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Operator Kernels for Irreducible Unitary Representations of Solvable Exponential Lie Groups

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Abstract. Let G be a connected, simply connected, exponential solvable Lie group. The irreducible unitary representations of G may be obtained by the Kirillov-Bernat orbit method. Let $l \in \mathfrak{g}^*$, $\mathfrak{p}$ a Pukanszky polarization associated to l, $P = \exp \mathfrak{p}$, χ_l the corresponding character of P and $\pi_l = \mathrm{ind}_P^G \chi_l$ the associated unitary representation. We show through an example that not all the functions of $\mathcal{C}_c^\infty(G/P, G/P, \chi_l)$ ($\mathcal{C}^\infty$-functions with compact support on $G/P \times G/P$ satisfying a certain covariance condition) are kernel functions of some operator of the form $\pi_l(f)$, $f \in L^1(G)$, even if the polarization is well chosen. This contradicts a result of Leptin ([5]). But if the polarization $\mathfrak{p}$ is an ideal of $\mathfrak{g}$, then the result of Leptin is true, the corresponding retract from $\mathcal{C}_c^\infty(G/P, G/P, \chi_l)$ into $L^1(G)$ exists and a construction algorithm of the function f may be indicated.
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0. Introduction

Let $G = \exp \mathfrak{g}$ be a connected, simply connected, exponential solvable Lie group. The characterization of all the unitary irreducible representations of G (up to unitary equivalence) is given by the Kirillov-Bernat orbit method and will be explained later on. If π is such a representation of G, then

$$\pi(f) = \int_G f(x)\pi(x)dx, \quad f \in L^1(G),$$

defines an irreducible $*$-representation of $L^1(G)$. Moreover the operators $\pi(f)$ are kernel operators on a certain function space which may be identified with an L^2-space. An important problem consists in characterizing as far as possible the different kernel-functions, and hence to get some useful information on the operators $\pi(f)$ themselves.

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For nilpotent Lie groups this problem has been widely solved by Howe ([3]). Let's briefly recall his result. Let $G = \exp \mathfrak{g}$ be a connected, simply connected, nilpotent Lie group. Let $l \in \mathfrak{g}^*$, $\mathfrak{p}$ a polarization for l in $\mathfrak{g}$, $P = \exp \mathfrak{p}$ and $\pi_l = \mathrm{ind}_P^G \chi_l$ the corresponding unitary representation that may be realized as follows: Let $d\dot{x}$ be a G-invariant measure on G/P. The Hilbert space $\mathfrak{H}_{\pi_l}$ of the representation π_l consists of all measurable functions ξ on G such that

$$\xi(xp) = \overline{\chi_l(p)}\xi(x), \quad \text{for almost all } x \in G, \forall p \in P$$
$$\|\xi\|_2^2 := \int_{G/P} |\xi(x)|^2 d\dot{x} < +\infty.$$

The group G acts on $\mathfrak{H}_{\pi_l}$ by

$$\pi_l(x)\xi(y) = \xi(x^{-1}y), \quad \forall x, y \in G, \forall \xi \in \mathfrak{H}_{\pi_l}.$$

If $\mathcal{B} = \{X_1, \ldots, X_k\}$ is a supplementary Malcev basis to $\mathfrak{p}$ in $\mathfrak{g}$, then the invariant measure may be identified with the Lebesgue measure on $\mathbb{R}^k$ and the space $\mathfrak{H}_{\pi_l}$ with $L^2(\mathbb{R}^k)$. For $f \in \mathcal{S}(G)$, the Schwartz algebra of G, the operator $\pi_l(f)$ is a kernel operator given by

$$\pi_l(f)\xi(x) := \int_G f(y)\big(\pi_l(y)\xi\big)(x)dy = \int_{G/P} \mathcal{K}_{\pi_l}(f)(x, y)\xi(y)d\dot{y}$$

where the kernel $\mathcal{K}_{\pi_l}(f)$ is obtained by

$$\mathcal{K}_{\pi_l}(f)(x, y) = \int_P f(xpy^{-1})\chi_l(p)dp, \quad \forall x, y \in G.$$

This kernel satisfies the following covariance condition:

$$\mathcal{K}_{\pi_l}(f)(xp, yp') = \overline{\chi_l(p)}\chi_l(p')\mathcal{K}_{\pi_l}(f)(x, y), \quad \forall x, y \in G, \forall p, p' \in P.$$

If not otherwise specified, the measures used in the previous integrals are the Haar measures on the groups G, P (identified with the Lebesgue measure if a Malcev basis has been fixed), resp. the invariant measure on G/P. The space $\mathcal{S}(G/P, G/P, \chi_l)$ may be defined as the space of all $\mathcal{C}^\infty$-functions on $G \times G$ satisfying:

a) $F(xp, yp') = \overline{\chi_l(p)}\chi_l(p')F(x, y), \quad \forall x, y \in G, \forall p, p' \in P,$

b) the function

$$\tilde{F}(s_1, \ldots, s_k; t_1, \ldots, t_k) := F(\exp s_1X_1 \cdots \exp s_kX_k; \exp t_1X_1 \cdots \exp t_kX_k)$$

is in the Schwartz space $\mathcal{S}(\mathbb{R}^k \times \mathbb{R}^k)$.

Then the result of Howe ([3]) states that the map

$$f \mapsto \mathcal{K}_{\pi_l}(f)$$

is an open continuous surjection from $\mathcal{S}(G)$ onto $\mathcal{S}(G/P, G/P, \chi_l)$, if the latter space is equipped with the topology of $\mathcal{S}(\mathbb{R}^k \times \mathbb{R}^k)$.
Let's finally notice that the previous definitions and results are independent of the choice of the basis.

For exponential solvable Lie groups the results are far from being so nice. In the multiplication formulas in the group and with changes of the basis, exponential functions appear in certain directions. To take this into account the Schwartz spaces $\mathcal{S}(G)$ and $\mathcal{S}(G/P, G/P, \chi_l)$ will certainly have to be replaced by generalized Schwartz spaces $\mathcal{ES}$ for which one requires exponential decay for the functions and their derivatives in certain directions (in the "nilpotent directions" polynomial decay will still be sufficient). Unfortunately this turns out to be insufficient. As a matter of fact the proofs (which are proofs by induction) require at several stages to take partial Fourier transforms, and the exponential decay conditions are not necessarily stable under such partial Fourier transforms. This observation led Ludwig ([7]) to define the generalized Schwartz spaces $\mathcal{ES}$ by requiring exponential decay in certain directions not only for the functions and their derivatives, but also for some of their partial Fourier transforms, provided the polarization and the bases have been well chosen. Under such hypotheses he proves that every function in the generalized Schwartz space $\mathcal{ES}(G/P, G/P, \chi_l)$ is the kernel of an operator of the form $\pi_l(f)$ for some $f \in \mathcal{ES}(G)$. Let's not insist on the precise definitions of these spaces which are rather technical. The results of Ludwig have been improved in ([6]) and ([1]), but even these improved statements crucially need some decay conditions on certain partial Fourier transforms.

These somehow unsatisfactory definitions of the function spaces have motivated Leptin ([5]) for further investigations. In his work he introduces special polarizations, called tame polarizations (zahme Polarisierungen) and pretends to prove that with this particular choice of the polarizations, every function in $\mathcal{C}_c^\infty(G/P, G/P, \chi_l)$ is the kernel of an operator of the form $\pi_l(f)$ for some $f \in L^1(G)$. As a matter of fact, his result even covers a slightly bigger function space whose definition doesn't require any condition on partial Fourier transforms. Unfortunately, Leptin's result is wrong, as can be seen by the counterexample we give in this paper. Hence the results of ([6]) and ([1]) are probably the best possible ones in the general situation.

However in the special case where the polarization $\mathfrak{p}$ is in fact an ideal in $\mathfrak{g}$, every function in $\mathcal{C}_c^\infty(G/P, G/P, \chi_l)$ is the kernel of an operator of the form $\pi_l(f)$ for some $f \in L^1(G)$. Moreover in that case a precise construction algorithm of the function f may be given, which is of course much more satisfactory than a simple proof of existence by induction.

1. Leptin's "result"

1.1. Let's first recall the construction of the unitary irreducible representations π_l for connected, simply connected, exponential solvable Lie groups $G = \exp \mathfrak{g}$, a construction which may for instance be found in ([2]). Let $l \in \mathfrak{g}^*$, $\mathfrak{p}$ a Pukanszky polarization for l, $P = \exp \mathfrak{p}$ and $\chi_l(p) = e^{-i\langle l, \log p\rangle}$ the associated character on P. Let dx and dp denote the (left) Haar measures on G and P, and Δ_G and Δ_P the corresponding modular functions. Hence

$$\Delta_G(x) = |\det \mathrm{Ad}(x)|^{-1} = e^{-\operatorname{tr} \ \mathrm{ad}_{\mathfrak{g}}(\log x)}.$$

Similarly for Δ_P. Let's define $\Delta_{P,G} : P \to \mathbb{R}_+$ by

$$\Delta_{P,G}(p) = \frac{\Delta_P(p)}{\Delta_G(p)} = e^{\operatorname{tr} \ \mathrm{ad}_{\mathfrak{g}/\mathfrak{p}}(\log p)}, \quad \forall p \in P.$$

Let's put $\Delta_{G,P} = \Delta_{P,G}^{-1}$ and consider the space

$$\begin{aligned} K(G,P) = \{f : G \to \mathbb{C} \mid\ & f \text{ continuous with compact support mod } P, \\ & f(xp) = \Delta_{P,G}(p) f(x),\ \forall x \in G,\ \forall p \in P\}. \end{aligned}$$

The group G acts by left translation on the space $K(G,P)$. There exists on $K(G,P)$ a unique (up to a constant) positive G-invariant linear functional noted by

$$\mu_{G,P}(F) = \oint_{G/P} F(x) d\dot{x}$$

and defined by

$$\oint_{G/P} F(x) d\dot{x} = \int_G f(x) dx$$

where f is any function in $\mathcal{C}_c(G)$ such that

$$F(x) = \int_P \Delta_{G,P}(p) f(xp) dp.$$

It is shown in ([2]) that $\oint_{G/P} F(x) d\dot{x}$ is well defined and that

$$\oint_{G/P} \Big(\int_P f(xp) \Delta_{G,P}(p) dp \Big) d\dot{x} = \int_G f(x) dx.$$

Let's now define $K_l(G,P)$ to be the space spanned by all the functions $\xi : G \mapsto \mathbb{C}$ which are continuous with compact support modulo P and which satisfy

$$\xi(xp) = \Delta_{P,G}^{\frac{1}{2}}(p) \overline{\chi_l}(p) \xi(x), \quad \forall x \in G, \forall p \in P.$$

This function space is equipped with the norm $\|\cdot\|_2$ defined by

$$\|\xi\|_2^2 = \oint_{G/P} |\xi(x)|^2 d\dot{x}.$$

The induced representation $\pi_l = \mathrm{ind}_P^G \chi_l$ is then defined by:
The space $\mathfrak{H}_{\pi_l}$ is the completion of $K_l(G,P)$ for the norm $\|\cdot\|_2$.

The action of G on $\mathfrak{H}_{\pi_l}$ is given by

$$\pi_l(x)\xi(y) = \xi(x^{-1}y), \quad \forall \xi \in \mathfrak{H}_{\pi_l}, \forall x, y \in G.$$

Let now $f \in L^1(G)$. The operator $\pi_l(f) = \int_G f(x)\pi_l(x)dx$ is the unique bounded operator on the Hilbert space $\mathfrak{H}_{\pi_l}$ such that

$$\langle \pi_l(f)\xi, \eta \rangle := \int_G \langle \pi_l(x)\xi, \eta \rangle f(x)dx$$

for all $\xi, \eta \in \mathfrak{H}_{\pi_l}$. It is a kernel operator of the form

$$\big(\pi_l(f)\xi\big)(x) = \oint_{G/P} \mathcal{K}_{\pi_l}(f)(x,y)\xi(y)d\dot{y}$$

with

$$\mathcal{K}_{\pi_l}(f)(x,y) = \int_P f(xpy^{-1})\Delta_G^{-1}(y)\Delta_{P,G}^{-\frac{1}{2}}(p)\chi_l(p)dp.$$

The kernel $\mathcal{K}_{\pi_l}(f)$ satisfies the following covariance condition:

$$\mathcal{K}_{\pi_l}(f)(xp, yp') = \Delta_{P,G}^{\frac{1}{2}}(p)\Delta_{P,G}^{\frac{1}{2}}(p')\overline{\chi_l(p)}\chi_l(p')\mathcal{K}_{\pi_l}(f)(x,y),$$

for almost all $(x,y) \in G \times G$, for all $(p,p') \in P \times P$.

Remarks 1.2. **a)** The absolute convergence of the previous integrals may be justified by the following arguments: Let's consider the left regular representation $\lambda_{G/P}$ of $L^1(G)$ on $L^2(G/P)$. For all $f \in L^1(G)$ and all $\xi, \eta \in L^2(G/P)$,

$$\begin{aligned}\oint_{G/P}\Big(\int_G |f(g)|\ |\xi(g^{-1}x)|\ |\overline{\eta(x)}|dg\Big)d\dot{x} &= \langle \lambda_{G/P}(|f|)|\xi|, |\eta| \rangle \\ &\leq \|f\|_1\ \|\xi\|_2\ \|\eta\|_2 \\ &< +\infty.\end{aligned}$$

Hence, by the analogue of Fubini's theorem (see [2]), $\int_G |f(g)|\ |\xi(g^{-1}x)|dg$ exists for almost all $x \in G/P$ and may be computed in the following manner:

$$\begin{aligned}\int_G |f(g)|\ |\xi(g^{-1}x)|dg &= \int_G |f(xg^{-1})|\ |\xi(g)|\Delta_G(g^{-1})dg \\ &= \oint_{G/P}\Big(\int_P |f(xp^{-1}y^{-1})|\ |\xi(yp)| \\ &\qquad \Delta_G(p^{-1})\Delta_G(y^{-1})\Delta_{G,P}(p)dp\Big)d\dot{y} \\ &= \oint_{G/P}\Big(\int_P |f(xpy^{-1})|\Delta_{P,G}^{-\frac{1}{2}}(p)\Delta_G^{-1}(y)|\chi_l(p)|dp\Big)|\xi(y)|d\dot{y}.\end{aligned}$$

So, for almost all $(x,y) \in G \times G$,

$$\mathcal{K}_{\pi_l}(f)(x,y) = \int_P f(xpy^{-1})\Delta_G^{-1}(y)\Delta_{P,G}^{-\frac{1}{2}}(p)\chi_l(p)dp$$

is an absolutely convergent integral and

$$\big(\pi_l(f)\xi\big)(x) = \oint_{G/P} \mathcal{K}_{\pi_l}(f)(x,y)\xi(y)d\dot{y}$$

is also given by an absolutely convergent integral.
b) If $\Delta_G|_P = \Delta_P$, then $\Delta_{G,P} \equiv 1$, the linear functional $\mu_{G,P}$ coincides with a G-invariant measure on G/P noted $\int_{G/P} f(x)d\dot{x}$ and the kernel $\mathcal{K}_{\pi_l}(f)$ is given by

$$\mathcal{K}_{\pi_l}(f)(x,y) = \int_P f(xpy^{-1})\Delta_G^{-1}(y)\chi_l(p)dp.$$

This is in particular the case when the polarization $\mathfrak{p}$ is an ideal in $\mathfrak{g}$.
c) There exists a slightly different way to introduce the induced representations, using a relatively G-invariant measure on G/P (see for instance [6]). Both definitions give unitarily equivalent representations of G. If $\Delta_G|_P = \Delta_P$, both definitions coincide.

1.3. Leptin ([5]) gives the following definition: A polarization $\mathfrak{p}$ of $l \in \mathfrak{g}^*$ is called *tame* if the following conditions are satisfied:
(i) There exists a Jordan-Hölder sequence $\mathcal{S} = \{\mathfrak{g}_i\}_{1\le i\le n}$ of $\mathfrak{g}$ containing the maximal nilpotent ideal $\mathfrak{n}$ of $\mathfrak{g}$ and containing an ideal $\mathfrak{h}$ with the following properties:

$$(\alpha)\ \ \mathfrak{g} = \mathfrak{g}(l) + \mathfrak{h}, \quad (\beta)\ \ \mathfrak{h} \cap (\mathfrak{g}(l) + \mathfrak{n}) = \mathfrak{n}$$

where $\mathfrak{g}(l) = \{U \in \mathfrak{g} \mid \langle l, [U,V]\rangle = 0\ \forall V \in \mathfrak{g}\}$.
(ii) $\mathfrak{p}$ is the Vergne polarization associated to $\mathcal{S}$, i.e.

$$\mathfrak{p} = \sum_{i=1}^{n} \mathfrak{g}_i(l_i)$$

with $l_i = l|_{\mathfrak{g}_i}$ and $\mathfrak{g}_i(l_i) = \{U \in \mathfrak{g}_i \mid \langle l_i, [U,V]\rangle = 0\ \forall V \in \mathfrak{g}_i\}$.
(iii)

$$\mathrm{tr}\ \mathrm{ad}_{\mathfrak{h}/\mathfrak{h}\cap\mathfrak{p}} X = 0, \quad \forall X \in \mathfrak{h} \cap \mathfrak{p}.$$

Leptin shows that tame polarizations always exist.

1.4. Let's now define the space $\mathcal{C}_c^\infty(G/P, G/P, \chi_l)$ to be the space of all continuous functions F from $G \times G$ to $\mathbb{C}$ satisfying:
a) $F(xp, yp') = \Delta_{P,G}^{\frac{1}{2}}(p)\Delta_{P,G}^{\frac{1}{2}}(p')\overline{\chi_l(p)}\chi_l(p')F(x,y), \quad \forall x, y \in G, \forall p, p' \in P$
b) If one identifies G/P with $\mathbb{R}^k$ thanks to any fixed coexponential basis to $\mathfrak{p}$ in $\mathfrak{g}$, then F is a smooth function with compact support in $\mathbb{R}^k \times \mathbb{R}^k$.
Leptin then pretends among others to prove the following result:

Let $G = \exp\mathfrak{g}$ be a connected, simply connected, exponential solvable Lie group. Let $l \in \mathfrak{g}^*$ and $\mathfrak{p}$ an arbitrary tame polarization for l. Then, for every $F \in \mathcal{C}_c^\infty(G/P, G/P, \chi_l)$ there exists $f \in L^1(G)$ such that $\mathcal{K}_{\pi_l}(f) = F$, i.e. such that

$$\pi_l(f)\xi(x) = \oint_{G/P} F(x,y)\xi(y)d\dot{y}.$$

As a matter of fact, Leptin proposes a proof for a slightly bigger class of functions.

1.5. Unfortunately, the result of Leptin is incorrect. This is shown by the counterexample of section 2.

2. A counterexample

2.1. Let $\mathfrak{g} = \langle X_1, X_2, X_3, X_4, X_5, X_6, X_7, X_8, X_9 \rangle$ be the Lie algebra whose generators satisfy the following non trivial relations:

$$\begin{array}{lll} [X_1, X_7] = -X_7 & [X_2, X_3] = -X_3 & [X_2, X_5] = X_5 \\ [X_1, X_8] = X_8 & [X_2, X_4] = X_4 & [X_2, X_6] = -X_6 \\ [X_1, X_2] = X_9 & [X_3, X_5] = X_9 & [X_4, X_6] = X_9 \end{array}$$

Let's give some comments on the construction of this Lie algebra: The presence of X_7, X_8 is not necessary for the following main argument. These generators are just introduced to get a tame polarization in the sense of Leptin. Forgetting about X_7, X_8, one sees that the algebra is obtained by acting with X_2 on the 5-dimensional Heisenberg algebra $\langle X_3, X_4, X_5, X_6, X_9 \rangle$ in the manner of the Boidol group. Finally the presence of X_1 produces another Heisenberg algebra $\langle X_1, X_2, X_9 \rangle$. The Jordan-Hölder basis $\mathcal{B} = \{X_1, \ldots, X_9\}$ defines the corresponding Jordan-Hölder sequence

$$\mathfrak{g}_{10} = \{0\} \subset \mathfrak{g}_9 = \langle X_9 \rangle \subset \cdots \subset \mathfrak{g}_i = \langle X_i, \ldots, X_9 \rangle \subset \cdots \subset \mathfrak{g}_1 = \mathfrak{g}.$$

The maximal nilpotent ideal $\mathfrak{n}$ is given by

$$\mathfrak{n} = [\mathfrak{g}, \mathfrak{g}] = \langle X_3, X_4, X_5, X_6, X_7, X_8, X_9 \rangle = \mathfrak{g}_3.$$

The corresponding Lie group $G = \exp \mathfrak{g}$ is connected, simply connected, exponential, completely solvable and unimodular.

2.2. Let now $l = X_9^* \in \mathfrak{g}^*$. The corresponding Vergne polarization is

$$\mathfrak{p}(l) = \sum_{i=1}^{10} \mathfrak{g}_i(l_i) = \langle X_2, X_5, X_6, X_7, X_8, X_9 \rangle$$

and the stabilizer equals

$$\mathfrak{g}(l) = \langle X_7, X_8, X_9 \rangle.$$

If we put $\mathfrak{h} := \mathfrak{g}$, then condition (i) of the definition of a tame polarization is satisfied. Moreover, if $P(l) = \exp \mathfrak{p}(l)$, $\Delta_G|_{P(l)} = \Delta_{P(l)} = 1$, $\Delta_G = 1$ and

$$\operatorname{tr}\ \operatorname{ad}_{\mathfrak{h}/\mathfrak{h}\cap\mathfrak{p}(l)} X = \operatorname{tr}\ \operatorname{ad}_{\mathfrak{g}/\mathfrak{p}(l)} X = 0, \quad \forall X \in \mathfrak{h} \cap \mathfrak{p}(l) = \mathfrak{p}(l).$$

This proves condition (iii) of the definition of a tame polarization. Hence $\mathfrak{p} := \mathfrak{p}(l)$ is a tame polarization associated to l.

2.3. Thanks to the given basis, the functions on G may be identified with functions on $\mathbb{R}^9$ by

$$\tilde{f}(x_1, x_2, \ldots, x_9) := f(\exp x_1 X_1 \cdot \exp x_2 X_2 \cdots \exp x_9 X_9)$$

and functions g on G/P may be identified with functions $\tilde{g}$ on $\mathbb{R}^3$ by

$$\tilde{g}(x_1, x_3, x_4) := g(\exp x_1 X_1 \cdot \exp x_3 X_3 \cdot \exp x_4 X_4).$$

Similarly for functions on $G/P \times G/P$. Finally, as $\Delta_G|_P = \Delta_P (= 1)$, the induced representation $\pi_l = \mathrm{ind}_P^G \chi_l$ may be realized on $L^2(\mathbb{R}^3)$ in the following way: For $\xi \in L^2(\mathbb{R}^3)$ and $(t_1, t_2, \ldots, t_9) \equiv \exp t_1 X_1 \cdot \exp t_2 X_2 \cdots \exp t_9 X_9 \in G$,

$$\begin{aligned} \pi(t_1, t_2, \ldots, t_9)\xi(x_1, x_3, x_4) = e^{-it_9} e^{i(x_1 - t_1)t_2} e^{i(e^{t_2} x_3 - t_3)t_5} e^{i(e^{-t_2} x_4 - t_4)t_6} \\ \cdot \xi(x_1 - t_1, x_3 e^{t_2} - t_3, x_4 e^{-t_2} - t_4). \end{aligned}$$

For every $f \in L^1(G)$, $\pi_l(f)$ is a kernel operator whose kernel is given by the formula

$$\begin{aligned} \mathcal{K}_{\pi_l}(f)(x_1, x_3, x_4; y_1, y_3, y_4) &= \int_P f\big((\exp x_1 X_1 \exp x_3 X_3 \exp x_4 X_4) p \\ &\qquad \cdot (\exp y_1 Y_1 \exp y_3 Y_3 \exp y_4 Y_4)^{-1}\big)\chi_l(p) dp \\ &= \int_{\mathbb{R}^6} \tilde{f}\big((x_1, x_3, x_4) \cdot (t_2, t_5, t_6, t_6, t_7, t_8, t_9) \\ &\qquad \cdot (y_1, y_3, y_4)^{-1}\big) e^{-it_9} dt_9 dt_8 dt_7 dt_6 dt_5 dt_2 \\ &= \int_{\mathbb{R}^6} \tilde{f}(x_1 - y_1, t_2, e^{t_2} x_3 - y_3, e^{-t_2} x_4 - y_4, t_5, t_6, \\ &\qquad e^{-y_1} t_7, e^{y_1} t_8, t_9 + y_1 t_2 + y_4 t_6 + y_3 t_5) \\ &\qquad \cdot e^{-it_9} dt_9 dt_8 dt_7 dt_6 dt_5 dt_2. \end{aligned}$$

Let's now write $\mathcal{F}_{i_1, i_2, \ldots, i_k} \tilde{f}$ for the partial Fourier transform of the function $\tilde{f}$ in the directions $i_1, i_2, \ldots, i_k$. Then the kernel of the operator $\pi_l(f)$ may be written as

$$\begin{aligned} \mathcal{K}_{\pi_l}(f)(x_1, x_3, x_4; y_1, y_3, y_4) = \int_{\mathbb{R}} \mathcal{F}_{9,8,7,6,5} \tilde{f}(x_1 - y_1, t_2, e^{t_2} x_3 - y_3, \\ e^{-t_2} x_4 - y_4, -y_3, -y_4, 0, 0, 1) e^{i y_1 t_2} dt_2. \end{aligned}$$

2.4. Let $0 \not\equiv \phi \in \mathcal{C}_c^\infty(\mathbb{R})$ be chosen such that

$$\begin{cases} \mathrm{supp}\phi \subset [-\frac{\pi}{4}, \frac{\pi}{4}], \\ \phi \geq 0 \\ \phi(-x) = \phi(x) \ \forall x \in \mathbb{R}, \\ \frac{1}{2\pi} \int_{\mathbb{R}} \hat{\phi}(u) du = \phi(0) = 1 \end{cases}$$

and let's define $F := \otimes^6 \phi \in \mathcal{C}_c^\infty(\mathbb{R}^6) \equiv \mathcal{C}_c^\infty(G/P, G/P, \chi_l)$. We shall show that F cannot be the kernel of an operator of the form $\pi_l(f)$, $f \in L^1(G)$. To do this, let's assume that there is $f \in L^1(G)$ such that $\mathcal{K}_{\pi_l}(f) = F$. Hence, through change of variables and inverse Fourier transforms, we get

$$\mathcal{F}_{9,8,7,6,5}\tilde{f}(x_1, y_1, x_2, y_2, y_3, y_4, 0, 0, 1) = \\ \frac{1}{2\pi} \int_{\mathbb{R}} F(x_1 + s, e^{-y_1}(x_2 - y_3), e^{y_1}(y_2 - y_4), s, -y_3, -y_4) e^{-isy_1} ds.$$

As $\tilde{f} \in L^1(\mathbb{R}^9)$, the function

$$g : (x_1, y_1, x_2, y_2, y_3, y_4) \mapsto \mathcal{F}_{9,8,7}\tilde{f}(x_1, y_1, x_2, y_2, y_3, y_4, 0, 0, 1)$$

belongs to $L^1(\mathbb{R}^6)$. As ϕ is even, the function g is given by the formula

$$\begin{aligned} g(x_1, y_1, x_2, y_2, y_3, y_4) &= \mathcal{F}_{9,8,7}\tilde{f}(x_1, y_1, x_2, y_2, y_3, y_4, 0, 0, 1) \\ &= \big(\frac{1}{2\pi}\big)^3 \int_{\mathbb{R}^3} F(x_1 + s, e^{-y_1}(x_2 - r), e^{y_1}(y_2 - t), s, -r, -t) \\ &\qquad \cdot e^{-isy_1} e^{iry_3} e^{ity_4} ds dr dt \\ &= \big(\frac{1}{2\pi}\big)^3 \int_{\mathbb{R}} \phi(x_1 + s)\phi(s) e^{-isy_1} ds \int_{\mathbb{R}} \phi(e^{-y_1}(x_2 - r)) \\ &\qquad \cdot \phi(r) e^{iry_3} dr \int_{\mathbb{R}} \phi(e^{y_1}(y_2 - t))\phi(t) e^{ity_4} dt. \end{aligned}$$

Let

$$E := \{(x_1, x_2, x_3, x_4, x_5, x_6) \in \mathbb{R}^6 \mid |x_6| \leq e^{x_2}\}$$

and let $\mathbf{1}_E$ be its characteristic function. As $g \in L^1(\mathbb{R}^6)$, we also have $h := (2\pi)^3 |g| \times \mathbf{1}_E \in L^1(\mathbb{R}^6)$ and

$$h(x_1, y_1, x_2, y_2, y_3, y_4) = \\ \Big|\int_{\mathbb{R}} \phi(x_1 + s)\phi(s) e^{-isy_1} ds\Big| \times \Big|\int_{\mathbb{R}} \phi(e^{-y_1}(x_2 - r))\phi(r) e^{iry_3} dr\Big| \times \alpha,$$

where

$$\begin{aligned} \alpha &= \mathbf{1}_E(x_1, y_1, x_2, y_2, y_3, y_4) \Big|\int_{\mathbb{R}} \phi(e^{y_1}(y_2 - t))\phi(t) e^{-i(y_2 - t)y_4} e^{iy_2 y_4} dt\Big| \\ &\geq \mathbf{1}_E(x_1, y_1, x_2, y_2, y_3, y_4) \Big|\, \mathfrak{Re}\Big(\int_{\mathbb{R}} \phi(e^{y_1}(y_2 - t))\phi(t) e^{-i(y_2 - t)y_4} dt\Big)\Big| \\ &= \mathbf{1}_{\{|y_4| \leq e^{y_1}\}}(x_1, y_1, x_2, y_2, y_3, y_4) \Big| \int_{\{t \in \mathbb{R} \ \mid \ |e^{y_1}(y_2 - t)| \leq \frac{\pi}{4}\}} \phi(e^{y_1}(y_2 - t))\phi(t) \\ &\qquad \cdot \cos(y_4(y_2 - t)) dt \Big|. \end{aligned}$$

But $|y_4| \leq e^{y_1}$ and $|e^{y_1}(y_2 - t)| \leq \frac{\pi}{4}$ implies that $|y_4(y_2 - t)| \leq \frac{\pi}{4}$ and $\cos(y_4(y_2 - t)) \geq \frac{1}{\sqrt{2}}$. Hence

$$\alpha \geq \frac{1}{\sqrt{2}} \mathbf{1}_E(x_1, y_1, x_2, y_2, y_3, y_4) \int_{\mathbb{R}} \phi(e^{y_1}(y_2 - t))\phi(t) dt.$$

Let's finally define $k \in L^1(\mathbb{R}^2)$ by the formula

$$k(x_1, y_1) = \int_{\mathbb{R}^4} h(x_1, y_1, x_2, y_2, y_3, y_4) dx_2 dy_2 dy_3 dy_4, \quad \forall (x_1, y_1) \in \mathbb{R}^2.$$

The previous computations show that

$$k(x_1, y_1) \geq |\int_{\mathbb{R}} \phi(x_1 + s)\phi(s) e^{-isy_1} ds| \times \beta \times \gamma$$

where

$$\beta = \int_{\mathbb{R}^2} |\int_{\mathbb{R}} \phi(e^{-y_1}(x_2 - r))\phi(r) e^{iry_3} dr| dx_2 dy_3,$$
$$\gamma = \frac{1}{\sqrt{2}} \int_{\{|y_4| \leq e^{y_1}\}} \Big(\int_{\mathbb{R}} \big(\int_{\mathbb{R}} \phi(e^{y_1}(y_2 - t))\phi(t) dt \big) dy_2 \Big) dy_4.$$

If we put $c := \int_{\mathbb{R}} \phi(t) dt > 0$, we get

$$\gamma = \frac{1}{\sqrt{2}} 2e^{y_1} \int_{\mathbb{R}} \big(\int_{\mathbb{R}} \phi(e^{y_1} y_2) dy_2 \big) \phi(t) dt = \sqrt{2} c^2$$

and

$$\beta \geq |\int_{\mathbb{R}^2} \int_{\mathbb{R}} \phi(e^{-y_1}(x_2 - r)) dx_2 \phi(r) e^{iry_3} dr dy_3| = ce^{y_1} |\int_{\mathbb{R}} \hat{\phi}(-y_3) dy_3| = 2\pi c e^{y_1},$$

as $\phi(0) = 1$. Hence

$$k(x_1, y_1) \geq 2\sqrt{2}\pi c^3 e^{y_1} |\int_{\mathbb{R}} \phi(x_1 + s)\phi(s) e^{-isy_1} ds|.$$

This implies that the function

$$(x_1, y_1) \mapsto e^{y_1} \int_{\mathbb{R}} \phi(x_1 + s)\phi(s) e^{-isy_1} ds$$

would belong to $L^1(\mathbb{R}^2)$ and that the function

$$y_1 \mapsto e^{y_1} \hat{\phi}(y_1)$$

would belong to $L^1(\mathbb{R})$ as

$$e^{y_1} \int_{\mathbb{R}} \big(\int_{\mathbb{R}} \phi(x_1 + s)\phi(s) e^{-isy_1} ds \big) dx_1 = e^{y_1} c \hat{\phi}(y_1).$$

Finally, as the function ϕ is even, the same is true for the function $\hat{\phi}$ and the map

$$y_1 \mapsto e^{|y_1|} \hat{\phi}(y_1)$$

would belong to $L^1(\mathbb{R})$. But this contradicts the following lemma and hence there doesn't exist any function $f \in L^1(G)$ such that the operator $\pi(f)$ admits $F = \otimes^6 \phi$ as a kernel.

Lemma . *Let $\psi \in \mathcal{C}_c^\infty(\mathbb{R})$. Then the function*

$$\Psi : t \mapsto e^{|t|}\hat{\psi}(t)$$

is in $L^1(\mathbb{R})$ if and only if $\psi \equiv 0$.

Proof. Let's assume $\Psi \in L^1(\mathbb{R})$. Hence the integral $\frac{1}{2\pi}\int_{\mathbb{R}} e^{izs}\hat{\psi}(s)ds$ is absolutely convergent for every $z = a + ib$ such that $|b| < 1$ and defines a holomorphic extension $\tilde{\psi}$ of ψ to $\mathbb{R}+i]-1,1[$. As the function $\tilde{\psi}$ is holomorphic and coincides with the $\mathcal{C}_c^\infty$-function ψ on $\mathbb{R}$, necessarily $\psi \equiv 0$. ∎

2.5. In this section we have hence constructed a function $F = \otimes^6\phi \in \mathcal{C}_c^\infty(\mathbb{R}^6) \equiv \mathcal{C}_c^\infty(G/P, G/P, \chi_l)$ which cannot be the kernel of an operator of the form $\pi_l(f)$ for some $f \in L^1(G)$, even though the polarization $\mathfrak{p}$ is tame. This contradicts the result of Leptin. In the next section we shall prove Leptin's result under the stronger assumption that the polarization $\mathfrak{p}$ is in fact an ideal of $\mathfrak{g}$. In this case the function f will effectively be constructed.

3. The ideal case

3.1. Some generalities: In this section $G = \exp\mathfrak{g}$ will always denote a connected, simply connected, exponential solvable Lie group and $l \in \mathfrak{g}^*$ will be fixed. We shall assume that $\mathfrak{p} \subset \mathfrak{g}$ is a Pukanszky polarization for l in $\mathfrak{g}$ that is also an ideal of $\mathfrak{g}$, i.e. that satisfies $[\mathfrak{g}, \mathfrak{p}] \subset \mathfrak{p}$. The following observations are trivial, respectively well known:

a) The algebra $\mathfrak{g}$ and the group G act on $\mathfrak{p}$, on $\mathfrak{p}^*$. Hence $\mathrm{tr}\ \mathrm{ad}_{\mathfrak{p}}(\log y)$ and $\delta(y) = e^{\mathrm{tr}\ \mathrm{ad}_{\mathfrak{g}}(\log y)} \cdot e^{-\mathrm{tr}\ \mathrm{ad}_{\mathfrak{p}}(\log y)}$ make sense for every $y \in G$.

b) Let $\Omega(l|_{\mathfrak{p}})$ denote the orbit of the action of G on $\mathfrak{p}^*$. As this action is exponential, the description of the orbit ([2], [8]) shows that $\Omega(l|_{\mathfrak{p}})$ is a submanifold of $\mathfrak{p}^*$, locally closed, regularly embedded in $\mathfrak{p}^*$. In particular, every $\mathcal{C}^\infty$-function defined on $\Omega(l|_{\mathfrak{p}})$ may locally be extended to a $\mathcal{C}^\infty$-function on an open subset of $\mathfrak{p}^*$. Moreover, using a partition of the unity, it is possible to glue together local extensions and hence to extend every $\mathcal{C}^\infty$-function with compact support in $\Omega(l|_{\mathfrak{p}})$ to a function belonging to $\mathcal{C}_c^\infty(\mathfrak{p}^*)$ ([4]).

c) As $\mathfrak{p}$ is a polarization for l in $\mathfrak{g}$, the stabilizer

$$\mathfrak{g}(l|_{\mathfrak{p}}) := \{X \in \mathfrak{g} \mid \langle l, [X, \mathfrak{p}]\rangle \equiv 0\}$$

coincides with $\mathfrak{p}$. Hence there is a diffeomorphism between G/P and $\Omega(l|_{\mathfrak{p}})$. Of course, G/P may be replaced by a smooth section in G (defined for instance thanks to a fixed basis). In what follows, G/P will stand for such a section.

d) The operator kernel computed in (1.1.) may be transformed to

$$\mathcal{K}_{\pi_l}(f)(x, y) = \delta(y) \int_P f(xy^{-1}p)\chi_{\mathrm{Ad}^*(y)(l)}(p)dp.$$

3.2. The retract problem: a) The aim is to solve the following problem: Given a function $F \in \mathcal{C}_c^\infty(G/P, G/P, \chi_l)$, construct $f \in L^1(G)$ such that $\pi_l(f)$ admits F as an operator kernel.
b) Let's first solve the problem if $[\mathfrak{p}, \mathfrak{p}] = 0$, i. e. if $\mathfrak{p}$ is abelian. In that case the Haar measure on P coincides with the Lebesgue measure on $\mathfrak{p}$, which is a finite-dimensional real vector space. We shall identify P with $\mathfrak{p}$.
c) Given F, let's define $\tilde{f} : G \times \Omega(l|_\mathfrak{p}) \to \mathbb{C}$ by

$$\tilde{f}(x, \mathrm{Ad}^*(y)(l)|_\mathfrak{p}) := \delta(y)^{-1} F(xy, y).$$

The function $\tilde{f}$ satisfies the covariance relation

$$\tilde{f}(xp, \mathrm{Ad}^*(y)(l)|_\mathfrak{p}) = \overline{\chi_{\mathrm{Ad}^*(y)(l)}(p)}\, \tilde{f}(x, \mathrm{Ad}^*(y)(l)|_\mathfrak{p})$$

and may hence be identified with a function in $\mathcal{C}_c^\infty(G/P \times \Omega(l|_\mathfrak{p}))$.
d) As F has compact support in $G/P \times G/P$, $\tilde{f}$ has compact support in $G/P \times \Omega(l|_\mathfrak{p})$. By (3.1.b)), there exists a function $g : \mathcal{C}_c^\infty(G/P \times \mathfrak{p}^*) \to \mathbb{C}$ such that $g|_{G/P \times \Omega(l|_\mathfrak{p})} = \tilde{f}$. The function g may be extended to a $\mathcal{C}^\infty$-function on all of $G \times \mathfrak{p}^*$ by using the covariance relation $g(xp, q) := \overline{\chi_q(p)} g(x, q)$.
e) We then define a function $f : G \to \mathbb{C}$ by the Fourier inversion formula

$$f(xp) := \Big(\frac{1}{2\pi}\Big)^n \int_{\mathfrak{p}^*} g(x, q) e^{i\langle q, \log p\rangle} dq$$

with $x \in G$, $p \in P \equiv \mathfrak{p}$. If we restrict to $x \in G/P$, we thus obtain a smooth L^1-function, as $g \in \mathcal{C}_c^\infty(G/P \times \mathfrak{p}^*)$. Moreover, the function f is well defined, as one may check that $f((xp_1)p_2) = f(x(p_1 p_2))$ for all $x \in G$, $p_1, p_2 \in P$. Using the Fourier inversion theorem, it is easy to show that the kernel $\mathcal{K}_{\pi_l}(f)$ of the operator $\pi_l(f)$ is the function F. This solves the problem if $\mathfrak{p}$ is abelian.
f) Let's now return to the general case where $\mathfrak{p}$ is not necessarily abelian. We factorize through $[\mathfrak{p}, \mathfrak{p}]$, resp. $[P, P]$ in order to reduce the problem to a question on $G_1 = G/[P, P]$, its Lie algebra $\mathfrak{g}_1 = \mathfrak{g}/[\mathfrak{p}, \mathfrak{p}]$, the linear form l_1 on $\mathfrak{g}_1$ defined by $\langle l_1, X + [\mathfrak{p}, \mathfrak{p}]\rangle := \langle l, X\rangle$, the corresponding polarization $\mathfrak{p}_1 = \mathfrak{p}/[\mathfrak{p}, \mathfrak{p}]$. This polarization is an abelian ideal of $\mathfrak{g}_1$, $G_1/P_1 \simeq G/P$ and the kernel function F may also be considered as an element of $\mathcal{C}_c^\infty(G_1/P_1, G_1/P_1, \chi_{l_1})$. So the previous construction gives a function $f_1 \in L^1(G_1) = L^1(G/[P, P])$, solution to the problem in G_1. Finally we notice that the map $\Phi : L^1(G) \to L^1(G/[P, P])$ defined by $\Phi(f)(u) := \int_{[P,P]} f(up)dp$, is a surjection. So any $f \in L^1(G)$ such that $\Phi(f) = f_1$ is such that $\pi_l(f)$ admits F as an operator kernel.

3.3. We have thus proven the following theorem:

Theorem . *Let $G = \exp \mathfrak{g}$ be a connected, simply connected, exponential solvable Lie group. Let $l \in \mathfrak{g}^*$ and assume that l admits a Pukanszky polarization $\mathfrak{p}$ which is an ideal in $\mathfrak{g}$. Let $\pi_l = \mathrm{ind}_P^G \chi_l$ be the corresponding irreducible unitary representation. Then, for every $F \in \mathcal{C}_c^\infty(G/P, G/P, \chi_l)$ there exists $f \in L^1(G)$ such that the operator $\pi_l(f)$ admits the function F as kernel.*

3.4. Examples. a) There is not always a polarization that is an ideal. Let $G = \exp \mathfrak{g}$ with $\mathfrak{g} = \langle T, X, Y, Z\rangle$ where

$$[T, X] = -X \quad [T, Y] = Y \quad [X, Y] = Z$$

be the Boidol group. Let $l = Z^*$. Then l does not admit a polarization which is an ideal.

b) Let $G = \exp \mathfrak{g}$ be the group of the counterexample studied in section 2. Let $l = X_9^*$. Let $\mathfrak{p} = \langle X_1, X_3, X_4, X_7, X_8, X_9\rangle$. Then $\mathfrak{p}$ is a Pukanszky polarization of l which is an ideal in $\mathfrak{g}$. Let $F \in \mathcal{C}_c^\infty(\mathbb{R}^6) \equiv \mathcal{C}_c^\infty(G/P, G/P, \chi_l)$. Let $\psi \in \mathcal{C}_c^\infty(\mathbb{R})$ such that $\psi(0) = 1$ be arbitrary. Let's define a function TF by

$$\begin{aligned} TF(x_1, \ldots, x_9) = F(x_1, &e^{x_2 - x_1} x_3, e^{x_1 - x_2} x_4, x_1 - x_2, \\ &e^{x_2 - x_1}(x_3 - x_5), e^{x_1 - x_2}(x_4 - x_6))\psi(x_7)\psi(x_8)\psi(x_9 - 1). \end{aligned}$$

Let's define

$$f = (\mathcal{F}_{9,8,7,4,3,1})^{-1}(TF) \in L^1(G).$$

It is then easy to check that the kernel of the operator $\pi_l(f)$ is equal to F.

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