

## Symmetry of Arthur Parameters under Aubert Involution

Dubravka Ban\*

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**Abstract.** For a generic irreducible representation  $\pi$  of the odd orthogonal group  $SO(2n+1, F)$  over a  $p$ -adic field  $F$ , we compute the Aubert involution  $\hat{\pi}$  and the corresponding  $L$ -parameter. We show that, among generic representations, only tempered representations are base points attached to  $A$ -parameters and prove that in this case the  $A$ -parameters of  $\pi$  and  $\hat{\pi}$  are symmetric. In addition, we consider  $A$ -parameters  $\psi$  of  $SO(2n+1, F)$  corresponding to certain nontempered representations and prove that  $\psi$  and  $\hat{\psi}$  are symmetric.

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### 1. Introduction

This paper studies the effects of the duality operator on generic representations of  $SO(2n+1, F)$  and corresponding  $L$ -parameters and  $A$ -parameters. It also deals with classes of nontempered representations arising from considerations of  $A$ -parameters of a certain type (see Theorem 3.1 for more details). In accordance with Arthur's conjectures [1, 2], attached to each  $A$ -parameter is a finite set of equivalence classes of irreducible admissible representations, called an  $A$ -packet. There is, however, a natural way to associate to each  $A$ -parameter a particular representation; we call it a base point. We study effects of the duality operator on  $A$ -parameters via base points. The proof relies on recent fundamental developments by Jiang-Soudry, Harris-Taylor and Henniart. It provides an interesting illustration of the Langlands-Arthur functoriality formalism. Recall that  $A$ -parameters and  $A$ -packets emerged from Arthur's work on the question of how nontempered representations should fit into the trace formula. There are very few examples of nontempered parameters for larger groups, where Arthur's formalism has been confirmed.

The duality operator is a generalization of the Zelevinsky involution. The Zelevinsky involution is an operator defined on the Grothendieck group of the category of all smooth finite length representations of the general linear group  $GL(n, F)$

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[32]. This involution has many important properties. It relates a discrete series representation to the corresponding Langlands quotient. The Zelevinsky involution on  $GL(n, F)$  preserves unitarity. Furthermore, its action on  $A$ -parameters can be precisely defined, as follows. Let

$$\psi : W_F \times SL(2, \mathbb{C}) \times SL(2, \mathbb{C}) \rightarrow GL(n, \mathbb{C})$$

be an  $A$ -parameter of  $GL(n, F)$ . Here,  $W_F$  denotes the Weil group of  $F$ . Let  $\pi$  be the representation of  $GL(n, F)$  associated to  $\psi$ . Denote by  $\hat{\pi}$  the Zelevinsky involution of  $\pi$  and by  $\hat{\psi}$  the  $A$ -parameter of  $\hat{\pi}$ . Then [32, 23, 29],

$$\hat{\psi}(w, x, y) = \psi(w, y, x). \quad (1)$$

In other words, the Zelevinsky involution acts on  $A$ -parameters by interchanging two copies of  $SL(2, \mathbb{C})$ . We say  $\psi$  and  $\hat{\psi}$  are symmetric.

The Zelevinsky involution allows generalizations to a connected reductive quasi-split algebraic group  $G$  defined over  $F$ . Aubert [3], Schneider and Stuhler [26], and Bernstein [8] have defined duality operators on the category of all smooth finite length representations of  $G$  and on its Grothendieck group. The involutions defined by Aubert and Schneider-Stuhler are the same on irreducible smooth representations, after having fixed the sign of the Aubert duality operator in order to get a positive element in the Grothendieck group. The Bernstein involution differs by taking contragredient.

The duality operator sends an irreducible representation to an irreducible representation. Other questions, related to important properties of the Zelevinsky involution, are still open. Barbasch conjectured that the duality operator sends an  $A$ -packet to an  $A$ -packet. If Barbasch's conjecture holds, we may consider the  $A$ -parameter associated to an  $A$ -packet and the  $A$ -parameter associated to the packet obtained by applying the duality operator on the original packet. This raises the question of the action of the involution on  $A$ -parameters. It is conjectured that, as for general linear groups, the involution acts on  $A$ -parameters of  $G$  by interchanging two copies of  $SL(2, \mathbb{C})$ . Although the conjecture was known previously, a precise statement is due to Hiraga [17]. In a joint work with Zhang [6], we proved that, for a generic discrete series representation  $\pi$  of  $SO(2n+1, F)$ , the  $A$ -parameters of  $\pi$  and  $\hat{\pi}$  are symmetric.

In this paper, we consider a generic representation  $\pi$  of  $SO(2n+1, F)$ . Let  $\phi$  be the  $L$ -parameter of  $\pi$  (defined by Jiang and Soudry in [19]). We compute the Aubert involution  $\hat{\pi}$  and the corresponding  $L$ -parameter (Theorem 5.3). Then we consider the  $A$ -parameters. We say that  $\psi$  is the  $A$ -parameter of  $\pi$  if  $\phi_\psi$  is the  $L$ -parameter of  $\pi$  (see section 3. for the definition). Not all generic representations have  $A$ -parameters in this sense. We show that, among generic representations, only tempered representations are attached to  $A$ -parameters (Theorem 5.4). In this case, we compare the  $A$ -parameters of  $\pi$  and  $\hat{\pi}$  and show that they are symmetric. This is a generalization of the work with Zhang [6] on generic discrete series representations. Symmetry of  $A$ -parameters has further consequences; for example, it implies that a generic tempered representation of a Levi subgroup of  $SO(2n+1, F)$  and its involution have the same  $R$ -group, as conjectured by Arthur (cf. [4, 5]).

We also consider certain classes of nontempered representations. Let  $\pi$  be the representation of  $SO(2\ell + 1, F)$  with the  $A$ -parameter

$$\psi = \phi \otimes S_k \otimes S_n \oplus \bigoplus_{i \in A} \phi_i \otimes S_1 \otimes S_1, \quad (2)$$

$k \geq 1$ ,  $n = 2, 3$ . (For precise definitions, see section 2. and Theorem 3.1). Then  $\pi$  is nontempered. Let  $\hat{\psi}$  be the  $A$ -parameter of  $\hat{\pi}$ . We prove that

$$\hat{\psi} = \phi \otimes S_n \otimes S_k \oplus \bigoplus_{i \in A} \phi_i \otimes S_1 \otimes S_1,$$

i.e.,  $\psi$  and  $\hat{\psi}$  are symmetric.

The base point associated to an  $L$ -parameter is determined from the work of Jiang and Soudry [19]. They deal with the groups  $SO(2n + 1, F)$  and in this paper we consider the same series of groups. In view of the recent work by Cogdell, Kim, Piatetski-Shapiro and Shahidi [12], we expect our methods can be applied to other series of classical  $p$ -adic groups.

We now give a short summary of the paper. In section 2., we recall some basic definitions. The  $A$ -parameters given by equation (2) are considered in section 3. We prove that  $\psi$  and  $\hat{\psi}$  are symmetric (Theorem 3.1). In section 4., we review Muić's classification of generic representations of  $SO(2n + 1, F)$ . In section 5., we study the effects of the duality operator on generic representations of  $SO(2n + 1, F)$  and corresponding  $L$ -parameters and  $A$ -parameters.

Let us mention that in the paper we are not assuming any conjecture. The conjectures described above are given to explain the motivation for the work done in this paper.

Before closing the introduction, I would like to thank the mathematicians who helped me during different stages of the project. I learned about the conjecture on involution and  $A$ -parameters from Anne-Marie Aubert and Peter Schneider in Luminy, 2002, and about the importance of the conjecture from James Arthur during the Clay Mathematics Institute Summer School at the Fields Institute, 2003. This paper has benefited from discussions with Dan Barbasch, Bob Fitzgerald, David Goldberg, Chris Jantzen, Gordan Savin and Freydoon Shahidi. I thank them all. Finally, I thank the referees for valuable comments.

## 2. Preliminaries

In this section, we recall some basic definitions. We consider the group  $G = SO(2n + 1, F)$  or  $G = GL(m, F)$  over a nonarchimedean local field  $F$  of characteristic zero. For both groups, we fix the Borel subgroup  $B \subset G$  consisting of all upper triangular matrices in  $G$  and the maximal torus  $T \subset B$  consisting of all diagonal matrices in  $G$ . Let  $\Delta$  denote the corresponding set of simple roots.

**Parabolic induction and segments** Let  $P$  be a standard parabolic subgroup of  $G$ , i.e., a parabolic subgroup containing  $B$ . Let  $M$  be the unique Levi subgroup of  $P$  containing  $T$ . We call such  $M$  a standard Levi subgroup of  $G$ . Denote by  $i_{G,M}$  the functor of parabolic induction and by  $r_{M,G}$  the Jacquet functor [9, 11].

For admissible representations  $\rho_i$  of  $GL(k_i, F)$ ,  $i = 1, 2$ , define

$$\rho_1 \times \rho_2 = i_{GL(k_1+k_2, F), GL(k_1, F) \times GL(k_2, F)}(\rho_1 \otimes \rho_2).$$

Similarly, if  $\rho$  is an admissible representation of  $GL(k, F)$  and  $\sigma$  admissible representation of  $SO(2\ell + 1, F)$ , define

$$\rho \rtimes \sigma = i_{SO(2(k+\ell)+1, F), GL(k, F) \times SO(2\ell+1, F)}(\rho \otimes \sigma).$$

Define  $\nu = |\det|$ . We say the pair  $(\rho, \sigma)$  satisfies  $(C\alpha)$  if  $\nu^{\pm\alpha}\rho \rtimes \sigma$  is reducible and  $\nu^\beta\rho \rtimes \sigma$  is irreducible for  $|\beta| \neq \alpha$ .

Let  $\rho$  be an irreducible supercuspidal representation of  $GL(k, F)$  and  $m \leq n$  integers. The set  $[\nu^m\rho, \nu^n\rho] = \{\nu^m\rho, \nu^{m+1}\rho, \dots, \nu^n\rho\}$  is called a segment [32]. The induced representation  $\nu^n\rho \times \nu^{n-1}\rho \times \dots \times \nu^m\rho$  has a unique irreducible subrepresentation, which we denote by  $\delta[\nu^m\rho, \nu^n\rho]$ , and a unique irreducible quotient, which we denote by  $\zeta[\nu^m\rho, \nu^n\rho]$ . A segment  $\Sigma$  is called balanced if it is of the form  $\Sigma = [\nu^{-m}\rho, \nu^m\rho]$ , with  $\rho$  unitary. The segment  $\Sigma$  is balanced if and only if  $\delta(\Sigma)$  is square integrable. In this paper, when we use the segment notation  $[\nu^m\rho, \nu^n\rho]$ , we always assume  $\rho$  is unitary.

Two segments  $\Sigma_1$  and  $\Sigma_2$  are said to be linked if  $\Sigma_1 \not\subseteq \Sigma_2$ ,  $\Sigma_2 \not\subseteq \Sigma_1$  and  $\Sigma_1 \cup \Sigma_2$  is a segment.

For a representation  $\sigma$ , we denote by  $\tilde{\sigma}$  the contragredient of  $\sigma$ .

**Aubert involution** Let  $R(G)$  be the Grothendieck group of the category of all smooth finite length representations of  $G$ . The Aubert duality operator  $D_G$  is defined on  $R(G)$  by

$$D_G = \sum_{\Phi \subset \Delta} (-1)^{|\Phi|} i_{G, M_\Phi} \circ r_{M_\Phi, G}$$

[3]. Here  $M_\Phi$  denotes the standard Levi subgroup corresponding to  $\Phi$ . If  $\pi$  is an irreducible admissible representation of  $G$ , we define  $\hat{\pi} = \pm D_G(\pi)$ , taking the sign  $+$  or  $-$  so that  $\hat{\pi}$  is a positive element in the Grothendieck group. We call  $\hat{\pi}$  the Aubert involution of  $\pi$ . It follows from [3] that  $\hat{\pi}$  is an irreducible representation.

**Langlands classification for  $SO(2n + 1, F)$**  Suppose  $\delta_i$  is a discrete series representation of  $GL(n_i, F)$ ,  $i = 1, \dots, k$  and  $\alpha_1 \geq \dots \geq \alpha_k > 0$  are real numbers. Let  $\tau$  be a tempered representation of  $SO(2\ell + 1, F)$ . Then the induced representation  $\nu^{\alpha_1}\delta_1 \times \dots \times \nu^{\alpha_k}\delta_k \rtimes \tau$  has a unique irreducible quotient, which we call the Langlands quotient and denote by  $L_q(\nu^{\alpha_1}\delta_1, \dots, \nu^{\alpha_k}\delta_k, \tau)$ . Equivalently, if  $\beta_1 \leq \dots \leq \beta_k < 0$ , then the induced representation  $\nu^{\beta_1}\delta_1 \times \dots \times \nu^{\beta_k}\delta_k \rtimes \tau$  has a unique irreducible subrepresentation, which we call the Langlands subrepresentation and denote by  $L_s(\nu^{\beta_1}\delta_1, \dots, \nu^{\beta_k}\delta_k, \tau)$ . The connection between the two classifications is given as follows: if  $\pi = L_q(\nu^{\alpha_1}\delta_1, \dots, \nu^{\alpha_k}\delta_k, \tau)$ , then  $\pi = L_s(\nu^{-\alpha_1}\tilde{\delta}_1, \dots, \nu^{-\alpha_k}\tilde{\delta}_k, \tau)$ . Note that we are allowed to work with square integrable representations  $\delta_i$  instead of tempered representations because of the irreducibility of induced-from-unitary representations of  $GL(m, F)$ . In particular, if  $\rho$  is a tempered representation of  $GL(m, F)$ , then  $\rho \cong \delta_1 \times \dots \times \delta_s$ , for some square integrable representations  $\delta_1, \dots, \delta_s$ .

**Irreducible representations of  $SL(2, \mathbb{C})$**  For each integer  $n \geq 1$  there exists up to equivalence a unique  $n$ -dimensional irreducible representation of  $SL(2, \mathbb{C})$ , and it can be described as follows. Let  $V = P_{n-1}^H[x, y]$  be the complex vector space of homogeneous polynomials of degree  $n - 1$  in variables  $x, y$ . Then  $SL(2, \mathbb{C})$  acts on  $V$  by change of variables. We denote this representation by  $S_n$ .

**Langlands parameters and base points** Let  $W_F$  be the Weil group of  $F$ . We take  $W_F \times SL(2, \mathbb{C})$  as the Weil-Deligne group [31, 21]. A Langlands parameter, or  $L$ -parameter, of  $SO(2n + 1, F)$  is a homomorphism

$$\phi : W_F \times SL(2, \mathbb{C}) \rightarrow Sp(2n, \mathbb{C})$$

such that  $\phi(W_F)$  consists of semi-simple elements in  $Sp(2n, \mathbb{C})$  and the restriction of  $\phi$  to  $SL(2, \mathbb{C})$  is analytic [10, 22, 21]. The parameter  $\phi$  is called tempered if the image  $\phi(W_F)$  is bounded. Two  $L$ -parameters are equivalent if they are conjugate in  $Sp(2n, \mathbb{C})$ . According to the Local Langlands Conjecture, each parameter  $\phi$  should parametrize a finite set of equivalence classes of irreducible admissible representations of  $SO(2n + 1, F)$ , called the  $L$ -packet of  $\phi$ . Jiang and Soudry in [19] defined a bijection

$$\phi \longleftrightarrow \pi = L_q(\nu^{\alpha_1} \delta_1, \dots, \nu^{\alpha_k} \delta_k, \tau), \quad \tau \text{ generic} \tag{3}$$

between the set of equivalence classes of  $L$ -parameters of  $SO(2n + 1, F)$  and the set of equivalence classes of irreducible admissible representations of the form  $\pi = L_q(\nu^{\alpha_1} \delta_1, \dots, \nu^{\alpha_k} \delta_k, \tau)$ , with  $\tau$  generic. The representation  $\pi$  is a member of the  $L$ -packet of  $\phi$  and plays an important role. We call it the base point representation in the  $L$ -packet of  $\phi$ .

Note that  $\phi$  is an arbitrary  $L$ -parameter of  $SO(2n + 1, F)$ , while the representation  $\pi$  is of specific type. If  $\pi$  is tempered, then  $\pi = \tau$  is generic. In general case,  $\pi$  is a representation such that the corresponding Langlands data are generic. Jiang and Soudry describe explicitly the bijection (3). For  $\pi$  generic, the description of  $\phi$  is based on Muić’s classification of irreducible generic representations of  $SO(2n + 1, F)$ . We will review the classification in section 4. The corresponding  $L$ -parameter is given in Theorem 5.3.

Now, we describe the  $L$ -parameter  $\phi$  associated to the representation  $\pi = L_q(\nu^{\alpha_1} \delta_1, \dots, \nu^{\alpha_k} \delta_k, \tau)$ . For  $i = 1, \dots, k$ , the representation  $\delta_i$  is of the form  $\delta_i = \delta(\Sigma_i)$ , where  $\Sigma_i$  is a balanced segment, so  $\nu^{\alpha_i} \delta_i = \delta[\nu^{c_i} \rho_i, \nu^{d_i} \rho_i]$ . We have

$$\begin{aligned} \pi &= L_q(\delta[\nu^{c_1} \rho_1, \nu^{d_1} \rho_1], \dots, \delta[\nu^{c_k} \rho_k, \nu^{d_k} \rho_k], \tau) \\ &= L_s(\delta[\nu^{-d_1} \tilde{\rho}_1, \nu^{-c_1} \tilde{\rho}_1], \dots, \delta[\nu^{-d_k} \tilde{\rho}_k, \nu^{-c_k} \tilde{\rho}_k], \tau). \end{aligned} \tag{4}$$

Let  $\varphi(\tau)$  denote the  $L$ -parameter of  $\tau$  and  $\phi_i$  the  $L$ -parameter of  $\rho_i$ . Then, by Theorem 6.1 and Proposition 6.1 of [19], the  $L$ -parameter of  $\pi$  is

$$\phi = \bigoplus_{i=1}^k (|\cdot|^{\frac{c_i+d_i}{2}} \phi_i \otimes S_{d_i-c_i+1} \oplus |\cdot|^{\frac{-c_i-d_i}{2}} \tilde{\phi}_i \otimes S_{d_i-c_i+1}) \oplus \varphi(\tau). \tag{5}$$

Observe that  $\frac{c_i+d_i}{2} = \alpha_i$  is positive.

### 3. Arthur parameters and Aubert involution

In this section, we first recall the definition and some properties of  $A$ -parameters. Then we consider a certain nontempered representation  $\pi$ . We compute its dual  $\hat{\pi}$ . We prove that the  $A$ -parameters  $\psi$  and  $\hat{\psi}$  corresponding to  $\pi$  and  $\hat{\pi}$  are symmetric (Theorem 3.1).

**Arthur parameters** An Arthur parameter, or  $A$ -parameter, of  $SO(2n + 1, F)$  is a homomorphism

$$\psi : W_F \times SL(2, \mathbb{C}) \times SL(2, \mathbb{C}) \rightarrow Sp(2n, \mathbb{C})$$

such that  $\psi(W_F)$  is bounded and included in the set of semi-simple elements of  $Sp(2n, \mathbb{C})$  and the restriction of  $\psi$  to the two copies of  $SL(2, \mathbb{C})$  is analytic [1, 2, 20]. In accordance with Arthur's conjectures, attached to each  $A$ -parameter  $\psi$  is a finite set of equivalence classes of irreducible admissible representations, called the  $A$ -packet of  $\psi$ . To any  $A$ -parameter  $\psi$ , Arthur associates an  $L$ -parameter  $\phi_\psi$  by

$$\phi_\psi(w, x) = \psi(w, x, \begin{pmatrix} |w|^{1/2} & \\ & |w|^{-1/2} \end{pmatrix}).$$

We say that  $\pi$  is the base point attached to  $\psi$  if  $\pi$  is the base point attached to  $\phi_\psi$  (see page 255). Contrary to  $L$ -packets,  $A$ -packets need not to be disjoint. A representation  $\pi$  may occur in more than one  $A$ -packet. An  $A$ -parameter  $\psi$  is called the  $A$ -parameter of  $\pi$  if  $\phi_\psi$  is the  $L$ -parameter of  $\pi$ . This definition is justified by noticing that  $\psi \mapsto \phi_\psi$  is injective [2]. If  $\psi$  is an  $A$ -parameter, we may decompose it into a direct sum

$$\psi = \bigoplus_{i=1}^k (\phi_i \otimes S_{m_i} \otimes S_{n_i}),$$

where  $m_i, n_i \in \mathbb{Z}^+$ ,  $\phi_i$  is a continuous homomorphism such that  $\phi_i(W_F)$  is bounded and consists of semisimple matrices and  $S_m$  is the  $m$  dimensional irreducible complex representation of  $SL(2, \mathbb{C})$ . Note that

$$\phi(w) \otimes S_n \left( \begin{pmatrix} |w|^{1/2} & \\ & |w|^{-1/2} \end{pmatrix} \right) = \bigoplus_{j=-(n-1)/2}^{(n-1)/2} \phi(w) |w|^j.$$

Therefore, for  $\psi = \phi \otimes S_m \otimes S_n$ , we have

$$\phi_\psi = \bigoplus_{j=-(n-1)/2}^{(n-1)/2} |\cdot|^j \phi \otimes S_m. \quad (6)$$

### Symmetry of Arthur parameters under Aubert involution

**Theorem 3.1.** *Let  $\rho$  be an irreducible unitary supercuspidal representation of  $GL(p, F)$  and  $\sigma$  an irreducible supercuspidal generic representation of  $SO(2q + 1, F)$ . Assume  $\rho \cong \tilde{\rho}$ . Let  $\bigoplus_{i \in A} \phi_i \otimes S_1$  be the  $L$ -parameter of  $\sigma$  and  $\phi$  be the  $L$ -parameter of  $\rho$ .*

*Let  $\pi$  be the base point corresponding to the  $A$ -parameter*

$$\psi = \phi \otimes S_k \otimes S_n \oplus \bigoplus_{i \in A} \phi_i \otimes S_1 \otimes S_1, \tag{7}$$

$k \geq 1, n = 2, 3$ . *Let  $\hat{\psi}$  be the  $A$ -parameter of  $\hat{\pi}$ . Then*

$$\hat{\psi} = \phi \otimes S_n \otimes S_k \oplus \bigoplus_{i \in A} \phi_i \otimes S_1 \otimes S_1.$$

**Proof.** Suppose  $\psi = \phi \otimes S_k \otimes S_2 \oplus \bigoplus_{i \in A} \phi_i \otimes S_1 \otimes S_1$ , where  $k \in \mathbb{Z}^+$ . From (6), the corresponding  $L$ -parameter  $\phi_\psi$  is equal to

$$\phi_\psi = |\cdot|^{\frac{1}{2}} \phi \otimes S_k \oplus |\cdot|^{-\frac{1}{2}} \phi \otimes S_k \oplus \bigoplus_{i \in A} \phi_i \otimes S_1.$$

Equations (4) and (5) implies that the base point representation attached to  $\phi_\psi$  is

$$\pi = L_s(\delta[\nu^{-\frac{k}{2}} \rho, \nu^{\frac{k}{2}-1} \rho], \sigma). \tag{8}$$

Next, suppose  $\psi = \phi \otimes S_k \otimes S_3 \oplus \bigoplus_{i \in A} \phi_i \otimes S_1 \otimes S_1$ . The corresponding  $L$ -parameter  $\phi_\psi$  is equal to  $\phi_\psi = |\cdot| \phi \otimes S_k \oplus \phi \otimes S_k \oplus |\cdot|^{-1} \phi \otimes S_k \oplus \bigoplus_{i \in A} \phi_i \otimes S_1$ . By (4),(5), the base point representation attached to  $\phi_\psi$  is

$$\pi = L_s(\delta[\nu^{-\frac{k}{2}-\frac{1}{2}} \rho, \nu^{\frac{k}{2}-\frac{3}{2}} \rho], \tau_0), \tag{9}$$

where  $\tau_0$  is the tempered generic representation with the  $L$ -parameter  $\phi \otimes S_k \oplus \bigoplus_{i \in A} \phi_i \otimes S_1$ .

Suppose  $\psi$  is given by equation (7). Then  $\psi$  is a homomorphism  $\psi : W_F \times SL(2, \mathbb{C}) \times SL(2, \mathbb{C}) \rightarrow GL(2\ell, \mathbb{C})$ . We say that  $\psi$  is symplectic (respectively, orthogonal) if  $\psi$  factors through  $Sp(2\ell, \mathbb{C})$  (respectively,  $SO(2\ell, \mathbb{C})$ ). We will give the conditions so that  $\psi$  is symplectic.

The  $L$ -parameter  $\phi$  of  $\rho$  is a homomorphism  $\phi : W_F \rightarrow GL(p, \mathbb{C})$ . For some representations  $r$  of  $GL(p, \mathbb{C})$ , local-global methods attach factors  $L(s, \rho, r)$ . Conjecturally, we have  $L(s, r \circ \phi) = L(s, \rho, r)$ , where the left hand side is the Artin  $L$ -function, while the right hand side is the Langlands  $L$ -function. The cases  $r = \wedge^2$  and  $r = Sym^2$  are significant, due to important results of Shahidi [28] and Henniart [16]. The result of Shahidi proves that exactly one of the two  $L$ -functions  $L(s, \rho, Sym^2)$  or  $L(s, \rho, \wedge^2)$  has a pole at  $s = 0$  ([28], Corollary 3.7, using  $\rho \cong \tilde{\rho}$ ). In addition,  $(\rho, \sigma)$  satisfies  $(C_{\frac{1}{2}})$  if and only if  $L(s, \rho, Sym^2)$  has a pole at  $s = 0$ . This follows from [27] and [28], and it is explicitly stated in [25], Lemma 2.3. On the other hand, Henniart proved the above equality of  $L$ -functions for  $r = \wedge^2$  and  $r = Sym^2$ . We have  $L(s, \rho, \wedge^2) = L(s, \wedge^2 \phi)$ ,  $L(s, \rho, Sym^2) = L(s, Sym^2 \phi)$ . In addition,  $L(s, \wedge^2 \phi)$  has a pole at  $s = 0$  if and only if  $\phi$  is symplectic, and  $L(s, Sym^2 \phi)$  has a pole at  $s = 0$  if and only if  $\phi$  is orthogonal. It follows that  $(\rho, \sigma)$  satisfies  $(C_{\frac{1}{2}})$  if and only if  $\phi$  is orthogonal.

If  $k$  is odd, then there is a basis of  $P_{k-1}^H[x, y]$  such that  $imS_k \subset SO(k, \mathbb{C})$ . If  $k$  is even, then there is a basis of  $P_{k-1}^H[x, y]$  such that  $imS_k \subset Sp(k, \mathbb{C})$ . Therefore, we have the following:

(C0), (C1) Assume  $\phi$  factors through  $Sp(p, \mathbb{C})$ . Then  $\psi$  factors through  $Sp(2\ell, \mathbb{C})$  for  $k$  even and  $n = 2$ , or  $k$  odd and  $n = 3$ . In this case,  $\nu^\alpha \rho \rtimes \sigma$  is reducible for  $\alpha = 0$  or  $1$ .

(C $\frac{1}{2}$ ) Assume  $\phi$  factors through  $SO(p, \mathbb{C})$ . Then  $\psi$  factors through  $Sp(2\ell, \mathbb{C})$  for  $k$  odd and  $n = 2$ , or  $k$  even and  $n = 3$ . In this case,  $\nu^\alpha \rho \rtimes \sigma$  is reducible for  $\alpha = \frac{1}{2}$ .

Suppose  $\nu^{\frac{1}{2}}\rho \rtimes \sigma$  is reducible. First, we consider

$$\psi = \phi \otimes S_{2m+1} \otimes S_2 \oplus \bigoplus_{i \in A} \phi_i \otimes S_1 \otimes S_1,$$

$m \geq 0$ . From (8),  $\pi = L_s(\delta[\nu^{-m-\frac{1}{2}}\rho, \nu^{m-\frac{1}{2}}\rho], \sigma)$ . Let  $\tau$  be the representation corresponding to the  $A$ -parameter  $\phi \otimes S_2 \otimes S_{2m+1} \oplus \bigoplus_{i \in A} \phi_i \otimes S_1 \otimes S_1$ . Then, by (6), the corresponding  $L$ -parameter is

$$\begin{aligned} & \bigoplus_{j=-m}^m |\cdot|^j \phi \otimes S_2 \oplus \bigoplus_{i \in A} \phi_i \otimes S_1 \\ &= \bigoplus_{j=1}^m (|\cdot|^j \phi \otimes S_2 \oplus |\cdot|^{-j} \phi \otimes S_2) \oplus \phi \otimes S_2 \oplus \bigoplus_{i \in A} \phi_i \otimes S_1 \end{aligned}$$

and

$$\tau = L_s(\delta[\nu^{-m-\frac{1}{2}}\rho, \nu^{-m+\frac{1}{2}}\rho], \delta[\nu^{-m+\frac{1}{2}}\rho, \nu^{-m+\frac{3}{2}}\rho], \dots, \delta[\nu^{-\frac{3}{2}}\rho, \nu^{-\frac{1}{2}}\rho], \delta(\nu^{\frac{1}{2}}\rho; \sigma)).$$

We have to prove  $\hat{\pi} = \tau$ . Let us consider the representation

$$\Pi = \zeta[\nu^{-m-\frac{1}{2}}\rho, \nu^{m-\frac{1}{2}}\rho] \rtimes \sigma.$$

We analyze  $\Pi$  using [18]. Note that  $\zeta[\nu^{-m-\frac{1}{2}}\rho, \nu^{m-\frac{1}{2}}\rho] = \nu^{-\frac{1}{2}}\zeta[\nu^{-m}\rho, \nu^m\rho]$ . The representation  $\zeta[\nu^{-m}\rho, \nu^m\rho]$  is the unique irreducible quotient of  $\nu^m\rho \times \nu^{m-1}\rho \times \dots \times \nu^{-m}\rho$ . Equivalently, it is defined as the unique irreducible subrepresentation of  $\nu^{-m}\rho \times \nu^{-m+1}\rho \times \dots \times \nu^m\rho$ . Therefore,  $\zeta[\nu^{-m}\rho, \nu^m\rho]$  is the representation  $\zeta(\rho, 2m+1)$  of [18] and we can write  $\Pi = \nu^\alpha \zeta(\rho, n) \rtimes \sigma$ , with  $\alpha = -\frac{1}{2}$  and  $n = 2m+1$ . It follows that  $\Pi$  has three irreducible subquotients,  $\pi_1, \pi_2, \pi_3$  given in Proposition 3.6 (2) of [18]. In particular,

$$\pi_3 = L_s([\nu^{-n+j+\frac{1}{2}}\rho, \nu^{-j-\frac{3}{2}}\rho], \nu^{-j}\delta(\rho, 2), \nu^{-j+1}\delta(\rho, 2), \dots, \nu^{-1}\delta(\rho, 2); \delta(\nu^{\frac{1}{2}}\rho; \sigma)),$$

where  $\delta(\rho, 2) = \delta[\nu^{-\frac{1}{2}}\rho, \nu^{\frac{1}{2}}\rho]$ ,  $j = \alpha + \frac{n}{2}$ . In our case,  $j = m$ , so the segment  $[\nu^{-n+j+\frac{1}{2}}\rho, \nu^{-j-\frac{3}{2}}\rho] = [\nu^{-m-\frac{1}{2}}\rho, \nu^{-m-\frac{3}{2}}\rho]$  is empty and

$$\pi_3 = L_s(\nu^{-m}\delta[\nu^{-\frac{1}{2}}\rho, \nu^{\frac{1}{2}}\rho], \nu^{-m+1}\delta[\nu^{-\frac{1}{2}}\rho, \nu^{\frac{1}{2}}\rho], \dots, \nu^{-1}\delta[\nu^{-\frac{1}{2}}\rho, \nu^{\frac{1}{2}}\rho], \delta(\nu^{\frac{1}{2}}\rho; \sigma))$$

is equal to  $\tau$ . Jacquet modules of  $\pi_1, \pi_2$  and  $\pi_3$  are given in part (c) of Proposition 3.6 (2) in [18]. We observe that only  $\pi_3$  does not have terms of the form  $\nu^{-m-\frac{1}{2}}\rho \otimes \dots$  in its Jacquet module.



On the other hand, the dual of  $\Pi$  in the Grothendieck group is

$$\delta[\nu^{-m+\frac{1}{2}}\rho, \nu^{m+\frac{1}{2}}\rho] \rtimes \sigma$$

and it has three components,  $\hat{\pi}_1, \hat{\pi}_2$  and  $\hat{\pi}_3$ . The Theorem in the Introduction of [30] tells us that the representation  $\delta[\nu^{-m+\frac{1}{2}}\rho, \nu^{m+\frac{1}{2}}\rho] \rtimes \sigma$  has two irreducible square-integrable subrepresentations. The third component is the Langlands quotient  $L_q(\delta[\nu^{-m+\frac{1}{2}}\rho, \nu^{m+\frac{1}{2}}\rho]; \sigma) = L_s(\delta[\nu^{-m-\frac{1}{2}}\rho, \nu^{m-\frac{1}{2}}\rho]; \sigma)$ . By Frobenius reciprocity, the square-integrable subrepresentations have terms of the form  $\nu^{m+\frac{1}{2}}\rho \otimes \dots$  in their Jacquet modules. Therefore,  $\hat{\pi}_1$  and  $\hat{\pi}_2$  are square integrable. It follows  $\hat{\pi}_3 = L_s(\delta[\nu^{-m-\frac{1}{2}}\rho, \nu^{m-\frac{1}{2}}\rho]; \sigma)$ . In other words,  $\hat{\tau} = \pi$  and  $\hat{\pi} = \tau$ .

Now, let  $\psi = \phi \otimes S_{2m} \otimes S_3 \oplus \bigoplus_{i \in A} \phi_i \otimes S_1 \otimes S_1$ ,  $m \geq 1$ . By (9),

$$\pi = L_s(\delta[\nu^{-m-\frac{1}{2}}\rho, \nu^{m-\frac{3}{2}}\rho], \delta([\nu^{\frac{1}{2}}\rho, \nu^{m-\frac{1}{2}}\rho]; \sigma)).$$

Let  $\tau$  correspond to  $\phi \otimes S_3 \otimes S_{2m} \oplus \bigoplus_{i \in A} \phi_i \otimes S_1 \otimes S_1$ . Then

$$\tau = L_s(\delta([\nu^{-m-\frac{1}{2}}\rho, \nu^{-m+\frac{3}{2}}\rho]), \dots, \delta([\nu^{-\frac{3}{2}}\rho, \nu^{\frac{1}{2}}\rho]), \sigma).$$

We will prove  $\hat{\pi} = \tau$ . By Theorems 6.1 and 7.1 (Case 3a) of [18],  $\tau$  is the unique irreducible subquotient of

$$\nu^{-1}\zeta[\nu^{-m+\frac{1}{2}}\rho, \nu^{m-\frac{1}{2}}\rho] \rtimes \zeta([\nu^{-\frac{1}{2}}\rho, \nu^{-m+\frac{1}{2}}\rho]; \sigma).$$

In particular, it is the unique irreducible quotient because it contains the unique copy of  $\nu\zeta[\nu^{-m+\frac{1}{2}}\rho, \nu^{m-\frac{1}{2}}\rho] \otimes \zeta([\nu^{\frac{1}{2}}\rho, \nu^{m-\frac{1}{2}}\rho]; \sigma)$  in the Jacquet module of the generalized degenerate principal series. Therefore,  $\hat{\tau}$  is a subquotient of the representation  $\nu^{-1}\delta[\nu^{-m+\frac{1}{2}}\rho, \nu^{m-\frac{1}{2}}\rho] \rtimes \delta([\nu^{-\frac{1}{2}}\rho, \nu^{-m+\frac{1}{2}}\rho]; \sigma)$ . In addition, it contains  $\nu^{-1}\delta[\nu^{-m+\frac{1}{2}}\rho, \nu^{m-\frac{1}{2}}\rho] \otimes \delta([\nu^{-\frac{1}{2}}\rho, \nu^{-m+\frac{1}{2}}\rho]; \sigma)$  (by Théorème 1.7 (b) of Aubert). This forces  $\hat{\tau} = \pi$ .

Now, suppose  $\nu\rho \rtimes \sigma$  or  $\rho \rtimes \sigma$  is reducible. The proofs are similar to the case  $(C\frac{1}{2})$ . For  $n = 2$ ,  $\pi = L_s(\delta[\nu^{-m}\rho, \nu^{m-1}\rho], \sigma)$  and the proof is based on consideration of the representation  $\Pi = \zeta[\nu^{-m}\rho, \nu^{m-1}\rho] \rtimes \sigma$ . The components of  $\Pi$  are given in Proposition 3.10 of [18] in the case  $(C1)$  and in Proposition 3.11 of [18] in the case  $(C0)$ . For  $n = 3$ , however, we obtain a new supercuspidal representation  $\sigma'$  in the Langlands data of  $\pi$ . Suppose  $\nu\rho \rtimes \sigma$  is reducible. Let

$$\psi = \phi \otimes S_{2m+1} \otimes S_3 \oplus \bigoplus_{i \in A} \phi_i \otimes S_1 \otimes S_1,$$

$m \geq 1$ . Reducibility of  $\nu\rho \rtimes \sigma$  implies  $\phi \cong \phi_k$ , for some  $k \in A$ . Let  $\sigma'$  be the supercuspidal generic representation of  $SO(2q' + 1, F)$  associated by [19] to the parameter

$$\bigoplus_{i \in A \setminus \{k\}} \phi_i \otimes S_1.$$

From (9),  $\pi = L_s(\delta[\nu^{-m-1}\rho, \nu^{m-1}\rho], \delta([\rho, \nu^m\rho]; \sigma'))$ . Let  $\tau$  be the representation corresponding to the  $A$ -parameter  $\phi \otimes S_3 \otimes S_{2m+1} \oplus \bigoplus_{i \in A} \phi_i \otimes S_1 \otimes S_1$ . Then, by

(6), the corresponding  $L$ -parameter is

$$\begin{aligned} & \bigoplus_{j=-m}^m |\cdot|^j \phi \otimes S_3 \oplus \bigoplus_{i \in A} \phi_i \otimes S_1 \\ &= \bigoplus_{j=1}^m (|\cdot|^j \phi \otimes S_3 \oplus |\cdot|^{-j} \phi \otimes S_3) \oplus \phi \otimes S_3 \oplus \phi \otimes S_1 \oplus \bigoplus_{i \in A \setminus \{k\}} \phi_i \otimes S_1 \end{aligned}$$

and

$$\tau = L_s(\delta[\nu^{-m-1}\rho, \nu^{-m+1}\rho], \delta[\nu^{-m}\rho, \nu^{-m+2}\rho], \dots, \delta[\nu^{-2}\rho, \rho], \delta([\rho, \nu\rho]; \sigma')).$$

Now, the proof  $\hat{\pi} = \tau$  is analogous to the case  $(C\frac{1}{2})$ , using the Theorem 3.4 (3)(d) of [7]. In the case when  $\rho \rtimes \sigma$  is reducible, we have  $\phi \not\cong \phi_i$ , for all  $i \in A$ , and  $\sigma'$  is the supercuspidal generic representation of  $SO(2q' + 1, F)$  associated by [19] to the parameter

$$\phi \otimes S_1 \oplus \bigoplus_{i \in A} \phi_i \otimes S_1.$$

Then  $\nu\rho \rtimes \sigma'$  is reducible. We apply exactly the same arguments as in the case  $(C\frac{1}{2})$ , based on composition factors described in Theorems 6.2 and 7.2 of [18]. This finishes the proof of Theorem 3.1. ■

#### 4. Classification

We review Muić's classification of generic representations of the group  $SO(2n + 1, F)$  given in [24].

(a) Let  $\sigma_0$  be a generic supercuspidal representation of  $SO(2n' + 1, F)$  and  $\Sigma_i = [\nu^{-a_i}\rho_i, \nu^{b_i}\rho_i]$ ,  $2b_i \in \mathbb{Z}_+$ ,  $2a_i \in \mathbb{Z}$ ,  $\rho_i \cong \tilde{\rho}_i$ ,  $i = 1, \dots, k$  a set of segments satisfying

- (i)  $b_i > a_i$ .
- (ii) If  $(\rho_i, \sigma_0)$  satisfies  $(C\frac{1}{2})$ , then  $b_i \in \frac{1}{2} + \mathbb{Z}$ ,  $a_i \geq -\frac{1}{2}$ .  
 If  $(\rho_i, \sigma_0)$  satisfies  $(C0)$ , then  $b_i \in \mathbb{Z}$ ,  $a_i \geq 0$ .  
 If  $(\rho_i, \sigma_0)$  satisfies  $(C1)$ , then  $b_i \in \mathbb{Z}$ ,  $a_i \geq -1$ ,  $a_i \neq 0$ .
- (iii) If  $\rho_i \cong \rho_j$  for  $i \neq j$ , then either  $b_i < a_j$  or  $b_j < a_i$ .

The representation  $\delta(\Sigma_1 \cap \tilde{\Sigma}_1) \times \dots \times \delta(\Sigma_k \cap \tilde{\Sigma}_k) \rtimes \sigma_0$  has a unique irreducible generic subrepresentation, denote it by  $\tau$ . The representation  $\delta(\Sigma_1 \setminus \tilde{\Sigma}_1) \times \dots \times \delta(\Sigma_k \setminus \tilde{\Sigma}_k) \rtimes \tau$  has a unique irreducible subrepresentation which we denote by  $\delta(\Sigma_1, \dots, \Sigma_k; \sigma_0)$ . The representation  $\delta(\Sigma_1, \dots, \Sigma_k; \sigma_0)$  is square integrable and generic.

Conversely, if  $\sigma$  is an irreducible square integrable generic representation of  $SO(2n + 1, F)$ , then there exists a unique  $\sigma_0$  and a unique set of segments  $\{\Sigma_1, \dots, \Sigma_k\}$  satisfying (i) - (iii) such that  $\sigma \cong \delta(\Sigma_1, \dots, \Sigma_k; \sigma_0)$ .

(b) Let  $\sigma$  be an irreducible generic square integrable representation. Write  $\sigma \cong \delta(\Sigma_1, \dots, \Sigma_k; \sigma_0)$  as in (a). Suppose  $\Sigma_{k+1}, \dots, \Sigma_l$  is a sequence of segments satisfying

- (iv) Segments  $\Sigma_{k+1}, \dots, \Sigma_l$  are balanced and mutually different.

(v)  $\delta(\Sigma_i) \rtimes \sigma$  is reducible, for  $i = k + 1, \dots, l$ .

Then, the representation  $\delta(\Sigma_{k+1}) \times \dots \times \delta(\Sigma_l) \rtimes \sigma$  has a unique irreducible generic subrepresentation, denote it by  $\sigma^{(ell)}$ . This representation is elliptic and tempered.

(c) Suppose that  $\Sigma_{l+1}, \dots, \Sigma_m$  is a sequence of segments satisfying

(vi) Segments  $\Sigma_i, \Sigma_j$  are not linked, for all  $l + 1 \leq i < j \leq m$ .

(vii) Segments  $\Sigma_i, \tilde{\Sigma}_j$  are not linked, for all  $l + 1 \leq i < j \leq m$ .

(viii)  $\delta(\Sigma_i) \rtimes \sigma^{(ell)}$  is irreducible, for  $i = l + 1, \dots, m$ .

Then, the representation  $\delta(\Sigma_{l+1}) \times \dots \times \delta(\Sigma_m) \rtimes \sigma^{(ell)}$  is irreducible, generic.

Conversely, let  $\pi$  be an irreducible generic representation of  $SO(2n + 1, F)$ . Then, there exist a square integrable representation  $\sigma$ , a sequence of segments  $\Sigma_{k+1}, \dots, \Sigma_l$  satisfying (iv)-(v), and a sequence of segments  $\Sigma_{l+1}, \dots, \Sigma_m$  satisfying (vi)-(viii), such that  $\pi \cong \delta(\Sigma_{l+1}) \times \dots \times \delta(\Sigma_m) \rtimes \sigma^{(ell)}$ . The representation  $\sigma$  is unique, the sequence  $\Sigma_{k+1}, \dots, \Sigma_l$  is unique up to a permutation, and the sequence  $\Sigma_{l+1}, \dots, \Sigma_m$  is unique up to a permutation and taking contragredient.

### 5. Generic representations

Let  $\pi$  be a generic representation of  $SO(2n + 1, F)$  and  $\phi$  the  $L$ -parameter of  $\pi$ . In this section, we compute the involution  $\hat{\pi}$  (Lemma 5.1) and the  $L$ -parameter  $\hat{\phi}$  of  $\hat{\pi}$  (Theorem 5.3). We show that  $\phi = \phi_\psi$ , for an  $A$ -parameter  $\psi$ , if and only if  $\pi$  is tempered. In this case,  $\hat{\pi}$  is attached to an  $A$ -parameter  $\hat{\psi}$ . The parameters  $\psi$  and  $\hat{\psi}$  are symmetric.

**Langlands data** We fix an irreducible generic representation  $\pi$  and associate to it a square integrable representation  $\sigma$  and segments  $\Sigma_{k+1}, \dots, \Sigma_l, \Sigma_{l+1}, \dots, \Sigma_m$  such that  $\pi \cong \delta(\Sigma_{l+1}) \times \dots \times \delta(\Sigma_m) \rtimes \sigma^{(ell)}$  as in section 4. Let

$$P = \{1, \dots, k\}, Q = \{k + 1, \dots, l\}, R = \{l + 1, \dots, m\}$$

and  $T = P \cup Q \cup R$ . For  $i \in T$ , let

$$\Sigma_i = [\nu^{-a_i} \rho_i, \nu^{b_i} \rho_i].$$

We may assume that  $b_i \geq 0, b_i \geq a_i$ , for all  $i \in T$ . This condition is satisfied for  $i \in P \cup Q$ . For  $i \in R$ , we can replace the segment  $\Sigma_i$  by its contragredient, if necessary. Denote by  $r$  the local Langlands reciprocity map for  $GL(F)$  [14, 15]. Let  $\phi_i$  be the Langlands parameter of  $\rho_i$ , i.e.,  $r(\phi_i) = \rho_i$ . Recall that  $r(\hat{\phi}_i) = \tilde{\rho}_i$ . For  $\alpha = 0, \frac{1}{2}, 1$ , define the following sets:

$$\begin{aligned} C_\alpha &= \{i \in P \mid (\rho_i, \sigma_0) \text{ satisfies } (C\alpha), a_i \geq 0\}, \\ C_{-1} &= \{i \in P \mid a_i = -1\}, \\ P_0 &= \{i \in P \mid a_i \geq 0\} = C_0 \cup C_{\frac{1}{2}} \cup C_1, \\ T_0 &= \{i \in T \mid 0 \in [-a_i, b_i]\}. \end{aligned}$$

Let  $\ell(\sigma_0) = \rho_{m+1} \times \cdots \times \rho_p$  be the local Langlands functorial lift of  $\sigma_0$  (Theorem 1.1 of [19]). Define

$$A = \{m + 1, \dots, p\},$$

$$A_0 = A \setminus \{j \in A \mid \rho_j \cong \rho_i \text{ for some } i \in C_{-1}\}.$$

**Lemma 5.1.** *Let  $\pi$  be an irreducible generic representation of  $SO(2n + 1, F)$ . Let  $\sigma$  be the generic square integrable representation and*

$$\Sigma_{k+1}, \dots, \Sigma_l, \Sigma_{l+1}, \dots, \Sigma_m$$

*the sequence of segments associated to  $\pi$  by section 4. For  $j_i \in [-a_i, b_i]$ ,  $j_i \neq 0$ , define*

$$\epsilon_{j_i} = \begin{cases} 1, & \text{if } j_i > 0, \\ -1, & \text{if } j_i < 0. \end{cases}$$

*Let  $\{(l_1, \epsilon_1), \dots, (l_t, \epsilon_t)\}$  be the multiset  $\{(|j_i|, \epsilon_{j_i}) \mid i \in T, j_i \in \{-a_i, -a_i + 1, \dots, b_i\} \setminus \{0\}\}$  written in a non-increasing order, with respect to the first coordinate. For  $l_s = |j_i|$ , let  $\rho_{l_s} = \rho_i$ . Let  $\tau_0$  be the unique generic component of  $\left(\times_{i \in T_0} \rho_i\right) \rtimes \sigma_0$ . Then  $\hat{\pi}$  is the Langlands quotient of the induced representation*

$$\nu^{l_1} \rho_{l_1}^{\epsilon_1} \times \cdots \times \nu^{l_t} \rho_{l_t}^{\epsilon_t} \rtimes \hat{\tau}_0,$$

*where  $\rho^\epsilon$  is defined by  $\rho^\epsilon = \begin{cases} \rho, & \text{if } \epsilon = 1, \\ \tilde{\rho}, & \text{if } \epsilon = -1. \end{cases}$*

**Remark 5.2.** The equivalence class of the irreducible representation  $\times_{i \in T_0} \rho_i$  does not depend on the order of  $\rho_i, i \in T_0$ .

**Proof.** The proof is similar to the proof of Lemma 5.1, [6]. Let

$$\pi_0 = (\nu^{b_1} \rho_1 \otimes \nu^{b_1-1} \rho_1 \otimes \cdots \otimes \nu^{-a_1} \rho_1) \otimes \cdots \otimes (\nu^{b_m} \rho_m \otimes \nu^{b_m-1} \rho_m \otimes \cdots \otimes \nu^{-a_m} \rho_m) \otimes \sigma_0.$$

Denote by  $M$  the standard Levi subgroup corresponding to  $\pi_0$ . We consider the induced representation  $i_{G,M}(\pi_0)$ . Then  $\pi$  is a subrepresentation of  $i_{G,M}(\pi_0)$  and, by Corollary 4.2 of [4],  $\hat{\pi}$  is a quotient of  $i_{G,M}(\pi_0)$ . Write  $\hat{\pi}$  as a Langlands quotient

$$\hat{\pi} = L_q(\nu^{\alpha_1} \delta_1, \dots, \nu^{\alpha_q} \delta_q, \tau_1), \tag{10}$$

$\alpha_1 \geq \cdots \geq \alpha_q > 0$  (see page 254). Then Lemma 4.2 [6] tells us that  $\delta_i, i = 1, \dots, q$  are supercuspidal unitary representations and  $\tau_1$  is a subrepresentation of

$$\delta_{q+1} \times \cdots \times \delta_r \rtimes \sigma_0,$$

where  $\delta_i, i = q + 1, \dots, r$  are supercuspidal unitary representations. Therefore,  $\hat{\pi}$  is a subquotient of the representation induced from

$$\pi_1 = \nu^{\alpha_1} \delta_1 \otimes \cdots \otimes \nu^{\alpha_q} \delta_q \otimes \delta_{q+1} \otimes \cdots \otimes \delta_r \otimes \sigma_0.$$

It follows from [11], Corollary 6.3.7 and from the description of the Weyl group for odd-orthogonal groups that  $\pi_1$  can be obtained from  $\pi_0$  by permutations and

taking contragredients. The condition on  $\alpha_1, \dots, \alpha_q$  implies that the sequence  $\alpha_1, \dots, \alpha_q$  are equal to  $l_1, \dots, l_t$  and  $\{\delta_1, \dots, \delta_q\} = \{\rho_{l_1}^{\epsilon_1}, \dots, \rho_{l_t}^{\epsilon_t}\}$ . In addition, the sequence  $\delta_{q+1}, \dots, \delta_r$  is up to a permutation equal to  $\rho_i^{\eta_i}, i \in T_0$ , with  $\eta_i = 1$  or  $-1$ . It follows from equation (10) that  $\hat{\pi}$  is the Langlands quotient

$$\hat{\pi} = L_q(\nu^{l_1} \rho_{l_1}^{\epsilon_1}, \dots, \nu^{l_t} \rho_{l_t}^{\epsilon_t}, \tau_1) \tag{11}$$

and  $\tau_1$  is a subrepresentation of  $\left(\times_{i \in T_0} \rho_i^{\eta_i}\right) \rtimes \sigma_0$ . We claim  $\left(\times_{i \in T_0} \rho_i^{\eta_i}\right) \rtimes \sigma_0 \cong \left(\times_{i \in T_0} \rho_i\right) \rtimes \sigma_0$ . To prove the claim, we first show that  $\rho_j \rtimes \sigma_0 \cong \tilde{\rho}_j \rtimes \sigma_0$ , for all  $j \in T_0$ . If  $\tilde{\rho}_j \cong \rho_j$ , this is clear. If  $\tilde{\rho}_j \not\cong \rho_j$ , then  $\rho_j \rtimes \sigma_0$  is irreducible, which implies  $\rho_j \rtimes \sigma_0 \cong \tilde{\rho}_j \rtimes \sigma_0$ . Therefore,  $\rho_j^{\eta_j} \rtimes \sigma_0 \cong \rho_j \rtimes \sigma_0$ , for all  $j \in T_0$ . Since  $\times_{i \in T_0} \rho_i^{\eta_i}$  is irreducible and the factors commute, we have, for  $j \in T_0$ ,

$$\left(\times_{i \in T_0} \rho_i^{\eta_i}\right) \rtimes \sigma_0 \cong \left(\times_{i \in T_0 \setminus \{j\}} \rho_i^{\eta_i}\right) \times \rho_j^{\eta_j} \rtimes \sigma_0 \cong \left(\times_{i \in T_0 \setminus \{j\}} \rho_i^{\eta_i}\right) \times \rho_j \rtimes \sigma_0$$

and the claim follows.

Now, equation (11) implies  $\pi$  is a component of  $(\nu^{l_1} \rho_{l_1}^{\epsilon_1} \times \dots \times \nu^{l_t} \rho_{l_t}^{\epsilon_t} \rtimes \tau_1)^\wedge$ . In the Grothendieck group, the Aubert involution commutes with parabolic induction ([3], Théorème 1.7). Therefore,  $\pi$  is a component of  $\nu^{l_1} \rho_{l_1}^{\epsilon_1} \times \dots \times \nu^{l_t} \rho_{l_t}^{\epsilon_t} \rtimes \hat{\tau}_1$ . Since  $\pi$  is generic, it follows from the properties of generic representations with respect to parabolic induction that  $\hat{\tau}_1$  is generic (cf. Lemma 4.1 of [6]). Therefore,  $\tau_1 = \hat{\tau}_0$ . ■

**Langlands parameters**

**Theorem 5.3.** *Let  $\pi$  be an irreducible generic representation of  $SO(2n+1, F)$ . Let  $\sigma$  be the generic square integrable representation and*

$$\Sigma_{k+1}, \dots, \Sigma_l, \Sigma_{l+1}, \dots, \Sigma_m$$

*the sequence of segments associated to  $\pi$  by section 4. Then the local Langlands parameter of  $\sigma$  is*

$$\varphi(\sigma) = \left(\bigoplus_{i \in P} \phi_i \otimes S_{2b_i+1}\right) \oplus \left(\bigoplus_{i \in P_0} \phi_i \otimes S_{2a_i+1}\right) \oplus \left(\bigoplus_{i \in A_0} \phi_i \otimes S_1\right),$$

where  $\phi_i$  is the Langlands parameter of  $\rho_i$ . The local Langlands parameter of  $\pi$  is

$$\varphi(\sigma) \oplus \bigoplus_{i \in Q \cup R} \left(|\cdot|^{\frac{b_i-a_i}{2}} \phi_i \otimes S_{a_i+b_i+1} \oplus |\cdot|^{\frac{a_i-b_i}{2}} \tilde{\phi}_i \otimes S_{a_i+b_i+1}\right),$$

which is equal to

$$\begin{aligned} &\varphi(\sigma) \oplus \left(\bigoplus_{i \in Q} (\phi_i \otimes S_{a_i+b_i+1} \oplus \phi_i \otimes S_{a_i+b_i+1})\right) \\ &\oplus \left(\bigoplus_{i \in R} \left(|\cdot|^{\frac{b_i-a_i}{2}} \phi_i \otimes S_{a_i+b_i+1} \oplus |\cdot|^{\frac{a_i-b_i}{2}} \tilde{\phi}_i \otimes S_{a_i+b_i+1}\right)\right). \end{aligned}$$

The local Langlands parameter of  $\hat{\pi}$  is

$$\left( \bigoplus_{i \in T} \bigoplus_{j=-a_i}^{b_i} (|\cdot|^j \phi_i \otimes S_1 \oplus |\cdot|^{-j} \tilde{\phi}_i \otimes S_1) \right) \oplus \left( \bigoplus_{i \in A} \phi_i \otimes S_1 \right).$$

**Proof.** The local Langlands parameter of  $\sigma$  follows from [19], the proof of Theorem 2.1. The description of  $\varphi(\sigma)$  is given in [6], Theorem 5.2.

The local Langlands parameter of  $\pi$  follows from [19]. First, according to [19], the proof of Theorem 3.1,

$$\varphi(\sigma^{(ell)}) = \varphi(\sigma) \oplus \left( \bigoplus_{i \in Q} (\phi_i \otimes S_{a_i+b_i+1} \oplus \phi_i \otimes S_{a_i+b_i+1}) \right).$$

Define  $R^* = \{i \in R \mid \Sigma_i \text{ is balanced}\}$  and  $R^{**} = R \setminus R^*$ . Let

$$\sigma^{(temp)} = \left( \times_{i \in R^*} \delta(\Sigma_i) \right) \rtimes \sigma^{(ell)}.$$

Then  $\sigma^{(temp)}$  is a tempered generic representation and

$$\begin{aligned} \varphi(\sigma^{(temp)}) = & \varphi(\sigma) \oplus \left( \bigoplus_{i \in Q} (\phi_i \otimes S_{a_i+b_i+1} \oplus \phi_i \otimes S_{a_i+b_i+1}) \right) \\ & \oplus \left( \bigoplus_{i \in R^*} (\phi_i \otimes S_{a_i+b_i+1} \oplus \tilde{\phi}_i \otimes S_{a_i+b_i+1}) \right), \end{aligned}$$

by [19], the proof of Theorem 4.1. Now,  $\pi \cong \left( \times_{i \in R^{**}} \delta(\Sigma_i) \right) \rtimes \sigma^{(temp)}$ . The proof of Theorem 5.2, [19], tells us that the Langlands parameter of  $\pi$  is

$$\varphi(\sigma^{(temp)}) \oplus \left( \bigoplus_{i \in R^{**}} (|\cdot|^{\frac{b_i-a_i}{2}} \phi_i \otimes S_{a_i+b_i+1} \oplus |\cdot|^{\frac{a_i-b_i}{2}} \tilde{\phi}_i \otimes S_{a_i+b_i+1}) \right),$$

which is equal to

$$\varphi(\sigma) \oplus \bigoplus_{i \in Q \cup R} \left( |\cdot|^{\frac{b_i-a_i}{2}} \phi_i \otimes S_{a_i+b_i+1} \oplus |\cdot|^{\frac{a_i-b_i}{2}} \tilde{\phi}_i \otimes S_{a_i+b_i+1} \right).$$

Let  $\{(l_1, \epsilon_1), \dots, (l_t, \epsilon_t)\}$  be the multiset  $\{(|j_i|, \epsilon_{j_i}) \mid i \in T, j_i \in \{-a_i, -a_i + 1, \dots, b_i\} \setminus \{0\}\}$  written in a non-increasing order, with respect to the first coordinate. For  $l_s = |j_i|$ , let  $\rho_{l_s} = \rho_i$ . Let  $\tau_0$  be the unique generic component of  $\left( \times_{i \in T_0} \rho_i \right) \rtimes \sigma_0$ . According to Lemma 5.1,  $\hat{\pi}$  is the Langlands quotient of the induced representation

$$\nu^{l_1} \rho_{l_1}^{\epsilon_1} \times \dots \times \nu^{l_t} \rho_{l_t}^{\epsilon_t} \rtimes \hat{\tau}_0.$$

In a similar way as in the proof of Theorem 5.2 [6], we prove that the parameter of  $\tau_0$  is

$$\varphi(\tau_0) = \left( \bigoplus_{i \in T_0} \phi_i \otimes S_1 \right) \oplus \left( \bigoplus_{i \in A} \phi_i \otimes S_1 \right) \oplus \left( \bigoplus_{i \in T_0} \tilde{\phi}_i \otimes S_1 \right).$$

The representation  $\tau_0$  is a component of  $\left(\prod_{i \in T_0} \rho_i\right) \rtimes \sigma_0$ . Since the representations  $\rho_i$ ,  $i \in T_0$  and  $\sigma_0$  are supercuspidal, it follows from the definition of the Aubert involution that  $\hat{\tau}_0$  is a component of  $\left(\prod_{i \in T_0} \rho_i\right) \rtimes \sigma_0$ . Therefore,  $\tau_0$  and  $\hat{\tau}_0$  are tempered representations induced from the same discrete series representation  $\left(\prod_{i \in T_0} \rho_i\right) \otimes \sigma_0$ . It follows  $\varphi(\hat{\tau}_0) = \varphi(\tau_0)$  ([6], page 340). By equations (4), (5), the Langlands parameter of  $\hat{\pi}$  is

$$\begin{aligned} & \left( \bigoplus_{s=1}^t (|\cdot|^{l_s r^{-1}}(\rho_{l_s}^{\epsilon_s}) \otimes S_1 \oplus |\cdot|^{-l_s r^{-1}}(\tilde{\rho}_{l_s}^{\epsilon_s}) \otimes S_1) \right) \oplus \varphi(\tau_0) \\ &= \left( \bigoplus_{i \in T} \bigoplus_{\substack{j=-a_i \\ j > 0}}^{b_i} (|\cdot|^{j} \phi_i \otimes S_1 \oplus |\cdot|^{-j} \tilde{\phi}_i \otimes S_1) \right) \\ & \oplus \left( \bigoplus_{i \in T} \bigoplus_{\substack{j=-a_i \\ j < 0}}^{b_i} (|\cdot|^{j} \tilde{\phi}_i \otimes S_1 \oplus |\cdot|^{-j} \phi_i \otimes S_1) \right) \oplus \varphi(\tau_0) \\ &= \left( \bigoplus_{i \in T} \bigoplus_{\substack{j=-a_i \\ j > 0}}^{b_i} (|\cdot|^{j} \phi_i \otimes S_1 \oplus |\cdot|^{-j} \tilde{\phi}_i \otimes S_1) \right) \\ & \oplus \left( \bigoplus_{i \in T} \bigoplus_{\substack{j=-a_i \\ j < 0}}^{b_i} (|\cdot|^{-j} \tilde{\phi}_i \otimes S_1 \oplus |\cdot|^{j} \phi_i \otimes S_1) \right) \oplus \varphi(\tau_0). \end{aligned}$$

This is equal to

$$\begin{aligned} & \left( \bigoplus_{i \in T} \bigoplus_{\substack{j=-a_i \\ j \neq 0}}^{b_i} (|\cdot|^{j} \phi_i \otimes S_1 \oplus |\cdot|^{-j} \tilde{\phi}_i \otimes S_1) \right) \\ & \oplus \left( \bigoplus_{i \in T_0} \phi_i \otimes S_1 \right) \oplus \left( \bigoplus_{i \in A} \phi_i \otimes S_1 \right) \oplus \left( \bigoplus_{i \in T_0} \tilde{\phi}_i \otimes S_1 \right) \\ &= \left( \bigoplus_{i \in T} \bigoplus_{j=-a_i}^{b_i} (|\cdot|^{j} \phi_i \otimes S_1 \oplus |\cdot|^{-j} \tilde{\phi}_i \otimes S_1) \right) \oplus \left( \bigoplus_{i \in A} \phi_i \otimes S_1 \right). \quad \blacksquare \end{aligned}$$

**Arthur parameters** Let  $\pi$  be an irreducible generic representation of  $SO(2n + 1, F)$  and  $\hat{\pi}$  the Aubert involution of  $\pi$ . Using the methods of Jiang and Soudry [19], we were able to compute the  $L$ -parameters of  $\pi$  and  $\hat{\pi}$  (Theorem 5.3). Now, we would like to compare the  $A$ -parameters of  $\pi$  and  $\hat{\pi}$  (if they exist). Our methods are restricted to consideration of base point representations. We show that, among generic representations, only tempered representations are base points

attached to  $A$ -parameters (Theorem 5.4). Suppose  $\pi$  is tempered and let  $\psi$  be the  $A$ -parameter of  $\pi$ . In this case,  $\hat{\pi}$  also has the  $A$ -parameter, denote it by  $\hat{\psi}$ . We show that  $\psi$  and  $\hat{\psi}$  are symmetric.

**Theorem 5.4.** *Let  $\pi$  be an irreducible generic representation of  $SO(2n+1, F)$  and  $\phi$  the  $L$ -parameter of  $\pi$ . Suppose  $\phi = \phi_\psi$ , for an  $A$ -parameter  $\psi : W_F \times SL(2, \mathbb{C}) \times SL(2, \mathbb{C}) \rightarrow Sp(2n, \mathbb{C})$ . Then  $\pi$  is tempered. Let  $\hat{\psi}$  be the  $A$ -parameter of  $\hat{\pi}$ . Then*

$$\hat{\psi}(w, x, y) = \psi(w, y, x).$$

**Proof.** We associate to  $\pi$  a generic supercuspidal representation  $\sigma_0$  and a sequence segments  $\Sigma_1, \dots, \Sigma_k, \Sigma_{k+1}, \dots, \Sigma_l, \Sigma_{l+1}, \dots, \Sigma_m$  as in section 4. The sets  $P = \{1, \dots, k\}$ ,  $Q = \{k + 1, \dots, l\}$ ,  $R = \{l + 1, \dots, m\}$ ,  $R^* = \{q \in R \mid \Sigma_q \text{ is balanced}\}$  and  $R^{**} = R \setminus R^*$  are defined as earlier. The sequence  $\Sigma_{l+1}, \dots, \Sigma_m$  is unique up to a permutation and taking contragredient, so we may assume

$$R^* = \{\Sigma_{l+1}, \dots, \Sigma_p\}, \quad R^{**} = \{\Sigma_{p+1}, \dots, \Sigma_m\}.$$

For the same reason, we may assume that the exponents  $c_q, d_q$  in the segments

$$\Sigma_q = [\nu^{c_q} \rho_q, \nu^{d_q} \rho_q], \quad q = p + 1, \dots, m,$$

satisfy  $\alpha_q = \frac{c_q + d_q}{2} > 0$  and  $\alpha_{p+1} \geq \dots \geq \alpha_m > 0$ . Then

$$\pi = L_q(\delta[\nu^{c_{p+1}} \rho_{p+1}, \nu^{d_{p+1}} \rho_{p+1}], \dots, \delta[\nu^{c_m} \rho_m, \nu^{d_m} \rho_m], \tau),$$

where the tempered representation  $\tau$  is the unique generic component of the representation  $(\times_{q \in P \cup Q \cup R^*} \delta(\Sigma_q)) \rtimes \sigma_0$ . The exponents  $a_q, b_q$  in the segments

$$\Sigma_q = [\nu^{-a_q} \rho_q, \nu^{b_q} \rho_q], \quad q = 1, \dots, p,$$

satisfy the conditions of section 4. Let  $\phi_q$  be the  $L$ -parameter of  $\rho_q$ ,  $q = 1, \dots, m$ . The  $L$ -parameter  $\phi$  of  $\pi$  is given by equation (5) and the  $L$ -parameter  $\varphi(\tau)$  of  $\tau$  is given by Theorem 5.3.

Now, suppose  $\phi_\psi = \phi$ , for the  $A$ -parameter  $\psi = \bigoplus_{i=1}^r (\phi'_i \otimes S_{m_i} \otimes S_{n_i})$ . We have

$$\begin{aligned} \phi_\psi &= \bigoplus_{i=1}^r \bigoplus_{j_i = -(n_i-1)/2}^{(n_i-1)/2} |\cdot|^{j_i} \phi'_i \otimes S_{m_i} \\ &= \bigoplus_{n_i \text{ even}} \bigoplus_{j_i = 1/2}^{(n_i-1)/2} |\cdot|^{j_i} \phi'_i \otimes S_{m_i} \oplus |\cdot|^{-j_i} \tilde{\phi}'_i \otimes S_{m_i} \\ &\oplus \bigoplus_{n_i \text{ odd}} \bigoplus_{j_i = 1}^{(n_i-1)/2} |\cdot|^{j_i} \phi'_i \otimes S_{m_i} \oplus |\cdot|^{-j_i} \tilde{\phi}'_i \otimes S_{m_i} \oplus \bigoplus_{n_i \text{ odd}} \phi'_i \otimes S_{m_i}. \end{aligned} \tag{12}$$

It follows from equations (5) and (12) that each  $\phi'_i$  is equal to some  $\phi_q$  and  $\varphi(\tau) = \bigoplus_{n_i \text{ odd}} \phi'_i \otimes S_{m_i}$ . We want to show  $\pi = \tau$ , i.e.,  $n_i = 1$ , for all  $i = 1, \dots, r$  and  $R^{**} = \emptyset$ .



Assume first, for some  $i$ ,  $n_i \geq 4$  and  $n_i$  is even. We take the terms in equation (12) coming from  $j_i = \frac{1}{2}, \frac{3}{2}$  and apply equations (4), (5) to find the corresponding segments. It follows the segments  $[\nu^{-\frac{m_i}{2}+1}\rho'_i, \nu^{\frac{m_i}{2}}\rho'_i], [\nu^{-\frac{m_i}{2}+2}\rho'_i, \nu^{\frac{m_i}{2}+1}\rho'_i]$  are in the Langlands data of  $\pi$ . These two segments are linked, which contradicts section 4. Similar arguments rule out the case  $n_i \geq 4$ ,  $n_i$  odd.

Therefore,  $n_i \leq 3$ , for all  $i$ . Assume now, for some fixed  $i$ ,  $1 < n_i \leq 3$ . Assume in addition  $(m_i, n_i) \neq (1, 3)$ . The terms in equation (12) coming from  $j_i = (n_i - 1)/2$  correspond to a certain imbalanced segment  $\Sigma_q = \Sigma = [\nu^c\rho, \nu^d\rho]$  in the Langlands data of  $\pi$ . More precisely,

$$\begin{aligned} \Sigma &= [\nu^{-\frac{m_i}{2}+1}\rho, \nu^{\frac{m_i}{2}}\rho], & n_i &= 2, \\ \Sigma &= [\nu^{-\frac{m_i}{2}+\frac{3}{2}}\rho, \nu^{\frac{m_i}{2}+\frac{1}{2}}\rho], & n_i &= 3. \end{aligned}$$

If  $\phi'_i \otimes S_{m_i} \otimes S_{n_i}$  is not symplectic, then  $\psi$  in addition contains the term  $\tilde{\phi}'_i \otimes S_{m_i} \otimes S_{n_i}$  (if  $\tilde{\phi}'_i \cong \phi'_i$ , then  $\phi'_i \otimes S_{m_i} \otimes S_{n_i}$  appears with even multiplicity). The term  $\tilde{\phi}'_i \otimes S_{m_i} \otimes S_{n_i}$  gives the segment  $\Sigma' = [\nu^c\tilde{\rho}, \nu^d\tilde{\rho}]$ . Then  $\Sigma$  and  $\tilde{\Sigma}' = [\nu^{-d}\rho, \nu^{-c}\rho]$  are linked, which contradicts condition (vii) in section 4. It follows that  $\phi'_i \otimes S_{m_i} \otimes S_{n_i}$  is symplectic.

Define  $\tau_0$  to be the tempered generic representation with the  $L$ -parameter  $\phi'_i \otimes S_{m_i} \oplus \varphi(\sigma_0)$ , for  $n_i = 3$ , and  $\tau_0 = \sigma_0$ , for  $n_i = 2$ . Define

$$\pi_0 = L_s(\delta(\tilde{\Sigma}), \tau_0).$$

Since  $\phi'_i \otimes S_{m_i} \otimes S_{n_i}$  is symplectic, the representation  $\pi_0$  is precisely the representation considered in Theorem 3.1. In particular, it follows from the proof of Theorem 3.1 that  $\delta(\Sigma) \rtimes \sigma_0$  is reducible.

On the other hand, we know from (viii) in section 4. that  $\delta(\Sigma) \rtimes \sigma^{(ell)}$  is irreducible. Then Theorem 4.2 of [24] tells us one of the following two conditions is satisfied

- (\*)  $\delta(\Sigma) \rtimes \sigma_0$  is irreducible, or
- (\*\*) there exists  $t \in C_{-1}$  such that the segments  $[\rho_t]$  and  $\Sigma$  are linked.

Note that  $\Sigma$  and  $[\rho']$  are not linked, for any unitary  $\rho'$ . Since  $\delta(\Sigma) \rtimes \sigma_0$  is reducible, we see that the assumption  $1 < n_i \leq 3$ ,  $(m_i, n_i) \neq (1, 3)$  leads to a contradiction.

It remains to consider  $(m_i, n_i) = (1, 3)$ . In this case,  $\Sigma = [\nu\rho]$ , so  $\Sigma$  and  $\tilde{\Sigma}' = [\nu^{-1}\rho]$  are not linked. Again, we have  $\delta(\Sigma) \rtimes \sigma^{(ell)}$  is irreducible, so one of the conditions (\*), (\*\*) holds. In addition, Theorem 4.2 of [24] implies

- (†) the segments  $\Sigma$  and  $\Sigma_t$  (respectively,  $\Sigma$  and  $\tilde{\Sigma}_t$ ),  $t \in Q \cup R^*$ , are not linked, and
- (‡) the segments  $\Sigma$  and  $[\nu^{-at}\rho_t, \nu^{at}\rho_t]$ ,  $t \in P$ , are not linked.

Notice that  $\varphi(\tau)$  contains  $\phi'_i \otimes S_1$ , because  $n_i$  is odd. From Theorem 5.3, we have

$$\varphi(\tau) = \varphi(\sigma) \oplus \left( \bigoplus_{t \in Q \cup R} \left( \phi_t \otimes S_{2b_t+1} \oplus \tilde{\phi}_t \otimes S_{2b_t+1} \right) \right),$$

$$\varphi(\sigma) = \left( \bigoplus_{i \in P} \phi_t \otimes S_{2b_t+1} \right) \oplus \left( \bigoplus_{t \in P_0} \phi_t \otimes S_{2a_t+1} \right) \oplus \left( \bigoplus_{t \in A_0} \phi_t \otimes S_1 \right).$$

We conclude from (†) that  $\phi'_i \otimes S_1$  is a part of  $\varphi(\sigma)$ ; otherwise, it would correspond to the segment  $[\rho]$ , which is linked to  $\Sigma$ . Similarly, (‡) implies that  $\phi'_i \otimes S_1$  does not belong to  $\bigoplus_{t \in P_0} \phi_t \otimes S_{2a_t+1}$ . Finally, the assumption  $\phi'_i \otimes S_1$  is a part of  $\bigoplus_{t \in A_0} \phi_t \otimes S_1$  contradicts both (\*) and (\*\*). Therefore,  $(m_i, n_i) \neq (1, 3)$ .

We have proved that  $\pi$  is tempered. Next, we will show that the  $A$ -parameters of  $\pi$  and  $\hat{\pi}$  are symmetric. The proof is just an extension by the terms corresponding to  $Q \cup R$  of the proof of Corollary 5.1 in [6]. From Theorem 5.3, we have

$$\begin{aligned} \psi = & \left( \bigoplus_{i \in P} \phi_i \otimes S_{2b_i+1} \otimes S_1 \right) \oplus \left( \bigoplus_{i \in P_0} \phi_i \otimes S_{2a_i+1} \otimes S_1 \right) \oplus \left( \bigoplus_{i \in A_0} \phi_i \otimes S_1 \otimes S_1 \right) \\ & \oplus \left( \bigoplus_{i \in Q \cup R} \left( \phi_i \otimes S_{2b_i+1} \otimes S_1 \oplus \tilde{\phi}_i \otimes S_{2b_i+1} \otimes S_1 \right) \right), \end{aligned}$$

using the fact that all the segments  $\Sigma_i = [\nu^{-a_i} \rho_i, \nu^{b_i} \rho_i] = [\nu^{-b_i} \rho_i, \nu^{b_i} \rho_i]$ ,  $i \in R$  are balanced. Define

$$\begin{aligned} \hat{\psi} = & \left( \bigoplus_{i \in P} \phi_i \otimes S_1 \otimes S_{2b_i+1} \right) \oplus \left( \bigoplus_{i \in P_0} \phi_i \otimes S_1 \otimes S_{2a_i+1} \right) \oplus \left( \bigoplus_{i \in A_0} \phi_i \otimes S_1 \otimes S_1 \right) \\ & \oplus \left( \bigoplus_{i \in Q \cup R} \left( \phi_i \otimes S_1 \otimes S_{2b_i+1} \oplus \tilde{\phi}_i \otimes S_1 \otimes S_{2b_i+1} \right) \right). \end{aligned}$$

Then  $\hat{\psi}(w, x, y) = \psi(w, y, x)$ . We will prove that  $\hat{\psi}$  is the  $A$ -parameter of  $\hat{\pi}$ . We have

$$\begin{aligned} \phi_{\hat{\psi}} = & \left( \bigoplus_{i \in P} \bigoplus_{j=-b_i}^{b_i} |\cdot|^j \phi_i \otimes S_1 \right) \oplus \left( \bigoplus_{i \in P_0} \bigoplus_{j=-a_i}^{a_i} |\cdot|^j \phi_i \otimes S_1 \right) \oplus \left( \bigoplus_{i \in A_0} \phi_i \otimes S_1 \right) \\ & \oplus \left( \bigoplus_{i \in Q \cup R} \bigoplus_{j=-b_i}^{b_i} \left( |\cdot|^j \phi_i \otimes S_1 \oplus |\cdot|^j \tilde{\phi}_i \otimes S_1 \right) \right) \\ = & \left( \bigoplus_{i \in T} \bigoplus_{j=-a_i}^{b_i} \left( |\cdot|^j \phi_i \otimes S_1 \oplus |\cdot|^{-j} \tilde{\phi}_i \otimes S_1 \right) \right) \oplus \left( \bigoplus_{i \in A} \phi_i \otimes S_1 \right). \end{aligned}$$

We recognize this as the  $L$ -parameter of  $\hat{\pi}$  given in Theorem 5.3, finishing the proof. ■

### References

- [1] Arthur, J., *Unipotent automorphic representations: conjectures*, Astérisque, **171-172** (1989), 13–71.
- [2] —, *On some problems suggested by the trace formula*, in: Lie group representations, II, 1–49, Lecture Notes in Math. **1041**, Springer, Berlin, 1984.

- [3] —, *Dualité dans le groupe de Grothendieck de la catégorie des représentations lisses de longueur finie d'un groupe réductif  $p$ -adique*, Trans. Amer. Math. Soc. **347** (1995), 2179–2189, and *Erratum*, *ibid.* **348** (1996), 4687–4690.
- [4] Ban, D., *The Aubert involution and  $R$ -groups*, Ann. Sci. École Norm. Sup. **35** (2002), 673–693.
- [5] —, *Linear independence of intertwining operators*, J. Algebra **271** (2004), 749–767.
- [6] Ban, D., and Y. Zhang, *Arthur  $R$ -Groups, Classical  $R$ -groups, and Aubert Involutions for  $SO(2n + 1)$* , Compositio Math **141** (2) (2005), 323–343.
- [7] Ban, D., and C. Jantzen, *Degenerate principal series for even orthogonal groups*, Representation Theory **7** (2003), 440–480.
- [8] Bernstein, J., “Representations of  $p$ -adic groups,” Lectures, Harvard University, 1992.
- [9] Bernstein, I. N., and A. V. Zelevinsky, *Induced representations of reductive  $p$ -adic groups, I*, Ann. Sci. École Norm. Sup. **10** (1977), 441–472.
- [10] Borel, A., *Automorphic  $L$ -functions*, in: Automorphic Forms, Representations, and  $L$ -functions, Part 2, Proc. Symp. Pure Math. **33** (1979), 27–61.
- [11] Casselman, W., *Introduction to the theory of admissible representations of  $p$ -adic reductive groups*, Preprint, available at <http://www.math.ubc.ca/~cass/research.html>.
- [12] Cogdell, J. W. , H. H. Kim, I. I. Piatetski-Shapiro, and F. Shahidi, *Functoriality for the Classical Groups*, Publ. Math. Inst. Hautes Études Sci. **99** (2004), 163–233.
- [13] Goldberg, D., *Reducibility of induced representations for  $Sp(2n)$  and  $SO(n)$* , Amer. J. Math. **116** (1994), 1101–1151.
- [14] Harris, M., and R. Taylor, “On the geometry and cohomology of some simple Shimura varieties,” Ann. of Math. Studies **151**, Princeton University Press, 2001.
- [15] Henniart, G., *Une preuve simple des conjectures de Langlands pour  $GL(n)$  sur un corps  $p$ -adique*, Inv. Math. **139** (2000), 439–455.
- [16] —, *Correspondance de Langlands et fonctions  $L$  des carrés extérieur et symétrique*, Prépublication IHES, Mars 2003.
- [17] Hiraga, K., *On functoriality of Zelevinski involutions*, Compos. Math. **140** (2004), 1625–1656.
- [18] Jantzen, C., *Degenerate principal series for symplectic and odd-orthogonal groups*, Mem. Amer. Math. Soc. **590** (1996), 1–100.
- [19] Jiang, D., and D. Soudry, *Generic representations and Local Langlands Reciprocity Law for  $p$ -adic  $SO_{2n+1}$* , Contributions to automorphic forms, geometry, and number theory, Johns Hopkins Univ. Press, Baltimore, MD, 2004, 457–519.
- [20] Kim, H. H., and F. Shahidi, *Quadratic unipotent Arthur parameters and residual spectrum of symplectic groups*, American J. Math. **118** (1996), 401–425.

- [21] Knapp, A. W., *Introduction to the Langlands program*, *Representation Theory and Automorphic Forms* (Edinburg, 1996), Proc. Symp. Pure Math. **61** (1997), 245–302.
- [22] Langlands, R., *On the classification of irreducible representations of real algebraic groups*, in: Representation Theory and Harmonic Analysis on Semisimple Lie Groups, Amer. Math. Soc. Mathematical Surveys and Monographs, **31** (1989), 101–170.
- [23] Mœglin, C. and J.L. Waldspurger, *Sur l'involution de Zelevinski*, J. Reine Angew. Math. **372** (1986), 136–177.
- [24] Muić, G., *On generic irreducible representations of  $Sp(n, F)$  and  $SO(2n+1, F)$* , Glasnik Mat. **33** (53) (1998), 19–31.
- [25] —, *Some results on square integrable representations; irreducibility of standard representations*, International Mathematics Research Notices **14** (1998), 705–726.
- [26] Schneider, P., and U. Stuhler, *Representation theory and sheaves on the Bruhat-Tits building*, Publ. Math. IHES **85** (1997), 97–191.
- [27] Shahidi, F., *A proof of Langlands' conjecture on Plancherel measures; Complementary series for  $p$ -adic groups*, Ann. of Math. **132** (1990), 273–330.
- [28] —, *Twisted endoscopy and reducibility of induced representations for  $p$ -adic groups*, Duke Math. J. **66** (1992), 1–41.
- [29] Tadić, M., *Classification of unitary representations in irreducible representations of general linear group (non-archimedean case)*, Ann. Sci. École Norm. Sup. **19** (1986), 335–382.
- [30] —, *A family of square integrable representations of classical  $p$ -adic groups in the case of general half-integral reducibilities*, Glas. Mat. Ser. III **37** (57) (2002), 21–57.
- [31] Tate, J., *Number theory background, in: Automorphic Forms, Representations, and  $L$ -functions, Part 2*, Proc. Symp. Pure Math. **33** (1979), 1–26.
- [32] Zelevinsky, A. V., *Induced representations of reductive  $p$ -adic groups, II: On irreducible representations of  $GL(n)$* , Ann. Sci. École Norm. Sup. **13** (1980), 165–210.

Dubravka Ban  
Department of Mathematics  
Southern Illinois University  
Carbondale, IL 62901  
USA, and  
Universität Münster  
Mathematisches Institut  
Einsteinstr. 62  
D-48149 Münster  
Germany  
dban@math.siu.edu

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