other hand, the orthogonal with respect to $\langle , \rangle$ of the ideal $[\mathcal{G}, \mathcal{G}]$ is given by

$$[\mathcal{G}, \mathcal{G}]^\perp = \bigcap_{u \in \mathcal{G}} \ker ad_u^t. \tag{18}$$

Let us introduce, for any $u \in \mathcal{G}$, the endomorphism

$$D_u = ad_u - A_u. \tag{19}$$

We have, by a straightforward calculation, the relations

$$\begin{aligned} D_u(v) &= \frac{1}{2}[u, v] + \frac{1}{2}\left(ad_u^t v + ad_v^t u\right), \\ D_u^t(v) &= \frac{1}{2}[u, v] + \frac{1}{2}\left(ad_u^t v - ad_v^t u\right). \end{aligned}$$

From these relations, we remark that, for any $u, v \in \mathcal{G}$, $D_u^t(v) = -D_v^t(u)$ and then

$$\forall u \in \mathcal{G},\ D_u^t(u) = 0. \tag{20}$$

We remark also that

$$D_u^t = D_u \quad \Leftrightarrow \quad \forall v \in \mathcal{G}, ad_v^t u = 0.$$

So, by (18), we get

$$[\mathcal{G}, \mathcal{G}]^\perp = \{u \in \mathcal{G}; D_u^t = D_u\}. \tag{21}$$

Now, we prove a sequence of results which will give a proof of the equivalence "1) ⇔ 5)" in Theorem 2.2.

Proposition 3.2. *Let* $(\mathcal{G}, [\ ,\ ], \langle , \rangle)$ *be a Riemann-Lie algebra. Then* $Z(\mathcal{G})^{\perp}$ *(*$Z(\mathcal{G})$ *is the center of* $\mathcal{G}$*) is an ideal of* $\mathcal{G}$ *which contains the ideal* $[\mathcal{G}, \mathcal{G}]$*. In particular,*

$$\mathcal{G} = Z(\mathcal{G}) \oplus Z(\mathcal{G})^{\perp}.$$

Proof. For any $u \in Z(\mathcal{G})$ and $v \in \mathcal{G}$, from (17) and the fact that A_u is skew-adjoint, $A_u v = -\frac{1}{2} ad_v^t u \in Z(\mathcal{G})^{\perp}$. By (8), for any $w \in \mathcal{G}$

$$[A_u v, w] = [A_w v, u] = 0,$$

so $A_u v \in Z(\mathcal{G})$ and then $A_u v = -\frac{1}{2} ad_v^t u = 0$ which shows that $u \in [\mathcal{G}, \mathcal{G}]^{\perp}$. So $Z(\mathcal{G}) \subset [\mathcal{G}, \mathcal{G}]^{\perp}$ and the proposition follows. ■

From this proposition and the fact that for a nilpotent Lie algebra $\mathcal{G}$ $Z(\mathcal{G}) \cap [\mathcal{G}, \mathcal{G}] \neq \{0\}$, we get the following lemma.

Lemma 3.3. *A nilpotent Lie algebra* $\mathcal{G}$ *carries a structure of Riemann-Lie algebra if and only if* $\mathcal{G}$ *is abelian.*

We can now get the following crucial result.

Lemma 3.4. *Let* $(\mathcal{G}, [\ ,\ ], \langle , \rangle)$ *be a Riemann-Lie algebra. Then the orthogonal Lie subalgebra* $S_{\langle , \rangle}$ *is abelian.*

Proof. By (17), $A_u v = \frac{1}{2}[u, v]$ for any $u, v \in S_{\langle , \rangle}$. So, by Proposition 3.1, $S_{\langle , \rangle}$ is a Riemann-subalgebra. By (9), we have, for any $u, v, w \in S_{\langle , \rangle}$,

$$[u, [v, w]] = [A_u v, w] + [v, A_u w] = \frac{1}{2}[[u, v], w] + \frac{1}{2}[v, [u, w]] = \frac{1}{2}[u, [v, w]]$$

and then $[S_{\langle , \rangle}, [S_{\langle , \rangle}, S_{\langle , \rangle}]] = 0$ i.e. $S_{\langle , \rangle}$ is a nilpotent Lie algebra and then abelian by Lemma 3.3. ■

Lemma 3.5. *Let* $(\mathcal{G}, [\ ,\ ], \langle , \rangle)$ *be a Riemann-Lie algebra. Then*

$$[\mathcal{G}, \mathcal{G}]^{\perp} = \{u \in \mathcal{G}; D_u = 0\}.$$

Proof. Firstly, we notice that, by (21), $[\mathcal{G}, \mathcal{G}]^{\perp} \supset \{u \in \mathcal{G}; D_u = 0\}$. On other hand, remark that the relation (8) can be rewritten

$$[D_u(v), w] + [v, D_u(w)] = 0$$

for any $u, v, w \in \mathcal{G}$. So, we can deduce immediately that $[\ker D_u, Im D_u] = 0$ for any $u \in \mathcal{G}$.

Now we observe that, for any $u \in [\mathcal{G}, \mathcal{G}]^{\perp}$, the endomorphism D_u is auto-adjoint and then diagonalizeable on $\mathbb{R}$. Let $u \in [\mathcal{G}, \mathcal{G}]^{\perp}$, $\lambda \in \mathbb{R}$ be an eigenvalue of D_u and $v \in \mathcal{G}$ an eigenvector associated with λ. We have

$$\langle D_u(v), v \rangle = \lambda \langle v, v \rangle \overset{(\alpha)}{=} -\langle A_v u, v \rangle \overset{(\beta)}{=} -\langle [v, u], v \rangle \overset{(\gamma)}{=} 0.$$

So $\lambda = 0$ and we obtain that D_u vanishes identically. Hence the lemma follows.

The equality (α) is a consequence of the definition of D_u, and the equality (β) follows from the definition of A. We observe that $v \in Im D_u$ and $u \in \ker D_u$ since $D_u(u) = D_u^t(u) = 0$ (see (20)) and the equality (γ) follows from the remark above. ■

Lemma 3.6. *Let $(\mathcal{G},[\ ,\ ],\langle,\rangle)$ be a Riemann-Lie algebra. Then*

$$S_{\langle,\rangle}=[\mathcal{G},\mathcal{G}]^{\perp}.$$

Proof. From Lemma 3.5, for any $u\in[\mathcal{G},\mathcal{G}]^{\perp}$, $A_u=ad_u$ and then ad_u is skew-adjoint. So $[\mathcal{G},\mathcal{G}]^{\perp}\subset S_{\langle,\rangle}$. To prove the second inclusion, we need to work harder than the first one.

Firstly, remark that one can suppose that $Z(\mathcal{G})=\{0\}$. Indeed, $\mathcal{G}=Z(\mathcal{G})\oplus Z(\mathcal{G})^{\perp}$ (see Proposition 3.2), $Z(\mathcal{G})^{\perp}$ is a Riemann-Lie algebra (see Proposition 3.1), $[\mathcal{G},\mathcal{G}]=[Z(\mathcal{G})^{\perp},Z(\mathcal{G})^{\perp}]$ and $S_{\langle,\rangle}=Z(\mathcal{G})\oplus S'_{\langle,\rangle}$ where $S'_{\langle,\rangle}$ is the orthogonal subalgebra associated to $(Z(\mathcal{G})^{\perp},\langle,\rangle)$.

We suppose now that $(\mathcal{G},[\ ,\ ],\langle,\rangle)$ is a Riemann-Lie algebra such that $Z(\mathcal{G})=\{0\}$ and we want to prove the inclusion $[\mathcal{G},\mathcal{G}]^{\perp}\supset S_{\langle,\rangle}$. Notice that it suffices to show that, for any $u\in S_{\langle,\rangle}$, $A_u=ad_u$.

The proof requires some preparation. Let us introduce the subalgebra K given by

$$K=\bigcap_{u\in S_{\langle,\rangle}}\ker ad_u.$$

Firstly, we notice that K contains $S_{\langle,\rangle}$ because $S_{\langle,\rangle}$ is abelian (see Lemma 3.4).

On other hand, we remark that, for any $u\in S_{\langle,\rangle}$, the endomorphism A_u leaves invariant K and $K^{\perp}$. Indeed, for any $v\in K$ and any $w\in S_{\langle,\rangle}$, we have

$$[w,A_uv]\overset{(\alpha)}{=}[w,A_vu]\overset{(\beta)}{=}-[A_wu,v]\overset{(\gamma)}{=}0$$

and then $A_uv\in K$, this shows that A_u leaves invariant K. Furthermore, A_u being skew-adjoint, we have $A_u(K^{\perp})\subset K^{\perp}$.

The equality (α) follows from the relation $A_uv=A_vu+[u,v]=A_uv$, the equality (β) follows from (8) and (γ) follows from the relation $A_wu=\frac{1}{2}[w,u]=0$.

With this observation in mind, we consider the representation $\rho:S_{\langle,\rangle}\longrightarrow so(K^{\perp})$ given by

$$\rho(u)=ad_{u|K^{\perp}}\qquad u\in S_{\langle,\rangle}.$$

It is clear that

$$\bigcap_{u\in S_{\langle,\rangle}}\ker\rho(u)=\{0\}.\tag{$*$}$$

This relation and the fact that $S_{\langle,\rangle}$ is abelian imply that $\dim K^{\perp}$ is even and that there is an orthonormal basis $(e_1,f_1,\ldots,e_p,f_p)$ of $K^{\perp}$ such that

$$\forall i\in\{1,\ldots,p\},\forall u\in S_{\langle,\rangle},\quad ad_ue_i=\lambda^i(u)f_i\quad\text{and}\quad ad_uf_i=-\lambda^i(u)e_i,\tag{$**$}$$

where $\lambda^i\in S^*_{\langle,\rangle}$.

Now, for any $u\in S_{\langle,\rangle}$, since A_u leaves $K^{\perp}$ invariant, we can write

$$\begin{aligned}A_ue_i&=\sum_{j=1}^p\left(\langle A_ue_i,e_j\rangle e_j+\langle A_ue_i,f_j\rangle f_j\right),\\ A_uf_i&=\sum_{j=1}^p\left(\langle A_uf_i,e_j\rangle e_j+\langle A_uf_i,f_j\rangle f_j\right).\end{aligned}$$

From (9), we have for any $v \in S_{\langle,\rangle}$ and for any $i \in \{1, \dots, p\}$,

$$\begin{aligned} [u, [v, e_i]] &= [A_u v, e_i] + [v, A_u e_i], \\ [u, [v, f_i]] &= [A_u v, f_i] + [v, A_u f_i]. \end{aligned}$$

Using the the equality $A_u v = 0$ and $(**)$ and substituting we get

$$\begin{aligned} -\lambda^i(u)\lambda^i(v)e_i &= \sum_{j=1}^{p} \lambda^j(v)\langle A_u e_i, e_j\rangle f_j - \sum_{j=1}^{p} \lambda^j(v)\langle A_u e_i, f_j\rangle e_j, \\ -\lambda^i(u)\lambda^i(v)f_i &= \sum_{j=1}^{p} \lambda^j(v)\langle A_u f_i, e_j\rangle f_j - \sum_{j=1}^{p} \lambda^j(v)\langle A_u f_i, f_j\rangle e_j. \end{aligned}$$

Now, it is clear from $(*)$ that, for any $i \in \{1, \dots, p\}$, there exists $v \in S_{\langle,\rangle}$ such that $\lambda^i(v) \neq 0$. Using this fact and the relations above, we get

$$A_u e_i = \lambda^i(u) f_i \quad \text{and} \quad A_u f_i = -\lambda^i(u) e_i.$$

So we have shown that, for any $u \in S_{\langle,\rangle}$,

$$A_{u|K^\perp} = ad_{u|K^\perp}.$$

Now, for any $u \in S_{\langle,\rangle}$ and for any $k \in K$, $ad_u(k) = 0$. So, to complete the proof of the lemma, we will show that, for any $u \in S_{\langle,\rangle}$ and for any $k \in K$, $A_u k = 0$. This will be done by showing that $A_u k \in Z(\mathcal{G})$ and conclude by using the assumption $Z(\mathcal{G}) = \{0\}$.

Indeed, for any $h \in K$, by (8)

$$[A_u k, h] = [A_h k, u].$$

Since $A_u(K) \subset K$ and since K is a subalgebra, $[A_u k, h] \in K$. Now, $K \subset \ker ad_u$ and ad_u is skew-adjoint so $[A_h k, u] \in Imad_u \subset K^\perp$. So $[A_u k, h] = 0$. On other hand, for any $f \in K^\perp$, we have, also from (8),

$$[A_u k, f] = [A_k u, f] = [A_f u, k] = 0$$

since $A_f u = [f, u] + A_u f = [f, u] + [u, f] = 0$.

We deduce that $A_u k \in Z(\mathcal{G})$ and then $A_u k = 0$. The proof of the lemma is complete. ∎

Lemma 3.7. *Let $(\mathcal{G}, [\ ,\ ], \langle,\rangle)$ be a Riemann-Lie algebra such that $Z(\mathcal{G}) = 0$. Then*

$$\mathcal{G} \neq [\mathcal{G}, \mathcal{G}].$$

Proof. Let $(\mathcal{G}, [\ ,\ ], \langle,\rangle)$ be a Riemann-Lie algebra such that $Z(\mathcal{G}) = 0$. We will show that the assumption $\mathcal{G} = [\mathcal{G}, \mathcal{G}]$ implies that the Killing form of $\mathcal{G}$ is strictly negative definite and then $\mathcal{G}$ is semi-simple and compact which is in contradiction with lemma 2.3.

Let $u \in \mathcal{G}$ fixed. Since A_u is skew-adjoint, there is an orthonormal basis $(a_1, b_1, \ldots, a_r, b_r, c_1, \ldots, c_l)$ of $\mathcal{G}$ and $(\mu_1, \ldots, \mu_r) \in \mathbb{R}^r$ such that, for any $i \in \{1, \ldots, r\}$ and any $j \in \{1, \ldots, l\}$,

$$A_u a_i = \mu_i b_i, \quad A_u b_i = -\mu_i a_i \quad \text{and} \quad A_u c_j = 0.$$

Moreover, $\mu_i > 0$ for any $i \in \{1, \ldots, r\}$.

By applying (9), we can deduce, for any $i, j \in \{1, \ldots, r\}$ and for any $k, h \in \{1, \ldots, l\}$, the relations:

$$\begin{aligned} [u,[a_i,a_j]] &= \mu_i[b_i,a_j] + \mu_j[a_i,b_j], \quad [u,[b_i,b_j]] = -\mu_j[b_i,a_j] - \mu_i[a_i,b_j], \\ [u,[a_i,b_j]] &= -\mu_j[a_i,a_j] + \mu_i[b_i,b_j], \quad [u,[b_i,a_j]] = -\mu_i[a_i,a_j] + \mu_j[b_i,b_j], \\ [u,[c_k,a_j]] &= \mu_j[c_k,b_j], \quad [u,[c_k,b_j]] = -\mu_j[c_k,a_j], \quad [u,[c_k,c_h]] = 0. \end{aligned}$$

From these relations we deduce

$$\begin{aligned} ad_u \circ ad_u([a_i,a_j]) &= -(\mu_i^2 + \mu_j^2)[a_i,a_j] + 2\mu_i\mu_j[b_i,b_j], \\ ad_u \circ ad_u([b_i,b_j]) &= 2\mu_i\mu_j[a_i,a_j] - (\mu_i^2 + \mu_j^2)[b_i,b_j], \\ ad_u \circ ad_u([b_i,a_j]) &= -(\mu_i^2 + \mu_j^2)[b_i,a_j] - 2\mu_i\mu_j[a_i,b_j], \\ ad_u \circ ad_u([a_i,b_j]) &= -2\mu_i\mu_j[b_i,a_j] - (\mu_i^2 + \mu_j^2)[a_i,b_j], \\ ad_u \circ ad_u([c_k,a_j]) &= -\mu_j^2[c_k,a_j], \\ ad_u \circ ad_u([c_k,b_j]) &= -\mu_j^2[c_k,b_j], \\ ad_u \circ ad_u([c_k,c_h]) &= 0. \end{aligned}$$

By an obvious transformation we obtain

$$\begin{aligned} ad_u \circ ad_u\left([a_i,a_j] + [b_i,b_j]\right) &= -(\mu_i - \mu_j)^2\left([a_i,a_j] + [b_i,b_j]\right), \\ ad_u \circ ad_u\left([a_i,a_j] - [b_i,b_j]\right) &= -(\mu_i + \mu_j)^2\left([a_i,a_j] - [b_i,b_j]\right), \\ ad_u \circ ad_u\left([b_i,a_j] + [a_i,b_j]\right) &= -(\mu_i + \mu_j)^2\left([b_i,a_j] + [a_i,b_j]\right), \\ ad_u \circ ad_u\left([b_i,a_j] - [a_i,b_j]\right) &= -(\mu_i - \mu_j)^2\left([b_i,a_j] - [a_i,b_j]\right), \\ ad_u \circ ad_u([c_k,a_j]) &= -\mu_j^2[c_k,a_j], \\ ad_u \circ ad_u([c_k,b_j]) &= -\mu_j^2[c_k,b_j], \\ ad_u \circ ad_u([c_k,c_h]) &= 0. \end{aligned}$$

Suppose now $\mathcal{G} = [\mathcal{G}, \mathcal{G}]$. Then the family of vectors $\{[a_i,a_j] + [b_i,b_j], [a_i,a_j] - [b_i,b_j], [b_i,a_j] + [a_i,b_j], [b_i,a_j] - [a_i,b_j], [c_k,a_i], [c_k,b_j], [c_k,c_h];\ i,j \in \{1,\ldots,r\}, h,k \in \{1,\ldots,l\}\}$ spans $\mathcal{G}$ and then $ad_u \circ ad_u$ is diagonalizeable and all its eigenvalues are non positive. Now its easy to deduce that $ad_u \circ ad_u = 0$ if and only if $ad_u = 0$. Since $Z(\mathcal{G}) = 0$ we have shown that, for any $u \in \mathcal{G} \setminus \{0\}$, $Tr(ad_u \circ ad_u) < 0$ and then the Killing form of $\mathcal{G}$ is strictly negative definite and then $\mathcal{G}$ is semi-simple compact. We can conclude with Lemma 2.3. ∎

Proof of the equivalence "1) $\Leftrightarrow$ 5)" in Theorem 2.2.

It is an obvious and straightforward calculation to show that 5) $\Rightarrow$ 1).

Conversely, let $(\mathcal{G}, [\ ,\ ], \langle, \rangle)$ be a Riemann-Lie algebra. By Proposition 3.2, we can suppose that $Z(\mathcal{G}) = \{0\}$.

We have, from Lemma 3.7 and Lemma 3.6, $\mathcal{G} \neq [\mathcal{G}, \mathcal{G}]$ which implies $S_{\langle,\rangle} \neq 0$ and $\mathcal{G} = S_{\langle,\rangle} \overset{\perp}{\oplus} [\mathcal{G}, \mathcal{G}]$. Moreover, $[\mathcal{G}, \mathcal{G}]$ is a Riemann-Lie algebra (see Proposition 3.1) and we can repeat the argument above to deduce that eventually $\mathcal{G}$ is solvable which implies that $[\mathcal{G}, \mathcal{G}]$ is nilpotent and then abelian by Lemma 3.3 and the implication follows.

Remark 3.8. The pseudo-Riemann-Lie algebras are completely different from the Riemann-Lie algebras. Indeed, the 3-dimensional Heisenberg Lie algebra which is nilpotent carries a Lorentzian Lie algebra structure. On other hand, the non trivial 2-dimensional Lie algebra carries a Lorentzian inner product whose curvature vanishes and does not carry any structure of a pseudo-Riemann-Lie algebra.

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