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Kazhdan and Haagerup Properties in algebraic groups over local fields

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Abstract. Given a Lie algebra $\mathfrak{s}$, we call Lie $\mathfrak{s}$-algebra a Lie algebra endowed with a reductive action of $\mathfrak{s}$. We characterize the minimal $\mathfrak{s}$-Lie algebras with a nontrivial action of $\mathfrak{s}$, in terms of irreducible representations of $\mathfrak{s}$ and invariant alternating forms.

As a first application, we show that if $\mathfrak{g}$ is a Lie algebra over a field of characteristic zero whose amenable radical is not a direct factor, then $\mathfrak{g}$ contains a subalgebra which is isomorphic to the semidirect product of $\mathfrak{sl}_2$ by either a nontrivial irreducible representation or a Heisenberg group (this was essentially due to Cowling, Dorofaeff, Seeger, and Wright). As a corollary, if G is an algebraic group over a local field $\mathbf{K}$ of characteristic zero, and if its amenable radical is not, up to isogeny, a direct factor, then $G(\mathbf{K})$ has Property (T) relative to a noncompact subgroup. In particular, $G(\mathbf{K})$ does not have Haagerup's property. This extends a similar result of Cherix, Cowling and Valette for connected Lie groups, to which our method also applies.

We give some other applications. We provide a characterization of connected Lie groups all of whose countable subgroups have Haagerup's property. We give an example of an arithmetic lattice in a connected Lie group which does not have Haagerup's property, but has no infinite subgroup with relative Property (T). We also give a continuous family of pairwise non-isomorphic connected Lie groups with Property (T), with pairwise non-isomorphic (resp. isomorphic) Lie algebras.

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1. Introduction

In the sequel, all Lie algebras are finite-dimensional over a field of characteristic zero, denoted by K, or $\mathbf{K}$ when it is a local field. If $\mathfrak{g}$ is a Lie algebra, denote by $\mathrm{rad}(\mathfrak{g})$ its radical and $Z(\mathfrak{g})$ its centre, $D\mathfrak{g}$ its derived subalgebra, and $\mathrm{Der}(\mathfrak{g})$ the Lie algebra of all derivations of $\mathfrak{g}$. If $\mathfrak{h}_1, \mathfrak{h}_2$ are Lie subalgebras of $\mathfrak{g}$, $[\mathfrak{h}_1, \mathfrak{h}_2]$ denotes the Lie subalgebra generated by the brackets $[h_1, h_2]$, $(h_1, h_2) \in \mathfrak{h}_1 \times \mathfrak{h}_2$.

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Let $\mathfrak{g}$ be a Lie algebra with radical $\mathrm{rad}(\mathfrak{g})=\mathfrak{r}$ and semisimple Levi factor $\mathfrak{s}$ (so that $\mathfrak{g}\simeq\mathfrak{s}\ltimes\mathfrak{r}$). We focus here on aspects of $\mathfrak{g}$ related to the action of $\mathfrak{s}$. This suggests the following definitions.

If $\mathfrak{s}$ is a Lie algebra, we define a *Lie $\mathfrak{s}$-algebra* to be a Lie algebra $\mathfrak{n}$ endowed with a morphism $i:\mathfrak{s}\to\mathrm{Der}(\mathfrak{n})$, defining a *completely reducible* action of $\mathfrak{s}$ on $\mathfrak{n}$. (This latter technical condition is empty if $\mathfrak{s}$ is semisimple.)

A Lie $\mathfrak{s}$-algebra naturally embeds in the semidirect product $\mathfrak{s}\ltimes\mathfrak{n}$, so that we write $i(s)(n)=[s,n]$ for $s\in\mathfrak{s}$, $n\in\mathfrak{n}$.

By the *trivial irreducible module* of $\mathfrak{s}$ we mean a one-dimensional space vector space endowed with a trivial action of $\mathfrak{s}$. We say that a module (over a Lie algebra or over a group) is *full* if is is completely reducible and does not contain the trivial irreducible module.

Definition 1.1. Let $\mathfrak{s}$ be a Lie algebra. We say that a Lie $\mathfrak{s}$-algebra $\mathfrak{n}$ is *minimal* if $[\mathfrak{s},\mathfrak{n}]\neq 0$, and for every $\mathfrak{s}$-subalgebra $\mathfrak{n}'$ of $\mathfrak{n}$, either $\mathfrak{n}'=\mathfrak{n}$ or $[\mathfrak{s},\mathfrak{n}']=0$.

It is clear that a Lie $\mathfrak{s}$-algebra $\mathfrak{n}$ satisfying $[\mathfrak{s},\mathfrak{n}]\neq 0$ contains a minimal $\mathfrak{s}$-subalgebra. We begin by a characterization of minimal $\mathfrak{s}$-algebras:

Theorem 1.2. *Let $\mathfrak{s}$ be a Lie algebra. A solvable Lie $\mathfrak{s}$-algebra $\mathfrak{n}$ is minimal if and only if it satisfies the following conditions 1), 2), 3), and 4):*

1) $\mathfrak{n}$ is 2-nilpotent (that is, $[\mathfrak{n},D\mathfrak{n}]=0$).

2) $[\mathfrak{s},\mathfrak{n}]=\mathfrak{n}$.

3) $[\mathfrak{s},D\mathfrak{n}]=0$.

4) $\mathfrak{n}/D\mathfrak{n}$ is irreducible as a $\mathfrak{s}$-module.

Definition 1.3. We call a solvable Lie $\mathfrak{s}$-algebra $\mathfrak{n}$ *almost minimal* if it satisfies conditions 1), 2), and 3) of Theorem 1.2.

This definition has the advantage to be invariant under field extensions. Note that an almost minimal solvable Lie $\mathfrak{s}$-algebra $\mathfrak{n}$ satisfies Condition 4'): $\mathfrak{n}/D\mathfrak{n}$ is a full $\mathfrak{s}$-module.

The classification of (almost) minimal solvable Lie $\mathfrak{s}$-algebras can be deduced from the classification of irreducible $\mathfrak{s}$-modules. Let $\mathfrak{v}$ be a full $\mathfrak{s}$-module (equivalently, an abelian Lie $\mathfrak{s}$-algebra satisfying $[\mathfrak{s},\mathfrak{v}]=\mathfrak{v}$). Recall that a bilinear form φ on $\mathfrak{v}$ is called $\mathfrak{s}$-invariant if it satisfies $\varphi([s,v],w)+\varphi(v,[s,w])=0$ for all $s\in\mathfrak{s}$, $v,w\in\mathfrak{v}$. Let $\mathrm{Bil}_{\mathfrak{s}}(\mathfrak{v})$ (resp. $\mathrm{Alt}_{\mathfrak{s}}(\mathfrak{v})$) denote the space of all $\mathfrak{s}$-invariant bilinear (resp. alternating bilinear) forms on $\mathfrak{v}$.

Definition 1.4. We define the Lie $\mathfrak{s}$-algebra $\mathfrak{h}(\mathfrak{v})$ as follows: as a vector space, $\mathfrak{h}(\mathfrak{v})=\mathfrak{v}\oplus\mathrm{Alt}_{\mathfrak{s}}(\mathfrak{v})^*$; it is endowed with the following bracket:

$$[(x,z),(x',z')]=(0,e_{x,x'})\qquad x,x'\in\mathfrak{v}\ \ z,z'\in\mathrm{Alt}_{\mathfrak{s}}(\mathfrak{v})^*\tag{1}$$

where $e_{x,x'}\in\mathrm{Alt}_{\mathfrak{s}}(\mathfrak{v})^*$ is defined by $e_{x,x'}(\varphi)=\varphi(x,x')$.

This is a 2-nilpotent Lie $\mathfrak{s}$-algebra under the action $[s,(x,z)]=([s,x],0)$, which is almost minimal. Other almost minimal Lie $\mathfrak{s}$-algebras can be obtained by taking the quotient by a linear subspace of the centre. The following theorem states that this is the only way to construct almost minimal solvable Lie $\mathfrak{s}$-algebras.

Theorem 1.5. *If $\mathfrak{n}$ is an almost minimal solvable Lie $\mathfrak{s}$-algebra, then it is isomorphic (as a $\mathfrak{s}$-algebra) to $\mathfrak{h}(\mathfrak{v})/Z$, for some full $\mathfrak{s}$-module $\mathfrak{v}$ and some subspace Z of $\mathrm{Alt}_{\mathfrak{s}}(\mathfrak{v})^*$. It is minimal if and only if $\mathfrak{v}$ is irreducible.*

Moreover, the almost minimal $\mathfrak{s}$-algebras $\mathfrak{h}(\mathfrak{v})/Z$ and $\mathfrak{h}(\mathfrak{v})/Z'$ are isomorphic if and only if Z' and Z are in the same orbit for the natural action of $\mathrm{Aut}_{\mathfrak{s}}(\mathfrak{v})$ on the Grassmannian of $\mathrm{Alt}_{\mathfrak{s}}(\mathfrak{v})^*$.

Remark 1.6. *If $\mathfrak{s}$ is semisimple, $\mathfrak{s} \ltimes \mathfrak{h}(\mathfrak{v})$ is the universal central extension of the perfect Lie algebra $\mathfrak{s} \ltimes \mathfrak{v}$.*

The case of $\mathfrak{sl}_2$ is essential, and there is a simple description for it. Recall that if $\mathfrak{s} = \mathfrak{sl}_2$, then, up to isomorphism, there exists exactly one irreducible $\mathfrak{s}$-module $\mathfrak{v}_n$ of dimension n for every $n \geq 1$. If $n = 2m$ is even, it has a central extension by a one-dimensional subspace, giving a Heisenberg Lie algebra, on which $\mathfrak{sl}_2$ acts naturally (see 2. for details), denoted by $\mathfrak{h}_{2m+1}$. Theorem 1.5 thus reduces as:

Proposition 1.7. *Up to isomorphism, the minimal solvable Lie $\mathfrak{sl}_2$-algebras are $\mathfrak{v}_n$ and $\mathfrak{h}_{2n-1}$ ($n \geq 2$).*

Let $\mathfrak{g}$ be a Lie algebra, $\mathfrak{r}$ its radical and $\mathfrak{s}$ a semisimple factor. Write $\mathfrak{s} = \mathfrak{s}_c \oplus \mathfrak{s}_{nc}$ by separating K-anisotropic and K-isotropic factors[1]. The ideal $\mathfrak{s}_c \ltimes \mathfrak{r}$ is sometimes called the *amenable radical* of $\mathfrak{g}$.

Definition 1.8. We call $\mathfrak{g}$ M-decomposed if $[\mathfrak{s}_{nc}, \mathfrak{r}] = 0$. Equivalently, $\mathfrak{g}$ is M-decomposed if the amenable radical is a direct factor of $\mathfrak{g}$.

Proposition 1.9. *Let $\mathfrak{g}$ be a Lie algebra, and keep notation as above. Suppose that $\mathfrak{g}$ is not M-decomposed. Then there exists a Lie subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ which is isomorphic to $\mathfrak{sl}_2 \ltimes \mathfrak{v}_n$ or $\mathfrak{sl}_2 \ltimes \mathfrak{h}_{2n-1}$ for some $n \geq 2$.*

This result is essentially due to Cowling, Dorofaeff, Seeger and Wright [6], where it is not explicitly stated, but it is actually proved in the proof of Proposition 8.2 there (under the assumption $K = \mathbf{R}$, but their argument generalizes to any field of characteristic zero). This was a starting point for the present paper.

Let G be a locally compact, σ-compact group. Recall that G has the Haagerup Property if it has a metrically proper isometric action on a Hilbert space; in contrast, G has Kazhdan's Property (T) if every isometric action of G on a Hilbert space has a fixed point. See 4. for a short reminder about Haagerup and Kazhdan Properties.

We provide corresponding statements for Proposition 1.9 in the realm of algebraic groups and connected Lie groups. As a consequence, we get the following theorem, which was the initial motivation for the results above. It was already proved, in a different way, for connected Lie groups by Cherix, Cowling and Valette [5, Chap. 4].

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[1] c and nc respectively stand for "non-compact" and "compact"; this is related to the fact that if S is a simple algebraic group defined over the local field $\mathbf{K}$, then its Lie algebra is $\mathbf{K}$-isotropic if and only if $S(\mathbf{K})$ is not compact.

Theorem 1.10. *Let G be either a connected Lie group, or $G = \mathbf{G}(\mathbf{K})$, where $\mathbf{G}$ is a linear algebraic group over the local field $\mathbf{K}$ of characteristic zero. Let $\mathfrak{g}$ be its Lie algebra. The following are equivalent.*

(i) G has Haagerup's property.

(ii) For every noncompact closed subgroup H of G, (G, H) does not have relative Property (T).

(iii) The following conditions are satisfied:

- *$\mathfrak{g}$ is M-decomposed.*
- *All simple factors of $\mathfrak{g}$ have $\mathbf{K}$-rank ≤ 1.*
- *(in the case of Lie groups or when $\mathbf{K} = \mathbf{R}$) No simple factor of $\mathfrak{g}$ is isomorphic to $\mathfrak{sp}(n, 1)$ $(n \geq 2)$ or $\mathfrak{f}_{4(-20)}$.*

(iv) $\mathfrak{g}$ contains no isomorphic copy of any one of the following Lie algebras

- *$\mathfrak{sl}_2 \ltimes \mathfrak{v}_n$ or $\mathfrak{sl}_2 \ltimes \mathfrak{h}_{2n-1}$ for some $n \geq 2$,*
- *(in the case of Lie groups or when $\mathbf{K} = \mathbf{R}$) $\mathfrak{sp}(2, 1)$.*

Remark 1.11. *The notion of M-decomposed (real) Lie algebras also appears in other contexts: heat kernel on Lie groups [12], Rapid Decay Property [7], weak amenability [6].*

We derive some other results with the help of Theorem 1.5.

Proposition 1.12. *There exists a continuous family $(\mathfrak{g}_t)$ of pairwise non-isomorphic real (or complex) Lie algebras satisfying the following properties:*
(i) $\mathfrak{g}_t$ is perfect, and
(ii) the simply connected Lie group corresponding to $\mathfrak{g}_t$ has Property (T).

Note that Proposition 1.12 with only (i) may be of independent interest; we do not know if it had already been observed. On the other hand, it is well-known that there exist continuously many pairwise non-isomorphic complex n-dimensional nilpotent Lie algebras if $n \geq 7$.

Proposition 1.13. *There exists a continuous family of pairwise non-isomorphic connected Lie groups with Property (T), and with isomorphic Lie algebras.*

We also give the classification, when $\mathbf{K} = \mathbf{R}$, of the minimal $\mathfrak{so}_3$-algebras (Proposition 2.3). We use it to prove (ii)$\Rightarrow$(i) in the following result (while the reverse implication is essentially due to Guentner, Higson and Weinberger [9, Theorem 5.1]).

Theorem 1.14. *Let G be a connected Lie group. Then the following are equivalent:*
(i) G is locally isomorphic to $\mathrm{SO}_3(\mathbf{R})^a \times \mathrm{SL}_2(\mathbf{R})^b \times \mathrm{SL}_2(\mathbf{C})^c \times R$, for a solvable Lie group R and integers a, b, c.
(ii) Every countable subgroup of G has Haagerup's property (when endowed with the discrete topology).

Remark 1.15. *Assertion (i) of Theorem 1.14 is equivalent to: (ii')* The complexification $\mathfrak{g}_{\mathbf{C}}$ of $\mathfrak{g}$ is an M-decomposed complex Lie algebra, and its semisimple part is isomorphic to $\mathfrak{sl}_2(\mathbf{C})^n$ for some n.

For instance, $\mathrm{SO}_3(\mathbf{R}) \ltimes \mathbf{R}^3$ has a countable subgroup which does not have Haagerup's property. An explicit example is given by $\mathrm{SO}_3(\mathbf{Z}[1/p]) \ltimes \mathbf{Z}[1/p]^3$. It can also be shown that this group has no infinite subgroup with relative Property (T). This answers an open question in [5, Section 7.1], and is in contrast with Theorem 1.10. This group is not finitely presented (this is a consequence of [1, Theorem 2.6.4]); we give a similar example in Remark 4.10 which is, in addition, finitely presented.

2. Lie algebras

2.1. Minimal subalgebras.

Proposition 2.1. *Let $\mathfrak{n}$ be a solvable Lie $\mathfrak{s}$-algebra.*

1) The Lie $\mathfrak{s}$-subalgebra $[\mathfrak{s},\mathfrak{n}]$ is an ideal in $\mathfrak{n}$ (and also in $\mathfrak{s}\ltimes\mathfrak{n}$), and $[\mathfrak{s},[\mathfrak{s},\mathfrak{n}]] = [\mathfrak{s},\mathfrak{n}]$.

2) If, moreover, $[\mathfrak{s}, D\mathfrak{n}] = 0$, then $[\mathfrak{s},\mathfrak{n}]$ is an almost minimal Lie algebra (see Definition 1.3).

Proof. 1) Let $\mathfrak{v}$ be the subspace generated by the brackets $[s,n]$, $(s,n) \in \mathfrak{s}\times\mathfrak{n}$. Since the action of $\mathfrak{s}$ is completely reducible (see the definition of Lie $\mathfrak{s}$-algebra), it is immediate that $[\mathfrak{s},\mathfrak{n}]$ and $[\mathfrak{s},[\mathfrak{s},\mathfrak{n}]]$ both coincide with the Lie subalgebra generated by $\mathfrak{v}$. Then, using Jacobi identity,

$$[\mathfrak{n},[\mathfrak{s},\mathfrak{n}]] \;=\; [\mathfrak{n},[\mathfrak{s},[\mathfrak{s},\mathfrak{n}]]] \;\subseteq\; [\mathfrak{s},[\mathfrak{n},[\mathfrak{s},\mathfrak{n}]]] + [[\mathfrak{s},\mathfrak{n}],[\mathfrak{s},\mathfrak{n}]] \;\subseteq\; [\mathfrak{s},[\mathfrak{n},\mathfrak{n}]] + [\mathfrak{s},\mathfrak{n}] \;\subseteq\; [\mathfrak{s},\mathfrak{n}].$$

2) Let $\mathfrak{z}$ be the linear subspace generated by the commutators $[v,w]$, $v,w \in \mathfrak{v}$. By Jacobi identity,

$$[\mathfrak{v},\mathfrak{z}] \;=\; [[\mathfrak{s},\mathfrak{v}],\mathfrak{z}] \;\subseteq\; [[\mathfrak{s},\mathfrak{z}],\mathfrak{v}] + [\mathfrak{s},[\mathfrak{v},\mathfrak{z}]] \;\subseteq\; [[\mathfrak{s},D\mathfrak{n}],\mathfrak{v}] + [\mathfrak{s},D\mathfrak{n}] = 0.$$

Thus, the subspace $\mathfrak{n}' = \mathfrak{v}\oplus\mathfrak{z}$ is a 2-nilpotent Lie $\mathfrak{s}$-subalgebra of $\mathfrak{n}$. The Lie subalgebra $[\mathfrak{s},\mathfrak{n}']$ contains $\mathfrak{v}$, hence also contains $\mathfrak{z}$, so $[\mathfrak{s},\mathfrak{n}']$ is equal to $\mathfrak{n}'$. Thus Conditions 1) and 2) of Definition 1.3 are satisfied, while Condition 3) follows immediately from the hypothesis $[\mathfrak{s}, D\mathfrak{n}] = 0$. ∎

Proof of Theorem 1.2. Suppose (ii). Condition 4 implies $\mathfrak{n} \neq 0$. Then Condition 2 implies $[\mathfrak{s},\mathfrak{n}] = \mathfrak{n} \neq 0$. Let $\mathfrak{n}' \subseteq \mathfrak{n}$ be a $\mathfrak{s}$-subalgebra. Then, by irreducibility (Condition 4), either $D\mathfrak{n}+\mathfrak{n}' = D\mathfrak{n}$ or $D\mathfrak{n}+\mathfrak{n}' = \mathfrak{n}$. In the first case, $\mathfrak{n}'$ centralizes $\mathfrak{s}$. In the second case, $\mathfrak{n} = [\mathfrak{s},\mathfrak{n}] = [\mathfrak{s},\mathfrak{n}'+D\mathfrak{n}] = [\mathfrak{s},\mathfrak{n}'] \subseteq \mathfrak{n}'$, using Conditions 1 and 2, and the fact that $\mathfrak{n}'$ is a $\mathfrak{s}$-subalgebra.

Conversely, suppose that (i) holds. Since $\mathfrak{n}$ is solvable, $D\mathfrak{n}$ is a proper $\mathfrak{s}$-subalgebra, so that, by minimality, $[\mathfrak{s},D\mathfrak{n}] = 0$. By Proposition 2.1, $[\mathfrak{s},\mathfrak{n}]$ is a nonzero almost minimal Lie $\mathfrak{s}$-subalgebra of $\mathfrak{n}$, hence satisfies 1), 2), 3). The minimality implies that 4) is also satisfied. ∎

Proof of Theorem 1.5. Let $\mathfrak{n}$ be an almost minimal solvable Lie $\mathfrak{s}$-algebra. Let $\mathfrak{v}$ be the subspace generated by the brackets $[s,n]$, $(s,n)\in\mathfrak{s}\times\mathfrak{n}$. Since $\mathfrak{n}$ is almost minimal, $\mathfrak{v}$ is a complementary subspace of $D\mathfrak{n}$, and is a full $\mathfrak{s}$-module. If $u\in D\mathfrak{n}^*$, consider the alternating bilinear form ϕ_u on $\mathfrak{v}$ defined by $\phi_u(x,y)=u([x,y])$. This defines a mapping $D\mathfrak{n}^*\to\mathrm{Alt}_{\mathfrak{s}}(\mathfrak{v})$ which is immediately seen to be injective. By duality, this defines a surjective linear map $\mathrm{Alt}_{\mathfrak{s}}(\mathfrak{v})^*\to D\mathfrak{n}$, whose kernel we denote by Z. It is immediate from the definition of $\mathfrak{h}(\mathfrak{v})$ that this map extends to a surjective morphism of Lie $\mathfrak{s}$-algebras $\mathfrak{h}(\mathfrak{v})\to\mathfrak{n}$ with kernel Z. This proves that $\mathfrak{n}$ is isomorphic to $\mathfrak{h}(\mathfrak{v})/Z$.

The second assertion is immediate.

The third assertion follows from the proof of the first one, where we made no choice. Namely, take an isomorphism $\psi:\mathfrak{h}(\mathfrak{v})/Z\to\mathfrak{h}(\mathfrak{v})/Z'$. It gives by restriction an $\mathfrak{s}$-automorphism φ of $\mathfrak{v}$, which induces a unique automorphism $\tilde{\varphi}$ of $\mathfrak{h}(\mathfrak{v})$. Let p and p' denote the natural projections in the following diagram of Lie $\mathfrak{s}$-algebras:

$$\begin{array}{ccc}\mathfrak{h}(\mathfrak{v}) & \xrightarrow{p} & \mathfrak{h}(\mathfrak{v})/Z\\ {\scriptstyle\tilde{\varphi}}\downarrow & & \downarrow{\scriptstyle\psi}\\ \mathfrak{h}(\mathfrak{v}) & \xrightarrow[p']{} & \mathfrak{h}(\mathfrak{v})/Z'\end{array}$$

This diagram is commutative: indeed, $p'\circ\tilde{\varphi}$ and $\psi\circ p$ coincide in restriction to $\mathfrak{v}$, and $\mathfrak{v}$ generates $\mathfrak{h}(\mathfrak{v})$ as a Lie algebra. This implies $Z=\mathrm{Ker}(\psi\circ p)=\mathrm{Ker}(p'\circ\tilde{\varphi})=\tilde{\varphi}^{-1}(Z')$. ■

2.2. The example $\mathfrak{sl}_2$.

If $\mathfrak{s}=\mathfrak{sl}_2(K)$, then, up to isomorphism, there exists exactly one irreducible $\mathfrak{s}$-module $\mathfrak{v}_n$ of dimension n for every $n\ge 1$.

Since $\mathfrak{v}_n$ is absolutely irreducible for all n, by Schur's Lemma, $\mathrm{Bil}_{\mathfrak{s}}(\mathfrak{v}_n)$ is at most one dimensional for all n. In fact, it is one-dimensional. Indeed, take the usual basis (H,X,Y) of $\mathfrak{sl}_2$ satisfying $[H,X]=2X$, $[H,Y]=-2Y$, $[X,Y]=H$, and take the basis $(e_0,\dots,e_{n-1})$ of $\mathfrak{v}_n$ so that $H.e_i=(n-1-2i)e_i$, $X.e_i=(n-i)e_{i-1}$, and $Y.e_i=(i+1)e_{i+1}$, with the convention $e_{-1}=e_n=0$. Then $Bil_{\mathfrak{s}}(\mathfrak{v}_n)$ is generated by the form φ_n defined by

$$\varphi_n(e_i,e_{n-1-i})=(-1)^i\binom{i}{n-1};\quad \varphi(e_i,e_j)=0 \text{ if } i+j\neq n-1.$$

For odd n, φ_n is symmetric so that $Alt_{\mathfrak{s}}(\mathfrak{v}_n)=0$; for even n, φ_n is symplectic and generates $\mathrm{Alt}_{\mathfrak{s}}(\mathfrak{v}_n)$. For even n, denote by $\mathfrak{h}_{n+1}$ the one-dimensional central extension $\mathfrak{h}(\mathfrak{v}_n)$, well-known as the $(n+1)$-dimensional Heisenberg Lie algebra.

Proof of Proposition 1.9. Since $\mathfrak{s}_{nc}$ is semisimple and isotropic, it is generated by its subalgebras K-isomorphic to $\mathfrak{sl}_2$. Since $[\mathfrak{s}_{nc},\mathfrak{r}]\neq 0$, this implies that there exists some subalgebra $\mathfrak{s}'$ of $\mathfrak{s}_{nc}$ which is K-isomorphic to $\mathfrak{sl}_2$ and such that $[\mathfrak{s}',\mathfrak{r}]\neq 0$. Then the result is clear from Proposition 1.7. Notice that the proof gives the following slight refinement: $\mathfrak{h}$ can be chosen so that $\mathrm{rad}(\mathfrak{h})\subseteq\mathrm{rad}(\mathfrak{g})$. ■

2.3. The example $\mathfrak{so}_3$.

We now study a more specific example. Let us deal with the field $\mathbf{R}$ of real numbers, and with $\mathfrak{s} = \mathfrak{so}_3$.

Since the complexification of $\mathfrak{so}_3$ is isomorphic to $\mathfrak{sl}_2(\mathbf{C})$, the irreducible complex $\mathfrak{s}$-modules make up a family $(\mathfrak{u}_n)$ $(n \geq 1)$; $\dim(\mathfrak{u}_n) = n$, which are the symmetric powers of the natural action of $\mathfrak{su}_2 = \mathfrak{so}_3$ on $\mathbf{C}^2$.

If $n = 2m+1$ is odd, then this is the complexification of a real $\mathfrak{so}_3$-module $\mathfrak{d}^{\mathbf{R}}_{2m+1}$ (of dimension n). If $n = 2m$ is even, $\mathfrak{d}_n$ is irreducible as a $4m$-dimensional real $\mathfrak{so}_3$-module, we call it $\mathfrak{u}_{4m}$.

These two families $(\mathfrak{d}^{\mathbf{R}}_{2n+1})$ and $(\mathfrak{u}_{4n})$ make up all irreducible real $\mathfrak{so}_3$-modules.

Proposition 2.2. *The irreducible real $\mathfrak{so}_3$-modules make up two families: a family $(\mathfrak{d}^{\mathbf{R}}_{2n+1})$ of $(2n+1)$-dimensional modules ($n \geq 0$), absolutely irreducible, and a family $(\mathfrak{u}_{4n})$ of $4n$-dimensional modules ($n \geq 1$), not absolutely irreducible, preserving a quaternionic structure.* ■

Since $(\mathfrak{d}^{\mathbf{R}}_{2n+1})$ is absolutely irreducible, the space of invariant bilinear forms on $(\mathfrak{d}^{\mathbf{R}}_{2n+1})$ is generated by a scalar product, so that $\mathrm{Alt}_{\mathfrak{so}_3}(\mathfrak{d}^{\mathbf{R}}_{2n+1}) = 0$

On the other hand, $\mathrm{Alt}_{\mathfrak{so}_3}(\mathfrak{u}_{4n})$ is three-dimensional, and is given by the imaginary part of an invariant quaternionic hermitian form.

In order to classify the minimal solvable $\mathfrak{so}_3$-algebras, we must determine the orbits of the natural action of $\mathrm{Aut}_{\mathfrak{so}_3}(\mathfrak{u}_{4n})$ on $\mathrm{Alt}_{\mathfrak{so}_3}(\mathfrak{u}_{4n})$. It is a standard fact that $\mathrm{Aut}_{\mathfrak{so}_3}(\mathfrak{u}_{4n})$ is isomorphic to the group of nonzero quaternions, that $\mathrm{Alt}_{\mathfrak{so}_3}(\mathfrak{u}_{4n})$ naturally identifies with the set of imaginary quaternions, and that the action of $\mathrm{Aut}_{\mathfrak{so}_3}(\mathfrak{u}_{4n})$ on $\mathrm{Alt}_{\mathfrak{so}_3}(\mathfrak{u}_{4n})$ is given by conjugation of quaternions. This implies that it acts transitively on each component of the Grassmannian.

For $i = 0, 1, 2, 3$, let Z_i be a fixed $(3-i)$-dimensional linear subspace of $\mathrm{Alt}_{\mathfrak{s}}(\mathfrak{v})^*$. Denote by $\mathfrak{hu}^i_{4n}$ the minimal Lie $\mathfrak{so}_3$-algebra $\mathfrak{h}(\mathfrak{u}_{4n})/Z_i$; of course, $\mathfrak{hu}^0_{4n} = \mathfrak{u}_{4n}$ and $\mathfrak{hu}^3_{4n} = \mathfrak{h}(\mathfrak{u}_{4n})$.

Proposition 2.3. *Up to isomorphism, the minimal solvable Lie $\mathfrak{so}_3(\mathbf{R})$-algebras are $\mathfrak{d}^{\mathbf{R}}_{2n+1}$ ($n \geq 1$) and $\mathfrak{hu}^i_{4n}$ ($n \geq 1$, $i = 0, 1, 2, 3$).* ■

There is an analogous statement to Proposition 1.9.

Proposition 2.4. *Let $\mathfrak{g}$ be a Lie algebra over $\mathbf{R}$. Suppose that $[\mathfrak{s}_c, \mathfrak{r}] \neq 1$. Then $\mathfrak{g}$ has a Lie subalgebra which is isomorphic to either $\mathfrak{so}_3 \ltimes \mathfrak{d}^{\mathbf{R}}_{2n+1}$ or $\mathfrak{so}_3 \ltimes \mathfrak{hu}^i_{4n}$ for some $i = 0, 1, 2, 3$ and some $n \geq 1$.* ■

3. Corresponding results for algebraic groups and connected Lie groups

3.1. Minimal algebraic subgroups.

We now give the corresponding statements and results for algebraic groups.

Let S be a reductive K-group. A K-S-group means a linear K-group endowed with a K-action of S by automorphisms.

Recall that the Lie algebra functor gives an equivalence of categories between the category of unipotent K-groups and the category of nilpotent Lie K-algebras. If S is semisimple and simply connected with Lie algebra $\mathfrak{s}$, it induces an equivalence of categories between the category of unipotent K-S-groups and the category of nilpotent Lie S-algebras over K. If S is not simply connected (in particular, if S is not semisimple), this is no longer an essentially surjective functor, but it remains fully faithful.

A minimal (resp. almost minimal) solvable S-group N is defined similarly as in the case of Lie algebras; it is automatically unipotent (since it satisfies $[S,N]=N$). Moreover, N is a minimal (resp. almost minimal) solvable K-S-group if and only if its Lie algebra $\mathfrak{n}$ is a minimal (resp. almost minimal) solvable Lie $\mathfrak{s}$-algebra. Proposition 2.1 and Theorem 1.2 also immediately carry over into the context of algebraic groups.

If S is reductive and V is a K-S-module, we define the unipotent K-S-group $H(V)$ as follows: as a variety, $H(V)=V\oplus \mathrm{Alt}_S(V)^*$; it is endowed with the following group law:

$$(x,z)(x',z')=(x+x',z+z'+e_{x,x'}) \qquad x,x'\in V \quad z,z'\in \mathrm{Alt}_S(V)^* \tag{2}$$

where $e_{x,x'}\in \mathrm{Alt}_S(V)^*$ is defined by $e_{x,x'}(\varphi)=\varphi(x,x')$. This is a K-S-group under the action $s.(x,z)=(s.x,z)$. It is clear that its Lie algebra is isomorphic as a Lie K-S-algebra to $\mathfrak{h}(\mathfrak{v})$, where $V=\mathfrak{v}$ viewed as a $\mathfrak{s}$-module. Here is the analog of Theorem 1.5.

Theorem 3.1. *If N is an almost minimal solvable K-S-group, then it is isomorphic (as a K-S-group) to $H(V)/Z$, for some full K-S-module V and some K-subspace Z of $\mathrm{Alt}_S(V)^*$. It is minimal if and only if V is irreducible.*

Moreover, the almost minimal K-S-groups $H(V)/Z$ and $H(V)/Z'$ are isomorphic if and only if Z' and Z are in the same orbit for the natural action of $\mathrm{Aut}_S(V)$ on the Grassmannian of $\mathrm{Alt}_S(V)^*$.

3.2. The example SL_2.

The simply connected K-group with Lie algebra $\mathfrak{sl}_2$ is SL_2. Denote by V_n and H_{2n-1} the SL_2-groups corresponding to $\mathfrak{v}_n$ and $\mathfrak{h}_{2n-1}$. These are the solvable minimal SL_2-groups over K. The only non-simply connected K-group with Lie algebra $\mathfrak{sl}_2$ is the adjoint group PGL_2; thus the minimal solvable PGL_2-groups over K are V_{2n-1} for $n\geq 2$.

Remark 3.2. *It is convenient, in algebraic groups, to deal with the unipotent radical rather than with the radical. It is straightforward to see that a reductive subgroup S of a linear algebraic group centralizes the radical if and only if it centralizes the unipotent radical. Indeed, suppose $[S,R_u]=1$. We always have $[S,R/R_u]=1$ since R/R_u is central in G^0/R_u and S is connected (G^0 denoting the unit component of G). This easily implies that S acts trivially*[2].

[2]Write, for $s\in S$ and $r\in R$, $s.r=ru(s,r)$, where $u(s,r)\in R_u$ and $u(s,r)=1$ if $r\in R_u$. Then, for all $s,t\in S$ and $r\in R$ $st.r=s.ru(t,r)=(s.r)(s.u(t,r))=ru(s,r)u(t,r)$, so that $u(st,r)=u(s,r)u(t,r)$. This implies that if $s\in D^nS=S$, then $u(s,r)\in D^nR_u$. Taking n sufficiently large, we obtain $u(s,r)=1$ for all $s\in S$ and $r\in R$, that is, $[S,R]=1$.

Let G be a linear algebraic group over the field K of characteristic zero, R its radical, S a Levi factor, decomposed as $S_{nc}S_c$ by separating K-isotropic and K-anisotropic factors.

Proposition 3.3. *Suppose that $[S_{nc}, R] \neq 1$. Then G has a K-subgroup which is K-isomorphic to either $\mathrm{SL}_2 \ltimes V_n$, $\mathrm{PGL}_2 \ltimes V_{2n-1}$, or $\mathrm{SL}_2 \ltimes H_{2n-1}$ for some $n \geq 2$.*

Let us mention the translation into the context of connected Lie groups, which is immediate from the Lie algebraic version.

Proposition 3.4. *Let G be a real Lie group. Suppose that $[S_{nc}, R] \neq 1$. Then there exists a Lie subgroup H of G which is locally isomorphic to $\mathrm{SL}_2(\mathbf{R}) \ltimes V_n(\mathbf{R})$ or $\mathrm{SL}_2(\mathbf{R}) \ltimes H_{2n-1}(\mathbf{R})$ for some $n \geq 2$.*

Remark 3.5. *1) An analogous result holds with complex Lie groups.*

2) The Lie subgroup H is not necessarily closed; this is due to the fact that $\widetilde{\mathrm{SL}_2(\mathbf{R})}$ and $H_{2n-1}(\mathbf{R})$ have noncompact centre. For instance, take an element z of the centre of H that generates an infinite discrete subgroup, and take the image of H in the quotient of $H \times \mathbf{R}/\mathbf{Z}$ by (z, α), where α is irrational.

3) It can be easily be shown that, if the Lie group G is linear, then the subgroup H is necessarily closed. In a few words, this is because the derived subgroup of the radical is unipotent, hence simply connected, and the centre of the semisimple part is finite.

3.3. The example SO_3. We go on with the notation of Proposition 2.2.

In the context of algebraic $\mathbf{R}$-groups as in the context of connected Lie groups, the simply connected group corresponding to $\mathfrak{so}_3(\mathbf{R})$ is $\mathrm{SU}(2)$. The only non-simply connected corresponding group is $\mathrm{SO}_3(\mathbf{R})$.

The irreducible $\mathrm{SU}(2)$-modules corresponding to $\mathfrak{d}_{2m+1}^{\mathbf{R}}$ and $\mathfrak{u}_{4n}$ are denoted by $D_{2n+1}^{\mathbf{R}}$ and U_{4n}. Among those, only $D_{2n+1}^{\mathbf{R}}$ provide $\mathrm{SO}_3(\mathbf{R})$-modules.

Denote by HU_{4n}^i the unipotent $\mathbf{R}$-group corresponding to $\mathfrak{hu}_{4n}^i$, $i = 0, 1, 2, 3$.

Remark 3.6. *It can be shown that the maximal unipotent subgroups of $\mathrm{Sp}(n, 1)$ are isomorphic to HU_{4n}^3.*

Proposition 3.7. *Up to isomorphism, the minimal solvable Lie $\mathrm{SO}_3(\mathbf{R})$-algebras are $D_{2n+1}^{\mathbf{R}}$ for $n \geq 1$; the other minimal solvable Lie $\mathrm{SU}(2)$-algebras are HU_{4n}^i, for $n \geq 1$, $i = 0, 1, 2, 3$.* ∎

Proposition 3.8. *Let G be a linear algebraic $\mathbf{R}$-group. Suppose that $[S_c, R] \neq 1$. Then G has a $\mathbf{R}$-subgroup which is $\mathbf{R}$-isomorphic to either $\mathrm{SU}(2) \ltimes D_{2n+1}^{\mathbf{R}}$, $\mathrm{SO}_3(\mathbf{R}) \ltimes D_{2n+1}^{\mathbf{R}}$, or $\mathrm{SU}(2) \ltimes HU_{4n}^i$ for some $i = 0, 1, 2, 3$ and some $n \geq 1$.*

Let G be a real Lie group. Suppose that $[S_c, R] \neq 1$. Then G has a Lie subgroup which is locally isomorphic to either $\mathrm{SU}(2) \ltimes D_{2n+1}^{\mathbf{R}}$ or $\mathrm{SU}(2) \ltimes HU_{4n}^i$ for some $i = 0, 1, 2, 3$ and some $n \geq 1$. ∎

4. Application to Haagerup and Kazhdan Properties

4.1. Reminder.

Recall [5, Chap. 1] that a locally compact, σ-compact group G has the *Haagerup Property* if there exists a metrically proper, isometric action of G on some affine Hilbert space.

If H is a subgroup of G, the pair (G, H) has *Kazhdan Property (T)*, or that H has Kazhdan's Property (T) relatively to G, if every isometric action of G on any affine Hilbert space has a fixed point in restriction to H. In the case when $H = G$, G is said to have Property (T) (see [10] or [2]).

As an immediate consequence of these definitions, if (G, H) has Property (T) and H is not relatively compact in G, then G does not have the Haagerup Property; this is a frequent obstruction to Haagerup Property, although it is not the only one (see Remark 4.10).

The class of groups with the Haagerup Property generalizes the class of amenable groups as a strong negation of Kazhdan's Property (T). For other motivations of the Haagerup Property and equivalent definitions, see [5].

In the following lemma, we summarize the hereditary properties of the Haagerup and Kazhdan Properties that we will use in the sequel.

Lemma 4.1. *The Haagerup Property for locally compact, σ-compact groups is closed under taking (H1) closed subgroups, (H2) finite direct products, (H3) direct limits [5, Proposition 6.1.1], (H4) extensions with amenable quotient [5, Example 6.1.6], and (H5) is inherited from lattices [5, Proposition 6.1.5].*

Relative Property (T) is inherited by dense images: if (G, H) has Property (T) and $f : G \to K$ is a continuous morphism, then $(K, \overline{f(H)})$ has Property (T).

4.2. Continuous families of Lie groups with Property (T).

Proof of Proposition 1.12. We must construct a continuous family of connected Lie groups with Property (T) and with perfect and pairwise non-isomorphic Lie algebras.

Consider $\mathfrak{s} = \mathfrak{sp}_{2n}(\mathbf{R})$ $(n \geq 2)$. Let $\mathfrak{v}_i$, $i = 1, 2, 3, 4$ be four nontrivial absolutely irreducible, $\mathfrak{s}$-modules which are pairwise non-isomorphic and all preserve a symplectic form[3]. Then $\mathfrak{v} = \bigoplus_{i=1}^4 \mathfrak{v}_i$ is a full $\mathfrak{s}$-module and $\mathrm{Aut}_{\mathfrak{s}}(\mathfrak{v}) = \prod_{i=1}^4 \mathrm{Aut}_{\mathfrak{s}}(\mathfrak{v}_i) \simeq (\mathbf{R}^*)^4$. In particular, $\mathrm{Alt}_{\mathfrak{s}}(\mathfrak{v})^* \simeq \mathbf{R}^4$ and $\mathrm{Aut}_{\mathfrak{s}}(\mathfrak{v})$ acts diagonally on it. The action on the 2-Grassmannian, which is 4-dimensional, is trivial on the scalars, so that its orbits are at most 3-dimensional. So there exists a continuous family (P_t) of 2-planes in $\mathrm{Alt}_{\mathfrak{s}}(\mathfrak{v})^*$ which are in pairwise distinct orbits for the action of $\mathrm{Aut}_{\mathfrak{s}}(\mathfrak{v})$. By Theorem 3.1, the Lie $\mathfrak{s}$-algebras $\mathfrak{h}(\mathfrak{v})/P_t$ are pairwise non-isomorphic, and so the Lie algebras $\mathfrak{s} \ltimes \mathfrak{h}(\mathfrak{v})/P_t$ are pairwise non-isomorphic. The Lie algebras $\mathfrak{g}_t$ are perfect, and the corresponding Lie groups G_t have Property (T): this immediately follows from Wang's classification [13, Theorem 1.9]. ■

[3]There exist infinitely many such modules, which can be obtained by taking large irreducible components of the odd tensor powers of the standard $2n$-dimensional $\mathfrak{s}$-module.

Remark 4.2. *These examples have 2-nilpotent radical. This is, in a certain sense, optimal, since there exist only countably many isomorphism classes of Lie algebras over $\mathbf{R}$ with abelian radical, and only a finite number for each dimension.*

Proof of Proposition 1.13. We must construct a continuous family of locally isomorphic, pairwise non-isomorphic connected Lie groups with Property (T). The proof is actually similar to that of Proposition 1.12. Use the same construction, but, instead of taking the quotient G_t by P_t, take the quotient H_t by a lattice Γ_t of P_t. If we take the quotient of H_t by its biggest compact normal subgroup P_t/Γ_t, we obtain G_t. Accordingly, the groups H_t are pairwise non-isomorphic. ∎

4.3. Characterization of groups with the Haagerup Property.

Proposition 4.3. *Let $\mathbf{K}$ be a local field of characteristic zero and $n \geq 1$. Then the pairs $(\mathrm{SL}_2(\mathbf{K}) \ltimes V_n(\mathbf{K}), V_n(\mathbf{K}))$, $(\mathrm{PGL}_2(\mathbf{K}) \ltimes V_n(\mathbf{K}), V_n(\mathbf{K}))$, $(\mathrm{SL}_2(\mathbf{K}) \ltimes H_n(\mathbf{K}), H_n(\mathbf{K}))$, $(\widetilde{\mathrm{SL}_2(\mathbf{R})} \ltimes V_n(\mathbf{R}), V_n(\mathbf{R}))$, and $(\widetilde{\mathrm{SL}_2(\mathbf{R})} \ltimes H_n(\mathbf{R}), H_n(\mathbf{R}))$ have Property (T).*

Proof. The first (and the fourth) case is well-known; it follows, for instance, from Furstenberg's theory [8] of invariant probabilities on projective spaces, which implies that $\mathrm{SL}_2(\mathbf{K})$ does not preserve any probability on $V_n(\mathbf{K})$ (more precisely, on its Pontryagin dual) other than the Dirac measure at zero. See, for instance, the proof [10, Chap. 2, Proposition 2]. The second case is an immediate consequence of the first. For the third (resp. fifth) case, we invoke [5, Proposition 4.1.4], with $S = \mathrm{SL}_2(\mathbf{K})$, $N = H_n(\mathbf{K})$, even if the hypotheses are slightly different (unless $\mathbf{K} = \mathbf{R}$ or $\mathbf{C}$): the only modification is that, since here $[N, S]$ is not necessarily connected, we must show that its image in the unitary group U_n is connected so as to justify Lie's Theorem. Otherwise, it would have a nontrivial finite quotient. This is a contradiction, since $[N, S]$ is generated by divisible elements; this is clear, since, as the group of $\mathbf{K}$-points of an unipotent group, it has a well-defined logarithm. ∎

Corollary 4.4. *Let G be either a connected Lie group, or $G = \mathbf{G}(\mathbf{K})$, where $\mathbf{G}$ is a linear algebraic group over the local field $\mathbf{K}$ of characteristic zero. Suppose that the Lie algebra $\mathfrak{g}$ of G contains a subalgebra $\mathfrak{h}$ isomorphic to either $\mathfrak{sl}_2 \ltimes \mathfrak{v}_n$ or $\mathfrak{sl}_2 \ltimes \mathfrak{h}_{2n-1}$ for some $n \geq 2$. Then G has a noncompact closed subgroup with relative Property (T). In particular, G does not have Haagerup's property.*

Proof. Let us begin by the case of algebraic groups. By [3, Chap. II, Corollary 7.9], since $\mathfrak{h}$ is perfect, it is the Lie algebra of a closed $\mathbf{K}$-subgroup H of G. Since H must be $\mathbf{K}$-isomorphic to either $\mathrm{SL}_2 \ltimes V_m$, $\mathrm{PGL}_2 \ltimes V_{2m-1}$, or $\mathrm{SL}_2 \ltimes H_{2m-1}$ for some $m \geq 2$, Proposition 4.3 implies that $G(\mathbf{K})$ has a noncompact closed subgroup with relative Property (T).

In the case of Lie groups, we obtain a Lie subgroup which is the image of an immersion i of $\widetilde{\mathrm{SL}_2(\mathbf{R})} \ltimes N$, where N is either $V_n(\mathbf{R})$ or $H_{2n-1}(\mathbf{R})$, for some $n \geq 2$, into G. By Proposition 4.3, $(G, \overline{i(N)})$ has Property (T). We claim that $\overline{i(N)}$ is not compact. Suppose the contrary. Then it is solvable and connected, hence it is a torus. It is normal in the closure H of $i(G)$. Since the automorphism group of a torus is totally disconnected, the action by conjugation of H on $\overline{i(N)}$ is trivial; that is, $i(N)$ is central in H. This is a contradiction. ∎

Proof of Theorem 1.10. As we already noticed in the reminder, (i)$\Rightarrow$(ii) is immediate from the definition. We are going to prove (ii)$\Rightarrow$(iv)$\Rightarrow$(iii)$\Rightarrow$(i).

For the implication (iii)$\Rightarrow$(i), in the algebraic case, G is isomorphic, up to a finite kernel, to $S_{nc}(\mathbf{K}) \times \mathrm{Mr}(\mathbf{K})$, where Mr denotes the amenable radical of $\mathbf{G}$. The group $\mathrm{Mr}(\mathbf{K})$ is amenable, hence has Haagerup's property. The group $S_{nc}(\mathbf{K})$ also has Haagerup's property: if $\mathbf{K}$ is Archimedean, it maps, with finite kernel, onto a product of groups isomorphic to $\mathrm{PSO}_0(n,1)$ or $\mathrm{PSU}(n,1)$ $(n \geq 2)$, and these groups have Haagerup's property, by a result of Faraut and Harzallah, see [2, Chap. 2]. If $\mathbf{K}$ is non-Archimedean, then $S_{nc}(\mathbf{K})$ acts properly on a product of trees (one for each simple factor) [4], and this also implies that it has Haagerup's property [2, Chap. 2].

The same argument also works for connected Lie groups when the semisimple part has finite centre; in particular, this is fulfilled for linear Lie groups and their finite coverings. The case when the semisimple part has infinite centre is considerably more involved, see [5, Chap. 4].

(ii)$\Rightarrow$(iv) Suppose that (iv) is not satisfied. If $\mathfrak{g}$ contains a copy of $\mathfrak{sl}_2 \ltimes \mathfrak{v}_n$ or $\mathfrak{sl}_2 \ltimes \mathfrak{h}_{2n-1}$ for some $n \geq 2$, then, by Corollary 4.4, G does not satisfy (ii). If $\mathbf{K} = \mathbf{R}$, we consider G as a Lie group with finitely many components. By a standard argument, since $\mathrm{Sp}(2,1)$ is simply connected with finite centre (of order 2), an embedding of $\mathfrak{sp}(2,1)$ into $\mathfrak{g}$ corresponds to a closed embedding of $\mathrm{Sp}(2,1)$ or $\mathrm{PSp}(2,1)$ into G. Since $\mathrm{Sp}(2,1)$ has Property (T) [2, Chap. 3], this contradicts (ii).

(iv)$\Rightarrow$(iii) If $\mathfrak{g}$ is not M-decomposed, then, by Proposition 1.9, it contains a copy of $\mathfrak{sl}_2 \ltimes \mathfrak{v}_n$ or $\mathfrak{sl}_2 \ltimes \mathfrak{h}_{2n-1}$ for some $n \geq 2$.

If $\mathfrak{g}$ has a simple factor $\mathfrak{s}$, then $\mathfrak{s}$ embeds in $\mathfrak{g}$ through a Levi factor. If $\mathfrak{s}$ has $\mathbf{K}$-rank ≥ 2, then it contains a subalgebra isomorphic to either $\mathfrak{sl}_3$ or $\mathfrak{sp}_4$ [11, Chap I, (1.6.2)], and such a subalgebra contains a subalgebra isomorphic to $\mathfrak{sl}_2 \ltimes \mathfrak{v}_2$ (resp. $\mathfrak{sl}_2 \ltimes \mathfrak{v}_3$) [2, 1.4 and 1.5].

Finally, if $\mathbf{K} = \mathbf{R}$ and $\mathfrak{s}$ is isomorphic to either $\mathfrak{sp}(n,1)$ for some $n \geq 2$ or $\mathfrak{f}_{4(-20)}$, then it contains a copy of $\mathfrak{sp}(2,1)$. ■

Remark 4.5. *Conversely,* $\mathfrak{sp}(n,1)$ *and* $\mathfrak{f}_{4(-20)}$ *do not contain any subalgebra isomorphic to* $\mathfrak{sl}_2 \ltimes \mathfrak{v}_\mathfrak{n}$ *or* $\mathfrak{sl}_2 \ltimes \mathfrak{h}_{2\mathfrak{n}-1}$ *for any* $n \geq 2$*; this can be shown using results of [6] about weak amenability.*

4.4. Subgroups of Lie groups.

Let us exhibit some subgroups in the groups above.

First observation. Let G denote $\mathrm{SL}_2 \ltimes V_n$, $\mathrm{PGL}_2 \ltimes V_{2n-1}$, or $\mathrm{SL}_2 \ltimes H_{2n-1}$ for some $n \geq 2$, and R its radical. Then, for every field K of characteristic zero, $G(K)$ contains $G(\mathbf{Z})$ as a subgroup. On the other hand, the pair $(G(\mathbf{Z}), R(\mathbf{Z}))$ has Property (T), this is because $G(\mathbf{Z})$ is a lattice in $G(\mathbf{R})$.

Second observation. Now, let G denote $\mathrm{SU}(2) \ltimes D^{\mathbf{R}}_{2n+1}$, $\mathrm{SO}_3(\mathbf{R}) \ltimes D^{\mathbf{R}}_{2n+1}$, or $\mathrm{SU}(2) \ltimes HU^i_{4n}$ for some $i = 0,1,2,3$. These groups all have a $\mathbf{Q}$-form: this is obvious at least for all but $\mathrm{SU}(2) \ltimes HU^i_{4n}$ for $i = 1,2$; for these two, this is because the subspace Z_i can be chosen rational in the definition of HU^i_{4n}.

Let R be the radical of G and S a Levi factor defined over $\mathbf{Q}$. Let F be a number field of degree three over $\mathbf{Q}$, not totally real. Let $\mathcal{O}$ be its ring of integers. Then $G(\mathcal{O})$ embeds diagonally as an irreducible lattice in $G(\mathbf{R})\times G(\mathbf{C})$. Its projection Γ in $G(\mathbf{R})$ does not have Haagerup's property, since otherwise $G(\mathbf{C})$ would also have Haagerup's property (by (H5) in Lemma 4.1), and this is excluded since it does not satisfy $[S_{nc},R]=1$, by Theorem 1.10 (noting that the anisotropic Levi factor becomes isotropic after complexification).

Proposition 4.6. *Let G be a real Lie group, R its radical, S a semisimple factor. Suppose that $[S,R]\neq 1$. Then G has a countable subgroup without Haagerup's property.*

Proof. First case: $[S_{nc},R]\neq 1$. Then, by Proposition 3.4, G has a Lie subgroup H isomorphic to a quotient of $\widetilde{H}=\widetilde{\mathrm{SL}_2(\mathbf{R})}\ltimes R(\mathbf{R})$ by a discrete central subgroup, where $R=V_n$ or H_{2n-1}, for some $n\geq 2$. Denote by $\widetilde{H}(\mathbf{Z})$ the inverse image of $\mathrm{SL}_2(\mathbf{Z})\ltimes R(\mathbf{Z})$ in $\widetilde{H}$. By the observation above, $(\widetilde{H}(\mathbf{Z}),R(\mathbf{Z}))$ has Property (T), so that its image in H, which we denote by $H(\mathbf{Z})$, satisfies $(H(\mathbf{Z}),R_G(\mathbf{Z}))$ has Property (T), where $R_G(\mathbf{Z})$ means the image of $R(\mathbf{Z})$ in G. Observe that $R_G(\mathbf{Z})$ is infinite: if $R=V_n$, this is $V_n(\mathbf{Z})$; if $R=H_{2n-1}$, this is a quotient of $H_{2n-1}(\mathbf{Z})$ by some central subgroup. Accordingly, $H(\mathbf{Z})$ does not have Haagerup's property.

Second case: $[S_c,R]\neq 1$. By Proposition 3.8, G has a Lie subgroup H isomorphic to a central quotient of $\mathrm{SU}(2)(\mathbf{R})\ltimes R$, where $R=D^{\mathbf{R}}_{2n+1}$ or HU^i_{4n}, for some $n\geq 1$ and $i=0,1,2,3$.

First suppose that the radical of H is simply connected. Then, by the second observation above, H has a subgroup without the Haagerup property.

Now, let us deal with the case when $H=\widetilde{H}/Z$, where Z is a discrete central subgroup. Then $\widetilde{H}$ has a subgroup Γ as above which does not have Haagerup's property. Let W denote the centre of $\widetilde{H}$. The kernel of the projection of Γ to H is given by $\Gamma\cap Z$. We use the following trick: we apply an automorphism α of $\widetilde{H}$ such that $\alpha(\Gamma)\cap Z$ is finite. It follows that the image of $\alpha(\Gamma)$ in H does not have Haagerup's property.

This allows to suppose that $\Gamma\cap Z$ is finite, so that the image of Γ in H does not have Haagerup's property. Let us construct such an automorphism.

Observe that the representations of $\mathrm{SU}(2)$ can be extended to the direct product $\mathbf{R}^*\times\mathrm{SU}(2)$ by making $\mathbf{R}^*$ act by scalar multiplication. This action lifts to an action of $\mathbf{R}^*\times\mathrm{SU}(2)$ on HU^i_{4n}, where the scalar a acts on the derived subgroup of HU^i_{4n} by multiplication by a^2.

Now, working in the unit component of the centre W of $\widetilde{H}$, which we treat as a vector space, we can take a so that $a^2\cdot(\Gamma\cap W)$ avoids $Z\cap W$ (a clearly exists, since Γ and Z are countable). ■

Definition 4.7. Let G be a locally compact group. We say that G has Haagerup's property if every σ-compact open subgroup of G does.

Remark 4.8. *In view of (H3) of Lemma 4.1, this is equivalent to: every compactly generated, open subgroup of G has Haagerup's property, and also equivalent to the existence of a C_0-representation with almost invariant vectors [5, Chap. 1].*

In particular, G having Haagerup's property and (G,H) having Property (T) still imply H relatively compact.

All properties of the class of groups with Haagerup's property claimed in Lemma 4.1 also clearly remain true for general locally compact groups.

If G is a topological group, denote by G_d the group G endowed with the discrete topology.

Proof of Theorem 1.14. We remind that we must prove, for a connected Lie group G, the equivalence between

(i) G is locally isomorphic to $\mathrm{SO}_3(\mathbf{R})^a \times \mathrm{SL}_2(\mathbf{R})^b \times \mathrm{SL}_2(\mathbf{C})^c \times R$, with R solvable and integers a, b, c, and

(ii) G_d has Haagerup's property.

The implication $(i) \Rightarrow (ii)$ is, essentially, a deep and recent result of Guentner, Higson, and Weinberger [9, Theorem 5.1], which implies that $(\mathrm{PSL}_2(\mathbf{C}))_d$ has Haagerup's property. Let G be as in (i), and S its semisimple factor. Then G/S is solvable, so that, by (H4) of Lemma 4.1, we can reduce to the case when $G = S$. Now, let Z be the centre of the semisimple group G, and embed G_d in $(G/Z)_d \times G$, where G_d means G endowed with the discrete topology. This is a discrete embedding. Since G has Haagerup's property, this reduces the problem to the case when G has trivial centre. So, we are reduced to the cases of $\mathrm{SO}_3(\mathbf{R})$, $\mathrm{PSL}_2(\mathbf{R})$, and $\mathrm{PSL}_2(\mathbf{C})$. The two first groups are contained in the third, so that the result follows from the Guentner-Higson-Weinberger Theorem.

Conversely, suppose that G does not satisfy (i). If $[S,R] \neq 1$, then, by Proposition 4.6, G_d does not have Haagerup's property. Otherwise, observe that the simple factors allowed in (i) are exactly those of geometric rank one (viewing $\mathrm{SL}_2(\mathbf{C})$ as a complex Lie group). Hence, S has a factor W which is not of geometric rank one. Then the result is provided by Lemma 4.9 below. ∎

Lemma 4.9. *Let S be a simple Lie group which is not locally isomorphic to $\mathrm{SO}_3(\mathbf{R})$, $\mathrm{SL}_2(\mathbf{R})$ or $\mathrm{SL}_2(\mathbf{C})$. Then S_d does not have Haagerup's property.*

Proof. Let Z be the centre of S, so that $S/Z \simeq G(\mathbf{R})$ for some $\mathbf{R}$-algebraic group G. By assumption, $G(\mathbf{C})$ has factors of higher rank, hence does not have Haagerup's property. Let F be a number field of degree three over $\mathbf{Q}$, not totally real. Let $\mathcal{O}$ be its ring of integers. Then $G(\mathcal{O})$ embeds diagonally as an irreducible lattice in $G(\mathbf{R}) \times G(\mathbf{C})$, and is isomorphic to its projection in $G(\mathbf{R})$. Let Γ be the inverse image in $S \times G(\mathbf{C})$ of $G(\mathcal{O})$. Then Γ is a lattice in $S \times G(\mathbf{C})$. Hence, by [5, Proposition 6.1.5], Γ does not have Haagerup's property. Note that the projection Γ' of Γ into S has finite kernel, contained in the centre of $G(\mathbf{C})$. So Γ' neither has Haagerup's property, and is a subgroup of S. ∎

Remark 4.10. In contrast with Theorem 1.10, Theorem 1.14 is no longer true if we replace the statement "G_d has Haagerup's property" by "G_d has no infinite subgroup with relative Property (T)". Indeed, let $G = K \ltimes V$, where K is locally isomorphic to $\mathrm{SO}_3(\mathbf{R})^n$ and V is a vector space on which K acts nontrivially. Suppose that (G_d, H) has Property (T) for some subgroup H. Then $(G_d/V, H/(H \cap V))$ has Property (T). In view of the Guentner-Higson-Weinberger

Theorem (see the proof of Theorem 1.14), $H/(H\cap V)$ is finite. On the other hand, since G has Haagerup's property, $H\cap V$ must be relatively compact, and this implies that $H\cap V=1$. Thus, H is finite.

Motivated by this example, it is easy to exhibit finitely generated groups without the Haagerup Property and do not have infinite subgroups with relative Property (T). For instance, let $n\geq 3$, and q be the quadratic form $\sqrt{2}\,x_0^2+x_1^2+x_2^2+\cdots+x_{n-1}^2$. Let $G(R)=\mathrm{SO}(q)(R)\ltimes R^n$ and write, for any commutative $\mathbf{Z}(\sqrt{2})$-algebra R, $H(R)=\mathrm{SO}(q)(R)$. Then $\Gamma=G(\mathbf{Z}[\sqrt{2}])$ is such an example. The fact that Γ has no infinite subgroup Λ with relative Property (T) can be seen without making use of the Guentner-Higson-Weinberger Theorem: first observe that $H(\mathbf{Z}[\sqrt{2}])$ is a cocompact lattice in $\mathrm{SO}(n-1,1)$, hence has Haagerup's property. So the projection of Λ in $H(\mathbf{Z}[\sqrt{2}])$ is finite. So, upon passing to a finite index subgroup, we can suppose that Λ is contained in the subgroup $\mathbf{Z}[\sqrt{2}]^n$ of $\Gamma=\mathrm{SO}(q)(\mathbf{Z}[\sqrt{2}])\ltimes\mathbf{Z}[\sqrt{2}]^n$. But then the closure L of Λ in the subgroup $\mathbf{R}^n$ of the amenable group $G(\mathbf{R})=\mathrm{SO}(q)(\mathbf{R})\ltimes\mathbf{R}^n$ is not compact, and $(G(\mathbf{R}),L)$ has Property (T). This is a contradiction.

On the other hand, Γ does not have Haagerup's property, since it is a lattice in $G(\mathbf{R})\ltimes G^\sigma(\mathbf{R})$ (use (H5) of Lemma 4.1), where σ is the nontrivial automorphism of $\mathbf{Q}(\sqrt{2})$, and $G^\sigma(\mathbf{R})\simeq\mathrm{SO}(n-1,1)\ltimes\mathbf{R}^n$ does not have Haagerup's property, by Theorem 1.10. Note that Γ, as a cocompact lattice in a connected Lie group, is finitely presented.

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References

[1] Abels, H., "Finite presentability of S-arithmetic groups. Compact presentability of solvable groups," Lecture Notes in Math. **1261**, Springer-Verlag, Berlin etc., 1987.

[2] Bekka, B., P. de la Harpe, A. Valette, "Kazhdan's Property (T)," forthcoming 2005, available at `http://name.math.univ-rennes1.fr/bachir.bekka`.

[3] Borel, A., "Linear Algebraic Groups," Benjamin, Inc., 1969 (second edition: Springer, 1991).

[4] Bruhat, F.,and J. Tits, *Groupes réductifs sur un corps local. I, Données radicielles valuées*, Publ. Math. Inst. Hautes Études Sci. **41** (1972), 5–251.

[5] Cherix, P.-A., M. Cowling, P. Jolissaint, and A. Valette, "Groups with the Haagerup Property," Birkhäuser, Progress in Mathematics 197, 2001.

[6] Cowling, M., B. Dorofaeff, A. Seeger, and J. Wright, *A Family of Singular Oscillatory Integral Operators and Failure of Weak Amenability*, Preprint 2004, to appear in Duke Math. J.

[7] Chatterji, I., C. Pittet, and L. Saloff-Coste, *Connected Lie groups and property RD*, Preprint, 2004.

[8] Furstenberg, H., *A note on Borel's density theorem*, Proc. Amer. Math. Soc. **55** (1976), 209–212.

[9] Guentner, E., N. Higson, and S. Weinberger, *The Novikov Conjecture for linear groups*, Preprint, 2003, to appear in Publ. Math. Inst. Hautes Études Sci.

[10] Harpe (de la), P., A. Valette, "La propriété (T) de Kazhdan pour les groupes localement compacts", Astérisque **175**, Soc. Math. France, 1989.

[11] Margulis, G., "Discrete Subgroups of Semisimple Lie Groups," Springer-Verlag, Berlin etc., 1991.

[12] Varopoulos, N., *Analysis on Lie groups*, Revista Math. Iberoamericana **12** (1996), 791–917.

[13] Wang, S. P., *On the Mautner phenomenon and groups with Property (T)*, Amer. J. Math. **104** (1982), 1191–1210.

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