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Analogues of Rossi's Map and E. Cartan's Classification of Homogeneous Strongly Pseudoconvex 3-Dimensional Hypersurfaces

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Abstract. We introduce analogues of a map due to Rossi and show how they can be used to explicitly determine all covers of certain homogeneous strongly pseudoconvex 3-dimensional hypersurfaces that appear in the classification obtained by E. Cartan in 1932.
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1. Introduction

In 1932 E. Cartan [3] classified all connected 3-dimensional homogeneous strongly pseudoconvex CR-manifolds – we reproduce his classification in detail below in Section 2. Most manifolds in the classification are given by explicit equations as hypersurfaces in either $\mathbb{C}^2$ or $\mathbb{CP}^2$. The exceptions (apart from lens spaces) are all possible covers of each of the following hypersurfaces:

$$\begin{aligned}
\chi &:= \{(z,w)\in\mathbb{C}^2 : x^2+u^2=1\},\\
\mu_\alpha &:= \{(z:w:\zeta)\in\mathbb{CP}^2 : |z|^2+|w|^2+|\zeta|^2=\alpha|z^2+w^2+\zeta^2|\},\ \alpha>1,\\
\nu_\alpha &:= \{(z,w)\in\mathbb{C}^2 : |z|^2+|w|^2-1=\alpha|z^2+w^2-1|\}\setminus\\
&\quad\ \{(x,u)\in\mathbb{R}^2 : x^2+u^2=1\},\ -1<\alpha<1,\\
\eta_\alpha &:= \Big\{(z,w)\in\mathbb{C}^2 : 1+|z|^2-|w|^2=\alpha|1+z^2-w^2|,\\
&\quad\ \mathrm{Im}(z(1+\overline{w}))>0\Big\},\ \alpha>1.
\end{aligned}$$

In the above formulas and everywhere below we set $z=x+iy$, $w=u+iv$.

Sometimes it is desirable to know all the covers explicitly. For example, such a need arises when one attempts to describe all Kobayashi-hyperbolic 2-dimensional complex manifolds with holomorphic automorphism group of dimension 3, since almost every hypersurface from Cartan's classification can be realized as an orbit of the automorphism group action on such a manifold [6] (see also [5] for motivations and related results).

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All covers of the hypersurface χ are easy to find – see Section 3. Next, we determine the covers of the hypersurfaces μ_α in Section 4. All of them are compact, and, since compact Levi non-degenerate homogeneous CR-manifolds have been extensively studied (see, e.g., [2] and references therein), all these covers have been discovered on many occasions in various guises (see, e.g. [4]). We show how the covers of μ_α can be found by using a map introduced by Rossi in [8] – see Section 4. Originally, Rossi defined it as a map from $\mathbb{CP}^2 \setminus \{0\}$ into $\mathbb{CP}^3$ and used it to construct strongly pseudoconvex 3-dimensional manifolds that do not bound any pseudoconvex analytic space. For our purposes, however, we will only be interested in the restriction of the map to $\mathbb{C}^2 \setminus \{0\}$ and call this restriction *Rossi's map* (see formula (4.1)).

Our main results are contained in Section 5. There we determine all covers of the hypersurfaces ν_α and η_α that, in contrast with those of μ_α, appear to have never been found explicitly. Interestingly, it turns out that this can be done by introducing a map (in fact, a sequence of maps) analogous to Rossi's map (see (5.1), (5.11)). While Rossi's map is associated with the action of SU_2 on $\mathbb{C}^2 \setminus \{0\}$, our analogues are associated with the action of $SU_{1,1}$ on $\mathbb{C}^2 \setminus \{(z,w) \in \mathbb{C}^2 : |z|^2 - |w|^2 = 0\}$ (which is a manifestation of the non-compactness of ν_α and η_α). While for μ_α only 2- and 4-sheeted covers can occur, there is an n-sheeted cover for every $n \geq 2$ as well as an infinite-sheeted cover, for each of ν_α, η_α. In fact, as in the cases of χ and μ_α, we explicitly find the covers of certain domains in $\mathbb{C}^2$, namely $D^\nu := \cup_{-1<\alpha<1}\nu_\alpha$ and $D^\eta := \cup_{\alpha>1}\eta_\alpha$. We equip the covers of these domains with complex structures obtained by pull-backs under explicit covering maps, the covers of ν_α, η_α with CR-structures induced by these complex structures, and find the groups of CR-automorphisms of the induced CR-structures explicitly. Hence the emphasis of the present paper is on the explicit determination of the covers of ν_α and η_α in the differential-geometric sense, and on group actions.

The main motivation for the present work was the above-mentioned problem of classifying 2-dimensional Kobayashi-hyperbolic manifolds with 3-dimensional automorphism groups (we call such manifolds (2,3)-*manifolds* for brevity). A complete explicit classification of (2,3)-manifolds was obtained in [6], and the particular realizations of the covers of ν_α and η_α (as well as those of χ and μ_α) constructed in the present paper turned out to be extremely useful for this purpose. In our construction, the covers $\nu_\alpha^{(N)}$ and $\eta_\alpha^{(N)}$ of ν_α and η_α with the same number of sheets N (where N can be infinite) are glued together into complex manifolds M_N^ν and M_N^η, respectively. On these manifolds certain connected 3-dimensional Lie groups $G_N^{\nu\,c}$ and $G_N^{\eta\,c}$ act by holomorphic transformations (these groups are in fact the connected identity components of the automorphism groups of M_N^ν and M_N^η), and the orbits of these actions are exactly the hypersurfaces $\nu_\alpha^{(N)}$ and $\eta_\alpha^{(N)}$, respectively (the groups $G_N^{\nu\,c}$ and $G_N^{\eta\,c}$ also coincide with the connected identity components of the groups of CR-automorphisms of $\nu_\alpha^{(N)}$ and $\eta_\alpha^{(N)}$) – see Theorem 5.1. The actions of $G_N^{\nu\,c}$ and $G_N^{\eta\,c}$ on M_N^ν and M_N^η are proper without singular orbits, and the orbit spaces $M_N^\nu/G_N^{\nu\,c}$ and $M_N^\eta/G_N^{\eta\,c}$ are homeomorphic to $\mathbb{R}$ (see [1]). The manifolds M_N^ν and M_N^η cover the domains D^ν and D^η by means of N-to-1 explicit covering maps $\mathfrak{P}_N^\nu$ and $\mathfrak{P}_N^\eta$ that are expressed in terms of our analogues of Rossi's map. The restrictions of $\mathfrak{P}_N^\nu$ to $\nu_\alpha^{(N)}$ and $\mathfrak{P}_N^\eta$ to $\eta_\alpha^{(N)}$ are N-to-1 covering maps onto ν_α and η_α, respectively.

Thus, our construction immediately gives a large number of examples of (2,3)-manifolds whose automorphism groups act with codimension 1 orbits: indeed, any domain in M_N^ν bounded by two $G_N^{\nu\,c}$-orbits and any domain in M_N^η bounded by two $G_N^{\eta\,c}$-orbits are such manifolds (this phenomenon takes place for other types of hypersurfaces from Cartan's classification as well).

Another feature of our construction utilized in [6] concerns the complex structures of M_N^ν and M_N^η. These structures are given by pull-backs under the maps $\mathfrak{P}_N^\nu$ and $\mathfrak{P}_N^\eta$ and, since the formulas for $\mathfrak{P}_N^\nu$ and $\mathfrak{P}_N^\eta$ are complicated, the resulting complex structures are not very explicit. Nevertheless, some properties of these structures can be derived from our construction. In [6], in order to classify (2,3)-manifolds whose automorphism groups act with codimension 2 orbits, we study complex curves in M_N^ν and M_N^η invariant under the actions of maximal compact subgroups of $G_N^{\nu\,c}$ and $G_N^{\eta\,c}$, respectively, for $N < \infty$. Namely, let $K \subset G_N^{\nu\,c}$ be a maximal compact subgroup (every such subgroup is isomorphic to the circle) and C a connected K-invariant non-singular complex curve in M_N^ν. Next, let G^ν be the connected identity component of the automorphism group of D^ν (in fact, G^ν is isomorphic to the connected identity component $SO_{2,1}(\mathbb{R})^c$ of $SO_{2,1}(\mathbb{R})$). The group $G_N^{\nu\,c}$ covers the group G^ν by means of an explicit N-to-1 covering homomorphism $\mathcal{P}_N^\eta$, the group $K' := \mathcal{P}_N^\nu(K)$ is a maximal compact subgroup of G^ν, and the restriction of $\mathcal{P}_N^\nu$ to K is an N-to-1 covering homomorphism onto K'. Then it turns out that $C' := \mathfrak{P}_N^\nu(C)$ is a K'-invariant non-singular complex curve in D^ν, and the restriction of $\mathfrak{P}_N^\nu$ to C is an N-to-1 covering map onto C'. Moreover, every connected non-singular K'-invariant complex curve in D^ν is obtained in this way. A similar statement holds for complex curves in M_N^η invariant under the actions of maximal compact subgroups of $G_N^{\eta\,c}$. In [6] we often relied on this property when studying (2,3)-manifolds with codimension 2 orbits.

Before proceeding, we would like to thank one of the anonymous referees for numerous suggestions that help improve the paper. These suggestions included, in particular, a group-theoretic interpretation of Rossi's map that we reproduce in Section 4. Similar constructions lead to group-theoretic interpretations of our analogues of Rossi's map – see Section 5.

2. E. Cartan's Classification

In this section we reproduce E. Cartan's classification of connected 3-dimensional homogeneous strongly pseudoconvex CR-manifolds from [3]. He shows that every such hypersurface is CR-equivalent to one of the manifolds on the list below. As does Cartan, we group the model manifolds into spherical and non-spherical ones.

The Spherical Case

(i) S^3,
(ii) $\mathcal{L}_m := S^3/\mathbb{Z}_m$, $m \in \mathbb{N}, n \geq 2$, (lens spaces),
(iii) $\sigma := \{(z,w) \in \mathbb{C}^2 : u = |z|^2\}$,
(iv) $\sigma_+ := \{(z,w) \in \mathbb{C}^2 : u = |z|^2, x > 0\}$,
(v) $\varepsilon_\alpha := \{(z,w) \in \mathbb{C}^2 : |z|^2 + |w|^\alpha = 1, w \neq 0\}$, $\alpha > 0$,
(vi) $\omega := \{(z,w) \in \mathbb{C}^2 : |z|^2 + e^u = 1\}$,
(vii) $\delta := \{(z,w) \in \mathbb{C}^2 : |w| = \exp(|z|^2)\}$,
(viii) $\nu_0 = S^3 \setminus \mathbb{R}^2$,
(ix) any cover of ν_0.

The groups of CR-automorphisms of the above hypersurfaces (except in case (ix)) are as follows:

$\underline{\mathrm{Aut}_{CR}(S^3) \simeq SU_{2,1}/(\text{center})}$:

$$\begin{pmatrix} z \\ w \end{pmatrix} \mapsto \frac{\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} z \\ w \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}}{c_1 z + c_2 w + d}, \tag{2.1}$$

with $Q \in SU_{2,1}$, where

$$Q := \begin{pmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ c_1 & c_2 & d \end{pmatrix}. \tag{2.2}$$

$\underline{\mathrm{Aut}_{CR}(\mathcal{L}_m) \simeq U_2/\mathbb{Z}_m}$:

$$\left[\begin{pmatrix} z \\ w \end{pmatrix}\right] \mapsto \left[U\begin{pmatrix} z \\ w \end{pmatrix}\right],$$

where $U \in U_2$ and $[(z,w)] \in \mathcal{L}_m$ denotes the equivalence class of $(z,w) \in S^3$ under the action of $\mathbb{Z}_m$ embedded in U_2 as a subgroup of scalar matrices.

$\underline{\mathrm{Aut}_{CR}(\sigma) \simeq CU_1 \ltimes H}$:

$$\begin{aligned} z &\mapsto \lambda e^{i\varphi} z + a, \\ w &\mapsto \lambda^2 w + 2\lambda e^{i\varphi}\overline{a} z + |a|^2 + i\gamma, \end{aligned}$$

where $\lambda \in \mathbb{R}^*$, $\varphi, \gamma \in \mathbb{R}$, $a \in \mathbb{C}$, CU_1 denotes the conformal unitary group given by the conditions $a = 0$, $\gamma = 0$, and H denotes the Heisenberg group given by the conditions $\lambda = 1$, $\varphi = 0$.

$\underline{\mathrm{Aut}_{CR}(\sigma_+) \simeq \mathbb{R} \ltimes \mathbb{R}^2}$:

$$\begin{aligned} z &\mapsto \lambda z + i\beta, \\ w &\mapsto \lambda^2 w - 2i\lambda\gamma z + \gamma^2 + i\gamma, \end{aligned}$$

where $\lambda > 0$, $\beta, \gamma \in \mathbb{R}$.

$\underline{\mathrm{Aut}_{CR}(\varepsilon_\alpha)}$:

$$z \mapsto e^{i\varphi}\frac{z-a}{1-\overline{a}z},$$

$$w \mapsto e^{i\psi}\frac{(1-|a|^2)^{1/\alpha}}{(1-\overline{a}z)^{2/\alpha}} w,$$

where $\varphi, \psi \in \mathbb{R}$, $a \in \mathbb{C}$, $|a| < 1$. We have $\mathrm{Aut}_{CR}(\varepsilon_\alpha) \simeq \widetilde{SO}_{2,1}(\mathbb{R})^c \times_{\mathrm{loc}} U_1$, if $\alpha \notin \mathbb{Q}$, where $\widetilde{SO}_{2,1}(\mathbb{R})^c$ is the universal cover of the connected identity component $SO_{2,1}(\mathbb{R})^c$ of the group $SO_{2,1}(\mathbb{R})$, and $\times_{\mathrm{loc}}$ denotes locally direct product; $\mathrm{Aut}_{CR}(\varepsilon_\alpha) \simeq SO_{2,1}(\mathbb{R})^{c\,(n)} \times_{\mathrm{loc}} U_1$, if $\alpha = n/k$, with $n, k \in \mathbb{Z}$, $n > 0$, $k \geq 0$, $(n,k) = 1$, where $SO_{2,1}(\mathbb{R})^{c\,(n)}$ is the n-sheeted cover of $SO_{2,1}(\mathbb{R})^c$.

$\underline{\mathrm{Aut}_{CR}(\omega) \simeq \widetilde{SO}_{2,1}(\mathbb{R})^c \times_{\mathrm{loc}} \mathbb{R}}$:

$$\begin{aligned} z &\mapsto e^{i\varphi}\frac{z-a}{1-\overline{a}z}, \\ w &\mapsto w + 2\ln\frac{\sqrt{1-|a|^2}}{1-\overline{a}z} + i\gamma, \end{aligned}$$

where $\varphi, \gamma \in \mathbb{R}$, $a \in \mathbb{C}$, $|a| < 1$.

$\underline{\mathrm{Aut}_{CR}(\delta) \simeq U_1 \times (U_1 \ltimes \mathbb{R}^2)}$:

$$\begin{aligned} z &\mapsto e^{i\varphi}z + a, \\ w &\mapsto e^{i\psi}\exp\Big(2e^{i\varphi}\overline{a}z + |a|^2\Big)w, \end{aligned}$$

where $\varphi, \psi \in \mathbb{R}$, $a \in \mathbb{C}$.

$\underline{\mathrm{Aut}_{CR}(\nu_0) \simeq SO_{2,1}(\mathbb{R})}$: This group consists of all maps of the form (2.1) with $Q \in SO_{2,1}(\mathbb{R})$, where Q is defined in (2.2).

The Non-Spherical Case

(i) $\tau_\alpha := \{(z,w) \in \mathbb{C}^2 : u = x^\alpha,\ x > 0\}$,
$\alpha \in (-\infty, -1] \cup (1,2) \cup (2,\infty)$,

(ii) $\xi := \{(z,w) \in \mathbb{C}^2 : u = x \cdot \ln x,\ x > 0\}$,

(iii) $\chi = \{(z,w) \in \mathbb{C}^2 : x^2 + u^2 = 1\}$,

(iv) any cover of χ,

(v) $\rho_\alpha := \{(z,w) \in \mathbb{C}^2 : r = e^{\alpha\varphi}\}$, $\alpha > 0$,
where (r,φ) denote the polar coordinates in the (x,u)-plane
with φ varying from $-\infty$ to ∞,

(vi) $\mu_\alpha = \{(z : w : \zeta) \in \mathbb{CP}^2 : |z|^2 + |w|^2 + |\zeta|^2 = \alpha|z^2 + w^2 + \zeta^2|\}$, $\alpha > 1$,

(vii) any cover of μ_α with $\alpha > 1$,

(viii) $\nu_\alpha = \{(z,w) \in \mathbb{C}^2 : |z|^2 + |w|^2 - 1 = \alpha|z^2 + w^2 - 1|\} \setminus$
$\{(x,u) \in \mathbb{R}^2 : x^2 + u^2 = 1\}$, $-1 < \alpha < 1$, $\alpha \neq 0$,

(ix) any cover of ν_α with $-1 < \alpha < 1$, $\alpha \neq 0$,

(x) $\eta_\alpha = \Big\{(z,w) \in \mathbb{C}^2 : 1 + |z|^2 - |w|^2 = \alpha|1 + z^2 - w^2|,$
$\mathrm{Im}(z(1+\overline{w})) > 0\Big\}$, $\alpha > 1$,

(xi) any cover of η_α with $\alpha > 1$.

Below we list the groups of CR-automorphisms of the above hypersurfaces excluding cases (iv), (vii), (ix), (xi). Note that it follows from the proof of Lemma 3.3 of [7] that the automorphism group of a connected non-spherical homogeneous hypersurface in a 2-dimensional complex manifold has at most two connected components.

$\underline{\mathrm{Aut}_{CR}(\tau_\alpha) \simeq \mathbb{R} \ltimes \mathbb{R}^2}$:

$$\begin{aligned} z &\mapsto \lambda z + i\beta, \\ w &\mapsto \lambda^\alpha w + i\gamma, \end{aligned}$$

where $\lambda > 0,\ \beta, \gamma \in \mathbb{R}$.

$\underline{\mathrm{Aut}_{CR}(\xi) \simeq \mathbb{R} \ltimes \mathbb{R}^2}$:

$$\begin{aligned} z &\mapsto \lambda z + i\beta, \\ w &\mapsto (\lambda \ln \lambda) z + \lambda w + i\gamma, \end{aligned}$$

where $\lambda > 0,\ \beta, \gamma \in \mathbb{R}$.

$\underline{\mathrm{Aut}_{CR}(\chi) \simeq O_2 \ltimes \mathbb{R}^2}$:

$$\begin{pmatrix} z \\ w \end{pmatrix} \mapsto A \begin{pmatrix} z \\ w \end{pmatrix} + i \begin{pmatrix} \beta \\ \gamma \end{pmatrix},$$

where $A \in O_2(\mathbb{R}),\ \beta, \gamma \in \mathbb{R}$.

$\underline{\mathrm{Aut}_{CR}(\rho_\alpha) \simeq \mathbb{R} \ltimes \mathbb{R}^2}$:

$$\begin{pmatrix} z \\ w \end{pmatrix} \mapsto e^{\alpha\psi} \begin{pmatrix} \cos\psi & \sin\psi \\ -\sin\psi & \cos\psi \end{pmatrix} \begin{pmatrix} z \\ w \end{pmatrix} + i \begin{pmatrix} \beta \\ \gamma \end{pmatrix},$$

where $\psi, \beta, \gamma \in \mathbb{R}$.

$\underline{\mathrm{Aut}_{CR}(\mu_\alpha) \simeq SO_3(\mathbb{R})}$:

$$\begin{pmatrix} z \\ w \\ \zeta \end{pmatrix} \mapsto A \begin{pmatrix} z \\ w \\ \zeta \end{pmatrix},$$

where $A \in SO_3(\mathbb{R})$.

$\underline{\mathrm{Aut}_{CR}(\nu_\alpha) \simeq SO_{2,1}(\mathbb{R})}$: This group consists of all maps of the form (2.1) with $Q \in SO_{2,1}(\mathbb{R})$, where Q is defined in (2.2).

$\underline{\mathrm{Aut}_{CR}(\eta_\alpha) \simeq SO_{2,1}(\mathbb{R})^c}$:

$$\begin{pmatrix} z \\ w \end{pmatrix} \mapsto \frac{\begin{pmatrix} a_{22} & b_2 \\ c_2 & d \end{pmatrix} \begin{pmatrix} z \\ w \end{pmatrix} + \begin{pmatrix} a_{21} \\ c_1 \end{pmatrix}}{a_{12} z + b_1 w + a_{11}}, \tag{2.3}$$

with $Q \in SO_{2,1}(\mathbb{R})^c$, where Q is defined in (2.2).

Thus, to obtain an explicit classification from the above lists, one needs to determine all possible covers of χ, μ_α, ν_α (including the spherical hypersurface ν_0), and η_α.

Let M be an arbitrary manifold, $\tilde{M}$ its universal cover, $\Pi : \tilde{M} \to M$ a covering map and Γ_Π the corresponding group of covering transformations of $\tilde{M}$. Then an arbitrary manifold that covers M is obtained from $\tilde{M}$ by factoring it by the action of a subgroup of Γ_Π. Hence, in order to find all covers of each of the hypersurfaces χ, μ_α, ν_α, η_α we need to determine their universal covers, the corresponding groups of covering transformations and all their subgroups.

3. The Covers of χ

Let $\Phi^\chi : \mathbb{C}^2 \to \mathbb{C}^2 \setminus \{x = 0,\, u = 0\}$ be the following map:

$$\begin{aligned} z &\mapsto e^x \cos y + iu, \\ w &\mapsto e^x \sin y + iv. \end{aligned}$$

Clearly, Φ^χ is an infinitely-sheeted covering map. Introduce on the domain of Φ^χ a complex structure so that Φ^χ becomes holomorphic (the pull-back complex structure under Φ^χ), and denote the resulting manifold by M^{Φ^χ}. Then $\tilde{\chi}$ coincides with the hypersurface

$$\chi^{(\infty)} := \left\{(z, w) \in M^{\Phi^\chi} : x = 0\right\},$$

equipped with the CR-structure induced by the complex structure of M^{Φ^χ}.

Clearly, Γ_{Φ^χ} consists of all transformations of the form

$$\begin{aligned} z &\mapsto z + 2\pi i k, \quad k \in \mathbb{Z}, \\ w &\mapsto w. \end{aligned}$$

Let $\Gamma \subset \Gamma_{\Phi^\chi}$ be a subgroup. Then there exists an integer $n \geq 0$ such that every element of Γ has the form

$$\begin{aligned} z &\mapsto z + 2\pi i n k, \quad k \in \mathbb{Z}, \\ w &\mapsto w. \end{aligned}$$

Suppose that $n \geq 1$ and consider the map Φ^χ_n from $\mathbb{C}^2 \setminus \{x = 0, u = 0\}$ onto itself defined as follows:

$$\begin{aligned} z &\mapsto \operatorname{Re}(x + iu)^n + iy, \\ w &\mapsto \operatorname{Im}(x + iu)^n + iv. \end{aligned}$$

Denote by $M^{\Phi^\chi_n}$ the domain of Φ^χ_n with the pull-back complex structure under Φ^χ_n. Then the hypersurface

$$\chi^{(n)} := \left\{(z, w) \in M^{\Phi^\chi_n} : x^2 + u^2 = 1\right\},$$

equipped with the CR-structure induced by the complex structure of $M^{\Phi^\chi_n}$ is an n-sheeted cover of χ corresponding to Γ with covering map $\chi^{(n)} \to \chi$ coinciding with $\Phi^\chi_n : M^{\Phi^\chi_n} :\to \mathbb{C}^2 \setminus \{x = 0,\, u = 0\}$ and factorization map $\chi^{(\infty)} \to \chi^{(n)}$ given by

$$\begin{aligned} z &\mapsto e^{x/n} \cos(y/n) + iu, \\ w &\mapsto e^{x/n} \sin(y/n) + iv. \end{aligned}$$

Thus, every cover of χ is CR-equivalent to either $\chi^{(\infty)}$ or $\chi^{(n)}$ for some $n \in \mathbb{N}$. The groups of CR-automorphisms of $\chi^{(\infty)}$ and $\chi^{(n)}$ are given below.

$\underline{\operatorname{Aut}_{CR}\left(\chi^{(\infty)}\right) \simeq \mathbb{R}^3 \rtimes \mathbb{Z}_2}$: This group is generated by the maps

$$\begin{aligned} z &\mapsto z + i\beta, \\ w &\mapsto w + a, \end{aligned}$$

where $\beta \in \mathbb{R}$, $a \in \mathbb{C}$, that form the connected identity component $\operatorname{Aut}_{CR}(\chi^{(\infty)})^c$ of $\operatorname{Aut}_{CR}\left(\chi^{(\infty)}\right)$, and the map

$$\begin{aligned} z &\mapsto \overline{z}, \\ w &\mapsto \overline{w}, \end{aligned}$$

which is a lift from $\mathbb{C}^2 \setminus \{x = 0, u = 0\}$ to M^{Φ^χ} of the following element of $\mathrm{Aut}_{CR}(\chi)$:

$$\begin{array}{rcl} z & \mapsto & z, \\ w & \mapsto & -w. \end{array} \tag{3.1}$$

$\underline{\mathrm{Aut}_{CR}\left(\chi^{(n)}\right) \simeq O_2(\mathbb{R}) \ltimes \mathbb{R}^2}$: This group is generated by the maps

$$\begin{array}{rcl} z & \mapsto & \cos\varphi \cdot x + \sin\varphi \cdot u + i(\cos(n\varphi) \cdot y + \sin(n\varphi) \cdot v + \beta), \\ w & \mapsto & -\sin\varphi \cdot x + \cos\varphi \cdot u + i(-\sin(n\varphi) \cdot y + \cos(n\varphi) \cdot v + \gamma), \end{array}$$

where $\varphi, \beta, \gamma \in \mathbb{R}$, that form the identity component of $\mathrm{Aut}_{CR}\left(\chi^{(n)}\right)$, and map (3.1).

4. The Covers of μ_α

All covers of μ_α can be found by using a map introduced by Rossi in [8]. Let Q_+ be the variety in $\mathbb{C}^3$ given by

$$z_1^2 + z_2^2 + z_3^2 = 1.$$

Consider the map $\Phi^\mu : \mathbb{C}^2 \setminus \{0\} \to Q_+$ defined by the formulas

$$\begin{array}{rcl} z_1 & = & -i(z^2 + w^2) + i\dfrac{z\overline{w} - w\overline{z}}{|z|^2 + |w|^2}, \\ z_2 & = & z^2 - w^2 - \dfrac{z\overline{w} + w\overline{z}}{|z|^2 + |w|^2}, \\ z_3 & = & 2zw + \dfrac{|z|^2 - |w|^2}{|z|^2 + |w|^2}. \end{array} \tag{4.1}$$

It is straightforward to verify that Φ^μ is a 2-to-1 covering map onto $Q_+ \setminus (Q_+ \cap \mathbb{R}^3)$ and that it satisfies

$$\Phi^\mu(g(z, w)) = \varphi^\mu(g)\Phi^\mu((z, w)), \tag{4.2}$$

for all $g \in SU_2$, $(z, w) \in \mathbb{C}^2 \setminus \{0\}$, where φ^μ is the standard 2-to-1 covering homomorphism from SU_2 onto $SO_3(\mathbb{R})$ defined as follows: for

$$g = \begin{pmatrix} a & b \\ -\overline{b} & \overline{a} \end{pmatrix} \in SU_2$$

(here $|a|^2 + |b|^2 = 1$), set

$$\varphi^\mu(g) := \begin{pmatrix} \mathrm{Re}\,(a^2 + b^2) & \mathrm{Im}\,(a^2 - b^2) & 2\mathrm{Im}(ab) \\ -\mathrm{Im}\,(a^2 + b^2) & \mathrm{Re}\,(a^2 - b^2) & 2\mathrm{Re}(ab) \\ 2\mathrm{Im}(a\overline{b}) & -2\mathrm{Re}(a\overline{b}) & |a|^2 - |b^2| \end{pmatrix}. \tag{4.3}$$

In formula (4.2) the actions of SU_2 on $\mathbb{C}^2$ and $SO_3(\mathbb{R})$ on $\mathbb{C}^3$ are standard.

In fact, the covering homomorphism φ^μ can be used to give a simple group-theoretic interpretation of Rossi's map Φ^μ (this interpretation was suggested to

us by one of the referees). First of all, we observe that the group $\mathbb{R}\times SU_2$ acts on $\mathbb{C}^2\setminus\{0\}$ simply transitively as follows:

$$(t,g)(z,w):=e^t\cdot g(z,w),$$

where $t\in\mathbb{R}$, $g\in SU_2$. On the other hand, the standard action of $SO_3(\mathbb{R})$ on $Q_+\setminus(Q_+\cap\mathbb{R}^3)$ can be extended to a simple transitive action of the group $\mathbb{R}\times SO_3(\mathbb{R})$ by diffeomorphisms. Indeed, $\mathbb{R}\times SO_3(\mathbb{R})$ acts simply transitively on

$$\hat{Q}_+:=\left\{\zeta=(\zeta_1,\zeta_2,\zeta_3)\in\mathbb{C}^3\setminus\{0\}:\zeta_1^2+\zeta_2^2+\zeta_3^2=0\right\}$$

as follows:

$$(t,g)\,\zeta:=e^t\cdot g\,\zeta,$$

where $t\in\mathbb{R}$, $g\in SO_3(\mathbb{R})$. The manifold $\hat{Q}_+$ is $SO_3(\mathbb{R})$-equivariantly diffeomorphic to $Q_+\setminus(Q_+\cap\mathbb{R}^3)$ by means of the map F_+ given by

$$\zeta\mapsto\zeta+\xi,$$

where $\xi\in\mathbb{R}^3$ is such that $\langle\xi,\xi\rangle_+=1$, $\langle\xi,\zeta\rangle_+=0$, $\det(\xi,\operatorname{Re}\zeta,\operatorname{Im}\zeta)>0$, and $\langle\cdot,\cdot\rangle_+$ denotes the standard Hermitian scalar product in $\mathbb{C}^3$. Using the $SO_3(\mathbb{R})$-equivariant diffeomorphism F_+, we can now push forward the action of $\mathbb{R}\times SO_3(\mathbb{R})$ on $\hat{Q}_+$ to a simple transitive action of $\mathbb{R}\times SO_3(\mathbb{R})$ on $Q_+\setminus(Q_+\cap\mathbb{R}^3)$ by diffeomorphisms.

Thus, as smooth manifolds $\mathbb{C}^3\setminus\{0\}$ and $Q_+\setminus(Q_+\cap\mathbb{R}^3)$ can be identified with $\mathbb{R}\times SU_2$ and $\mathbb{R}\times SO_3(\mathbb{R})$, respectively. Then the map $(t,g)\mapsto(t,\varphi^\mu(g))$ is precisely Rossi's map Φ^μ if we choose $(1,0)\in\mathbb{C}^2\setminus\{0\}$ and $(-i,1,1)\in Q_+\setminus(Q_+\cap\mathbb{R}^3)$ as basepoints. Hence formulas (4.1) – that may look somewhat mysterious at first sight – are a simple consequence of (4.3).

Next, we introduce on the domain of Φ^μ the pull-back complex structure under Φ^μ and denote the resulting complex manifold by M^{Φ^μ}. This complex structure is invariant under the ordinary action of SU_2 on M^{Φ^μ}. It follows from (4.2) that Φ^μ maps every SU_2-orbit in M^{Φ^μ} (all such orbits are diffeomorphic to S^3) onto an $SO_3(\mathbb{R})$-orbit in $Q_+\setminus(Q_+\cap\mathbb{R}^3)$ (we note in passing that $Q_+\cap\mathbb{R}^3$ is also an $SO_3(\mathbb{R})$-orbit in Q_+; it has dimension 2 and does not lie in the range of the map Φ^μ). Specifically, Φ^μ maps the SU_2-orbit $\left\{(z,w)\in M^{\Phi^\mu}:|z|^2+|w|^2=r^2\right\}$, $r>0$, onto the $SO_3(\mathbb{R})$-orbit

$$\mu^{(2)}_{2r^4+1}:=\left\{(z_1,z_2,z_3)\in\mathbb{C}^3:|z_1|^2+|z_2|^2+|z_3|^2=2r^4+1\right\}\cap Q_+.$$

Further, consider a holomorphic map $\Psi^\mu:Q_+\setminus(Q_+\cap\mathbb{R}^3)\to\mathbb{CP}^2\setminus\mathbb{RP}^2$ defined as

$$(z_1,z_2,z_3)\mapsto(z_1:z_2:z_3).$$

Clearly, Ψ^μ is a 2-to-1 covering map, and $\Psi^\mu\left(\mu^{(2)}_\alpha\right)=\mu_\alpha$ for every $\alpha>1$. Thus, we have shown that $\tilde{\mu}_\alpha$ coincides with the hypersurface

$$\mu^{(4)}_\alpha:=\left\{(z,w)\in M^{\Phi^\mu}:|z|^2+|w|^2=\sqrt{(\alpha-1)/2}\right\},$$

with the CR-structure induced from M^{Φ^μ}; the 4-to-1 covering map $\mu^{(4)}_\alpha\to\mu_\alpha$ is the composition $\Psi^\mu\circ\Phi^\mu$.

Next, a straightforward calculation shows that $\Gamma_{\Psi^\mu\circ\Phi^\mu}$ is a cyclic group of order 4 generated by the map f^μ defined as:

$$\begin{aligned} z &\mapsto i\frac{z(|z|^2+|w|^2)-\overline{w}}{\sqrt{1+(|z|^2+|w|^2)^2}}, \\ w &\mapsto i\frac{w(|z|^2+|w|^2)+\overline{z}}{\sqrt{1+(|z|^2+|w|^2)^2}}. \end{aligned} \tag{4.4}$$

The only non-trivial subgroup of $\Gamma_{\Psi^\mu\circ\Phi^\mu}$ is then a cyclic subgroup of order 2 generated by $(f^\mu)^2$. The cover of μ_α corresponding to this subgroup is the hypersurface $\mu_\alpha^{(2)}$ with covering map $\mu_\alpha^{(2)}\to\mu_\alpha$ coinciding with $\Psi^\mu : Q_+\setminus(Q_+\cap\mathbb{R}^3)\to\mathbb{CP}^2\setminus\mathbb{RP}^2$ and factorization map $\mu_\alpha^{(4)}\to\mu_\alpha^{(2)}$ coinciding with $\Phi^\mu : M^{\Phi^\mu}\to Q_+\setminus(Q_+\cap\mathbb{R}^3)$.

Thus, every non-trivial (that is, not 1-to-1) cover of μ_α is CR-equivalent to either $\mu_\alpha^{(4)}$ or $\mu_\alpha^{(2)}$. The groups of CR-automorphisms of $\mu_\alpha^{(4)}$ and $\mu_\alpha^{(2)}$ are as follows.

$\underline{\mathrm{Aut}_{CR}\left(\mu_\alpha^{(4)}\right)\simeq SU_2\times_{\mathrm{loc}}\mathbb{Z}_4}$: This group is generated by the maps

$$\begin{pmatrix} z \\ w \end{pmatrix}\mapsto A\begin{pmatrix} z \\ w \end{pmatrix},$$

where $A\in SU_2$, that form $\mathrm{Aut}_{CR}\left(\mu_\alpha^{(4)}\right)^c$, and the map f^μ defined in (4.4), which is a lift from $Q_+\setminus\mathbb{R}^3$ to M^{Φ^μ} of the following element of $\mathrm{Aut}_{CR}\left(\mu_\alpha^{(2)}\right)$:

$$\begin{aligned} z_1 &\mapsto -z_1, \\ z_2 &\mapsto -z_2, \\ z_3 &\mapsto -z_3. \end{aligned} \tag{4.5}$$

$\underline{\mathrm{Aut}_{CR}\left(\mu_\alpha^{(2)}\right)\simeq O_3(\mathbb{R})}$:

$$\begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix}\mapsto A\begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix}, \tag{4.6}$$

where $A\in O_3(\mathbb{R})$.

5. The Covers of ν_α and η_α

In order to find all covers of ν_α and η_α we introduce an analogue of Rossi's map. Instead of the Hermitian form $|z|^2+|w|^2$ it is associated with the form $|z|^2-|w|^2$. Let Q_- be the variety in $\mathbb{C}^3$ given by

$$z_1^2+z_2^2-z_3^2=1.$$

Set $\Omega := \{(z,w) \in \mathbb{C}^2 : |z|^2 - |w|^2 \neq 0\}$ and consider the map $\Phi : \Omega \to Q_-$ defined by the formulas

$$\begin{aligned} z_1 &= -i(z^2+w^2) - i\frac{z\overline{w}+w\overline{z}}{|z|^2-|w|^2}, \\ z_2 &= z^2 - w^2 + \frac{z\overline{w}-w\overline{z}}{|z|^2-|w|^2}, \\ z_3 &= -2izw - i\frac{|z|^2+|w|^2}{|z|^2-|w|^2}. \end{aligned} \tag{5.1}$$

It is straightforward to verify that the range of Φ is $Q_- \setminus (Q_- \cap \mathcal{W})$, where

$$\begin{aligned} \mathcal{W} : = \; & i\mathbb{R}^3 \cup \mathbb{R}^3 \cup \\ & \left\{(z_1,z_2,z_3) \in \mathbb{C}^3 \setminus \mathbb{R}^3 : |iz_1+z_2| = |iz_3-1|,\, |iz_1-z_2| = |iz_3+1|\right\}, \end{aligned}$$

and that the restrictions of Φ to the domains

$$\begin{aligned} \Omega^> &:= \{(z,w) \in \mathbb{C}^2 : |z|^2 - |w|^2 > 0\}, \\ \Omega^< &:= \{(z,w) \in \mathbb{C}^2 : |z|^2 - |w|^2 < 0\} \end{aligned}$$

are 2-to-1 covering maps onto $\Phi(\Omega^>)$ and $\Phi(\Omega^<)$, respectively (note that $\Phi(\Omega^<)$ is obtained from $\Phi(\Omega^>)$ by applying the transformation $z_1 \mapsto -z_1$, $z_2 \mapsto z_2$, $z_3 \mapsto -z_3$).

The map Φ satisfies

$$\Phi(g(z,w)) = \varphi(g)\Phi((z,w)), \tag{5.2}$$

for all $g \in SU_{1,1}$, $(z,w) \in \Omega$, where φ is the standard 2-to-1 covering homomorphism from $SU_{1,1}$ onto $SO_{2,1}(\mathbb{R})^c$, defined as follows: for

$$g = \begin{pmatrix} a & b \\ \overline{b} & \overline{a} \end{pmatrix} \in SU_{1,1}$$

(here $|a|^2 - |b|^2 = 1$), set

$$\varphi(g) := \begin{pmatrix} \mathrm{Re}\,(a^2+b^2) & \mathrm{Im}\,(a^2-b^2) & 2\mathrm{Re}(ab) \\ -\mathrm{Im}\,(a^2+b^2) & \mathrm{Re}\,(a^2-b^2) & -2\mathrm{Im}(ab) \\ 2\mathrm{Re}(a\overline{b}) & 2\mathrm{Im}(a\overline{b}) & |a|^2+|b^2| \end{pmatrix}. \tag{5.3}$$

The actions of $SU_{1,1}$ on $\mathbb{C}^2$ and $SO_{2,1}(\mathbb{R})^c$ on $\mathbb{C}^3$ in formula (5.2) are standard.

Hence Φ maps every $SU_{1,1}$-orbit in Ω onto an $SO_{2,1}(\mathbb{R})^c$-orbit in $Q_- \setminus (Q_- \cap \mathcal{W})$. Note that $SO_{2,1}(\mathbb{R})^c$ has exactly four orbits in Q_- that do not lie in the range of the map Φ:

$$\begin{aligned} O_1 &:= i\mathbb{R}^3 \cap Q_- = \{(z_1,z_2,z_3) \in \mathbb{C}^3 : |z_1|^2 + |z_2|^2 - |z_3|^2 = -1\} \cap Q_-, \\ O_2 &:= \mathbb{R}^3 \cap Q_-, \\ O_3 &:= \Big\{(z_1,z_2,z_3) \in \mathbb{C}^3 \setminus \mathbb{R}^3 : |iz_1+z_2| = |iz_3-1|, \\ &\qquad |iz_1-z_2| = |iz_3+1|,\, \mathrm{Im}\, z_3 < 0\Big\} \cap Q_-, \\ O_4 &:= \Big\{(z_1,z_2,z_3) \in \mathbb{C}^3 \setminus \mathbb{R}^3 : |iz_1+z_2| = |iz_3-1|, \\ &\qquad |iz_1-z_2| = |iz_3+1|,\, \mathrm{Im}\, z_3 > 0\Big\} \cap Q_-. \end{aligned}$$

The orbits O_1, O_2 are 2-dimensional, the orbits O_3, O_4 are 3-dimensional, and O_2, O_3, O_4 lie in the set

$$\mathfrak{S} := \left\{(z_1, z_2, z_3) \in \mathbb{C}^3 : |z_1|^2 + |z_2|^2 - |z_3|^2 = 1\right\} \cap Q_-.$$

In fact, we have

$$\mathfrak{S} = O_2 \cup O_3 \cup O_4 \cup O_5 \cup O_6,$$

where O_5, O_6 are the following 3-dimensional $SO_{2,1}(\mathbb{R})^c$-orbits in Q_-:

$$\begin{aligned} O_5 &:= \Big\{(z_1, z_2, z_3) \in \mathbb{C}^3 \setminus \mathbb{R}^3 : |iz_1 + z_2| = |iz_3 + 1|, \\ &\qquad |iz_1 - z_2| = |iz_3 - 1|,\ \operatorname{Im} z_3 < 0\Big\} \cap Q_-, \\ O_6 &:= \Big\{(z_1, z_2, z_3) \in \mathbb{C}^3 \setminus \mathbb{R}^3 : |iz_1 + z_2| = |iz_3 + 1|, \\ &\qquad |iz_1 - z_2| = |iz_3 - 1|,\ \operatorname{Im} z_3 > 0\Big\} \cap Q_-. \end{aligned}$$

In contrast with O_2, O_3, O_4, however, the orbits O_5, O_6 lie in the range of Φ and are the images under Φ of the sets $\{(z, w) \in \mathbb{C}^2 : |z|^2 - |w|^2 = 1\}$, $\{(z, w) \in \mathbb{C}^2 : |z|^2 - |w|^2 = -1\}$, respectively.

From now on we will only consider the restriction $\Phi^>$ of Φ to $\Omega^>$. The range of $\Phi^>$ is $D^> := \Sigma^\nu \cup \Sigma^\eta \cup O_5$, where

$$\Sigma^\nu := \Big\{(z_1, z_2, z_3) \in \mathbb{C}^3 : -1 < |z_1|^2 + |z_2|^2 - |z_3|^2 < 1,\ \operatorname{Im} z_3 < 0\Big\} \cap Q_-,$$

and

$$\Sigma^\eta := \Big\{(z_1, z_2, z_3) \in \mathbb{C}^3 : |z_1|^2 + |z_2|^2 - |z_3|^2 > 1,\ \operatorname{Im}(z_2(\overline{z_1} + \overline{z_3})) > 0\Big\} \cap Q_-.$$

The covering homomorphism φ defined in (5.3) can be used to give a group-theoretic interpretation of the map $\Phi^>$ analogous to that of Rossi's map Φ^μ from the previous section, where the homomorphism φ^μ defined in (4.3) was utilized. The group $\mathbb{R} \times SU_{1,1}$ acts on $\Omega^>$ simply transitively as follows:

$$(t, g)(z, w) := e^t \cdot g(z, w),$$

where $t \in \mathbb{R}$, $g \in SU_{1,1}$. On the other hand, the standard action of $SO_{2,1}(\mathbb{R})^c$ on $D^>$ can be extended to a simple transitive action of the group $\mathbb{R} \times SO_{2,1}(\mathbb{R})^c$ by diffeomorphisms. Indeed, $\mathbb{R} \times SO_{2,1}(\mathbb{R})^c$ acts simply transitively on each of the two connected components of the set

$$\hat{Q}_- := \left\{\zeta = (\zeta_1, \zeta_2, \zeta_3) \in \mathbb{C}^3 : \zeta_1^2 + \zeta_2^2 + \zeta_3^2 = 0,\ \langle \zeta, \zeta \rangle_- > 0\right\},$$

where $\langle \zeta, \zeta' \rangle_- := \zeta_1 \overline{\zeta_1'} + \zeta_2 \overline{\zeta_2'} - \zeta_3 \overline{\zeta_3'}$. The action is given as follows:

$$(t, g)\, \zeta := e^t \cdot g\, \zeta,$$

where $t \in \mathbb{R}$, $g \in SO_{2,1}(\mathbb{R})^c$. Let $\hat{Q}_-^0$ be the connected component of $\hat{Q}_-$ that contains the point $(-i, 1, 0)$. The manifold $\hat{Q}_-^0$ is $SO_{2,1}(\mathbb{R})^c$-equivariantly diffeomorphic to $D^>$ by means of the map F_- defined as

$$\zeta \mapsto \zeta + i\xi,$$

where $\xi \in \mathbb{R}^3$ is such that $\langle \xi, \xi \rangle_- = -1$, $\langle \xi, \zeta \rangle_- = 0$, $\det(\xi, \operatorname{Re}\zeta, \operatorname{Im}\zeta) < 0$. Using the $SO_{2,1}(\mathbb{R})^c$-equivariant diffeomorphism F_-, we can now push forward the action of $\mathbb{R} \times SO_{2,1}(\mathbb{R})^c$ on $\hat{Q}^0_-$ to a simple transitive action of $\mathbb{R} \times SO_{2,1}(\mathbb{R})^c$ on $D^>$ by diffeomorphisms.

Thus, as smooth manifolds $\Omega^>$ and $D^>$ can be identified with $\mathbb{R} \times SU_{1,1}$ and $\mathbb{R} \times SO_{2,1}(\mathbb{R})^c$, respectively. Then the map $(t, g) \mapsto (t, \varphi(g))$ is exactly the map $\Phi^>$ if we choose $(1, 0) \in \Omega^>$ and $(-i, 1, -i) \in D^>$ as basepoints.

We will now concentrate on the domains in $\Omega^>$ lying above $Q_- \setminus (O_1 \cup \mathfrak{S})$. Let Φ^ν, Φ^η denote the restrictions of $\Phi^>$ to the domains

$$\Omega^\nu := \{(z, w) \in \mathbb{C}^2 : 0 < |z|^2 - |w|^2 < 1\}$$

and

$$\Omega^\eta := \{(z, w) \in \mathbb{C}^2 : |z|^2 - |w|^2 > 1\},$$

respectively. The maps Φ^ν and Φ^η are 2-to-1 covering maps onto Σ^ν and Σ^η. Introduce on Ω^ν, Ω^η the pull-back complex structures under the maps Φ^ν, Φ^η, respectively, and denote the resulting complex manifolds by M^{Φ^ν}, M^{Φ^η}. These complex structures are invariant under the ordinary action of $SU_{1,1}$. The map Φ^ν takes the $SU_{1,1}$-orbit

$$\nu^{(2)}_{2r^4-1} := \left\{(z, w) \in M^{\Phi^\nu} : |z|^2 - |w|^2 = r^2\right\} \tag{5.4}$$

onto the $SO_{2,1}(\mathbb{R})^c$-orbit

$$\nu'_{2r^4-1} := \left\{(z_1, z_2, z_3) \in \Sigma^\nu : |z_1|^2 + |z_2|^2 - |z_3|^2 = 2r^4 - 1\right\},$$

where $0 < r < 1$. Similarly, the map Φ^η takes the $SU_{1,1}$-orbit

$$\eta^{(4)}_{2r^4-1} := \left\{(z, w) \in M^{\Phi^\eta} : |z|^2 - |w|^2 = r^2\right\} \tag{5.5}$$

onto the $SO_{2,1}(\mathbb{R})^c$-orbit

$$\eta^{(2)}_{2r^4-1} := \left\{(z_1, z_2, z_3) \in \Sigma^\eta : |z_1|^2 + |z_2|^2 - |z_3|^2 = 2r^4 - 1\right\}, \tag{5.6}$$

where $r > 1$.

Observe now that $z_3 \neq 0$ on Σ^ν and consider the holomorphic map $\Psi^\nu : \Sigma^\nu \to \mathbb{C}^2$ defined as follows:

$$\begin{aligned} z &= z_1/z_3, \\ w &= z_2/z_3. \end{aligned}$$

This map is 1-to-1, it takes Σ^ν onto

$$D^\nu := \left\{(z, w) \in \mathbb{C}^2 : -|z^2 + w^2 - 1| < |z|^2 + |w|^2 - 1 < |z^2 + w^2 - 1|\right\},$$

and establishes equivalence between ν'_α and ν_α for $-1 < \alpha < 1$. Next, we note that $z_1 \neq 0$ on Σ^η and consider the holomorphic map $\Psi^\eta : \Sigma^\eta \to \mathbb{C}^2$ defined by

$$\begin{aligned} z &= z_2/z_1, \\ w &= z_3/z_1. \end{aligned}$$

It is easy to see that Ψ^η is a 2-to-1 covering map onto

$$\begin{aligned} D^\eta \;:=\; & \Big\{(z,w)\in\mathbb{C}^2 : 1+|z|^2-|w|^2>|1+z^2-w^2|,\ \mathrm{Im}(z(1+\overline{w}))>0\Big\}\setminus \\ & \Big\{(z,w)\in\mathbb{C}^2 : 1+z^2-w^2=0,\ \mathrm{Im}(z(1+\overline{w}))>0\Big\}, \end{aligned}$$

and realizes $\eta^{(2)}_\alpha$ as a 2-sheeted cover of η_α for $\alpha>1$.

Let $\Lambda:\mathbb{C}\times\Delta\to\Omega^>$ be the following covering map:

$$\begin{aligned} z &:= e^s, \\ w &:= e^s t, \end{aligned}$$

where $s\in\mathbb{C}$, $t\in\Delta$ and Δ is the unit disk. Further, define

$$\begin{aligned} U^\nu &:= \left\{(s,t)\in\mathbb{C}^2 : |t|<1,\ \exp(2\mathrm{Re}\, s)(1-|t|^2)<1\right\}, \\ U^\eta &:= \left\{(s,t)\in\mathbb{C}^2 : |t|<1,\ \exp(2\mathrm{Re}\, s)(1-|t|^2)>1\right\}. \end{aligned}$$

Denote by Λ^ν, Λ^η the restrictions of Λ to U^ν, U^η, respectively. Clearly, U^ν covers M^{Φ^ν} by means of Λ^ν, and U^η covers M^{Φ^η} by means of Λ^η. Introduce now on U^ν, U^η the pull-back complex structures under the maps Λ^ν, Λ^η, respectively, and denote the resulting complex manifolds by M^{Λ^ν}, M^{Λ^η}. Then the simply-connected hypersurface

$$\left\{(s,t)\in M^{\Lambda^\nu} : r^2\exp(-2\mathrm{Re}\, s)+|t|^2=1\right\}$$

covers $\nu^{(2)}_{2r^4-1}$ by means of the map Λ^ν for $0<r<1$, and the simply-connected hypersurface

$$\left\{(s,t)\in M^{\Lambda^\eta} : r^2\exp(-2\mathrm{Re}\, s)+|t|^2=1\right\}$$

covers $\eta^{(4)}_{2r^4-1}$ by means of the map Λ^η for $r>1$.

Thus, $\tilde\nu_\alpha$ for $-1<\alpha<1$ coincides with

$$\nu^{(\infty)}_\alpha := \left\{(s,t)\in M^{\Lambda^\nu} : \sqrt{(\alpha+1)/2}\exp(-2\mathrm{Re}\, s)+|t|^2=1\right\}, \tag{5.7}$$

with the CR-structure induced from the complex structure of M^{Λ^ν}, and $\tilde\eta_\alpha$ for $\alpha>1$ coincides with

$$\eta^{(\infty)}_\alpha := \left\{(s,t)\in M^{\Lambda^\eta} : \sqrt{(\alpha+1)/2}\exp(-2\mathrm{Re}\, s)+|t|^2=1\right\}, \tag{5.8}$$

with the CR-structure induced from the complex structure of M^{Λ^η}. The covering maps $\nu^{(\infty)}_\alpha\to\nu_\alpha$ and $\eta^{(\infty)}_\alpha\to\eta_\alpha$ are respectively $\Psi^\nu\circ\Phi^\nu\circ\Lambda^\nu : U^\nu\to D^\nu$ and $\Psi^\eta\circ\Phi^\eta\circ\Lambda^\eta : U^\eta\to D^\eta$.

Next, the group $\Gamma_{\Psi^\nu\circ\Phi^\nu\circ\Lambda^\nu}=\Gamma_{\Phi^\nu\circ\Lambda^\nu}$ consists of the maps

$$\begin{aligned} s &\mapsto s+\pi i k, \quad k\in\mathbb{Z}, \\ t &\mapsto t. \end{aligned} \tag{5.9}$$

Let $\Gamma\subset\Gamma_{\Phi^\nu\circ\Lambda^\nu}$ be a subgroup. Then there exists an integer $n\geq 0$ such that every element of Γ has the form

$$\begin{aligned} s &\mapsto s+\pi i n k, \quad k\in\mathbb{Z}, \\ t &\mapsto t. \end{aligned} \tag{5.10}$$

Suppose that $n \geq 2$ and set

$$\Omega^{\nu\,(n)} := \left\{(z,w) \in \mathbb{C}^2 : 0 < |z|^n - |z|^{n-2}|w|^2 < 1\right\}$$

(note that $\Omega^{\nu\,(2)} = \Omega^{\nu}$). Consider the map Φ_n^{ν} from $\Omega^{\nu\,(n)}$ to Σ^{ν} defined as follows:

$$\begin{aligned} z_1 &= -i(z^n + z^{n-2}w^2) - i\frac{z\overline{w} + w\overline{z}}{|z|^2 - |w|^2}, \\ z_2 &= z^n - z^{n-2}w^2 + \frac{z\overline{w} - w\overline{z}}{|z|^2 - |w|^2}, \\ z_3 &= -2iz^{n-1}w - i\frac{|z|^2 + |w|^2}{|z|^2 - |w|^2}. \end{aligned} \tag{5.11}$$

This map is a generalization of map (5.1) introduced at the beginning of the section and also can be viewed as an analogue of Rossi's map (4.1). The extension of this map by the same formula to all of $\Omega^{>}$ admits a group-theoretic interpretation analogous to those given above for the maps Φ^{μ} and $\Phi^{>}$. It uses an n-to-1 covering homomorphism $SO_{2,1}(\mathbb{R})^{c\,(n)} \to SO_{2,1}(\mathbb{R})^{c}$, where the n-sheeted cover $SO_{2,1}(\mathbb{R})^{c\,(n)}$ of $SO_{2,1}(\mathbb{R})^{c}$ is realized as the group of maps of the form (5.17) that will appear below, acting on $\Omega^{>}$ (note that this group reduces to the group $SU_{1,1}$ for $n = 2$). We do not provide a detailed construction here since it is very similar to that for the map $\Phi^{>}$.

Denote by $M^{\Phi_n^{\nu}}$ the domain $\Omega^{\nu\,(n)}$ with the pull-back complex structure under the map Φ_n^{ν} (note that $M^{\Phi_2^{\nu}} = M^{\Phi^{\nu}}$). Then the hypersurface

$$\nu_{\alpha}^{(n)} := \left\{(z,w) \in M^{\Phi_n^{\nu}} : |z|^n - |z|^{n-2}|w|^2 = \sqrt{(\alpha+1)/2}\right\}, \tag{5.12}$$

equipped with the CR-structure induced by the complex structure of $M^{\Phi_n^{\nu}}$ is an n-sheeted cover of ν_{α} corresponding to Γ with covering map $\nu_{\alpha}^{(n)} \to \nu_{\alpha}$ coinciding with $\Psi^{\nu} \circ \Phi_n^{\nu} : M^{\Phi_n^{\nu}} \to D^{\nu}$ and factorization map $\nu_{\alpha}^{(\infty)} \to \nu_{\alpha}^{(n)}$ given by

$$\begin{aligned} z &\mapsto e^{2s/n}, \\ w &\mapsto e^{2s/n}t, \end{aligned} \tag{5.13}$$

(observe that for $n = 2$ formula (5.12) coincides with (5.4)). Thus, every non-trivial cover of ν_{α} is CR-equivalent to either $\nu_{\alpha}^{(\infty)}$ or $\nu_{\alpha}^{(n)}$ for some $n \in \mathbb{N}$, $n \geq 2$. The groups of CR-automorphisms of $\nu_{\alpha}^{(\infty)}$ and $\nu_{\alpha}^{(n)}$ are given below.

(A) $\underline{\mathrm{Aut}_{CR}\left(\nu_{\alpha}^{(\infty)}\right) \simeq \widetilde{SO}_{2,1}(\mathbb{R})^{c} \rtimes_{\mathrm{loc}} \mathbb{Z}}$: This group is generated by the following maps (they form the subgroup $\mathrm{Aut}_{CR}\left(\nu_{\alpha}^{(\infty)}\right)^{c}$):

$$\begin{aligned} s &\mapsto s + \ln(a + bt), \\ t &\mapsto \frac{\overline{b} + \overline{a}t}{a + bt}, \end{aligned} \tag{5.14}$$

where ln is any branch of the logarithm and $|a|^2-|b|^2=1$, and the map

$$\begin{aligned} s &\mapsto \overline{s}+\ln'\left(-\frac{1+e^{2s}t(1-|t|^2)}{\sqrt{1-\exp(4\mathrm{Re}\, s)\,(1-|t|^2)^2}}\right),\\ t &\mapsto -\frac{\overline{t}+e^{2s}(1-|t|^2)}{1+e^{2s}t(1-|t|^2)}, \end{aligned} \tag{5.15}$$

for some branch $\ln'$ of the logarithm. Map (5.15) is a lift from Σ^ν to M^{Λ^ν} of the following element of $\mathrm{Aut}_{CR}(\nu'_\alpha)$:

$$\begin{aligned} z_1 &\mapsto -z_1,\\ z_2 &\mapsto z_2,\\ z_3 &\mapsto z_3. \end{aligned} \tag{5.16}$$

Here $\rtimes_{\mathrm{loc}}$ denotes local semidirect product.

(B) $\underline{\mathrm{Aut}_{CR}\left(\nu_\alpha^{(n)}\right)\simeq SO_{2,1}(\mathbb{R})^{c\,(n)}\rtimes_{\mathrm{loc}}\mathbb{Z}_{2n}}$: This group is generated by the following maps (they form the subgroup $\mathrm{Aut}_{CR}\left(\nu_\alpha^{(n)}\right)^c$):

$$\begin{aligned} z &\mapsto z\sqrt[n]{(a+b\,w/z)^2},\\ w &\mapsto z\,\frac{\overline{b}+\overline{a}\,w/z}{a+b\,w/z}\sqrt[n]{(a+b\,w/z)^2}, \end{aligned} \tag{5.17}$$

where $\sqrt[n]{\ }$ is any branch of the nth root and $|a|^2-|b|^2=1$, and the map

$$\begin{aligned} z &\mapsto \overline{z}\left(\sqrt[n]{\frac{\left(1+z^{n-1}w(1-|w|^2/|z|^2)\right)^2}{1-|z|^{2n}(1-|w|^2/|z|^2)^2}}\right)',\\ w &\mapsto -\frac{\overline{w}/\overline{z}+z^n(1-|w|^2/|z|^2)}{1+z^{n-1}w(1-|w|^2/|z|^2)}\times\\ &\qquad \overline{z}\left(\sqrt[n]{\frac{\left(1+z^{n-1}w(1-|w|^2/|z|^2)\right)^2}{1-|z|^{2n}(1-|w|^2/|z|^2)^2}}\right)', \end{aligned} \tag{5.18}$$

for some branch $\left(\sqrt[n]{\ }\right)'$ of $\sqrt[n]{\ }$. Map (5.18) is a lift from Σ^ν to $M^{\Phi^\nu_n}$ of map (5.16) (recall that $SO_{2,1}^{c\,(n)}(\mathbb{R})$ is the n-sheeted cover of $SO_{2,1}(\mathbb{R})^c$).

Further, the group $\Gamma_{\Psi^\eta\circ\Phi^\eta\circ\Lambda^\eta}$ is generated by all maps of the form (5.9) and the map f^η defined as follows:

$$\begin{aligned} s &\mapsto 2s+\overline{s}+\ln'\left(i\frac{1-|t|^2+e^{-2s}\overline{t}}{\sqrt{\exp(4\mathrm{Re}\, s)\,(1-|t|^2)^2-1}}\right),\\ t &\mapsto \frac{1+e^{2s}t(1-|t|^2)}{\overline{t}+e^{2s}(1-|t|^2)}, \end{aligned} \tag{5.19}$$

for some fixed branch $\ln'$ of the logarithm. The map f^η is a lift from Σ^η to M^{Λ^η} of the element of $\mathrm{Aut}_{CR}\left(\eta_\alpha^{(2)}\right)$ given by formula (4.5). At the same time, f^η is a lift from M^{Φ^η} to M^{Λ^η} of the map

$$\begin{aligned} z &\mapsto i\frac{z(|z|^2-|w|^2)+\overline{w}}{\sqrt{(|z|^2-|w|^2)^2-1}}, \\ w &\mapsto i\frac{w(|z|^2-|w|^2)+\overline{z}}{\sqrt{(|z|^2+|w|^2)^2-1}}. \end{aligned} \tag{5.20}$$

Since the square of map (5.20) is

$$\begin{aligned} z &\mapsto -z, \\ w &\mapsto -w, \end{aligned} \tag{5.21}$$

it follows that $(f^\eta)^2$ is a lift from M^{Φ^η} to M^{Λ^η} of map (5.21) and thus has the form (5.9) with k odd. Since f^η clearly commutes with all maps (5.9), the group $\Gamma_{\Psi^\eta\circ\Phi^\eta\circ\Lambda^\eta}$ is isomorphic to $\mathbb{Z}$ and is generated by $f^\eta\circ g$, where g has the form (5.9).

Let $\Gamma\subset\Gamma_{\Psi^\eta\circ\Phi^\eta\circ\Lambda^\eta}$ be a subgroup. It then follows that Γ is generated by either a map of the form (5.9) or by $f^\eta\circ h$, where h has the form (5.9). In the first case there exists an integer $n\geq 0$ such that every element of Γ has the form (5.10). Suppose that $n\geq 2$ and set

$$\Omega^{\eta\,(n)} := \left\{(z,w)\in\mathbb{C}^2 : |z|^n-|z|^{n-2}|w|^2>1\right\}$$

(note that $\Omega^{\eta\,(2)}=\Omega^\eta$). Consider the map Φ_n^η from $\Omega^{\eta\,(n)}$ to Σ^η defined by formula (5.11). Denote by $M^{\Phi_n^\eta}$ the domain $\Omega^{\eta\,(n)}$ with the pull-back complex structure under the map Φ_n^η (note that $M^{\Phi_2^\eta}=M^{\Phi^\eta}$). Then the hypersurface

$$\eta_\alpha^{(2n)} := \left\{(z,w)\in M^{\Phi_n^\eta} : |z|^n-|z|^{n-2}|w|^2=\sqrt{(\alpha+1)/2}\right\}, \tag{5.22}$$

equipped with the CR-structure induced by the complex structure of $M^{\Phi_n^\eta}$ is a $2n$-sheeted cover of η_α corresponding to Γ with covering map $\eta_\alpha^{(2n)}\to\eta_\alpha$ coinciding with $\Psi^\eta\circ\Phi_n^\eta : M^{\Phi_n^\eta}\to D^\eta$ and factorization map $\eta_\alpha^{(\infty)}\to\eta_\alpha^{(2n)}$ given by formula (5.13); note that for $n=2$ formula (5.22) coincides with (5.5). For $n=1$ we obtain the hypersurface $\eta_\alpha^{(2)}$ defined in (5.6) that covers η_α by means of the 2-to-1 map $\Psi^\eta:\Sigma^\eta\to D^\eta$; the factorization map $\eta_\alpha^{(\infty)}\to\eta_\alpha^{(2)}$ is $\Phi^\eta\circ\Lambda^\eta : M^{\Lambda^\eta}\to\Sigma^\eta$.

We will now describe finitely-sheeted covers of η_α of odd orders. They arise in the case when Γ is generated by $f^\eta\circ h$, where h has the form (5.9). In this case the group Γ can be represented as $\Gamma=\Gamma_0\cup\left((f^\eta\circ h)\circ\Gamma_0\right)$, where Γ_0 is a subgroup that consists of maps of the form (5.10) for an odd positive integer n. We will first factor M^{Φ^η} with respect to the action of the subgroup of Γ_0 corresponding to even k. Namely, M^{Φ^η} by means of the map

$$\begin{aligned} z &\mapsto e^{s/n}, \\ w &\mapsto e^{s/n}t, \end{aligned} \tag{5.23}$$

covers the manifold $M^{\Phi^\eta_{2n}}$ and, accordingly, $\eta^{(\infty)}_\alpha$ covers $\eta^{(4n)}_\alpha$. In order to obtain the cover of η_α corresponding to the group Γ, the hypersurface $\eta^{(4n)}_\alpha$ must be further factored by the action of the cyclic group of four elements generated by the following automorphism f^η_n of $M^{\Phi^\eta_{2n}}$:

$$\begin{aligned} z &\mapsto iz^2\overline{z}\left(\sqrt[n]{\frac{1-|w|^2/|z|^2+z^{-2n}\overline{w}/\overline{z}}{\sqrt{|z|^{4n}(1-|w|^2/|z|^2)^2-1}}}\right)', \\ w &\mapsto i\frac{1+z^{2n-1}w(1-|w|^2/|z|^2)}{\overline{w}/\overline{z}+z^{2n}(1-|w|^2/|z|^2)}\times \\ &\qquad z^2\overline{z}\left(\sqrt[n]{\frac{1-|w|^2/|z|^2+z^{-2n}\overline{w}/\overline{z}}{\sqrt{|z|^{4n}(1-|w|^2/|z|^2)^2-1}}}\right)', \end{aligned}$$

for some branch $\left(\sqrt[n]{\ }\right)'$ of $\sqrt[n]{\ }$. Let $\hat{M}^{\Phi^\eta_n}$ denote the manifold arising from $M^{\Phi^\eta_{2n}}$ by means of this factorization and $\Pi^\eta_n : M^{\Phi^\eta_{2n}} \to \hat{M}^{\Phi^\eta_n}$ denote the corresponding 4-to-1 factorization map. Next, define

$$\eta^{(n)}_\alpha := \Pi^\eta_n\left(\eta^{(4n)}_\alpha\right). \tag{5.24}$$

The hypersurface $\eta^{(n)}_\alpha$ with the CR-structure induced from the complex structure of $\hat{M}^{\Phi^\eta_n}$ is an n-sheeted cover of η_α corresponding to Γ with the factorization map $\eta^{(\infty)}_\alpha \to \eta^{(n)}_\alpha$ coinciding with the composition of map (5.23) and Π^η_n. Note that for $n = 1$ the map $f^\eta_n = f^\eta_1$ coincides with the map defined in (5.20), and $\Pi^\eta_n = \Pi^\eta_1$ coincides with $\Psi^\eta \circ \Phi^\eta$. Both Π^η_n and the covering map $\eta^{(n)}_\alpha \to \eta_\alpha$ (which extends to a covering map $\hat{M}^{\Phi^\eta_n} \to D^\eta$) can be computed explicitly for any odd $n \in \mathbb{N}$, but, since the resulting formulas are quite lengthy and not very instructive, we omit them.

We will now write down the groups of CR-automorphisms of $\eta^{(\infty)}_\alpha$ and $\eta^{(n)}_\alpha$.

(C) $\underline{\mathrm{Aut}_{CR}\left(\eta^{(\infty)}_\alpha\right) \simeq \widetilde{SO}_{2,1}(\mathbb{R})^c \times_{\mathrm{loc}} \mathbb{Z}}$: This group is generated by its connected identity component that consists of all maps of the form (5.14), and the map f^η defined in (5.19).

(D) $\underline{\mathrm{Aut}_{CR}\left(\eta^{(2)}_\alpha\right) \simeq SO_{2,1}(\mathbb{R})^c \times \mathbb{Z}_2}$: This group is generated by its connected identity component that consists of all maps of the form (4.6), where $A \in SO_{2,1}(\mathbb{R})^c$ and the map given by formula (4.5).

(D') $\underline{\mathrm{Aut}_{CR}\left(\eta^{(2n)}_\alpha\right) \simeq SO_{2,1}(\mathbb{R})^{c\,(n)} \times_{\mathrm{loc}} \mathbb{Z}_{4n},\ n \geq 2}$: This group is generated by its connected identity component that consists of all maps of the form (5.17), and the map

$$
\begin{aligned}
z &\mapsto z^2\overline{z}\left(\sqrt[n]{\frac{\left(1-|w|^2/|z|^2+z^{-n}\overline{w}/\overline{z}\right)^2}{|z|^{2n}(1-|w|^2/|z|^2)^2-1}}\right)', \\
w &\mapsto \frac{1+z^{n-1}w(1-|w|^2/|z|^2)}{\overline{w}/\overline{z}+z^n(1-|w|^2/|z|^2)}\times \\
&\qquad z^2\overline{z}\left(\sqrt[n]{\frac{\left(1-|w|^2/|z|^2+z^{-n}\overline{w}/\overline{z}\right)^2}{|z|^{2n}(1-|w|^2/|z|^2)^2-1}}\right)',
\end{aligned}
$$

for some branch $\left(\sqrt[n]{\ }\right)'$ of $\sqrt[n]{\ }$.

(E) $\underline{\mathrm{Aut}_{CR}\left(\eta_\alpha^{(2n+1)}\right)\simeq SO_{2,1}(\mathbb{R})^{c\,(2n+1)}}$: This group is connected and consists of all lifts from D^η to $\hat{M}^\eta_{2n+1}$ of maps (2.3).

Thus, we have proved the following theorem:

Theorem 5.1.
(i) *A non-trivial cover of a hypersurface ν_α with $-1<\alpha<1$ is CR-equivalent to either $\nu_\alpha^{(\infty)}$ or $\nu_\alpha^{(n)}$ with $n\geq 2$ defined in* (5.7), (5.12)*; the groups of CR-automorphisms of these covers are described in* **(A)**-**(B)**, *in particular, for every $N\in\{2,3,\dots,\infty\}$ there exists a complex 2-dimensional manifold M^ν_N on which a connected 3-dimensional Lie group G^ν_N acts by holomorphic transformations, such that for every $-1<\alpha<1$ the hypersurface $\nu_\alpha^{(N)}$ is a G^ν_N-orbit in M^ν_N and the group $\mathrm{Aut}_{CR}\big(\nu_\alpha^{(N)}\big)$ consists of the restrictions of the elements of G^ν_N to $\nu_\alpha^{(N)}$.*

(ii) *A non-trivial cover of a hypersurface η_α with $\alpha>1$ is CR-equivalent to either $\eta_\alpha^{(\infty)}$ or $\eta_\alpha^{(n)}$ with $n\geq 2$ defined in* (5.6), (5.8), (5.22), (5.24)*; the groups of CR-automorphisms of the above covers are described in* **(C)**-**(E)**, *in particular, for every $N\in\{2,3,\dots,\infty\}$ there exists a complex 2-dimensional manifold M^η_N on which a connected 3-dimensional Lie group G^η_N acts by holomorphic transformations, such that for every $\alpha>1$ the hypersurface $\eta_\alpha^{(N)}$ is a G^η_N-orbit in M^η_N and the group $\mathrm{Aut}_{CR}\big(\eta_\alpha^{(N)}\big)$ consists of the restrictions of the elements of G^η_N to $\eta_\alpha^{(N)}$.*

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