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Local Coefficient Matrices of Metaplectic Groups

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Abstract. The principal series representations of the n-fold metaplectic covers of the general linear group $\mathrm{GL_r}(\mathbb{F})$ were described in the foundational paper "Metaplectic Forms," by Kazhdan and Patterson (1984). In this paper, we study the local coefficient matrices for a certain class of principal series representations over $\mathrm{GL}_2(\mathbb{F})$, where $\mathbb{F}$ is a nonarchimedean local field. The local coefficient matrices can be described in terms of the intertwining operators and Whittaker functionals associated to such representations in a standard way. We characterize the nonsingularity of local coefficient matrices in terms of the nonvanishing of certain local ζ-functions by computing the determinant of the local coefficient matrices explicitly. Using these results, it can be shown that for any divisor d of n, the irreducibility of the given principal series representation on the n-fold metaplectic cover of $\mathrm{GL}_2(\mathbb{F})$ is intimately related to the irreducibility of its d-fold counterpart.
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Introduction

Suppose that $\mathbb{F}$ is a nonarchimedean local field containing the full group of n^{th} roots of unity μ_n. One can define the n-fold metaplectic group $\widetilde{\mathrm{GL}}_r(\mathbb{F})$ to be a nontrivial central extension of $\mathrm{GL_r}(\mathbb{F})$ by μ_n, where multiplication is defined by a 2-cocycle in $Z^2(\mathrm{GL_r}(\mathbb{F});\mu_\mathrm{n})$. Such a cocycle was explicitly constructed for $\mathrm{GL}_2(\mathbb{F})$ by Kubota [9] and extended to $\mathrm{GL_r}(\mathbb{F})$ for $r>2$ by Matsumoto [10]. In [8], Kazhdan and Patterson described the principal series representations of the metaplectic covers of $\mathrm{GL_r}(\mathbb{F})$ and characterized their irreducibility. While such representations do not in general have unique Whittaker models, there is a finite-dimensional space of Whittaker functionals. The "local coefficient matrices" of this paper are formed by considering the action of the standard intertwining operator on a certain canonical basis of the functionals.

In the case $G=\mathrm{GL}_2(\mathbb{F})$, the nonsingularity of local coefficient matrices is shown to be equivalent to irreducibility of a particular class of representations in the unramified principal series. By considering these representations over different

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coverings of G, it is then shown that a generalization of the local Shimura correspondence preserves irreducibility. Such a local correspondence was previously described by Flicker (Section 5.2, [7]) for genuine irreducible admissible representations of n-fold coverings of G (Shimura's correspondence is the case $n = 2$). In particular, Flicker handled the correspondence for the principal series in Section 2.1 of [7].

The paper proceeds as follows. In Section 1, the definitions and notations to be used throughout the rest of the paper are given. For a nonzero $s \in \mathbb{C}$ and a fixed covering n, the unramified principal series representation (π_s, V_s) of $\widetilde{G}$ is constructed in Section 2 (we later use the notation $(\pi_s, V_s)^{(n)}$ when n is no longer fixed). The local coefficients associated with such representations are described in Section 3 following the derivation of Kazhdan and Patterson (Section 1.3, [8]).

The new results are contained in Section 4 and are an extension of the results obtained in [4]. Here, we arrange the local coefficients into an $n^2 \times n^2$ matrix and explicitly compute its determinant. The determinant of the local coefficient matrix depends upon s and the covering n. In Theorem 4.2, we prove the local correspondence: if d divides n and $|n|_{\mathbb{F}} = 1$, then

$$(\pi_s, V_s)^{(n)} \text{ irreducible} \iff (\pi_{sn/d}, V_{sn/d})^{(d)} \text{ irreducible.}$$

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1. Preliminaries

Throughout this paper, $\mathbb{F}$ will denote a nonarchimedean local field containing μ_n, the full group of n^{th} roots of unity. We assume that $\mathbb{F}$ is a finite extension of the $p-$adic field $\mathbb{Q}_p$ and fix an embedding $\epsilon : \mu_n \hookrightarrow \mathbb{C}^\times$, identifying μ_n with the group of n^{th} roots of unity in $\mathbb{C}^\times$. For any such field, let $v : \mathbb{F}^\times \to \mathbb{Z}$ be a normalized valuation. As usual, we extend v to all of $\mathbb{F}$ by setting $v(0) = \infty$ (with the convention that $\infty + z = \infty = z + \infty$ for all $z \in \mathbb{Z}$).

Let $\mathcal{O}$ denote the ring of integers $\{x \in \mathbb{F} \mid v(x) \geq 0\}$ in $\mathbb{F}$ and let $\mathfrak{p}$ denote the (unique) maximal ideal $\{x \in \mathbb{F} \mid v(x) \geq 1\}$ in $\mathcal{O}$. We will let ϖ denote a prime element in $\mathfrak{p}$ (that is, an element such that $v(\varpi) = 1$). The order of the residue field $\mathcal{O}/\mathfrak{p}$ is given by $q = p^f$ for some $f \in \mathbb{N}$. The absolute value on $\mathbb{F}$ is then given by $|x|_v = q^{-v(x)}$ and the local ζ-function at v is defined by

$$\zeta_v(s) = (1 - q^{-s})^{-1}.$$

The existence of the Hilbert symbol over $\mathbb{F}$ is well known (see [12], Chap. VIII sec. 5). The n^{th} order Hilbert symbol is a map

$$(\cdot, \cdot)_{\mathbb{F}} : \mathbb{F}^\times \times \mathbb{F}^\times \longrightarrow \mu_n$$

that satisfies the following conditions for all $a, a', b \in \mathbb{F}^\times$:

$$(aa', b)_{\mathbb{F}} = (a, b)_{\mathbb{F}}\ (a', b)_{\mathbb{F}},$$
$$(a, b)_{\mathbb{F}}\ (b, a)_{\mathbb{F}} = 1,$$
$$(a, -a)_{\mathbb{F}} = 1 = (a, 1-a)_{\mathbb{F}} \quad \text{when } a \neq 1, \text{ and}$$
$$\{x \in \mathbb{F}^\times \mid (x, y)_{\mathbb{F}} = 1, \text{ for all } y \in \mathbb{F}^\times\} = \mathbb{F}^{\times n},$$

where

$$\mathbb{F}^{\times n} := \{x \in \mathbb{F}^\times \mid x = y^n, \text{ for some } y \in \mathbb{F}^\times\}.$$

For our purposes, we assume that the Hilbert symbol is unramified: $(x, y)_{\mathbb{F}} = 1$ for all $x, y \in \mathcal{O}^\times$. It is known that this condition is equivalent to $|n|_{\mathbb{F}} = 1$.

For $G = \mathrm{GL}_2(\mathbb{F})$, the n-fold ($0-$twisted) metaplectic group $\widetilde{G} = \widetilde{\mathrm{GL}}_2(\mathbb{F})$ is a nontrivial central extension of G by μ_n:

$$1 \longrightarrow \mu_n \xrightarrow{\ \mathbf{i}\ } \widetilde{G} \xrightarrow{\ \mathbf{p}\ } G \longrightarrow 1. \tag{1}$$

As a set, the elements of $\widetilde{G}$ are of the form (g, ζ) where $g \in G$ and $\zeta \in \mu_n$. Multiplication is given by

$$(g, \zeta)(g', \zeta') = (gg',\ \zeta\zeta'\ \sigma(g, g')),$$

where σ is a nontrivial 2-cocycle in $Z^2(G; \mu_n)$ that satisfies the cocycle relation

$$\sigma(g, g')\sigma(gg', g'') = \sigma(g, g'g'')\sigma(g', g''), \tag{2}$$

for all $g, g', g'' \in G$. It is an easy exercise to check that (2) is equivalent to associativity of multiplication in $\widetilde{G}$. The cocycle σ was first described by Kubota [9] and is given by

$$\sigma(g, g') = \left(\frac{X(gg')}{X(g)}, \frac{X(gg')}{X(g')\det(g)}\right)_{\mathbb{F}}$$

where

$$X\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \begin{cases} c & \text{if } c \neq 0 \\ d & \text{if } c = 0 \end{cases}.$$

In the short exact sequence (1), the projection map $\mathbf{p} : \widetilde{G} \longrightarrow G$ is given by $(g, \zeta) \mapsto g$ and the inclusion map $\mathbf{i} : \mu_n \longrightarrow \widetilde{G}$ is given by $\zeta \mapsto (1, \zeta)$. We will also use the preferred section $\mathbf{s} : G \longrightarrow \widetilde{G}$, given by $g \mapsto (g, 1)$.

For every $c \in \mathbb{Z}/n\mathbb{Z}$, there exists a $c-$twisted cover $\widetilde{G}^{(c)} = \widetilde{\mathrm{GL}}_2^{(c)}(\mathbb{F})$, defined by the short exact sequence

$$1 \longrightarrow \mu_n \xrightarrow{\ \mathbf{i}\ } \widetilde{G}^{(c)} \xrightarrow{\ \mathbf{p}^{(c)}\ } G \longrightarrow 1,$$

where the cocycle $\sigma^{(c)}$ is given by

$$\sigma^{(c)}(g, g') = \sigma(g, g')(\det(g), \det(g'))_{\mathbb{F}}^c$$

Finally, we fix a nontrivial additive (unitary) character $\psi : \mathbb{F} \longrightarrow \mathbb{C}^\times$ that is unramified. For every $i \in \mathbb{Z}/n\mathbb{Z}$, the unnormalized n^{th} order Gauss sum is defined by

$$\mathfrak{g}_\psi^{(i)} := q \int\limits_{x \in \mathcal{O}^\times} (\varpi, x)_{\mathbb{F}}^i \psi(\varpi^{-1}x)\ dx,$$

where dx is the unique additive Haar measure satisfying $\mathrm{Vol}(\mathcal{O}; dx) = 1$. A well known property of Gauss sums that will be needed in Section 4 is

$$\mathfrak{g}_\psi^{(i)} \mathfrak{g}_\psi^{(-i)} = q(\varpi, \varpi)_{\mathbb{F}}^i.$$

2. Principal Series Representations

In this section, we describe the class of principal series representations to be used throughout the rest of the paper. For the construction of our class of representations, we need to consider several important subgroups of $\widetilde{G} = \widetilde{\mathrm{GL}}_2(\mathbb{F})$. Let T denote the subgroup of diagonal matrices (the torus) in G and denote by $\widetilde{T}^{(c)} := (\mathbf{p}^{(c)})^{-1}(T)$, the metaplectic preimage of T. Similarly, define the Borel subgroup B of upper triangular matrices and denote by $\widetilde{B}^{(c)} := (\mathbf{p}^{(c)})^{-1}(B)$, its metaplectic preimage. It will be convenient to leave off the superscript (c), but its dependence should not be forgotten.

The group B can be decomposed as $B = TN$ where N is the unipotent radical of B (the subgroup of B with 1's along the diagonal). The group $\widetilde{G}$ splits canonically over N via the $\mathbf{p}$-section $\mathbf{s}$. So we define $N^* := \mathbf{s}(N)$ and obtain the decomposition $\widetilde{B} = \widetilde{T}N^*$.

The group $\widetilde{G}$ also splits over the maximal compact subgroup $K = \mathrm{GL}_2(\mathcal{O})$ of G (see [11]). If we let

$$\mathbf{k} : K \longrightarrow \widetilde{G}$$

denote the canonical splitting, then by Proposition 0.1.3 of [8], $\mathbf{k}$ satisfies the relations

$$\mathbf{k}|_{T\cap K} = \mathbf{s}|_{T\cap K}, \quad \mathbf{k}|_W = \mathbf{s}|_W, \quad \text{and} \quad \mathbf{k}|_{N\cap K} = \mathbf{s}|_{N\cap K},$$

where W denotes the Weyl subgroup of G consisting of the identity matrix and

$$w = \mathbf{s}\begin{pmatrix} & 1 \\ 1 & \end{pmatrix}.$$

Let $K^* := \mathbf{k}(K)$ and for every $m \geq 0$, define $K_m^* := \mathbf{k}(K_m)$ where

$$K_m := \{k \in K \mid k \equiv I \pmod{\mathfrak{p}^m}\}.$$

The topology of $\widetilde{G}$ is defined by taking the collection K_m^* as a basis of open, compact neighborhoods of the identity in $\widetilde{G}$.

To obtain a principal series representation of $\widetilde{G}$, we induce from an irreducible representation of a subgroup of $\widetilde{B}$. We begin with a quasicharacter on the center of $\widetilde{T}$. As was noted in Section 1.1 of [8], the center of $\widetilde{T}$ is given by $\widetilde{T}^n Z(\widetilde{B})$ where

$$\widetilde{T}^n = \mathbf{p}^{-1}(T^n) := \mathbf{p}^{-1}\left\{\left.\begin{pmatrix} x & \\ & y \end{pmatrix}\right| x, y \in \mathbb{F}^{\times n}\right\}$$

and the center of $\widetilde{B}$, $Z(\widetilde{B})$, is given by

$$\mathbf{p}^{-1}\left\{\left(\begin{smallmatrix}\lambda & \\ & \lambda\end{smallmatrix}\right)\middle|\ \lambda^{1+4c}\in\mathbb{F}^{\times n}\right\}.$$

Let δ_B denote the modular (quasi)character of the Borel subgroup. We will consider δ to be the quasicharacter δ_B on $\widetilde{T}^n Z(\widetilde{B})$ that is genuine. In other words,

$$\delta\left(\left(\left(\begin{smallmatrix}x & \\ & y\end{smallmatrix}\right),\zeta\right)\right)=\zeta\left|\frac{x}{y}\right|_{\mathbb{F}}.$$

This quasicharacter can be extended to a maximal abelian subgroup $\widetilde{T}_*$ of $\widetilde{T}$ in the obvious way.

Our choice of maximal abelian subgroup was described in [2]. It is defined by

$$\widetilde{T}_* := \mathbf{p}^{-1}\left\{\left(\begin{smallmatrix}x & \\ & y\end{smallmatrix}\right)\middle|\ x,y\in\mathbb{F}_*\right\}$$

where

$$\mathbb{F}_* := \{x\in\mathbb{F}^\times \mid v(x)\equiv 0 \pmod{n}\} = \varpi^{n\mathbb{Z}}\mathcal{O}.$$

The quasicharacter δ is extended to $\widetilde{B}_* := \widetilde{T}_* N^*$ by making it trivial on N^*.

For any fixed nonzero $s\in\mathbb{C}$ and any unramified quasicharacter

$$\chi : \mathbb{F}^\times \longrightarrow \mathbb{C}^\times,$$

define the (normalized) induced representation

$$(\pi_s, V_s) := \mathrm{Ind}_{\widetilde{B}_*}^{\widetilde{G}}((\chi\circ\det\circ\mathbf{p})\otimes\delta^s)$$

where

$$V_s = \left\{ f\in C^\infty(\widetilde{G}) \,\middle|\, f(bg)=\chi(\det(\mathbf{p}(b)))\cdot\delta^{s+1/2}(b)\cdot f(g),\ \text{for all } b\in\widetilde{B}_*,\ g\in\widetilde{G}\right\}$$

and π_s acts by right translation:

$$(\pi_s(g)f)(g') = f(g'g).$$

Stone and von-Neumann's Theorem (see [5] or [6]) guarantees that the isomorphism class of a representation constructed in this way depends only on the central quasicharacter $(\chi\circ\det\circ\mathbf{p})\otimes\delta^s$, not on the choice of maximal abelian subgroup nor on the quasicharacter's extension.

3. Local Coefficients

To define the local coefficients of (π_s, V_s), we first consider the intertwining operators and Whittaker functionals associated to such representations. The standard intertwining operator

$$\mathbb{I}_s : V_s \longrightarrow V_{-s}$$

satisfies

$$\mathbb{I}_s(\pi_s(g)f) = \pi_{-s}(g)(\mathbb{I}_s f) \quad \text{for all } g\in\widetilde{G}, f\in V_s$$

and is given by the absolutely convergent integral

$$\mathbb{I}_s f(g) = \int\limits_{n\in N^*} f(w^{-1}ng)\ dn$$

where $f \in V_s$ and dn is the unique Haar measure that satisfies $\mathrm{Vol}(N^* \cap K^*; dn) = 1$.

To define a Whittaker functional of V_s, we first fix a nontrivial additive (unitary) character

$$\psi : \mathbb{F} \longrightarrow \mathbb{C}^\times.$$

We also assume that ψ is unramified and extend it to a multiplicative character

$$\psi_{N^*} : \mathbb{N}^* \longrightarrow \mathbb{C}^\times$$

by setting

$$\psi_{N^*}\left(\mathbf{s}\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}\right) = \psi(x).$$

It simplifies notation to denote ψ_{N^*} by ψ and this should cause no confusion. A $\psi-$Whittaker functional is a functional λ in the dual space V_s' of V_s that satisfies

$$\lambda(\pi_s(n)f) = \psi(n)\lambda(f) \tag{3}$$

for all $n \in N^*$ and $f \in V_s$.

For $\eta \in \widetilde{T}$, let $\lambda_\eta \in V_s'$ be given by

$$\lambda_\eta f = \int\limits_{n\in N^*} f(\eta w n)\ \overline{\psi}(n)\ dn. \tag{4}$$

Kazhdan and Patterson (Lemma 1.2.3 of [8]) showed that the integral (4) is absolutely convergent for $\mathrm{Re}(s) > 0$ and extends holomorphically to all of $\mathbb{C}$. One can check that λ_η satisfies (3) and that

$$\{\lambda_\eta \mid \eta \in \widetilde{T}_* \backslash \widetilde{T}\}$$

is a basis for $\mathrm{Wh}(V_s)$, the space of $\psi-$Whittaker functionals of V_s. We will use the set

$$\Omega := \left\{ \mathbf{s}\begin{pmatrix} \varpi^i & \\ & \varpi^j \end{pmatrix} \middle|\ i, j \in \mathbb{Z}/n\mathbb{Z} \right\}$$

as representatives of $\widetilde{T}_* \backslash \widetilde{T}$.

The local coefficients $\tau_s(\eta, \eta')$ of (π_s, V_s) are defined by the equation

$$\lambda_\eta \mathbb{I}_s = \sum_{\eta' \in \Omega} \tau_s(\eta, \eta')\lambda_{\eta'}. \tag{5}$$

Since Ω has cardinality n^2, there are n^4 local coefficients.

To evaluate the local coefficients, we follow [8], Section 1.3, starting with an appropriate set of "test" functions in V_s. Let

$$f_{\eta'}(g) = \begin{cases} \chi(\det(\mathbf{p}(b))) \cdot \delta^{s+1/2}(b) & \text{if} \quad g = b\eta' l w,\ b \in \widetilde{B}_*,\ l \in L \\ 0 & \qquad\qquad \text{otherwise} \end{cases}$$

where L is an open, compact subgroup of $\widetilde{G}$ that is taken to be sufficiently small.

Theorem 3.1. *The local coefficients are given by*

$$\tau_s(\eta,\eta') = \begin{cases} (-1,-1)_{\mathbb{F}}^{c} \cdot \frac{\epsilon_1 \cdot \zeta_v(-2ns\epsilon_1)}{\zeta_v(1-2ns\epsilon_2)} & \text{if} \quad \eta = \eta' \\ c_1 \cdot \mathfrak{g}_{\psi}^{(j'-j)} & \text{if} \quad \eta \neq \eta' \text{ and } \begin{array}{l} j \equiv i'+1 \ (mod\ n) \\ j' \equiv i+1 \ (mod\ n) \end{array} \\ 0 & \text{otherwise} \end{cases}$$

where

$$c_1 = q^{(i'-i-j'+j)(s+1/2)}\chi(\varpi)^{i-i'+j-j'}(-1,-1)_{\mathbb{F}}^{c}\ q^{-2s(j-i-1)-1}(\varpi,\varpi)_{\mathbb{F}}^{j(j-i-1)},$$

$$\epsilon_1 = \begin{cases} 1 & \text{if} \quad j \leq i \\ -1 & \text{if} \quad i < j, \end{cases}$$

$$\epsilon_2 = \begin{cases} 1 & \text{if} \quad j \equiv i+1 \ (mod\ n) \\ 0 & \text{if} \quad j \not\equiv i+1 \ (mod\ n). \end{cases}$$

Proof. The proof of Theorem 3.1 is omitted as it is essentially the content of the proof of Lemma 1.3.3 in [8] applied to the representation (π_s, V_s). ■

4. Local Coefficient Matrices

So far, the nonzero complex number s and the natural number n have been fixed. Now, we treat s as a variable and in Theorem 4.2, we will consider the representation (π_s, V_s) over different values of $n \in \mathbb{N}$ where $|n|_{\mathbb{F}} = 1$. Define the matrix $M_n(s) := (\tau_s(\eta,\eta'))$ by placing $\tau_s(\eta,\eta')$ in the $(in+j+1)^{th}$ row and the $(i'n+j'+1)^{th}$ column, where

$$\eta = \mathbf{s}\begin{pmatrix} \varpi^i & \\ & \varpi^j \end{pmatrix} \quad \text{and} \quad \eta' = \mathbf{s}\begin{pmatrix} \varpi^{i'} & \\ & \varpi^{j'} \end{pmatrix}$$

We should note some important properties of the matrix $M_n(s)$. First, for every nonzero off-diagonal entry $\tau_s(\eta,\eta')$, there is a symmetric (but not equal) nonzero entry $\tau_s(\eta',\eta)$. This follows from the fact that the nonzero off-diagonal entries only occur when the symmetric system of congruences

$$\begin{cases} j \equiv i'+1 \pmod{n} \\ j' \equiv i+1 \pmod{n} \end{cases}$$

is satisfied. Also, from this system we see that no two nonzero off-diagonal entries occur in the same row or the same column.

Theorem 4.1. *The matrix $M_n(s)$ has determinant*

$$(-1)^{\frac{n^2-n}{2}} \frac{\zeta_v(2ns)^{\frac{n^2+n}{2}}\zeta_v(-2ns)^{\frac{n^2-n}{2}}}{\zeta_v(1-2ns)^{\frac{n^2+n}{2}}\zeta_v(1+2ns)^{\frac{n^2-n}{2}}}$$

and this determinant is zero if and only if

$$s = \pm\frac{1}{2n} + \frac{i\pi k}{n\log q}, \quad k \in \mathbb{Z}.$$

Proof. We begin by conjugating $M_n(s)$ by an appropriate matrix so that the resulting matrix has just 2×2 and 1×1 nonzero blocks along the diagonal. Let W_{n^2} denote the subgroup of $\mathrm{GL}_{n^2}(\mathbb{C})$ consisting of matrices that contain a single 1 in every row and every column, and zeros elsewhere. Also let $w_{r_1,r_2} \in W_{n^2}$ be the element formed by taking the identity matrix in $\mathrm{GL}_{n^2}(\mathbb{C})$ and interchanging the r_1^{th} row with the r_2^{th} row.

If $M_n(s)$ is conjugated by $w_{in+j+1,i'n+j'+1}$, the resulting matrix has

$$\begin{pmatrix} \tau_s(\eta,\eta) & \tau_s(\eta,\eta') \\ \tau_s(\eta',\eta) & \tau_s(\eta',\eta') \end{pmatrix}$$

as a 2×2 block along the diagonal.

Now let $w_o \in W_{n^2}$ be the element given by

$$w_0 := \prod_{R(i,j,i',j')} w_{in+j+1,i'n+j'+1}$$

where

$$R(i,j,i',j') = \{0 \le i,i',j,j' < n \mid j \equiv i'+1 \pmod{n} \text{ and } j' \equiv i+1 \pmod{n}\}.$$

The matrix $\overline{M}_n(s) = w_0^{-1}(M_n(s))w_0$ has only 2×2 and 1×1 nonzero blocks along the diagonal and has the same determinant as $M_n(s)$.

To compute the determinant of $\overline{M}_n(s)$, it helps to define an equivalence relation on our choice of representatives $\eta \in \Omega$ that provides a distinction between the 2×2 and 1×1 blocks. Let $\eta \sim \eta'$ if and only if

$$\eta = \eta' \quad \text{or} \quad \begin{cases} j \equiv i'+1 \pmod{n} \\ j' \equiv i+1 \pmod{n} \end{cases}.$$

We will let $\overline{\eta}$ denote the equivalence class of η and $|\overline{\eta}|$ denote the cardinality of $\overline{\eta}$.

First, we take care of the cases where $|\overline{\eta}| = 1$. For such an η, there are no nonzero off-diagonal entries in the same row or column as $\tau_s(\eta,\eta')$. These are exactly the cases in which $j \equiv i+1 (\mathrm{mod}\ n)$ and we have the following solutions:

$$\begin{array}{rccccccc} i: & 0 & 1 & 2 & \dots & n-2 & n-1 \\ j: & 1 & 2 & 3 & \dots & n-1 & 0 \end{array}.$$

In the one case where $i = n-1$ and $j = 0$, there is the contribution of the factor

$$(-1,-1)_{\mathbb{F}}^{c} \frac{\zeta_v(2ns)}{\zeta_v(1-2ns)} \tag{6}$$

to $\det(\overline{M}_n(s))$. For each of the other $n-1$ cases, there is a contribution of

$$-(-1,-1)_{\mathbb{F}}^{c} \frac{\zeta_v(-2ns)}{\zeta_v(1-2ns)}. \tag{7}$$

Now we consider the $|\overline{\eta}| = 2$ blocks. We count each 2×2 block by counting the entries in the upper triangle ($i' > i$). In the first such case, we have $j = 0$, $i' = n-1$, and i and j' are given by the possibilities

$$\begin{array}{rcccccc} i: & 0 & 1 & 2 & \dots & n-2 \\ j': & 1 & 2 & 3 & \dots & n-1 \end{array}.$$

There are $n-1$ blocks of this form and they all satisfy

$$\det\begin{pmatrix} \tau_s(\eta,\eta) & \tau_s(\eta,\eta') \\ \tau_s(\eta',\eta) & \tau_s(\eta',\eta') \end{pmatrix} = \frac{(\zeta_v(2ns))^2}{\zeta_v(1-2ns)\zeta_v(1+2ns)}. \tag{8}$$

Note that the case $j=1$ and $i'=0$ does not occur since we are counting the entries in the upper triangle. The remaining cases are given by

$$\begin{array}{r|c|cc|ccc|c|cccc} j: & 2 & 3 & 3 & 4 & 4 & 4 & \ldots & n-1 & n-1 & \ldots & n-1 \\ i': & 1 & 2 & 2 & 3 & 3 & 3 & \ldots & n-2 & n-2 & \ldots & n-2 \\ i: & 0 & 0 & 1 & 0 & 1 & 2 & \ldots & 0 & 1 & \ldots & n-3 \\ j': & 1 & 1 & 2 & 1 & 2 & 3 & \ldots & 1 & 2 & \ldots & n-2 \end{array}$$

and there are $(n-2)(n-1)/2$ such cases. For each of these blocks, we have a contribution of

$$\det\begin{pmatrix} \tau_s(\eta,\eta) & \tau_s(\eta,\eta') \\ \tau_s(\eta',\eta) & \tau_s(\eta',\eta') \end{pmatrix} = \frac{-\zeta_v(2ns)\zeta_v(-2ns)}{\zeta_v(1-2ns)\zeta_v(1+2ns)} \tag{9}$$

to the determinant of $\overline{M}_n(s)$.

Putting together (6), (7), (8), and (9), we have

$$\det(\overline{M}_n(s)) = (-1)^{\frac{n^2-n}{2}} \frac{\zeta_v(2ns)^{\frac{n^2+n}{2}}\zeta_v(-2ns)^{\frac{n^2-n}{2}}}{\zeta_v(1-2ns)^{\frac{n^2+n}{2}}\zeta_v(1+2ns)^{\frac{n^2-n}{2}}},$$

and this determinant is zero if and only if $q^{2ns-1}=1$ or $q^{-2ns-1}=1$. ■

Up to this point, we have assumed that the covering n is fixed. Now we denote our representation by $(\pi_s, V_s)^{(n)}$ and consider different values of n for which $|n|_{\mathbb{F}}=1$. We have the following local correspondence.

Theorem 4.2. *If d is any divisor of n and $|n|_{\mathbb{F}}=1$, then*

$$(\pi_s, V_s)^{(n)} \text{ irreducible} \iff (\pi_{sn/d}, V_{sn/d})^{(d)} \text{ irreducible.}$$

Proof. Since both $\mathbb{I}_s$ and $\mathbb{I}_{-s}$ are intertwining operators, so is their composition $\mathbb{I}_{-s}\circ\mathbb{I}_s$, giving us the following commutative diagram

$$\begin{array}{ccccc} V_s & \xrightarrow{\mathbb{I}_s} & V_{-s} & \xrightarrow{\mathbb{I}_{-s}} & V_s \\ {\scriptstyle \pi_s(g)}\big\downarrow & & \big\downarrow{\scriptstyle \pi_{-s}(g)} & & \big\downarrow{\scriptstyle \pi_s(g)} \\ V_s & \xrightarrow[\mathbb{I}_s]{} & V_{-s} & \xrightarrow[\mathbb{I}_{-s}]{} & V_s \end{array}$$

for all $g\in\widetilde{G}$. By Proposition 1.2.2 of [8], the space of intertwining operators $\mathrm{Hom}(V_s,V_s)$ has dimension less than or equal to 1. However, the identity map $Id: V_s \longrightarrow V_s$ is an intertwining operator and hence, the composition map is just

$$\mathbb{I}_{-s}\circ\mathbb{I}_s = \alpha\cdot Id : V_s \longrightarrow V_s$$

for some constant $\alpha\in\mathbb{C}$. Applying Theorem 1.2.9 of [8] to our representation, we have that the image of the map $\mathbb{I}_{-s}\circ\mathbb{I}_s$ is the unique irreducible subrepresentation of V_s. Hence, $(\pi_s,V_s)^{(n)}$ is irreducible if and only if $\alpha\neq 0$.

For $s \neq 0$, we may view $\tau_s(\eta, \eta')$ as a function of s. Using all of the Whittaker functionals $V_s \longrightarrow \mathbb{C}$, we form the map $\Delta_{s,\eta'} : V_s \longrightarrow \mathbb{C}^{n^2}$ given by

$$f \mapsto (0, \dots, 0, \lambda_{\eta'} f, 0, \dots 0)$$

where $\lambda_{\eta'} f$ is in the $(i'n + j' + 1)^{th}$ position. Next, we define the map

$$\Delta_s : V_s \longrightarrow \mathbb{C}^{n^2}$$

by

$$\Delta_s(f) := \sum_{\eta' \in \Omega} \Delta_{s,\eta'}(f).$$

The map

$$\Delta_{-s} : V_{-s} \longrightarrow \mathbb{C}^{n^2}$$

is defined in a similar manner.

It can be shown that the space V_s can be decomposed (see [1]) as

$$V_s = \mathrm{Span}\langle f_{\eta'} \rangle \oplus \mathrm{Ker}\langle \lambda_{\eta'} \rangle.$$

The map $\mathrm{Im}(\Delta_{s,\eta'}) \longrightarrow V_s$ taking elements back to their preimages under $\Delta_{s,\eta'}$ is not well-defined, but any two preimages of the same element differ by an element of $\mathrm{Ker}\langle \lambda_{\eta'} \rangle$. By property (3) of Whittaker functionals, we have that

$$\mathrm{Ker}\langle \lambda_{\eta'} \rangle = \mathrm{Span}\langle \pi_s(n) f - \psi(n) f \rangle.$$

The intertwining operator $\mathbb{I}_s$ takes elements of $\mathrm{Ker}\langle \lambda_{\eta'} \rangle$ to elements of $\mathrm{Ker}\langle \lambda_{\eta} \rangle$ since

$$\mathbb{I}_s(\pi_s(n) f - \psi(n) f) = \pi_{-s}(n)(\mathbb{I}_s f) - \psi(n)(\mathbb{I}_s f).$$

Using the linearity of the Whittaker functionals, we obtain the commutative diagram

$$\begin{array}{ccccc} V_s & \xrightarrow{\mathbb{I}_s} & V_{-s} & \xrightarrow{\mathbb{I}_{-s}} & V_s \\ \Delta_s \downarrow & & \downarrow \Delta_{-s} & & \downarrow \Delta_s \\ \mathbb{C}^{n^2} & \xrightarrow[M_n(s)]{} & \mathbb{C}^{n^2} & \xrightarrow[M_n(-s)]{} & \mathbb{C}^{n^2} \end{array}$$

and the composition

$$M_n(s) \cdot M_n(-s) = \alpha \cdot Id : \mathbb{C}^{n^2} \longrightarrow \mathbb{C}^{n^2},$$

so that $\alpha \cdot Id$ and $M_n(s) \cdot M_n(-s)$ have the same determinant.

If d is any divisor of n, then $|n|_{\mathbb{F}} = 1 \implies |d|_{\mathbb{F}} = 1$. Thus, the relationship between the irreducibility of (π_s, V_s) on different coverings follows from

$$\det(M_n(s)) = 0 \iff \det(M_d(sn/d)) = 0,$$

from which we obtain the claim of Theorem 4.2. ∎

In particular, notice that the case $d = 1$ corresponds to the nonmetaplectic group G, illustrating a generalization of the local Shimura correspondence like that described by Flicker in Section 2.1 of [7]. With some work, these results should extend to the metaplectic covers of $\mathrm{GL_r}(\mathbb{F})$ for $r > 2$. However, Kubota's cocycle no longer applies when $r > 2$. Instead, the block-compatible metaplectic cocyle described in [3] can be used and this generalization is saved for future work.

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