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Mathematica and Heisenberg type superalgebras

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Abstract. With the support of software *Mathematica* 4.0 we obtain important properties of Heisenberg type superalgebras, this type of superalgebras corresponds to the family of Lie superalgebras that generalize Heisenberg algebras. In particular, we obtain concrete classifications for arbitrary dimension of even part and dimension of odd part up to three.
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1. Introduction

The theory of Lie superalgebras has many important applications in mathematics and physics in general, nuclear physics, genetics and molecular biophysics, see [4].

The first comprehensive description of the mathematical theory of Lie superalgebras is due to Kac [7] in 1977, who establishes the classification of all finite-dimensional simple Lie superalgebras over an algebraically closed field of characteristic zero.

However, nilpotent Lie superalgebras are practically unknown, so far [2]. This paper is placed within this context. As Heisenberg algebra plays a fundamental role in quantum mechanics [1], [8], the aim of this work is to study the Lie superalgebras that generalize Heisenberg algebra. These (nilpotent) Lie superalgebras are called Heisenberg superalgebras.

2. Preliminary

A *Lie superalgebra*, $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ is a superalgebra over a base field $\mathbf{K} = \mathbf{R}$ or $\mathbf{C}$ with an operation [,] satisfying the following axioms:

(*i*) $[X,Y] = -(-1)^{\alpha\cdot\beta}[Y,X] \quad \forall X \in \mathfrak{g}_\alpha, \forall Y \in \mathfrak{g}_\beta.$

(*ii*) $(-1)^{\gamma\cdot\alpha}[X,[Y,Z]] + (-1)^{\alpha\cdot\beta}[Y,[Z,X]] + (-1)^{\beta\cdot\gamma}[Z,[X,Y]] = 0.$
for all $X \in \mathfrak{g}_\alpha, Y \in \mathfrak{g}_\beta, Z \in \mathfrak{g}_\gamma$ with $\alpha,\beta,\gamma \in \mathbf{Z}_2$.

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This property is called *graded Jacobi identity* and we will denote it by $J_g(X, Y, Z)$.

$\mathfrak{g}_{\bar{0}}$ is called the even part, and it is a Lie algebra, and $\mathfrak{g}_{\bar{1}}$ is called the odd part and it is an $\mathfrak{g}_{\bar{0}}$-module by restriction of the adjoint representation [9].

We say that two Lie superalgebras $\mathfrak{g}$, $\mathfrak{g}'$ are isomorphic if there exists a $\mathbf{Z}_2$-graded vector spaces isomorphism $\Phi : \mathfrak{g} \longrightarrow \mathfrak{g}'$ satisfying $\Phi([X, Y]) = [\Phi(X), \Phi(Y)]$ for all $X, Y \in \mathfrak{g}$ (and then Φ is called an isomorphism of Lie superalgebras). We recall that an isomorphism (homomorphism, automorphism) of $\mathbf{Z}_2$-graded vector spaces are homogeneous linear mappings of degree zero. Particularly, changes of basis are isomorphisms of Lie superalgebras.

The *descending central sequence* of a Lie superalgebra $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ is defined by $\mathcal{C}^0(\mathfrak{g}) = \mathfrak{g}$, $\mathcal{C}^{k+1}(\mathfrak{g}) = [\mathcal{C}^k(\mathfrak{g}), \mathfrak{g}]$ for all $k \geq 0$. If $\mathcal{C}^k(\mathfrak{g}) = \{0\}$ for some k, Lie superalgebra is called *nilpotent*. The smallest integer k such as $\mathcal{C}^k(\mathfrak{g}) = \{0\}$ is called the nilindex of $\mathfrak{g}$.

Engel's theorem for Lie algebras and its direct consequences remain valid for Lie superalgebras, and the proof is the same for Lie algebras [6].

Proposition 2.1. *A Lie superalgebra $\mathfrak{g}$ is nilpotent if and only if $ad(X)$ is nilpotent for every homogeneous element X of $\mathfrak{g}$.*

Analogous to Lie algebras, *Goze's invariant* (or *pair of characteristic sequences*), [5], appears for Lie superalgebras and it is defined by the following pair of sequences

$$gz(\mathfrak{g}) = \Big(\max gz_0(X_1) \,\big|\, \max gz_1(X_2) \Big) \qquad \forall X_1, X_2 \in \mathfrak{g}_0 - [\mathfrak{g}_0, \mathfrak{g}_0]$$

with $gz_{\bar{i}}(X)$, $i = 0, 1$ the characteristic sequence formed by the dimensions of Jordan's boxes, in decreasing for lexicografic order, of the matrix associated to operator $ad(X)$ restricted to $\mathfrak{g}_{\bar{i}}$.

Corollary 2.2. *Let V be a vector space of dimension m and let $\mathfrak{h}$ be a set of nilpotent endomorphism of V. Then there exists a descending sequence of vector subspaces $V_m, \ldots, V_1, V_0$ of V, with dimensions $m, m-1, \ldots 0$, respectively, and such that $h(V_{i+1}) \subseteq V_i$ $\forall h \in \mathfrak{h}$ $i = 0, 1, \ldots, m-1$.*

Remark 2.3. Let $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ be a nilpotent Lie superalgebra. If we consider $V = \mathfrak{g}_{\bar{1}}$ (taking $\mathfrak{g}_{\bar{1}}$ as vector space) and $\mathfrak{h}$ the operator ad restricted to $\mathfrak{g}_{\bar{0}}$, the conditions of the above corollary are satisfied, and then we have a descending sequence of subspaces $V = V_m \supset \ldots \supset V_1 \supset V_0$ of dimensions $m, m-1, \ldots 0$, such that $[\mathfrak{g}_{\bar{0}}, V_{i+1}] \subseteq V_i$.

We denote by $\mathcal{L}^{n+m}$ the set of the Lie superalgebras $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ with $\dim(\mathfrak{g}_{\bar{0}}) = n$ and $\dim(\mathfrak{g}_{\bar{1}}) = m$.

By taking an homogeneous basis $\{X_0, X_1, \ldots, X_{n-1}, Y_1, \ldots, Y_m\}$ in $\mathfrak{g}$ (with $\mathfrak{g} \in \mathcal{L}^{n+m}$), the superalgebra is completely determined by its structure constants, that is, by the set of constants $\{C_{ij}^k, D_{ij}^k, E_{ij}^k\}_{i,j,k}$.

From the graded Jacobi identity we have the following restrictions for the structure constants:

$$
(S)\begin{cases}
\sum_{l=0}^{n-1}(C_{ij}^{l}C_{kl}^{s}+C_{jk}^{l}C_{il}^{s}+C_{ki}^{l}C_{jl}^{s})=0 & 0\leq i<j<k\leq n-1,\\
 & s=0,\ldots,n-1\\
\sum_{l=1}^{m}(D_{jk}^{l}D_{il}^{s}-D_{jl}^{s}D_{ik}^{l})-\sum_{l=0}^{n-1}C_{ij}^{l}D_{lk}^{s}=0 & 0\leq i<j\leq n-1,\\
 & k=1,\ldots,m, s=1,\ldots,m
\end{cases}
$$

$$
(S)\begin{cases}
\sum_{l=1}^{m}(D_{ij}^{l}E_{lk}^{s}+D_{ik}^{l}E_{jl}^{s})-\sum_{l=0}^{n-1}E_{jk}^{l}C_{il}^{s}=0 & i=0,\ldots,n-1,\\
 & 1\leq j\leq k\leq m, s=0,\ldots,n-1\\
\sum_{l=0}^{n-1}(E_{ij}^{l}D_{lk}^{s}+E_{ik}^{l}D_{lj}^{s}+E_{jk}^{l}D_{li}^{s})=0 & 1\leq i\leq j\leq m, 0\leq k\leq n-1\\
 & s=1,\ldots,m
\end{cases}
$$

One could think of a program of classifying all Lie superalgebras by considering the above equations to be solved for unknown structure constants. This turns out to be a very complicated problem because of the non-linearity of the equations.

In this paper we are going to solve (S) for some special classes of nilpotent Lie superalgebras (which we call Heisenberg type superalgebras) using changes of basis, nilindex, and the software *Mathematica 4.0*.

3. Heisenberg type superalgebras.

Definition 3.1. A nilpotent Lie superalgebra $\mathfrak{g}=\mathfrak{g}_{\overline{0}}\oplus\mathfrak{g}_{\overline{1}}$ is a Heisenberg type superalgebra (HSA) if $\mathfrak{g}_{\overline{0}}=\mathcal{H}_r$, where $\mathcal{H}_r$ is the Heisenberg Lie algebra of dimension $2r+1$ and law

$$(\mathcal{B}_{\mathcal{H}_r})\qquad [X_{2i},X_{2i+1}]=X_{2r},\qquad 0\leq i\leq r-1$$

in a certain basis denoted by $\{X_0,X_1,\ldots,X_{2r}\}$. The all other commutators of $\mathfrak{g}_{\overline{0}}$ are zero.

In what follows, when we use the laws of a HSA we omit the commutators $\mathcal{B}_{\mathcal{H}_r}$. Also we omit all commutators which are zero, for instances in the theorems 3.2., 3.3., 3.5.,. . .

In this section, with aid of the above program, we present the classification of the HSA with arbitrary dimension of even part and dimension of odd part three. The study the existence of adapted basis for HSA with dimension of odd part up to three can see in [3].

Theorem 3.2. *[3] If* $\mathfrak{g}=\mathfrak{g}_{\overline{0}}\oplus\mathfrak{g}_{\overline{1}}$ *is a HSA with* $\dim(\mathfrak{g}_{\overline{0}})=2r+1$ *and* $\dim(\mathfrak{g}_{\overline{1}})=1$*, then*

1. *there exists a suitable homogeneous basis of* $\mathfrak{g}$*,* $\{X_0,X_1,\ldots,X_{2r},Y_1\}$ *such that the commutators can be expressed as in* $\mathcal{B}_{\mathcal{H}_r}$
2. $\mathfrak{g}$ *is isomorphic to* $\mathfrak{g}^1$*, whose law can be expressed by* $\mathcal{B}_{\mathcal{H}_r}$ *and* $[Y_1,Y_1]=X_{2r}$

Theorem 3.3. *[3] If* $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ *is a HSA with* $\dim(\mathfrak{g}_{\bar{0}}) = 2r+1$ *and* $\dim(\mathfrak{g}_{\bar{1}}) = 2$*, then*

1. *there exists a suitable homogeneous basis of* $\mathfrak{g}$*,* $\{X_0, X_1, \ldots, X_{2r}, Y_1, Y_2\}$*, with* $\{X_0, X_1, \ldots, X_{2r}\}$ *a basis of* $\mathfrak{g}_{\bar{0}}$ *and* $\{Y_1, Y_2\}$ *a basis of* $\mathfrak{g}_{\bar{1}}$*, such that* $\mathcal{B}_{\mathcal{H}_r}$ *and* $[X_0, Y_1] = \epsilon Y_2, \quad \epsilon \in \{0,1\}$.

2. $\mathfrak{g}$ *is isomorphic to one of the superalgebras, pairwise non-isomorphic, whose laws can be expressed by*

$\mathfrak{g}^1 \oplus \mathbf{C}$: $\left\{ [Y_1, Y_1] = X_{2r} \right.$

$\mathfrak{g}^2_1$: $\left\{ [Y_1, Y_2] = X_{2r} \right.$

$\mathfrak{g}^2_2$: $\left\{ \begin{array}{l} [X_0, Y_1] = Y_2 \\ [Y_1, Y_1] = X_{2r} \end{array} \right.$

$\mathfrak{g}^2_3$: $\left\{ \begin{array}{l} [X_0, Y_1] = Y_2 \\ [Y_1, Y_1] = 2X_1 \\ [Y_1, Y_2] = X_{2r} \end{array} \right.$

Note that we omit the commutators $\mathcal{B}_{\mathcal{H}_r}$.

Theorem 3.4. *If* $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ *is a HSA with* $\dim(\mathfrak{g}_{\bar{0}}) = 2r+1$ *and* $\dim(\mathfrak{g}_{\bar{1}}) = 3$*, then there exists a suitable homogeneous basis of* $\mathfrak{g}$*,* $\{X_0, X_1, \ldots, X_{2r}, Y_1, Y_2, Y_3\}$*, with* $\{X_0, X_1, \ldots, X_{2r}\}$ *a basis of* $\mathfrak{g}_{\bar{0}}$ *and* $\{Y_1, Y_2, Y_3\}$ *a basis of* $\mathfrak{g}_{\bar{1}}$*, such that*

$$(*)\left\{ \begin{array}{l} [X_0, Y_1] = \epsilon_1 Y_2 \\ [X_0, Y_2] = \epsilon_1\epsilon_2 Y_3, \qquad \epsilon_1, \epsilon_2 \in \{0,1\} \end{array} \right. \quad \textit{or} \quad (**)\left\{ \begin{array}{l} [X_0, Y_1] = Y_3 \\ [X_0, Y_2] = Y_3 \end{array} \right.$$

The proofs of these theorems can be consulted in [3].

In the follows, when we use the label $(**)$ we omit the commutators $[X_0, Y_1] = Y_3$ and $[X_0, Y_2] = Y_3$.

In this paper, we have obtained the classifications for some concrete dimensions with the aid of a program. Afterwards, we have induced the expressions for the general case. Finally, we have proved the general result.

Thanks to Theorem 3.4, we can reduce the problem of classification of HSA with arbitrary dimension of even part and dimension of odd part three to the following cases.

The family $(*)$ verify that the second part of Goze invariant is $(1,1,1)$, $(2,1)$ or (3) if (ϵ_1, ϵ_2) is $(0,-)$, $(1,0)$ or $(1,1)$.

Case 1. Goze invariant $(2,1,\ldots,1|1,1,1)$

$$\left\{ \begin{array}{l} [Y_i, Y_j] = \sum_{k=0}^{2r} E_{ij}^k X_k, \quad 1 \le i,j \le 3, (i,j) \ne (3,3), \\ [Y_3, Y_3] = E_{33}^{2r} X_{2r}. \end{array} \right. \tag{1}$$

Case 2. Goze invariant $(2,1,\ldots,1|2,1)$

2.1. With adapted basis $[X_0, Y_1] = Y_2$, $[X_0, Y_2] = 0$

$$\left\{ \begin{array}{ll} [X_0, Y_1] = Y_2, & \\ [X_j, Y_1] = D_{j,1}^2 Y_2 + D_{j,1}^3 Y_3, & 1 \le j \le 2r-1, \\ [X_{2r}, Y_1] = D_{2r,1}^3 Y_3, & \\ [X_j, Y_2] = D_{j,2}^3 Y_3, & 1 \le j \le 2r-1, \\ [Y_i, Y_j] = \sum_{k=0}^{2r} E_{ij}^k X_k, & 1 \le i,j \le 3, (i,j) \ne (3,3), \\ [Y_3, Y_3] = E_{33}^{2r} X_{2r}. & \end{array} \right. \tag{2}$$

with the following restrictions for the parameters

$$\begin{cases} D^2_{j,1}D^3_{j,2} = 0 & 1 \leq j \leq 2r-1 \\ D^2_{j,1}D^3_{j+1,2} = 0 & 1 \leq j \leq 2r-2 \\ D^2_{j,1}D^3_{j-1,2} = 0 & 2 \leq j \leq 2r-1 \end{cases}$$

2.2. With adapted basis $[X_0, Y_1] = Y_3,\ [X_0, Y_2] = Y_3$

$$\begin{cases} (**) \\ [X_j, Y_1] = D^3_{j,1}Y_3, & 1 \leq j \leq 2r, \\ [X_j, Y_2] = D^3_{j,2}Y_3, & 1 \leq j \leq 2r-1, \\ [Y_i, Y_j] = \sum_{k=0}^{2r} E^k_{ij}X_k, & 1 \leq i,j \leq 3, (i,j) \neq (3,3), \\ [Y_3, Y_3] = E^{2r}_{33}X_{2r}. \end{cases} \tag{3}$$

Case 3. Goze invariant $(2,1,\ldots,1|3)$

$$\begin{cases} [X_0, Y_1] = Y_2, \\ [X_0, Y_2] = Y_3, \\ [X_j, Y_1] = D^2_{j,1}Y_2 + D^3_{j,1}Y_3, & 1 \leq j \leq 2r-1, \\ [X_{2r}, Y_1] = D^3_{2r,1}Y_3, \\ [X_j, Y_2] = D^3_{j,2}Y_3, & 1 \leq j \leq 2r-1, \\ [Y_i, Y_j] = \sum_{k=0}^{2r} E^k_{ij}X_k, & 1 \leq i,j \leq 3, (i,j) \neq (3,3), \\ [Y_3, Y_3] = E^{2r}_{33}X_{2r}. \end{cases}$$

The Lie superalgebras that we obtain in the different cases are non isomorphic because they have different Goze's invariant.

Notation. The superalgebras obtained from case j will be designed as $\mathfrak{g}^3_{(j,i)}$.Thus, those from case 2 as $\mathfrak{g}^3_{(2,i)}$, with $1 \leq i \leq 10$, and $\mathfrak{g}^{3,\alpha,\beta}_{(2,11)}$, $\mathfrak{g}^{3,\alpha}_{(2,12)}$, $\mathfrak{g}^3_{(2,13)}$, $\mathfrak{g}^{3,\alpha,\beta}_{(2,14)}$, $\mathfrak{g}^{3,\alpha}_{(2,15)}$ and $\mathfrak{g}^3_{(2,16)}$; those from case 3 as $\mathfrak{g}^3_{(3,i)}$, $1 \leq i \leq 9$. In case 1 there only exists one non-split and non degenerate superalgebra. To simplify we will design it as $\mathfrak{g}^3_1$, instead of $\mathfrak{g}^3_{(1,1)}$.

Now, we will study the HSA with dimension of even part $2r+1$ and dimension of odd part equal three. We classify those minimal Goze invariant $(2,1,\ldots,1|\ 1,1,1)$ and maximal $(2,1,\ldots,1|\ 3)$; from this Goze invariant $(2,1,\ldots,1|\ 2,1)$ we obtain the generic family and the list of 2-nilpotent.

3.1. Minimal Goze invariant $(2,1,\ldots,1|\ 1,1,1)$.

Theorem 3.5. *If $\mathfrak{g}$ is a non degenerate HSA with dimension of even part $2r+1$, dimension of odd part 3 and Goze invariant $(2,1,\ldots,1|\ 1,1,1)$, then it is isomorphic to one of the following three superalgebras, pairwise non-isomorphic, whose laws can be expressed in a suitable basis,*

$$(\{X_0, X_1, \ldots, X_{2r-1}, X_{2r}, Y_1, Y_2, Y_3\}),\ by$$

$\mathfrak{g}^1 \oplus \mathbf{C}^2$: $\{\ [Y_1, Y_1] = X_{2r}$

$\mathfrak{g}^2_1 \oplus \mathbf{C}$: $\{\ [Y_1, Y_2] = X_{2r}$

$\mathfrak{g}^3_1$: $\begin{cases} [Y_1, Y_1] = X_{2r} \\ [Y_2, Y_3] = X_{2r} \end{cases}$

Proof. It is easy to prove that the above superalgebras are HSA, with dimension of even part $2r+1$, dimension of odd part 3 and Goze invariant

$$(2,1,\dots,1|\, 1,1,1).$$

Moreover, these superalgebras are pairwise non-isomorphic. The split superalgebras $\mathfrak{g}^1 \oplus \mathbf{C}^2$, $\mathfrak{g}_1^2 \oplus \mathbf{C}$ are non-isomorphic and $\mathfrak{g}_1^3$ is not split.

It remains to prove that the given superalgebras are all the Lie superalgebras verifying the desired conditions.

We can consider the following family

$$\begin{cases} [X_i, Y_j] = 0, & 0 \le i \le 2r,\ 1 \le j \le 3 \\ [Y_i, Y_j] = \sum_{k=0}^{2r} E_{ij}^k X_k, & 1 \le i, j \le 3, (i,j) \ne (3,3) \\ [Y_3, Y_3] = E_{33}^{2r} X_{2r} \end{cases}$$

We obtain that $E_{ij}^{2k+1} = 0$ (from $J_g(X_{2k}, Y_i, Y_j)$) and $E_{ij}^{2k} = 0$ (from $J_g(X_{2k+1}, Y_i, Y_j)$), $0 \le k \le r-1,\ 1 \le i, j \le 3$.

The family to consider is the following,

$$\begin{cases} [Y_i, Y_j] = E_{ij}^{2r} X_{2r}, \quad 1 \le i, j \le 3 \end{cases}$$

We always can suppose that $E_{33}^{2r} = 0$, because

- If $E_{22}^{2r} = 0$, we make the change of basis $Y_2' = Y_3,\ Y_3' = Y_2$.
- If $E_{22}^{2r} \ne 0$, we make the change of basis $Y_2' = Y_2,\ Y_3' = cY_2 + Y_3$, where c is one solution of the equation $E_{22}^{2r} c^2 + 2E_{23}^{2r} c + E_{33}^{2r} = 0$.

Moreover, we can also suppose that $E_{22}^{2r} = 0$, because

- If $E_{11}^{2r} = 0$, we make the change of basis $Y_1' = Y_2,\ Y_2' = Y_1$.
- If $E_{11}^{2r} \ne 0$, we make the change of basis $Y_1' = Y_1,\ Y_2' = cY_1 + Y_2$, where c is one solution of the equation $E_{11}^{2r} c^2 + 2E_{12}^{2r} c + E_{22}^{2r} = 0$.

The family to classify is

$$\begin{cases} [Y_1, Y_1] = E_{11}^{2r} X_{2r} \\ [Y_1, Y_2] = E_{12}^{2r} X_{2r} \\ [Y_1, Y_3] = E_{13}^{2r} X_{2r} \\ [Y_2, Y_3] = E_{23}^{2r} X_{2r} \end{cases}$$

Analogously, we can suppose that $E_{13}^{2r} = 0$, because

- If $E_{12}^{2r} = 0$, we make the change of basis $Y_1' = Y_1,\ Y_2' = Y_3,\ Y_3' = Y_2$.
- If $E_{12}^{2r} \ne 0$:
  - $E_{23}^{2r} \ne 0$, making the change $Y_1' = Y_1 - \frac{E_{13}^{2r}}{E_{23}^{2r}} Y_2 - \frac{E_{12}^{2r}}{E_{23}^{2r}} Y_3$, we obtain that $[Y_1', Y_2'] = [Y_1', Y_3'] = 0$
  - $E_{23}^{2r} = 0$, making the change $Y_3' = Y_3 - \frac{E_{13}^{2r}}{E_{12}^{2r}} Y_2$, we obtain that $[Y_1', Y_3'] = 0$

and the family

$$\left\{\begin{array}{l}[Y_1,Y_1]=E_{11}^{2r}X_{2r}\\ [Y_1,Y_2]=E_{12}^{2r}X_{2r}\\ [Y_2,Y_3]=E_{23}^{2r}X_{2r}\end{array}\right.$$

Case $E_{23}^{2r}=0$**:**

- If $E_{12}^{2r}=0$, then $E_{11}^{2r}\neq 0$ (if $E_{11}^{2r}=0$ we have a Lie algebra) we obtain $\mathfrak{g}^1\oplus\mathbf{C}^2$, a split HSA.
- If $E_{12}^{2r}\neq 0$, we can suppose that $E_{11}^{2r}=0$, making a change of basis, $Y_1'=Y_1-\frac{E_{11}^{2r}}{2E_{12}^{2r}}Y_2$, we find $\mathfrak{g}_1^2\oplus\mathbf{C}$, a split HSA.

Case $E_{23}^{2r}\neq 0$.

We can suppose that $E_{12}^{2r}=0$, making the following change $Y_1'=Y_1-\frac{E_{12}^{2r}}{E_{23}^{2r}}Y_3$. If $E_{11}^{2r}=0$, we obtain $\mathfrak{g}_1^2\oplus\mathbf{C}$ by the change of basis $Y_1'=Y_2,\ Y_2'=Y_3,\ Y_3'=Y_1$. If $E_{11}^{2r}\neq 0$, we find $\mathfrak{g}_1^3$ by an obvious change of basis. ■

3.2. Goze invariant $(2,1,\ldots,1|\ 2,1)$**.** In this case, we have two subfamilies of HSA. A family with adapted basis $[X_0,Y_1]=Y_2,\ [X_0,Y_2]=0$ and the other family with adapted basis $[X_0,Y_1]=Y_3,\ [X_0,Y_2]=Y_3$. We will fully classify the first family and we will obtain from the second family the generic expression and the list of 2-nilpotent we will given.

Theorem 3.6. *If* $\mathfrak{g}$ *is a non degenerate HSA with dimension of even part* $2r+1$*, dimension of odd part* 3*, Goze invariant* $(2,1,\ldots,1|\ 2,1)$ *such that it admits an adapted basis* $\{X_0,X_1,\ldots,X_{2r-1},X_{2r},Y_1,Y_2,Y_3\}$ *with* $[X_0,Y_1]=Y_2$ *and* $[X_0,Y_2]=0$*, then it is isomorphic to one of the following superalgebras, pairwise non-isomorphic, whose laws can be expressed by*

$\mathfrak{g}_2^2\oplus\mathbf{C}$:
$$\left\{\begin{array}{l}[X_0,Y_1]=Y_2\\ [Y_1,Y_1]=X_{2r}\end{array}\right.$$

$\mathfrak{g}_3^2\oplus\mathbf{C}$:
$$\left\{\begin{array}{l}[X_0,Y_1]=Y_2\\ [Y_1,Y_1]=2X_1\\ [Y_1,Y_2]=X_{2r}\end{array}\right.$$

$\mathfrak{g}_{(2,1)}^3$:
$$\left\{\begin{array}{l}[X_0,Y_1]=Y_2\\ [X_1,Y_1]=Y_3\\ [Y_1,Y_1]=X_{2r}\end{array}\right.$$

$\mathfrak{g}_{(2,2)}^3$:
$$\left\{\begin{array}{l}[X_0,Y_1]=Y_2\\ [X_3,Y_1]=Y_3\\ [Y_1,Y_1]=X_{2r}\end{array}\right.$$

$\mathfrak{g}_{(2,3)}^3$:
$$\left\{\begin{array}{l}[X_0,Y_1]=Y_2\\ [X_3,Y_1]=Y_3\\ [Y_1,Y_1]=X_1-2X_2\\ [Y_1,Y_2]=\frac{1}{2}X_{2r}\\ [Y_1,Y_3]=X_{2r}\end{array}\right.$$

$\mathfrak{g}_{(2,4)}^3$:
$$\left\{\begin{array}{l}[X_0,Y_1]=Y_2\\ [X_3,Y_1]=Y_3\\ [Y_1,Y_1]=-2X_2\\ [Y_1,Y_3]=X_{2r}\end{array}\right.$$

$\mathfrak{g}_{(2,5)}^3$:
$$\left\{\begin{array}{l}[X_0,Y_1]=Y_2\\ [Y_1,Y_1]=X_1\\ [Y_1,Y_2]=\frac{1}{2}X_{2r}\\ [Y_1,Y_3]=X_{2r}\end{array}\right.$$

$\mathfrak{g}_{(2,6)}^3$:
$$\left\{\begin{array}{l}[X_0,Y_1]=Y_2\\ [Y_1,Y_1]=X_1\\ [Y_1,Y_2]=\frac{1}{2}X_{2r}\\ [Y_3,Y_3]=X_{2r}\end{array}\right.$$

$\mathfrak{g}_{(2,7)}^3$:
$$\left\{\begin{array}{l}[X_0,Y_1]=Y_2\\ [Y_1,Y_1]=X_{2r}\\ [Y_1,Y_3]=X_1\\ [Y_2,Y_3]=X_{2r}\end{array}\right.$$

$\mathfrak{g}_{(2,8)}^3$:
$$\left\{\begin{array}{l}[X_0,Y_1]=Y_2\\ [Y_1,Y_3]=X_{2r}\end{array}\right.$$

$\mathfrak{g}_{(2,9)}^3$:
$$\left\{\begin{array}{l}[X_0,Y_1]=Y_2\\ [Y_1,Y_3]=X_1\\ [Y_2,Y_3]=X_{2r}\end{array}\right.$$

$\mathfrak{g}_{(2,10)}^3$:
$$\left\{\begin{array}{l}[X_0,Y_1]=Y_2\\ [Y_3,Y_3]=X_{2r}\end{array}\right.$$

Proof. It is easy to prove that the above superalgebras are HSA, with dimension of even part $2r+1$, dimension of odd part 3 and Goze invariant $(2,1,\ldots,1|\,2,1)$. Moreover, these superalgebras are pairwise non-isomorphic.

The HSA $\mathfrak{g}^3_{(2,6)}$, $\mathfrak{g}^3_{(2,7)}$ and $\mathfrak{g}^3_{(2,9)}$ are non-isomorphic by generic change of basis and the remaining superalgebras can be easily observed in the table below, where $\mathcal{C}$ is the centralizer operator.

	$\dim(\mathcal{Z}(\mathfrak{g}))$	$\dim(\mathcal{C}^1(\mathfrak{g}))$	$\dim(\mathcal{C}_{\mathfrak{g}}(\mathfrak{g}_1))$	$\dim[\mathcal{C}_{\mathfrak{g}_0}(\mathcal{C}_{\mathfrak{g}_0}(\mathfrak{g}_1)),\mathcal{C}_{\mathfrak{g}_0}(\mathcal{C}_{\mathfrak{g}_0}(\mathfrak{g}_1))]$
$\mathfrak{g}^3_{(2,1)}$	3	3	2r+1	1
$\mathfrak{g}^3_{(2,2)}$	3	3	2r+1	0
$\mathfrak{g}^3_{(2,3)}$	1	4	-	-
$\mathfrak{g}^3_{(2,4)}$	2	4	2r	-
$\mathfrak{g}^3_{(2,5)}$	1	3	2r+1	-
$\mathfrak{g}^3_{(2,6)}$	1	3	2r	-
$\mathfrak{g}^3_{(2,7)}$	1	3	2r	-
$\mathfrak{g}^3_{(2,8)}$	2	2	2r+1	-
$\mathfrak{g}^3_{(2,9)}$	1	3	2r	-
$\mathfrak{g}^3_{(2,10)}$	2	2	2r+2	-

It remains to prove that the given superalgebras are all the Lie superalgebras verifying the desired conditions. Thus, we consider the following family

$$\begin{cases} [X_0,Y_1]=Y_2 \\ [X_0,Y_2]=0 \\ [X_j,Y_1]=D^2_{j,1}Y_2+D^3_{j,1}Y_3, & 1\le j\le 2r-1 \\ [X_{2r},Y_1]=D^3_{2r,1}Y_3 \\ [X_j,Y_2]=D^3_{j,2}Y_3, & 1\le j\le 2r-1 \\ [Y_i,Y_j]=\sum\limits_{k=0}^{2r}E^k_{ij}X_k, & 1\le i,j\le 3,\,(i,j)\ne(3,3) \\ [Y_3,Y_3]=E^{2r}_{33}X_{2r} \end{cases}$$

with the restrictions

$$\begin{cases} D^2_{j,1}D^3_{j,2}=0 & 1\le j\le 2r-1 \\ D^2_{j,1}D^3_{j+1,2}=0 & 1\le j\le 2r-2 \\ D^2_{j,1}D^3_{j-1,2}=0 & 2\le j\le 2r-1 \end{cases}$$

We can suppose that $D^2_{j,1}=0$, $1\le j\le 2r-1$, making the following change of basis

$$\begin{cases} X_1'=-D^2_{1,1}X_0+X_1-\sum\limits_{k=1}^{r-1}D^2_{2k+1,1}X_{2k}+\sum\limits_{k=1}^{r-1}D^2_{2k,1}X_{2k+1} \\ X_j'=-D^2_{j,1}X_0+X_j & 2\le j\le 2r-1 \end{cases}$$

We compute the graded Jacobi identity for the vectors of basis (with the help of the program) and we obtain the following family

$$\begin{cases} [X_0, Y_1] = Y_2 \\ [X_j, Y_1] = D^3_{j,1} Y_3, & 1 \leq j \leq 2r \\ [Y_1, Y_1] = \sum_{k=1}^{2r-1} E^k_{11} X_k + E^{2r}_{11} X_{2r} \\ [Y_1, Y_2] = \frac{1}{2} E^1_{11} X_{2r} \\ [Y_1, Y_3] = E^1_{13} X_1 + E^{2r}_{13} X_{2r} \\ [Y_2, Y_3] = E^1_{13} X_{2r} \\ [Y_3, Y_3] = E^{2r}_{33} X_{2r} \end{cases}$$

with the restrictions

$$\begin{array}{ll} \sum_{k=1}^{2r-1} E^k_{11} D^3_{k,1} = 0 \\ E^{2i}_{11} + 2D^3_{2i+1,1} E^{2r}_{13} = 0, & 1 \leq i \leq r-1 \\ E^{2i+1}_{11} - 2D^3_{2i,1} E^{2r}_{13} = 0, & 1 \leq i \leq r-1 \\ D^3_{j,1} E^{2r}_{33} = 0, & 1 \leq j \leq 2r-1 \\ D^3_{j,1} E^k_{13} = 0, & 1 \leq j \leq 2r-1, \ 1 \leq k \leq 2r-1 \\ D^3_{1,1} E^{2r}_{33} = D^3_{1,1} E^{2r}_{13} = 0 \end{array}$$

In this moment, making a study similar to the precedent cases we obtain the superalgebras of the theorem.

We can suppose that $D^3_{2i,1} = 0$, $1 \leq i \leq r-1$ and, by restrictions, $E^{2j+1}_{11} = 0$ with $1 \leq j \leq r-1$, making the following change of basis

- For each k, $1 \leq k \leq r-1$ such that $D^3_{2k+1,1} \neq 0$

$$\begin{cases} X'_{2k} = D^3_{2k+1,1} X_{2k} - D^3_{2k,1} X_{2k+1} \\ X'_{2k+1} = \dfrac{1}{D^3_{2k+1,1}} X_{2k+1} \end{cases}$$

- For each k, $1 \leq k \leq r-1$ such that $D^3_{2k+1,1} = 0$

$$\begin{cases} X'_{2k} = -X_{2k+1} \\ X'_{2k+1} = X_{2k} \end{cases}$$

So the family of laws is reduced to

$$\begin{cases} [X_0, Y_1] = Y_2 \\ [X_{2j+1}, Y_1] = D^3_{2j+1,1} Y_3, & 0 \leq j \leq r-1 \\ [Y_1, Y_1] = E^1_{11} X_1 - 2E^{2r}_{13} \sum_{k=1}^{r-1} D^3_{2k+1,1} X_{2k} + E^{2r}_{11} X_{2r} \\ [Y_1, Y_2] = \frac{1}{2} E^1_{11} X_{2r} \\ [Y_1, Y_3] = E^1_{13} X_1 + E^{2r}_{13} X_{2r} \\ [Y_2, Y_3] = E^{2r}_{13} X_{2r} \\ [Y_3, Y_3] = E^{2r}_{33} X_{2r} \end{cases}$$

with the restrictions

$$\begin{array}{ll} D^3_{11}E^1_{11} = D^3_{11}E^{2r}_{13} = 0 & \\ D^3_{2j+1,1}E^1_{13} = 0, & 0 \leq j \leq r-1 \\ D^3_{2j+1,1}E^{2r}_{33} = 0, & 0 \leq j \leq r-1 \end{array}$$

Now, a detailed study (similar the preceding case) of the possible values of the parameters determine:

$$E^1_{13} = 0 \left\{ \begin{array}{l} E^1_{11} = 0 \left\{ \begin{array}{l} E^{2r}_{33} = 0 \left\{ \begin{array}{ll} E^{2r}_{13} = 0: & \mathfrak{g}^2_2 \oplus \mathbf{C},\ \mathfrak{g}^3_{(2,1)} \\ E^{2r}_{13} \neq 0: & \mathfrak{g}^3_{(2,4)},\ \mathfrak{g}^3_{(2,8)} \end{array} \right. \\ E^{2r}_{33} \neq 0: \quad \mathfrak{g}^3_{(2,10)} \end{array} \right. \\ E^1_{11} \neq 0 \left\{ \begin{array}{l} D^3_{(2j+1,1)} = 0,\ \forall j \in \{1, \dots, r-1\} \left\{ \begin{array}{l} E^{2r}_{33} = 0: \ \mathfrak{g}^2_3 \oplus \mathbf{C},\ \mathfrak{g}^3_{(2,5)} \\ E^{2r}_{33} \neq 0: \ \mathfrak{g}^3_{(2,6)} \end{array} \right. \\ \exists j \in \{1, \dots, r-1\} / D^3_{(2j+1,1)} \neq 0 \left\{ \begin{array}{l} E^{2r}_{13} = 0: \ \mathfrak{g}^3_{(2,4)} \\ E^{2r}_{13} \neq 0: \ \mathfrak{g}^3_{(2,3)} \end{array} \right. \end{array} \right. \end{array} \right.$$

$$E^1_{13} \neq 0 \left\{ \begin{array}{l} E^{2r}_{11} = 0: \ \mathfrak{g}^3_{(2,9)} \\ \\ E^{2r}_{11} \neq 0: \mathfrak{g}^3_{(2,7)} \end{array} \right.$$

■

Now, we obtain the generic family for the HSA with dimension of even part $2r+1$, dimension of odd part 3, Goze invariant $(2,1,\dots,1|2,1)$ and adapted basis $\{X_0, X_1, \dots, X_{2r},\ Y_1,\ Y_2,\ Y_3\}$ with $[X_0,Y_1] = Y_3$ and $[X_0,Y_2] = Y_3$ $((**))$. Among them, we classify the 2-nilpotent HSA.

Lemma 3.7. *If $\mathfrak{g}$ is a non degenerate HSA with dimension of even part $2r+1$, dimension of odd part 3, Goze invariant $(2,1,\dots,1|\ 2,1)$ such that it admits an adapted basis $\{X_0, X_1, \dots, X_{2r-1}, X_{2r}, Y_1, Y_2, Y_3\}$ with $[X_0,Y_1] = Y_3$ and $[X_0,Y_2] = Y_3$, then $\mathfrak{g}$ is in the following family of superalgebras expressed by*

$$\left\{ \begin{array}{ll} (**) & \\ [X_1,Y_2] = D^3_{12}Y_3 & \\ [X_2,Y_2] = \varepsilon Y_3, & \varepsilon \in \{0,1\} \\ [Y_1,Y_1] = E^1_{11}X_1 + E^{2r}_{11}X_{2r} & \\ [Y_1,Y_2] = -\dfrac{1}{2}D^3_{12}E^1_{11}X_0 + E^1_{12}X_1 + \dfrac{\varepsilon}{2}E^1_{11}X_3 + E^{2r}_{12}X_{2r} & \\ [Y_1,Y_3] = \dfrac{1}{2}E^1_{11}X_{2r} & \\ [Y_2,Y_2] = D^3_{12}(E^1_{11} - 2E^1_{12})X_0 + (2E^1_{12} - E^1_{11})X_1 + 2\varepsilon(E^1_{12} - \dfrac{1}{2}E^1_{11})X_3 + E^{2r}_{22}X_{2r} & \\ [Y_2,Y_3] = (E^1_{12} - \dfrac{1}{2}E^1_{11})X_{2r} & \end{array} \right.$$

Proof. In this case, we have already seen that the family of laws of superalgebras is

$$\left\{ \begin{array}{ll} (**) & \\ [X_j,Y_1] = D^3_{j,1}Y_3, & 1 \leq j \leq 2r \\ [X_j,Y_2] = D^3_{j,2}Y_3, & 1 \leq j \leq 2r-1 \\ [Y_i,Y_j] = \displaystyle\sum_{k=0}^{2r} E^k_{ij}X_k, & 1 \leq i,j \leq 3, (i,j) \neq (3,3) \\ [Y_3,Y_3] = E^{2r}_{33}X_{2r} & \end{array} \right.$$

with certain restrictions. By an analogous reasoning of the preceding cases we obtain the generic family. ■

Lemma 3.8. *If $\mathfrak{g}$ belongs to the generic family of HSA in the above lemma and 2-nilpotent, then it is isomorphic to one of the following three superalgebras, or it belongs to one of the parametric families of superalgebras*

$\mathfrak{g}^{3,\alpha,\beta}_{(2,11)}$:

$$\begin{cases} (**) \\ [X_1,Y_2]=Y_3 \\ [Y_1,Y_1]=X_{2r} \\ [Y_1,Y_2]=\alpha X_{2r}, \quad \alpha\in\mathbf{C} \\ [Y_2,Y_2]=\beta X_{2r}, \quad \beta\in\mathbf{C} \end{cases}$$

$\mathfrak{g}^{3,\alpha}_{(2,12)}$:

$$\begin{cases} (**) \\ [X_1,Y_2]=Y_3 \\ [Y_1,Y_2]=X_{2r} \\ [Y_2,Y_2]=\alpha X_{2r}, \quad \alpha\in\mathbf{C} \end{cases}$$

$\mathfrak{g}^{3}_{(2,13)}$:

$$\begin{cases} (**) \\ [X_1,Y_2]=Y_3 \\ [Y_2,Y_2]=X_{2r} \end{cases}$$

$\mathfrak{g}^{3,\alpha,\beta}_{(2,14)}$:

$$\begin{cases} (**) \\ [X_2,Y_2]=Y_3 \\ [Y_1,Y_1]=X_{2r} \\ [Y_1,Y_2]=\alpha X_{2r}, \quad \alpha\in\mathbf{C} \\ [Y_2,Y_2]=\beta X_{2r}, \quad \beta\in\mathbf{C} \end{cases}$$

$\mathfrak{g}^{3,\alpha}_{(2,15)}$:

$$\begin{cases} (**) \\ [X_2,Y_2]=Y_3 \\ [Y_1,Y_2]=X_{2r} \\ [Y_2,Y_2]=\alpha X_{2r}, \quad \alpha\in\mathbf{C} \end{cases}$$

$\mathfrak{g}^{3}_{(2,16)}$:

$$\begin{cases} (**) \\ [X_2,Y_2]=Y_3 \\ [Y_2,Y_2]=X_{2r} \end{cases}$$

Proof. We will differentiate four cases in the generic family of the above lemma.

$$\begin{cases} D^3_{12}=0 \begin{cases} \varepsilon=0 & \text{(Case 1)} \\ \varepsilon=1 & \text{(Case 2)} \end{cases} \\ D^3_{12}\neq 0 \begin{cases} \varepsilon=0 & \text{(Case 3)} \\ \varepsilon=1 & \text{(Case 4)} \end{cases} \end{cases}$$

In the case 1, making the following change of basis $Y_1'=Y_1-Y_2$ and renaming the vectors Y_i adequately, we obtain a family of superalgebras that would be included in the case studied in Theorem 3.6.

Case 4 is the same as case 3, making the following change of basis

$$\begin{cases} X_0'=D^3_{12}X_0-X_3 \\ X_2'=D^3_{12}X_2-X_1 \\ X_{2i}'=D^3_{12}X_{2i}, & 2\leq i\leq r \\ Y_3'=D^3_{12}Y_3 \end{cases}$$

Case 2. $D^3_{12}=0$, $\varepsilon=1$.

A study similar to the preceding theorems permits to differentiate these relations among the parameters, and we obtain the following superalgebras

$$E^1_{11}=E^1_{12}=0 \begin{cases} E^{2r}_{11}=0 \begin{cases} E^{2r}_{12}=0 \Longrightarrow E^{2r}_{22}\neq 0 \quad \mathfrak{g}^3_{(2,16)} \\ E^{2r}_{12}\neq 0 \Longrightarrow \mathfrak{g}^{3,\alpha}_{(2,15)} \end{cases} \\ E^{2r}_{11}\neq 0 \Longrightarrow \mathfrak{g}^{3,\alpha,\beta}_{(2,14)} \end{cases}$$

It is easy to prove that these superalgebras have nilindex 2 and we obtain superalgebras with nilindex 3 for any other possibility of the constant structures.

Case 3. $D_{12}^3 \neq 0,\ \varepsilon = 0$.

A study similar to preceding theorems permits to differentiate this relation among the parameters, and we obtain the following superalgebras and family of superalgebras

$$E_{11}^1 = E_{12}^1 = 0 \begin{cases} E_{11}^{2r} = 0 \begin{cases} E_{12}^{2r} = 0 \Longrightarrow E_{22}^{2r} \neq 0 \quad \mathfrak{g}_{(2,13)}^3 \\ 5 \\ E_{12}^{2r} \neq 0 \Longrightarrow \mathfrak{g}_{(2,12)}^{3,\alpha} \end{cases} \\ E_{11}^{2r} \neq 0 \Longrightarrow \mathfrak{g}_{(2,11)}^{3,\alpha,\beta} \end{cases}$$

It is easy to prove that these superalgebras have nilindex 2 and we obtain superalgebras with nilindex 4 for any other possibility of the constant structures. ■

3.3. Maximal Goze invariant $(2,1,\dots,1|\,3)$.

Theorem 3.9. *If $\mathfrak{g}$ is a non degenerate HSA with dimension of even part $2r+1$, dimension of odd part 3 and Goze invariant $(2,1,\dots,1|\,3)$, then it is isomorphic to one of the following three superalgebras, pairwise non-isomorphic, whose laws can be expressed in a suitable basis, $(\{X_0, X_1, \dots, X_{2r-1}, X_{2r}, Y_1, Y_2, Y_3\})$, by*

$\mathfrak{g}_{(3,1)}^3$: $\begin{cases} [X_0,Y_1] = Y_2 \\ [X_0,Y_2] = Y_3 \\ [X_1,Y_1] = Y_3 \\ [Y_1,Y_1] = X_{2r} \end{cases}$

$\mathfrak{g}_{(3,2)}^3$: $\begin{cases} [X_0,Y_1] = Y_2 \\ [X_0,Y_2] = Y_3 \\ [X_1,Y_1] = Y_3 \\ [Y_1,Y_1] = X_1 \\ [Y_1,Y_2] = \frac{1}{2}X_{2r} \end{cases}$

$\mathfrak{g}_{(3,3)}^3$: $\begin{cases} [X_0,Y_1] = Y_2 \\ [X_0,Y_2] = Y_3 \\ [X_3,Y_1] = Y_3 \\ [Y_1,Y_1] = X_{2r} \end{cases}$

$\mathfrak{g}_{(3,4)}^3$: $\begin{cases} [X_0,Y_1] = Y_2 \\ [X_0,Y_2] = Y_3 \\ [X_3,Y_1] = Y_3 \\ [Y_1,Y_1] = X_1 \\ [Y_1,Y_2] = \frac{1}{2}X_{2r} \end{cases}$

$\mathfrak{g}_{(3,5)}^3$: $\begin{cases} [X_0,Y_1] = Y_2 \\ [X_0,Y_2] = Y_3 \\ [X_3,Y_1] = -\frac{1}{2}Y_3 \\ [Y_1,Y_1] = X_2 \\ [Y_1,Y_3] = X_{2r} \\ [Y_2,Y_2] = -X_{2r} \end{cases}$

$\mathfrak{g}_{(3,6)}^3$: $\begin{cases} [X_0,Y_1] = Y_2 \\ [X_0,Y_2] = Y_3 \\ [Y_1,Y_1] = X_{2r} \end{cases}$

$\mathfrak{g}_{(3,7)}^3$: $\begin{cases} [X_0,Y_1] = Y_2 \\ [X_0,Y_2] = Y_3 \\ [Y_1,Y_1] = X_1 \\ [Y_1,Y_2] = \frac{1}{2}X_{2r} \end{cases}$

$\mathfrak{g}_{(3,8)}^3$: $\begin{cases} [X_0,Y_1] = Y_2 \\ [X_0,Y_2] = Y_3 \\ [Y_1,Y_1] = X_1 \\ [Y_1,Y_2] = \frac{1}{2}X_{2r} \\ [Y_1,Y_3] = X_{2r} \\ [Y_2,Y_2] = -X_{2r} \end{cases}$

$\mathfrak{g}_{(3,9)}^3$: $\begin{cases} [X_0,Y_1] = Y_2 \\ [X_0,Y_2] = Y_3 \\ [Y_1,Y_3] = X_{2r} \\ [Y_2,Y_2] = -X_{2r} \end{cases}$

Proof. It is easy to prove that the above superalgebras are HSA, with dimension of even part $2r+1$, dimension of odd part 3 and Goze invariant $(2,1,\dots,1|\,3)$. Moreover, these superalgebras are pairwise non-isomorphic, just observe the table below.

	$\dim(\mathcal{Z}(\mathfrak{g}))$	$\dim(\mathcal{C}^1(\mathfrak{g}))$	$\dim(\mathcal{C}_{\mathfrak{g}}(\mathfrak{g}_1))$	$\dim[\mathcal{C}_{\mathfrak{g}_0}(\mathcal{C}_{\mathfrak{g}_0}(\mathfrak{g}_1)),\mathcal{C}_{\mathfrak{g}_0}(\mathcal{C}_{\mathfrak{g}_0}(\mathfrak{g}_1))]$
$\mathfrak{g}^3_{(3,1)}$	2	3	2r+1	1
$\mathfrak{g}^3_{(3,2)}$	2	4	2r	1
$\mathfrak{g}^3_{(3,3)}$	2	3	2r+1	0
$\mathfrak{g}^3_{(3,4)}$	2	4	2r	0
$\mathfrak{g}^3_{(3,5)}$	1	3	2r-1	-
$\mathfrak{g}^3_{(3,6)}$	2	3	2r+2	-
$\mathfrak{g}^3_{(3,7)}$	2	4	2r+1	-
$\mathfrak{g}^3_{(3,8)}$	1	4	2r	-
$\mathfrak{g}^3_{(3,9)}$	1	3	2r	-

It remains to prove that the given superalgebras $\mathfrak{g}^3_{(3,i)}$, $1 \leq i \leq 9$ are all the Lie superalgebras verifying the desired conditions.

We have already seen that the family of laws of superalgebras is

$$\begin{cases} [X_0, Y_1] = Y_2 \\ [X_0, Y_2] = Y_3 \\ [X_j, Y_1] = D^2_{j,1}Y_2 + D^3_{j,1}Y_3, & 1 \leq j \leq 2r-1 \\ [X_{2r}, Y_1] = D^3_{2r,1}Y_3 \\ [X_j, Y_2] = D^3_{j,2}Y_3, & 1 \leq j \leq 2r-1 \\ [Y_i, Y_j] = \sum\limits_{k=0}^{2r} E^k_{ij}X_k, & 1 \leq i,j \leq 3, (i,j) \neq (3,3) \\ [Y_3, Y_3] = E^{2r}_{33}X_{2r} \end{cases}$$

We can suppose that $D^3_{j,2} = 0,\ 1 \leq j \leq 2r-1$, making this change of basis

$$\begin{cases} X'_1 = -D^3_{12}X_0 + X_1 - \sum\limits_{k=1}^{r-1} D^3_{2k+1,2}X_{2k} + \sum\limits_{k=1}^{r-1} D^3_{2k,2}X_{2k+1} \\ X'_j = -D^3_{j,2}X_0 + X_j & 2 \leq j \leq 2r-1 \end{cases}$$

We compute the graded Jacobi identity for the vectors of basis (with aid of the program) and we obtain the following family

$$\begin{cases} [X_0, Y_1] = Y_2 \\ [X_0, Y_2] = Y_3 \\ [X_j, Y_1] = D^3_{j,1}Y_3, & 1 \leq j \leq 2r-1 \\ [Y_1, Y_1] = \sum\limits_{k=1}^{2r} E^k_{11}X_k \\ [Y_1, Y_2] = \frac{1}{2}E^1_{11}X_{2r} \\ [Y_1, Y_3] = E^{2r}_{13}X_{2r} \\ [Y_2, Y_2] = -E^{2r}_{13}X_{2r} \end{cases}$$

with the restrictions

$$\begin{cases} D^3_{11}E^{2r}_{13}=0 \\ E^{2i+1}_{11}-2D^3_{2i,1}E^{2r}_{13}=0, \quad 1\le i\le r-1 \\ E^{2i}_{11}+2D^3_{2i+1,1}E^{2r}_{13}=0, \quad 1\le i\le r-1 \\ \sum_{k=1}^{2r-1} D^3_{k,1}E^{k}_{11}=0 \end{cases}$$

Case $D^3_{11}=0$

We can suppose that $D^3_{2i,1}=0,\ 1\le i\le r-1$,

- For each k, such that $D^3_{2k+1,1}\neq 0$, we make the following change

$$\begin{cases} X'_{2k} = D^3_{2k+1,1}X_{2k}-D^3_{2k,1}X_{2k+1} \\ X'_{2k+1} = \dfrac{1}{D^3_{2k+1,1}}X_{2k+1} \end{cases}$$

- For each k, such that $D^3_{2k+1,1}=0$, we make the following change

$$\begin{cases} X'_{2k} = -X_{2k+1} \\ X'_{2k+1} = X_{2k} \end{cases}$$

and by the restrictions $E^{2i+1}_{11}=0$.

The family to classify is

$$\begin{cases} [X_0,Y_1]=Y_2 \\ [X_0,Y_2]=Y_3 \\ [X_{2i+1},Y_1]=D^3_{2i+1,1}Y_3, & 1\le i\le r-1 \\ [Y_1,Y_1]=E^1_{11}X_1+E^2_{11}X_2+E^4_{11}X_4+\ldots+E^{2r}_{11}X_{2r} \\ [Y_1,Y_2]=\frac{1}{2}E^1_{11}X_{2r} \\ [Y_1,Y_3]=E^{2r}_{13}X_{2r} \\ [Y_2,Y_2]=-E^{2r}_{13}X_{2r} \end{cases}$$

with restrictions $E^{2i}_{11}+2D^3_{2i+1,1}E^{2r}_{13}=0,\ 1\le i\le r-1$.

Case 1: $E^{2r}_{13}=0$. In this case we have that $E^{2i}_{11}=0,\ 1\le i\le r-1$, and the family is

$$\begin{cases} [X_0,Y_1]=Y_2 \\ [X_0,Y_2]=Y_3 \\ [X_{2j+1},Y_1]=D^3_{2j+1,1}Y_3, & 1\le j\le r-1 \\ [Y_1,Y_1]=E^1_{11}X_1+E^{2r}_{11}X_{2r} \\ [Y_1,Y_2]=\frac{1}{2}E^1_{11}X_{2r} \end{cases}$$

A detailed study of the possible values of the parameters determine:

$$\begin{cases} D^3_{2i+1,1}=0,\ 1\le i\le r-1 \begin{cases} E^1_{11}=0\Rightarrow E^{2r}_{11}\neq 0:\ \mathfrak{g}^3_{(3,6)} \\ E^1_{11}\neq 0\Rightarrow E^{2r}_{11}=0:\ \mathfrak{g}^3_{(3,7)} \end{cases} \\ \exists i,\ 1\le i\le r-1,\ D^3_{2i+1,1}\neq 0 \begin{cases} E^1_{11}=0\Rightarrow E^{2r}_{11}\neq 0:\ \mathfrak{g}^3_{(3,3)} \\ E^1_{11}\neq 0\Rightarrow E^{2r}_{11}=0:\ \mathfrak{g}^3_{(3,4)} \end{cases} \end{cases}$$

Case 2: $E_{13}^{2r} \neq 0$. The family to classify is

$$\begin{cases} [X_0, Y_1] = Y_2 \\ [X_0, Y_2] = Y_3 \\ [X_{2i+1}, Y_1] = -\frac{E_{11}^{2i}}{2E_{13}^{2r}} Y_3, & 1 \leq i \leq r-1 \\ [Y_1, Y_1] = E_{11}^1 X_1 + E_{11}^2 X_2 + E_{11}^4 X_4 + \ldots + E_{11}^{2r} X_{2r} \\ [Y_1, Y_2] = \frac{1}{2} E_{11}^1 X_{2r} \\ [Y_1, Y_3] = E_{13}^{2r} X_{2r} \\ [Y_2, Y_2] = -E_{13}^{2r} X_{2r} \end{cases}$$

Analogously, we have that the relations among the parameters are

$$\begin{cases} E_{11}^1 = 0 \begin{cases} E_{11}^{2i} = 0,\ 1 \leq i \leq r-1 \Rightarrow E_{11}^{2r} = 0 : \ \mathfrak{g}_{(3,9)}^3 \\ \exists i,\ E_{11}^{2i} \neq 0,\ 1 \leq i \leq r-1 : \ \mathfrak{g}_{(3,5)}^3 \end{cases} \\ E_{11}^1 \neq 0 \begin{cases} E_{11}^{2i} = 0,\ 1 \leq i \leq r-1 : \ \mathfrak{g}_{(3,8)}^3 \\ \exists i,\ E_{11}^{2i} \neq 0,\ 1 \leq i \leq r-1 : \ \mathfrak{g}_{(3,5)}^3 \end{cases} \end{cases}$$

Case $D_{11}^3 \neq 0$

We have that $E_{13}^{2r} = 0$, and the family is

$$\begin{cases} [X_0, Y_1] = Y_2 \\ [X_0, Y_2] = Y_3 \\ [X_j, Y_1] = D_{j,1}^3 Y_3, & 1 \leq j \leq 2r-1 \\ [Y_1, Y_1] = E_{11}^1 X_1 + E_{11}^{2r} X_{2r} \\ [Y_1, Y_2] = \frac{1}{2} E_{11}^1 X_{2r} \end{cases}$$

with $D_{11}^3 \neq 0$.

Analogously to the preceding case (making a change of basis), we can suppose $D_{2i,1}^3 = 0,\ 1 \leq i \leq r-1$. Moreover, the following change

$$\begin{cases} X_0' = D_{11}^3 X_0 + \sum_{i=1}^{r-1} D_{2i+1,1}^3 X_{2i} \\ X_1' = \frac{1}{D_{11}^3} X_1 \\ X_{2j}' = \frac{1}{D_{11}^3} X_{2j} & 1 \leq j \leq r-1 \\ X_{2j+1}' = D_{11}^3 X_{2j+1} - D_{2j+1,1}^3 X_1 & 1 \leq j \leq r-1 \end{cases}$$

permits to suppose that $D_{2j+1,1}^3 = 0,\ 1 \leq j \leq r-1$.

We can resume the relations among the parameters in the table below

$$\begin{cases} E_{11}^1 = 0 \Rightarrow E_{11}^{2r} \neq 0 : \ \mathfrak{g}_{(3,1)}^3 \\ E_{11}^1 \neq 0 \Rightarrow E_{11}^{2r} = 0 : \ \mathfrak{g}_{(3,2)}^3 \end{cases}$$

■

References

[1] Bäuerle, G. G. A., and E. A. De Kerf, "Lie Algebras, Part 1," Studies in Mathematical Physics I. Elsevier, 1990.

[2] Bauwens, K., and L. Le Bruyn, *Some remarks on solvable Lie superalgebras*, J. of Pure and App. Alg. **99** (1995), 113–134.

[3] Camacho, L. M., J. R. Gómez, R. M. Navarro, and I. Rodríguez, *Mathematica and Nilpotent Lie superalgebras*, in "Computer Algebra in Scientific Computing CASC," V. G. Ganzha, E. W. Mayr, and E. V. Vorozhtsov, Eds., Springer (2001), 91–105.

[4] Evans, J. M., and J. O. Madsen, *Dynkin diagrams and integrable models based on Lie superalgebras*, Nuclear Physics B. **503** (1997), 715–746.

[5] Goze, M., and Y. Khakimdjanov, "Nilpotent Lie Algebras," Kluwer Academic Publishers, 1996.

[6] Jacobson, N., "Lie Algebras," Interscience Publishers, Wiley. New York. 1962.

[7] Kac, V. G., *Lie Superalgebras*, Advances in Mathematics **26** (1977), 8–96.

[8] Sattinger, D. H., and O. L. Weaver, "Lie Groups and Algebras with Applications to Physics, Geometry, and Mechanics," Springer-Verlag, 1986.

[9] Scheunert, M., "The Theory of Lie Superálgebras," Springer-Verlag, 1979.

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