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On exceptional completions of symmetric varieties

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Abstract. Let G be a simple group with an exceptional involution σ having H as fixed point set. We study the embedding of G/H in the projective space $\mathbb{P}(V)$ for a simple G–module V with a line fixed by H but having no nonzero vector fixed by H. For a certain class of such modules V we describe the closure of G/H proving in particular that it is a smooth variety.
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Introduction

Let G be a simple and simply connected algebraic group and σ an involution of G with set of fixed point G^σ. We denote by H the normalizer of G^σ. In this paper we describe some special completions of the symmetric variety G/H.

If V is an irreducible representation of G we say that it is quasi-spherical if there exists a line in V stable by the action of H. If V is quasi-spherical and $h_V \in \mathbb{P}(V)$ is a point fixed by H we have a map from G/H to $\mathbb{P}(V)$ defined by $gH \mapsto g \cdot h_V$. We denote the closure of the image of this map by X_V (as shown in [2] the line h_V is unique so X_V depends only on V). These varieties are of some interest: one may ask, for example, whether they are smooth or normal (see [3]).

We say that an involution is exceptional if there exists an irreducible representation V of G and a vector v in V such that H stabilizes the line through v but v is not fixed by G^σ. As shown in the first section, in this situation H is equal to G^σ and it is a Levi corresponding to a maximal parabolic associated to a simple root which appears with multiplicity 1 in the highest root θ. Let ω be the fundamental weight corresponding to this simple root and consider the quasi-spherical irreducible representation V of highest weight $n\omega + \theta$, with n a positive integer. We give a description of X_V proving in particular that it is a smooth variety.

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1. Exceptional involutions

Let $\mathfrak{g}$ be a simple Lie algebra over an algebraically closed field $\Bbbk$ of characteristic zero, and let σ be an order 2 automorphism of $\mathfrak{g}$. Denote by $\mathfrak{h}$ the subalgebra of fixed points of σ in $\mathfrak{g}$.

Let G be a connected and simply connected group with Lie algebra $\mathfrak{g}$. The involution σ induces an involution on G that we still denote by σ. Let G^σ be the set of points fixed by σ in G and H be the normalizer of G^σ. It is well known that G^σ is reductive and connected and that H is the maximal subgroup of G having G^σ as identity component (see [1]).

An irreducible representation V is called *spherical* if V has a nonzero vector fixed by G^σ and it is called *quasi–spherical* if there is a point in $\mathbb{P}(V)$ fixed by H. It is easy to see that if V is spherical then the line pointwise fixed by G^σ is unique (see [1]). Notice also that a spherical representation V is quasi–spherical. Indeed if $v \in V$ is a nonzero vector fixed by G^σ and $h \in H = N(G^\sigma)$, then $h^{-1}ghv = v$, hence also hv is a vector fixed by G^σ. But, as noted, the space of vectors fixed by G^σ is at most one–dimensional, this shows that the point $[v]$ in $\mathbb{P}(V)$ is fixed by H. Hence V is quasi–spherical.

We say that the involution σ is *exceptional* if there exists a quasi–spherical representation which is not spherical. (There are other equivalent definitions of exceptional involution, see for example [2].)

Let V be such an irreducible representation and let v be a nonzero vector which spans a line that is stable under the action of H but not pointwise fixed by G^σ. In particular G^σ acts on $\Bbbk\, v$ by a one–dimensional character. This implies that the center of G^σ contains a non trivial torus Z; we denote by $\mathfrak{z}$ its Lie algebra and by χ the character of Z such that $c \cdot v = \chi(c)v$ for $c \in Z$.

Notice that $\mathfrak{g}$ is spherical and, since we know that the vector fixed by G^σ is unique up to scalar, we have that $\mathfrak{z}$ is one–dimensional.

Now we want to choose a suitable positive root system, this is done in two steps.

First step. We begin choosing a maximal toral subalgebra of $\mathfrak{h}$ containing $\mathfrak{z}$. It is known that we can extend this subalgebra to a σ–stable maximal toral subalgebra $\mathfrak{t}$ of $\mathfrak{g}$, let Φ be its associated root system, and we can take a σ–stable system of positive roots. Let $\rho^\vee$ the sum of positive coroots and notice that $\sigma(\rho^\vee) = \rho^\vee$. We denote by T the torus corresponding to $\mathfrak{t}$ and we notice that Z is a subtorus of T, hence we can choose a non zero element $z \in \mathfrak{z}$ that is real when evaluated on the roots.

Step two. Now we can choose the positive system as the subset Φ^+ of roots which are positive on $z + \epsilon\rho^\vee$ for a small positive ϵ. So we have $\sigma(\Phi^+) \subset \Phi^+$ and $(z, \beta) \geq 0$ for all positive roots β. In particular the first condition implies that $\sigma(\beta)$ is a simple root if β is a simple root

The choice of $\Phi^+ \subset \Phi$ determines simple roots, fundamental weights and dominant weights that we will consider fixed from now on. In particular if α is a root we denote by $\mathfrak{g}_\alpha \subset \mathfrak{g}$ the root space corresponding to α and if α is a simple root we denote by ω_α the corresponding fundamental weight. Also if λ is a dominant weight we denote by V_λ the irreducible representation of highest weight λ and we denote by v_λ a highest weight vector of this module. In what follows θ will be the highest root of Φ^+ and w_0 the longest element of the Weyl group

$N_G(T)/T$ w.r.t. Φ^+.

We have the following characterization of exceptional involutions.

Proposition 1.1. *If σ is exceptional then H is connected (hence it is equal to G^σ) and it is the Levi subgroup of a maximal parabolic corresponding to a simple root α which appears with multiplicity 1 in the highest root θ, moreover $w_0(\omega_\alpha) \neq -\omega_\alpha$.*

Conversely if α is a simple root which appears with multiplicity 1 in the highest root θ and $w_0(\omega_\alpha) \neq -\omega_\alpha$ then there exists an involution σ such that H, the normalizer of G^σ, is equal to the Levi of the maximal parabolic corresponding to α; in particular σ is exceptional.

Proof. We notice first that if L is a Levi of a maximal parabolic corresponding to a simple root α then $N_G(L) = L$ if and only if $w_0(\omega_\alpha) \neq -\omega_\alpha$. Indeed if $w_0(\omega_\alpha) = -\omega_\alpha$ let $g \in N_G(T)$ be such that $gT = w_0 \in N_G(T)/T$. Then g belongs to $N_G(L)$ but not to L.

Conversely assume $w_0(\omega_\alpha) \neq -\omega_\alpha$ and let $g \in N_G(L)$. Since T is a maximal torus of L we can assume that $g \in N_G(T)$. Consider the element w defined by g in the Weyl group, since $g \in N_G(L)$ we must have that w preserves the space orthogonal to the roots associates to L so $w(\omega_\alpha) = \pm\omega_\alpha$. Now if $w(\omega_\alpha) = \omega_\alpha$ we can assume that w permutes the simple roots different from α, up to multiplying g by a suitable element in $N_L(T) \subset L$. Now by $w(\omega_\alpha) = \omega_\alpha$ we deduce also that $w(\alpha) \in \Phi^+$, hence $w(\Phi^+) = \Phi^+$. So $w = id$ and $g \in L$. Similarly in the case $w(\omega_\alpha) = -\omega_\alpha$ we see that we can assume $w(\Phi^+) = -\Phi^+$. So $w = w_0$, but this is in contradiction with $w_0(\omega_\alpha) \neq -\omega_\alpha$.

Now we prove the second claim of the Proposition, so suppose that α is a simple root which appears with multiplicity 1 in the highest root θ and let Ψ be the root subsystem generated by the simple roots different from α. Consider the involution σ defined by

$$\sigma|_{\mathfrak{t}} = \mathrm{id}_{\mathfrak{t}} \quad \text{and } \sigma|_{\mathfrak{g}_\beta} = \begin{cases} \mathrm{id}_{\mathfrak{g}_\beta} & \text{if } \beta \in \Psi; \\ -\mathrm{id}_{\mathfrak{g}_\beta} & \text{if } \beta \notin \Psi. \end{cases} \tag{1}$$

Using that α appears with multiplicity 1 in θ it is easy to see that σ is a well defined involution of Lie algebras and that it has the claimed properties. Finally notice that the representation V_{ω_α} is quasi–spherical but not spherical since the only line fixed by H in V_{ω_α} is the line spanned by the highest weight vector, which is not fixed pointwise. In particular this shows that σ is exceptional.

Conversely suppose that σ is exceptional and let V be an irreducible module with a nonzero vector v which spans an H–fixed line not G^σ–pointwise fixed. We are now going to use all objects introduced above Proposition 1.1: the subtorus Z, it Lie algebra $\mathfrak{z}$, the "real" non zero element $z \in \mathfrak{z}$, the positive system Φ^+.

We begin the proof of the first claim of the Proposition by proving that $\mathfrak{h} = Z_{\mathfrak{g}}(\mathfrak{z})$; the inclusion $\mathfrak{h} \subset Z_{\mathfrak{g}}(\mathfrak{z})$ is clear so we have to prove the other inclusion.

Notice that by our choice of Φ^+, $Z_{\mathfrak{g}}(\mathfrak{z})$ is the Levi subalgebra of $\mathfrak{g}$ generated by the root vectors x_β of weight β for β simple root such that $(z, \beta) = 0$; so our claim is equivalent to $\sigma(x_\beta) = x_\beta$ for all such simple roots β.

Notice also that, $Z_{\mathfrak{g}}(\mathfrak{h})$ being one dimensional, all the simple roots are orthogonal to z but one that we denote with α.

Now we prove that σ is the identity on the torus $\mathfrak{t}$. Suppose the contrary and choose a simple root β as close as possible (and maybe equal) to α with the property $\sigma(\beta) \neq \beta$. Let $\alpha_1 = \beta, \alpha_2, \ldots, \alpha_m = \alpha$ be a minimal connected simple root string from β to α in the Dynkin diagram. By minimality we have $\sigma(\alpha_k) = \alpha_k$ for all $k = 2, \ldots, m$. Now let $\gamma = \alpha_1 + \alpha_2 + \cdots + \alpha_m$ and notice that $(z, \gamma) = (z, \alpha) > 0$, furthermore $\sigma(\gamma) = \sigma(\beta) + \alpha_2 + \cdots + \alpha_m \neq \gamma$. Now consider a root vector x_γ of weight γ, define $y = x_\gamma + \sigma(x_\gamma)$ and notice that $[z, y] \neq 0$ and $\sigma(y) = y$, that is impossible since $\mathfrak{h} \subset Z_{\mathfrak{g}}(z)$. This proves that σ is the identity on the torus.

In particular $\sigma(x_\beta) = \pm x_\beta$ for all roots β, since σ is an involution.

Now assume that there exists β simple and different from α such that $\sigma(x_\beta) \neq x_\beta$. We may choose β as close as possible to α and let, as above, $\beta = \alpha_1, \alpha_2, \ldots, \alpha_m = \alpha$ be a simple root connected minimal string from β to α; further let γ be the sum of these roots. By minimality $\sigma(x_{\alpha_k}) = x_{\alpha_k}$ for all $k = 2, \ldots, m-1$ and also $\sigma(x_\alpha) = -x_\alpha$. But, on one hand we have $x_\gamma = [x_{\alpha_1}, [x_{\alpha_2}, [\cdots [x_{\alpha_{m-1}}, x_{\alpha_m}] \cdots]]]$, hence $\sigma(x_\gamma) = x_\gamma$ and on the other hand $[z, x_\gamma] \neq 0$ and this is impossible since $\mathfrak{h} \subset Z_{\mathfrak{g}}(z)$.

So we have proved that $Z_{\mathfrak{g}}(\mathfrak{z}) = \mathfrak{h}$ as claimed. In particular $\mathfrak{h}$ is the Levi subalgebra of the maximal parabolic associated to the simple root α and σ must be defined as in equations (1). Now the fact that it is a morphism of algebras implies that α appears with multiplicity 1 in θ.

By the remark at the beginning of this proof it remains to prove only that H is connected or equivalently that H is in the centralizer of Z. First notice that H is the normalizer of G^σ and Z is the identity component of the center of G^σ, hence H normalizes also Z. So if we take elements $g \in H$ and $c \in Z$ we know that $gcg^{-1} \in Z$, and what we want to show is $gcg^{-1} = c$. But Z being a one dimensional torus and χ a nontrivial character, our claim is equivalent to $\chi(gcg^{-1}) = \chi(c)$. By our hypothesis on v we know that $g^{-1}v$ is in the line spanned by v, hence $\chi(gcg^{-1})v = gc(g^{-1}v) = g\chi(c)g^{-1}v = \chi(c)v$ and the proof is finished. ■

As a direct consequence of the Proposition 1.1 above, we notice that if σ is an exceptional involution then the root system of G is simply laced since $w_0 \neq -1$, hence ω_α is a minuscule weight since the simple root α appears with coefficient 1 in the highest root θ.

2. Exceptional symmetric varieties

From now on we fix an exceptional involution σ and we denote by α the corresponding simple root and P_α the associated maximal parabolic as in Proposition 1.1. Also we denote by $\mathfrak{p}_\alpha$ the Lie algebra of P_α and by ω the fundamental weight ω_α dual to $\alpha^\vee$. We also keep the notation introduced in the proof of Proposition 1.1. So $\mathfrak{z} \subset \mathfrak{h}$ is the center of $\mathfrak{h}$ and we recall that $N_G(\mathfrak{z}) = H$.

In the irreducible module V_ω of highest weight ω, $\Bbbk v_\omega$ is the unique line fixed by P_α. Notice that if we take the natural G–equivariant map

$$\mathfrak{g} \otimes V_\omega^{\otimes n} \longrightarrow V_{n\omega+\theta},$$

the image h of the line $\mathfrak{z}\otimes v_\omega^{\otimes n}$ is a line fixed by H; we want to study the variety $X_{n\omega+\theta} := \overline{Gh} \subset \mathbb{P}(V_{n\omega+\theta})$ proving the following theorem:

Theorem 2.1. *If σ is an exceptional involution (of a simple group) then the variety $X_{n\omega+\theta}$ is smooth and the morphism $\jmath : G/H \longrightarrow X_{n\omega+\theta}$ defined by $gH \mapsto g\cdot h$ is an open immersion.*

We begin considering the variety $\mathcal{P}$ of the parabolic subalgebras in $\mathfrak{g}$ conjugated to $\mathfrak{p}_\alpha$. Let Y be the subvariety in $\mathcal{P}\times\mathbb{P}(\mathfrak{g})$ consisting of pairs $(\mathfrak{p}, l)$ with l a line in the solvable radical $\mathfrak{p}^r$ of $\mathfrak{p}$. It is clear that

$$Y \simeq G\times_{P_\alpha}\mathbb{P}(\mathfrak{p}_\alpha^r),$$

so that in particular, Y is a smooth variety. We are going to show that for each $n\geq 1$, $X_{n\omega+\theta}$ is G–isomorphic to Y.

Let us start with some preliminary observations about the structure of Y. First of all notice that the unipotent radical $\mathfrak{n}_\alpha$ of $\mathfrak{p}_\alpha$ is a hyperplane in $\mathfrak{p}_\alpha^r$ complementary to $\mathfrak{z}$. Since $\mathfrak{n}_\alpha$ is an ideal in $\mathfrak{p}_\alpha$, Y contains the G–stable divisor $D := G\times_{P_\alpha}\mathbb{P}(\mathfrak{n}_\alpha)$ which is just the variety of pairs $(\mathfrak{p}, l)$ with l a line in the nilpotent radical of $\mathfrak{p}$. The root space $\mathfrak{g}_\theta$ is contained in $\mathfrak{n}_\alpha$ and we shall also consider the G–orbit $\mathcal{O}\subset D$ of the pair $(\mathfrak{p}_\alpha, \mathfrak{g}_\theta)$.

Lemma 2.2.

(1) *$Y\smallsetminus D$ is the G–orbit of $(\mathfrak{p}_\alpha, \mathfrak{z})$;*

(2) *$\mathcal{O}$ is the unique closed G–orbit in Y.*

Proof. (1) In order to show our claim it suffices to see that $P_\alpha(\mathfrak{z})$ equals $\{l\in\mathbb{P}(\mathfrak{g})\,|\,l\subset\mathfrak{p}_\alpha^r \text{ and } l\not\subset\mathfrak{n}_\alpha\}$, which choosing a non zero element $z\in\mathfrak{z}$ we can identify with $z+\mathfrak{n}_\alpha$.

Notice that, since ω is minuscule, $[\mathfrak{n}_\alpha,\mathfrak{n}_\alpha]=0$, so that $\exp(\mathrm{ad}_x)z = z+[x,z]$ for each $x\in\mathfrak{n}_\alpha$. We deduce that,

$$P_\alpha(z) = \exp(\mathfrak{n}_\alpha)H(z) = \exp(\mathfrak{n}_\alpha)(z) = z+[\mathfrak{n}_\alpha, z].$$

Since $\mathfrak{h}\cap\mathfrak{n}_\alpha = 0$, we have $[\mathfrak{n}_\alpha, z]=\mathfrak{n}_\alpha$ proving our claim.

(2) Notice that in $\mathfrak{n}_\alpha\subset\mathfrak{g}$ there is a unique line fixed by B, namely $\mathfrak{g}_\theta$; hence the B–variety $\mathbb{P}(\mathfrak{n}_\alpha)$ has a unique point fixed by B. So our claim follows at once by Borel fixed point Theorem. ■

We now want to construct a morphism $\varphi : Y\to X_{n\omega+\theta}$. As usual we identify $\mathcal{P}$ with the G–orbit of the highest weight line $\Bbbk v_{n\omega}$ in $\mathbb{P}(V_{n\omega})$. It follows that $Y\subset G/P_\alpha\times\mathbb{P}(\mathfrak{g})\subset\mathbb{P}(V_{n\omega})\times\mathbb{P}(\mathfrak{g})\subset\mathbb{P}(V_{n\omega}\otimes\mathfrak{g})$ where the last inclusion is given by the Segre embedding. In this way Y is identified with the closure of the G–orbit of the line $v_{n\omega}\otimes\mathfrak{z}$. Denote by W the unique G–stable complement of $V_{n\omega+\theta}$ in $V_{n\omega}\otimes\mathfrak{g}$ and consider the rational G–equivariant projection $\pi:\mathbb{P}(V_{n\omega}\otimes\mathfrak{g})\to\mathbb{P}(V_{n\omega+\theta})$ which is defined on the complement U of $\mathbb{P}(W)$. We have

Lemma 2.3.

(1) *Y is contained in the open set U so that π is defined on Y;*

(2) *$\pi(Y) = X_{n\omega+\theta}$;*

(3) *$D \subset \mathbb{P}(V_{n\omega+\theta}) \subset \mathbb{P}(V_{n\omega} \otimes \mathfrak{g})$, so in particular the restriction of π to D is an isomorphism.*

Proof. (1) U is a G–stable open set in $\mathbb{P}(V_{n\omega} \otimes \mathfrak{g})$, so we only need to show that π is defined on the point $(\mathfrak{p}_\alpha, \mathfrak{g}_\theta)$ whose orbit is the unique closed G–orbit in Y. This point maps to $v_{n\omega} \otimes \mathfrak{g}_\theta \in \mathbb{P}(V_{n\omega} \otimes \mathfrak{g})$ which in turn is mapped to the point representing the highest weight line in $\mathbb{P}(V_{n\omega+\theta})$. Our claim is proved.

(2) Clearly $\pi(Y)$ is the closure of the G–orbit of $\pi(v_{n\omega} \otimes \mathfrak{z}) = h_{n\omega+\theta}$ which is $X_{n\omega+\theta}$.

(3) By the definition of D we need to show that the subspace $v_{n\omega} \otimes \mathfrak{n}_\alpha$ is contained in $V_{n\omega+\theta}$. We know that $v_{n\omega} \otimes \mathfrak{g}_\theta \subset V_{n\omega+\theta}$.

Since ω is minuscule, it easily follows that for each $\mathfrak{g}_\beta \subset \mathfrak{n}_\alpha$ we can find a sequence of positive roots $\gamma_1, \ldots, \gamma_m$ not having α in their support with the property that $\mathfrak{g}_\beta = f_{\gamma_1} \cdots f_{\gamma_m}(\mathfrak{g}_\theta)$, f_γ denoting a non zero element in $\mathfrak{g}_{-\gamma}$.

On the other hand recall that for each i, $f_{\gamma_i} \in \mathfrak{p}_\alpha$, so that $f_{\gamma_i} v_{n\omega} = 0$. We deduce that

$$v_{n\omega} \otimes \mathfrak{g}_\beta = v_{n\omega} \otimes f_{\gamma_1} \cdots f_{\gamma_m}(\mathfrak{g}_\theta) = f_{\gamma_1} \cdots f_{\gamma_m}(v_{n\omega} \otimes \mathfrak{g}_\theta) \subset V_{n\omega+\theta}.$$

■

Let us denote by $\varphi : Y \to X_{n\omega+\theta}$ the restriction of π to Y. We have,

Lemma 2.4. *φ is an isomorphism.*

Proof. First we claim that $\varphi(D)$ does not intersect the orbit $\varphi(Y \setminus D)$. Indeed, if we suppose otherwise $\varphi(D)$ would contain that orbit and, by Lemma 2.3(3), $D = G \times_{P_\alpha} \mathbb{P}(\mathfrak{n}_\alpha)$ would contain a point fixed by H. Then, since the projection of D to $\mathbb{P}(\mathfrak{g})$ is contained in the projectification of the nilpotent cone, we would get the existence of a H–fixed line consisting of nilpotent elements in $\mathfrak{g}$. But $H \supset T$ so that such a line is a root space $\mathfrak{g}_\beta$ for some root β. Now notice that ω being minuscule immediately implies that there exists a simple root $\gamma \neq \alpha$ such that either $[\mathfrak{g}_\gamma, \mathfrak{g}_\beta] \neq 0$ or $[\mathfrak{g}_{-\gamma}, \mathfrak{g}_\beta] \neq 0$. This gives a contradiction.

Since we have seen that the restriction of φ to D is an isomorphism, the fact that $\varphi(D) \cap \varphi(Y \setminus D) = \varnothing$ clearly implies that φ is finite and that it is an isomorphism if and only if its differential $d\varphi_y$ in the point $y = (\mathfrak{p}_\alpha, \mathfrak{g}_\theta) \in \mathcal{O}$ is injective.

Notice that we can identify $T_y\mathbb{P}(V_{n\omega} \otimes \mathfrak{g})$ with the unique T–stable complement of the line $v_{n\omega} \otimes \mathfrak{g}_\theta$ in $V_{n\omega} \otimes \mathfrak{g}$ and similarly we can identify $T_y\mathbb{P}(V_{n\omega+\theta})$ with the unique T–stable complement of the highest weight line in $V_{n\omega+\theta}$, i.e. with the intersection $V_{n\omega+\theta} \cap T_y\mathbb{P}(V_{n\omega} \otimes \mathfrak{g})$. Using these identifications we get

$$T_yY = \mathfrak{n}_\alpha^- \cdot (v_{n\omega} \otimes \mathfrak{g}_\theta) \oplus v_{n\omega} \otimes \tilde{\mathfrak{p}}_\alpha^r \subset T_y\mathbb{P}(V_{n\omega} \otimes \mathfrak{g})$$

where $\mathfrak{n}_\alpha^-$ denotes the nilpotent radical of the parabolic opposite to $\mathfrak{p}_\alpha$ and $\tilde{\mathfrak{p}}_\alpha^r$ the unique T–stable complement of $\mathfrak{g}_\theta$ in $\mathfrak{p}_\alpha^r$. Furthermore $d\varphi_y$ is just the restriction to T_yY of the natural G–equivariant projection $\tilde{\pi} : V_{n\omega} \otimes V_\theta \to V_{n\omega+\theta}$.

Write $\tilde{\mathfrak{n}}_\alpha = \tilde{\mathfrak{p}}_\alpha \cap \mathfrak{n}_\alpha$ and $\tilde{\mathfrak{p}}_\alpha^r = \mathfrak{z} \oplus \tilde{\mathfrak{n}}_\alpha$. Notice that

$$T_yD = \mathfrak{n}_\alpha^- \cdot (v_{n\omega} \otimes \mathfrak{g}_\theta) \oplus v_{n\omega} \otimes \tilde{\mathfrak{n}}_\alpha^r$$

and that $T_yY = T_yD \oplus v_{n\omega} \otimes \mathfrak{z}$. By Lemma 2.3 $d\varphi_y|_{T_yD}$ is injective, so the only thing we need to show is that $d\varphi_y(v_{n\omega} \otimes \mathfrak{z}) \notin d\varphi_y(T_yD)$. Since the differential is T–equivariant and $\mathfrak{z}$ has weight zero, it suffices to show that $d\varphi_y(v_{n\omega} \otimes \mathfrak{z}) \notin d\varphi_y(T_yD[n\omega])$, where $T_yD[n\omega]$ denotes the subspace of weight $n\omega$ in T_y. We have $T_yD[n\omega] = \mathfrak{g}_{-\theta} \cdot (v_{n\omega} \otimes \mathfrak{g}_\theta)$, so we are reduced to show that the two lines

$$d\varphi_y(v_{n\omega} \otimes \mathfrak{z}) = \tilde{\pi}(v_{n\omega} \otimes \mathfrak{z}) \quad \text{and} \quad d\varphi_y(\mathfrak{g}_{-\theta} \cdot (v_{n\omega} \otimes \mathfrak{g}_\theta)) = \tilde{\pi}(\mathfrak{g}_{-\theta} \cdot (v_{n\omega} \otimes \mathfrak{g}_\theta))$$

are distinct.

We distinguish two cases. First of all let us assume that $\theta - \alpha$ is not a root. Since the considered root systems have always rank bigger or equal to 2 (otherwise $w_0 = -\mathrm{id}$ against the condition in Proposition 1.1) we have that $\theta - \alpha \neq 0$ hence $[\mathfrak{g}_{-\theta}, \mathfrak{g}_\alpha] = 0$. We get

$$\mathfrak{g}_\alpha \tilde{\pi}(\mathfrak{g}_{-\theta} \cdot (v_{n\omega} \otimes \mathfrak{g}_\theta)) = \tilde{\pi}(\mathfrak{g}_\alpha \mathfrak{g}_{-\theta} \cdot (v_{n\omega} \otimes \mathfrak{g}_\theta)) = \tilde{\pi}(\mathfrak{g}_{-\theta} \mathfrak{g}_\alpha \cdot (v_{n\omega} \otimes \mathfrak{g}_\theta)) = \tilde{\pi}(\mathfrak{g}_{-\theta} 0) = 0.$$

On the other hand, since $\mathfrak{g}_\alpha \not\subset \mathfrak{h}$ we have $[\mathfrak{g}_\alpha, \mathfrak{z}] = \mathfrak{g}_\alpha \neq 0$ so that

$$\mathfrak{g}_\alpha \tilde{\pi}(v_{n\omega} \otimes \mathfrak{z}) = \tilde{\pi}(v_{n\omega} \otimes [\mathfrak{g}_\alpha, z]) = \tilde{\pi}(v_{n\omega} \otimes \mathfrak{g}_\alpha) \neq 0$$

since we can consider $\tilde{\pi}$ as the multiplication of the sections of two line bundles on G/B.

Assume now that $\beta = \theta - \alpha$ is a root. Then it is a positive root and $[\mathfrak{g}_{-\theta}, \mathfrak{g}_\beta] = \mathfrak{g}_{-\alpha}$. We deduce

$$\begin{aligned}
\mathfrak{g}_\beta \tilde{\pi}(\mathfrak{g}_{-\theta} \cdot (v_{n\omega} \otimes \mathfrak{g}_\theta)) &= \tilde{\pi}(\mathfrak{g}_\beta \mathfrak{g}_{-\theta} \cdot (v_{n\omega} \otimes \mathfrak{g}_\theta)) \\
&= \tilde{\pi}(\mathfrak{g}_{-\theta} \mathfrak{g}_\beta \cdot (v_{n\omega} \otimes \mathfrak{g}_\theta) + \mathfrak{g}_{-\alpha} \cdot (v_{n\omega} \otimes \mathfrak{g}_\theta)) \\
&= \tilde{\pi}(\mathfrak{g}_{-\alpha} \cdot (v_{n\omega} \otimes \mathfrak{g}_\theta)) \\
&= \mathfrak{g}_{-\alpha}(\mathbb{C} v_{n\omega+\theta}) \\
&\neq 0
\end{aligned}$$

since the weight $n\omega + \theta$ is not orthogonal to α.

On the other hand notice that, α being minuscule, β is a sum of simple roots different from α, hence $\mathfrak{g}_\beta$ is contained in $\mathfrak{h}$. From the fact that $\mathfrak{g}_\beta$ consists of nilpotent elements and the line $\tilde{\pi}(v_{n\omega} \otimes \mathfrak{z})$ is preserved by H it follows that

$$\mathfrak{g}_\beta \tilde{\pi}(v_{n\omega} \otimes \mathfrak{z}) = 0$$

proving our claim. ∎

We can now prove our Theorem

Proof. [of Theorem 2.1] The smoothness of $X_{n\omega+\theta}$ follows by Lemma 2.4. To prove that $\jmath$ is injective we observe that it is an equivariant morphism and that the stabilizer of $\mathfrak{z} \in \mathbb{P}(\mathfrak{g})$ is H, hence also that of $(\mathfrak{p}_\alpha, \mathfrak{z})$ is H. ■

As the referee pointed out to us Theorem 2.1 holds in more generality. If G^σ has a non trivial center one may replace H by G^σ; indeed in such case G^σ has all the properties stated in Proposition 1.1 for H but now $w_0(\omega_\alpha) = -\omega_\alpha$, whereas in our setting $w_0(\omega_\alpha) \neq -\omega_\alpha$. However all the proofs remain valid since we have never used this last condition in Section 2.

Moreover notice that, as recalled in the Introduction, the H–fixed line in a simple quasi-spherical module is unique whereas the line fixed by G^σ is not, in general, unique if G^σ has a non trivial center. However the explicit construction of $X_{n\omega+\theta}$ given before the statement of Theorem 2.1, does'nt use this uniqueness property.

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