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Invariant Pseudo-Kähler Metrics in Dimension Four

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Abstract. Four dimensional simply connected Lie groups admitting a pseudo-Kähler metric are determined. The corresponding Lie algebras are modelled and the compatible pairs (J, ω) are parametrized up to complex isomorphism (where J is a complex structure and ω is a symplectic structure). Such structure gives rise to a pseudo-Riemannian metric g, for which J is a parallel. It is proved that most of these complex homogeneous spaces admit a compatible pseudo-Kähler Einstein metric. Ricci flat and flat metrics are determined. In particular Ricci flat unimodular pseudo-Kähler Lie groups are flat in dimension four. Other algebraic and geometric features are treated. A general construction of Ricci flat pseudo-Kähler structures in higher dimension on some affine Lie algebras is given. Walker and hypersymplectic metrics are compared.
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1. Introduction

Simply connected real Lie groups endowed with a left invariant pseudo-Riemannian Kähler metric are in correspondence with Kähler Lie algebras. Kähler Lie algebras are real Lie algebras $\mathfrak{g}$ equipped with a pair (J, ω) consisting of a complex structure J and a compatible symplectic structure ω. A Kähler structure on a Lie algebra determines a pseudo-Riemannian metric g defined as

$$g(x, y) = \omega(Jx, y) \qquad x, y \in \mathfrak{g}$$

not necessarily definite, and for which J is parallel. The Lie algebra $(\mathfrak{g}, J, g)$ is also known as a pseudo-Kähler Lie algebra or indefinite Kähler Lie algebra. Kähler Lie algebras are special cases of symplectic Lie algebras and of pseudometric Lie algebras and therefore tools of both fields can be used to their study.

Lie algebras (resp. homogenous manifolds) admitting a definite Kähler metric were exhaustive studied by many authors. Indeed the condition of the pseudometric to be definite impose restrictions on the structure of the Lie algebra

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[7] [13] [12] [17]. In the nilpotent case the metric associated to a pair (J, ω) cannot be definite positive [6]. However this is not the case in general for solvable Lie algebras.

In this paper we describe Kähler four dimensional Lie algebras. Since four dimensional symplectic Lie algebras must be solvable [9], our results concern all possibilities in this dimension. Similar studies in the six dimensional nilpotent case were recently given in [11].

We prove that four dimensional completely solvable Kähler Lie algebras and $\mathfrak{aff}(\mathbb{C})$ are modelled on one of the following short exact sequences of Lie algebras:

$$0 \longrightarrow \mathfrak{h} \longrightarrow \mathfrak{g} \longrightarrow J\mathfrak{h} \longrightarrow 0 \qquad \text{orthogonal sum}$$

$$0 \longrightarrow \mathfrak{h} \longrightarrow \mathfrak{g} \longrightarrow \mathfrak{k} \longrightarrow 0 \qquad \mathfrak{h} \text{ and } \mathfrak{k}\ J\text{-invariant subspaces,}$$

where in both cases $\mathfrak{h}$ is an ω-lagrangian ideal of $\mathfrak{g}$ and (hence abelian) and $J\mathfrak{h}$ and $\mathfrak{k}$ are ω-isotropic subalgebras. While the first sequence splits, the second one does not necessarily split. There are also three kind of non completely solvable four dimensional Lie algebras admitting a Kähler structure which can be modelled on other sequences. In all cases the compatible pairs (J, ω) are parametrized up to complex isomorphism.

The next step is to do a geometric study of these spaces. We compute metrics, curvature and Ricci curvature tensors. Making use of the above sequences it is possible to get information about totally geodesic subspaces. The results from this study are summarized in the following:

- the neutral metric on the Lie algebras satisfying the second short exact sequence are Walker metrics;

- in 8 of the 11 families of Kähler Lie algebras there exists an Einstein representative among the compatible pseudo-Kähler metrics.

- in the unimodular case there is an equivalence between Ricci flat and flat metrics in dimension four.

- aside from the hypersymplectic Lie algebras [15] [1], any Ricci flat metric is provided either by $(\mathbb{R} \times \mathfrak{e}(2), J)$, with $\mathfrak{e}(2)$ the Lie algebra of the group of rigid motions of $\mathbb{R}^2$ or by $(\mathfrak{aff}(\mathbb{C}), J_2)$, the real Lie algebra underlying the Lie algebra of the affine motions group of $\mathbb{C}$. Furthermore the Ricci flat pseudometrics are deformations of flat pseudo-Kähler metrics.

If we look at the Lie algebras admitting abelian complex structures we prove that a Lie algebra which admits this kind of complex structures and it is symplectic then it is also Kähler. Moreover if this is the case, $(\mathfrak{g}, J)$ is Kähler if and only if J is abelian. For instance the Lie algebra $\mathfrak{aff}(\mathbb{C})$ has both abelian and non abelian complex structures; however only the abelian ones admit a compatible symplectic form.

Finally we try to generalize our results. We construct Kähler structures on affine Lie algebras $\mathfrak{aff}(A)$, where A is a commutative algebra. This type of Lie algebras cover all cases of four dimensional Lie algebras having abelian complex structures [5]. We give examples in higher dimensions of Ricci flat pseudo-Riemannian metrics by generalizing the Kähler structure of $(\mathfrak{aff}(\mathbb{C}), J_2)$ to affine Lie algebras $\mathfrak{aff}(A)$ where A is a commutative complex associative algebra. It is proved that a Walker Kähler metric on a Lie algebra $\mathfrak{g}$ can be hypersymplectic whenever some extra condition is satisfied. In particular a Walker metric compatible with the canonical complex structure of $\mathfrak{aff}(\mathbb{C})$ is shown.

In a final section we compute the obtained pseudo-Riemannian metrics in global coordinates.

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2. Preliminaries

Kähler Lie algebras are endowed with a pair (J,ω) consisting of a complex structure J and a compatible symplectic structure ω: $\omega(Jx,Jy)=\omega(x,y)$, namely a Kähler structure on $\mathfrak{g}$.

Recall that an *almost complex* structure on a Lie algebra $\mathfrak{g}$ is an endomorphism $J:\mathfrak{g}\to\mathfrak{g}$ satisfying $J^2=-I$, where I is the identity map. The almost complex structure J is said to be integrable if $N_J\equiv 0$ where N_J is the tensor given by

$$N_J(x,y)=[Jx,Jy]-[x,y]-J[Jx,y]-J[x,Jy] \qquad \text{for all } x,y\in\mathfrak{g}. \tag{1}$$

An integrable almost complex structure J is called a *complex structure* on $\mathfrak{g}$.

An equivalence relation is defined among Lie algebras endowed with complex structures. The Lie algebras with complex structures $(\mathfrak{g}_1,J_1)$ and $(\mathfrak{g}_2,J_2)$ are equivalent if there exists an isomorphism of Lie algebras $\alpha:\mathfrak{g}_1\to\mathfrak{g}_2$ such that $J_2\circ\alpha=\alpha\circ J_1$.

Examples of special classes of complex structures are the abelian ones and those that determine a complex Lie bracket on $\mathfrak{g}$.

A complex structure J is said to be *abelian* if it satisfies $[JX,JY]=[X,Y]$ for all $X,Y\in\mathfrak{g}$. A complex structure J introduces on $\mathfrak{g}$ a structure of complex Lie algebra if $J\circ ad_X=ad\circ JX$ for all $X\in\mathfrak{g}$, and so $(\mathfrak{g},J)$ is a *complex Lie algebra*, and that means that the corresponding simply connected Lie group is also complex, that is, left and right multiplication by elements of the Lie group are holomorphic maps.

A *symplectic structure* on a 2n-dimensional Lie algebra $\mathfrak{g}$ is a closed 2-form $\omega\in\Lambda^2(\mathfrak{g}^*)$ such that ω has maximal rank, that is, $\omega^n\neq 0$. Lie algebras (groups) admitting symplectic structures are called *symplectic* Lie algebras (resp. Lie groups).

The existence problem of compatible pairs (J,ω) on a Lie algebra $\mathfrak{g}$ is set up to complex isomorphism. In other words to search for Kähler structures on $\mathfrak{g}$ it is sufficient to determine the compatibility condition between any symplectic structure and a representative of the class of complex structures. In fact, assume that there is a complex structure J_1 for which there exists a symplectic structure ω satisfying $\omega(J_1X,J_1Y)=\omega(X,Y)$ for all $X,Y\in\mathfrak{g}$ and assume that J_2 is equivalent to J_1. Thus there exists an automorphism $\sigma\in\operatorname{Aut}(\mathfrak{g})$ such that $J_2=\sigma_*^{-1}J_1\sigma_*$. Then it holds

$$\omega(X,Y)=\sigma^{*-1}\sigma^*\omega(X,Y)=\sigma^{*-1}\omega(J_1\sigma_*X,J_1\sigma_*Y)=\omega(J_2X,J_2Y).$$

Kähler Lie algebras belong to the class of symplectic Lie algebras. Special objects on a symplectic Lie algebra $(\mathfrak{g}, \omega)$ are the isotropic and lagrangian subspaces. Recall that a subspace $W \subset \mathfrak{g}$ is called ω-isotropic if and only if $\omega(W, W) = 0$ and is said to be ω-lagrangian if it is ω-isotropic and $\omega(W, y) = 0$ implies $y \in W$.

Lemma 2.1. *Let $(\mathfrak{g}, J, \omega)$ be a Kähler Lie algebra. If $\mathfrak{h}$ is a isotropic ideal, then:*

- *$\mathfrak{h}$ is abelian*
- *$J(\mathfrak{h})$ is a isotropic subalgebra of $\mathfrak{g}$.*

Thus $\mathfrak{h}+J\mathfrak{h}$ is a subalgebra of $\mathfrak{g}$ and the sum is not necessarily direct. Furthermore $\mathfrak{h} \cap J\mathfrak{h}$ is a J-invariant ideal of $\mathfrak{h} + J\mathfrak{h}$.

Proof. Since $\mathfrak{h}$ is a isotropic ideal, the first assertion follows ¿from the condition of ω of being closed.

The integrability condition of J restricted to $\mathfrak{h}$, which was proved to be abelian, implies

$$[Jx, Jy] = J([Jx, y] + [x, Jy])$$

showing that $J\mathfrak{h}$ is a subalgebra of $\mathfrak{g}$. The compatibility between J and ω says that $\omega(Jx, Jy) = \omega(x, y) = 0$ for $x, y \in \mathfrak{h}$, and so $J\mathfrak{h}$ is isotropic. Furthermore if $\mathfrak{h}$ is ω-lagrangian, then $J\mathfrak{h}$ is ω-lagrangian, and the second assertion is proved. ■

A Kähler structure on a Lie algebra determines a pseudo-Riemannian metric g defined as

$$g(x, y) = \omega(Jx, y) \qquad x, y \in \mathfrak{g} \tag{2}$$

for which J is parallel with respect to the Levi Civita connection for g. Note that g is not necessarily definite; the signature is $(2k, 2l)$ with $2(k + l) = \dim \mathfrak{g}$.

Conversely if $(\mathfrak{g}, J, g)$ is a Lie algebra endowed with a complex structure J compatible with the pseudometric g then (2) defines a 2-form compatible with J which is closed if and only if J is parallel [16]. Hence the Lie algebra $(\mathfrak{g}, J, g)$ is called a Kähler Lie algebra with pseudo-(Riemannian) Kähler metric g.

Let g be a pseudo-Riemannian metric on $\mathfrak{g}$. For a given subspace W of $\mathfrak{g}$, the orthogonal subspace $W^\perp$ is defined as usual by

$$W^\perp = \{x \in \mathfrak{g} \,/\, g(x, y) = 0, \text{ for all } y \in W\}.$$

The subspace W is said to be isotropic if $W \subset W^\perp$ and is called totally isotropic if $W = W^\perp$.

Lemma (2.1) can be rewritten in terms of the pseudo-Riemannian metric g as in the following one.

Lemma 2.2. *Let $(\mathfrak{g}, J, g)$ be Kähler Lie algebra. Assume that an ideal $\mathfrak{h} \subset \mathfrak{g}$ satisfies $J\mathfrak{h} \subset \mathfrak{h}^\perp$. Then*

- *$\mathfrak{h}$ is abelian and*
- *$J(\mathfrak{h})$ is a subalgebra of $\mathfrak{g}$ with $\mathfrak{h} \subset (J\mathfrak{h})^\perp = J(\mathfrak{h}^\perp) := J\mathfrak{h}^\perp$.*

Thus $\mathfrak{h} + J\mathfrak{h}$ is a subalgebra of $\mathfrak{g}$ invariant by J and the sum is not necessarily direct. However $\mathfrak{h} \cap J\mathfrak{h}$ is a J invariant ideal of $\mathfrak{h} + J\mathfrak{h}$.

Proof. The subspace $\mathfrak{h}$ is ω-isotropic if and only if $J\mathfrak{h} \subset \mathfrak{h}^{\perp}$. Hence $\mathfrak{h}$ is ω-lagrangian if and only if $J\mathfrak{h} = \mathfrak{h}^{\perp}$. These remarks prove the assertions. ■

In [12] it is proved that if $\mathfrak{g}$ is a Kähler Lie algebra whose respective metric is positive definite then $\mathfrak{g}$ is isomorphic to $\mathfrak{h} \rtimes J\mathfrak{h}$ when $\mathfrak{g}$ admits an ideal $\mathfrak{h}$ such that $\mathfrak{h}^{\perp} = J\mathfrak{h}$.

2.1. On four dimensional solvable Lie algebras. It is known that a four dimensional symplectic Lie algebra must be solvable [9]. Let us recall the classification of four dimensional solvable real Lie algebras. For a proof see for instance [2]. Notations used along this paper are compatible with the following table.

Proposition 2.3. *Let $\mathfrak{g}$ be a four dimensional solvable real Lie algebra. Then if $\mathfrak{g}$ is not abelian, it is equivalent to one and only one of the Lie algebras listed below:*

$\mathfrak{rh}_3$:	$[e_1, e_2] = e_3$
$\mathfrak{rr}_3$:	$[e_1, e_2] = e_2,\ [e_1, e_3] = e_2 + e_3$
$\mathfrak{rr}_{3,\lambda}$:	$[e_1, e_2] = e_2, [e_1, e_3] = \lambda e_3 \qquad \lambda \in [-1, 1]$
$\mathfrak{rr}'_{3,\gamma}$:	$[e_1, e_2] = \gamma e_2 - e_3, [e_1, e_3] = e_2 + \gamma e_3 \qquad \gamma \geq 0$
$\mathfrak{r}_2\mathfrak{r}_2$:	$[e_1, e_2] = e_2,\ [e_3, e_4] = e_4$
$\mathfrak{r}'_2$:	$[e_1, e_3] = e_3,\ [e_1, e_4] = e_4,\ [e_2, e_3] = e_4,\ [e_2, e_4] = -e_3$
$\mathfrak{n}_4$:	$[e_4, e_1] = e_2,\ [e_4, e_2] = e_3$
$\mathfrak{r}_4$:	$[e_4, e_1] = e_1,\ [e_4, e_2] = e_1 + e_2, [e_4, e_3] = e_2 + e_3$
$\mathfrak{r}_{4,\mu}$:	$[e_4, e_1] = e_1,\ [e_4, e_2] = \mu e_2,\ [e_4, e_3] = e_2 + \mu e_3 \qquad \mu \in \mathbb{R}$
$\mathfrak{r}_{4,\alpha,\beta}$:	$[e_4, e_1] = e_1,\ [e_4, e_2] = \alpha e_2,\ [e_4, e_3] = \beta e_3,$
	with $-1 < \alpha \leq \beta \leq 1,\ \alpha\beta \neq 0$, or $-1 = \alpha \leq \beta \leq 0$
$\mathfrak{r}'_{4,\gamma,\delta}$:	$[e_4, e_1] = e_1,\ [e_4, e_2] = \gamma e_2 - \delta e_3,\ [e_4, e_3] = \delta e_2 + \gamma e_3 \quad \gamma \in \mathbb{R}, \delta > 0$
$\mathfrak{d}_4$:	$[e_1, e_2] = e_3,\ [e_4, e_1] = e_1,\ [e_4, e_2] = -e_2$
$\mathfrak{d}_{4,\lambda}$:	$[e_1, e_2] = e_3,\ [e_4, e_3] = e_3,\ [e_4, e_1] = \lambda e_1,\ [e_4, e_2] = (1 - \lambda) e_2 \quad \lambda \geq \frac{1}{2}$
$\mathfrak{d}'_{4,\delta}$:	$[e_1, e_2] = e_3,\ [e_4, e_1] = \frac{\delta}{2} e_1 - e_2, [e_4, e_3] = \delta e_3,\ [e_4, e_2] = e_1 + \frac{\delta}{2} e_2 \ \ \delta \geq 0$
$\mathfrak{h}_4$	$[e_1, e_2] = e_3,\ [e_4, e_3] = e_3,\ [e_4, e_1] = \frac{1}{2} e_1,\ [e_4, e_2] = e_1 + \frac{1}{2} e_2$

Remark 2.4. Observe that $\mathfrak{r}_2\mathfrak{r}_2$ is the Lie algebra $\mathfrak{aff}(\mathbb{R}) \times \mathfrak{aff}(\mathbb{R})$, where $\mathfrak{aff}(\mathbb{R})$ is the Lie algebra of the Lie group of affine motions of $\mathbb{R}$, $\mathfrak{r}'_2$ is the real Lie algebra underlying on the complex Lie algebra $\mathfrak{aff}(\mathbb{C})$, $\mathfrak{r}'_{3,0}$ is the trivial extension of $\mathfrak{e}(2)$, the Lie algebra of the Lie group of rigid motions of $\mathbb{R}^2$; $\mathfrak{r}_{3,-1}$ is the Lie algebra $\mathfrak{e}(1, 1)$ of the group of rigid motions of the Minkowski 2-space; $\mathfrak{rh}_3$ is the trivial extension of the three-dimensional Heisenberg Lie algebra denoted by $\mathfrak{h}_3$.

A Lie algebra is called *unimodular* if $\mathrm{tr}(\mathrm{ad}_x)=0$ for all $x \in \mathfrak{g}$, where tr denotes the trace of the map. The unimodular four-dimensional solvable Lie algebras algebras are: $\mathbb{R}^4$, $\mathfrak{rh}_3$, $\mathfrak{rr}_{3,-1}$, $\mathfrak{rr}'_{3,0}$, $\mathfrak{n}_4$, $\mathfrak{r}_{4,-1/2}$, $\mathfrak{r}_{4,\mu,-1-\mu}$ $(-1 < \mu \leq -1/2)$, $\mathfrak{r}'_{4,\mu,-\mu/2}$, $\mathfrak{d}_4$, $\mathfrak{d}'_{4,0}$.

Recall that a solvable Lie algebra is *completely solvable* when ad_x has real eigenvalues for all $x \in \mathfrak{g}$.

Invariant complex structures in the four dimensional solvable real case were classified by J. Snow [22] and G. Ovando [20]. The following propositions show all Lie algebras of dimension four admitting special kinds of complex structures, making use of notations in (2.3).

Proposition 2.5. *If $\mathfrak{g}$ is a four dimensional Lie algebra admitting an abelian complex structure, then $\mathfrak{g}$ is isomorphic to one of the following Lie algebras $\mathbb{R}^4$, $\mathbb{R}\times\mathfrak{h}_3$, $\mathbb{R}^2\times\mathfrak{aff}(\mathbb{R})$, $\mathfrak{aff}(\mathbb{R})\times\mathfrak{aff}(\mathbb{R})$, $\mathfrak{aff}(\mathbb{C})$, $\mathfrak{d}_{4,1}$.*

Proof. If $\mathfrak{g}$ is a four dimensional Lie algebra admitting abelian complex structures then $\mathfrak{g}$ must be solvable and its commutator has dimension at most two (see [5]). Let $\mathfrak{g}$ be a four dimensional Lie algebra satisfying these conditions. The first case is the abelian one which clearly possesses an abelian complex structure. If $\dim\mathfrak{g}'=1$ then $\mathfrak{g}$ is isomorphic either to $\mathbb{R}\times\mathfrak{h}_3$ or to $\mathbb{R}\times\mathfrak{aff}(\mathbb{R})$, both admitting abelian complex structures (see [22] or [4]). If the commutator is two dimensional then it must be abelian and therefore $\mathfrak{g}$ must satisfy the following splitting short exact sequence of Lie algebras:

$$0\longrightarrow\mathbb{R}^2\longrightarrow\mathfrak{g}\longrightarrow\mathfrak{h}\longrightarrow 0$$

with $\mathfrak{h}\simeq\mathfrak{aff}(\mathbb{R})$ or $\mathbb{R}^2$. The solvable four dimensional Lie algebras which satify these conditions are: $\mathbb{R}^4$, $\mathfrak{rh}_3$, $\mathfrak{rr}_{3,\lambda}$, $\mathfrak{rr}_3$, $\mathfrak{rr}'_{3,\lambda}$, $\mathfrak{r}_2\mathfrak{r}_2$, $\mathfrak{r}'_2$, $\mathfrak{d}_{4,1}$ (see for example [2]). The Lie algebras $\mathfrak{rr}_3$, $\mathfrak{r}'_{3,\lambda}, \mathfrak{r}_{3,\lambda}$ $\lambda\neq 0$ do not admit abelian complex structures and the other Lie algebras admit such kind of complex structures (see [22]). ∎

Proposition 2.6. *Let $\mathfrak{g}$ be a solvable four dimensional Lie algebra such that $(\mathfrak{g},J)$ is a complex Lie algebra, then $\mathfrak{g}$ is either $\mathbb{R}^4$ or $\mathfrak{aff}(\mathbb{C})=\mathfrak{r}'_2$.*

Proof. Let $(\mathfrak{g},J)$ be Lie algebra with a complex structure J satisfying $J[x,y]=[Jx,y]$ for all $x,y\in\mathfrak{g}$. Then $J\mathfrak{g}'\subset\mathfrak{g}'$ and hence $\dim\mathfrak{g}'=2$ or 4. Assume now that $\mathfrak{g}$ is solvable but not abelian and let x,Jx be a basis of $\mathfrak{g}'$. Let y,Jy not in $\mathfrak{g}'$ such that $\{x,Jx,y,Jy\}$ is a basis of $\mathfrak{g}$. Then $[Jy,y]=0=[x,Jx]$ and the action of y,Jy restricted to $\mathfrak{g}'$ has the form

$$\mathrm{ad}_y=\begin{pmatrix}a & -b\\ b & a\end{pmatrix}\qquad \mathrm{ad}_{Jy}=\begin{pmatrix}b & a\\ -a & b\end{pmatrix}$$

where a and b are real numbers such that $a^2+b^2\neq 0$. This implies that $\mathfrak{g}\simeq\mathfrak{aff}(\mathbb{C})$. In fact taking $y'=\frac{1}{a^2+b^2}(ay+bJy)$ then $\{y',Jy',x,Jx\}$ is a basis of $\mathfrak{g}$ satisfying the Lie bracket relations of $\mathfrak{r}'_2$ in (2.3). ∎

3. Four dimensional Kähler Lie algebras

In this section we determine all four dimensional Kähler Lie algebras and we parametrize their compatible pairs (J,ω).

Most Kähler Lie algebras can be found in a constructive way. In fact, according to [21] any symplectic Lie algebra (g,J,ω) which is either completely solvable or isomorphic to $\mathfrak{aff}(\mathbb{C})$ admits a ω-lagrangian ideal or equivalently in terms of the pseudometric g admits an ideal $\mathfrak{h}$ with $J\mathfrak{h}=\mathfrak{h}^{\perp}$.

In four dimensional Kähler Lie algebras admitting such ideal $\mathfrak{h}$ there are two possibilities for $\mathfrak{h}\cap J\mathfrak{h}$: it is trivial or coincides with $\mathfrak{h}$. If it is trivial then $\mathfrak{g}$ is isomorphic to $\mathfrak{h}\rtimes J\mathfrak{h}$. Hence we have the following splitting short exact sequence of Lie algebras

$$0\longrightarrow\mathfrak{h}\longrightarrow\mathfrak{g}\longrightarrow J\mathfrak{h}\longrightarrow 0. \tag{3}$$

If $\mathfrak{h} \cap J\mathfrak{h}$ is not trivial, then $J\mathfrak{h} = \mathfrak{h}$. So $\mathfrak{g}$ can be decomposed as $\mathfrak{h} \oplus \mathfrak{k}$, where $\mathfrak{h}$ and $\mathfrak{k}$ are J-invariant totally isotropic subspaces and one has the following short exact sequence of Lie algebras, which does not necessarily splits:

$$0 \longrightarrow \mathfrak{h} \longrightarrow \mathfrak{g} \longrightarrow \mathfrak{k} \longrightarrow 0. \tag{4}$$

In every case $\mathfrak{h}$ is abelian (2.1) and therefore will be identified with $\mathbb{R}^2$.

These facts will help us to construct four dimensional Kähler Lie algebras. The results of the following propositions can be verified with Table (3.3).

Proposition 3.1. *Let $(\mathfrak{g}, J, g)$ be a four dimensional Kähler Lie algebra. Assume that there exists an abelian ideal $\mathfrak{h}$ such that the following splitting exact sequence holds*

$$0 \longrightarrow \mathfrak{h} \longrightarrow \mathfrak{g} \longrightarrow J\mathfrak{h} \longrightarrow 0.$$

where the sum is orthogonal. Then $\mathfrak{g}$ is isomorphic to: $\mathbb{R}^4$, $\mathbb{R} \times \mathfrak{h}_3$, $\mathbb{R}^2 \times \mathfrak{aff}(\mathbb{R})$, $\mathfrak{aff}(\mathbb{C})$, $\mathfrak{aff}(\mathbb{R}) \times \mathfrak{aff}(\mathbb{R})$, $\mathfrak{r}_{4,-1,-1}$, $\mathfrak{d}_{4,1}$, $\mathfrak{d}_{4,2}$, $\mathfrak{d}_{4,1/2}$.

Proof. Let J be an almost complex structure on $\mathfrak{g}$ compatible with the pseudo-Riemannian metric g. The splitting short exact sequence (3) is equivalent to one of the following short exact sequences of Lie algebras

$$0 \longrightarrow \mathbb{R}^2 \longrightarrow \mathfrak{g} \longrightarrow \mathbb{R}^2 \longrightarrow 0. \tag{5}$$

$$0 \longrightarrow \mathbb{R}^2 \longrightarrow \mathfrak{g} \longrightarrow \mathfrak{aff}(\mathbb{R}) \longrightarrow 0. \tag{6}$$

The pseudometric g restricted to $\mathfrak{h}$ defines a pseudo-Riemannian metric on the Euclidean two dimensional ideal. Up to equivalence on $\mathbb{R}^2$ there exist two pseudo-Riemannian metrics: the canonical one and the indefinite one of signature (1,1).

Case (5): If $\mathfrak{g}$ is a Lie algebra satisfying the short sequence (5) then the almost complex structure J is integrable if and only if it satisfies

$$[Jx, y] = [Jy, x] \qquad \text{for all } x, y \in \mathfrak{h} \tag{7}$$

and J is parallel with respect to the Levi Civita connection for g if and only if

$$g([Jx, z], y) = g([Jy, z], x) \qquad \text{for all } x, y, z \in \mathfrak{h}. \tag{8}$$

While for the canonical metric the Lie algebras satisfying conditions (7) and (8) are $\mathbb{R}^4$, $\mathbb{R}^2 \times \mathfrak{aff}(\mathbb{R})$, $\mathfrak{aff}(\mathbb{R}) \times \mathfrak{aff}(\mathbb{R})$, for the neutral metric one gets the Lie algebras $\mathbb{R} \times \mathfrak{h}_3$, $\mathbb{R}^2 \times \mathfrak{aff}(\mathbb{R})$,$\mathfrak{aff}(\mathbb{R}) \times \mathfrak{aff}(\mathbb{R})$, $\mathfrak{aff}(\mathbb{C})$ and $\mathfrak{d}_{4,1}$.

Case (6): If $\mathfrak{g}$ is a Lie algebra where (6) holds, then the almost complex structure J is integrable if and only if

$$e_2 = [Je_1, e_2] - [Je_2, e_1] \tag{9}$$

where $\operatorname{span}\{e_1, e_2\} = \mathfrak{h} \simeq \mathbb{R}^2$, and J is parallel with respect to the Levi Civita connection for g if and only if

$$g(Je_2, Je_k) = g([Je_2, e_k], e_1) - g([Je_1, e_k], e_2) \qquad \text{for } k = 1, 2 \tag{10}$$

By considering the conditions (9) and (10) one gets the Lie algebras $\mathfrak{d}_{4,1/2}$, $\mathfrak{d}_{4,2}$ for the canonical metric and the Lie algebras $\mathfrak{r}_{4,-1,-1}$, $\mathfrak{d}_{4,1/2}$, $\mathfrak{d}_{4,2}$ for the neutral metric. ■

Proposition 3.2. *Let $(\mathfrak{g}, J, g)$ be a four dimensional Kähler Lie algebra. Assume that there exists an abelian ideal $\mathfrak{h}$ such that the short exact sequence of Lie algebras (4) holds*

$$0 \longrightarrow \mathfrak{h} \longrightarrow \mathfrak{g} \longrightarrow \mathfrak{k} \longrightarrow 0,$$

where $\mathfrak{h}$ and $\mathfrak{k}$ are J-invariant totally isotropic subspaces. Then $\mathfrak{g}$ is isomorphic to: $\mathbb{R} \times \mathfrak{h}_3$, $\mathfrak{aff}(\mathbb{C})$, $\mathfrak{r}_{4,-1,-1}$, $\mathfrak{d}_{4,1}$, $\mathfrak{d}_{4,2}$.

Proof. At the algebraic level, the sequence (4) takes the form (5) or (6), where $\mathfrak{h} = \text{span}\{e_1, Je_1\} \simeq \mathbb{R}^2$ and $\mathfrak{k} = \text{span}\{e_2, Je_2\} \simeq \mathbb{R}^2$ in (5) or $\mathfrak{aff}(\mathbb{R})$ in (6). Let J be a complex structure on $\mathfrak{g}$ and let ω be a 2-form compatible with J. Then ω being closed is equivalent to:

$$\omega([e_2, Je_2], x) + \omega([x, e_2], Je_2) + \omega([Je_2, x], e_2) = 0.$$

If (4) splits then in the case (5) one gets the Lie algebra $\mathfrak{aff}(\mathbb{C})$, and in the case (6) one gets the Lie algebras $\mathfrak{r}_{4,-1,-1}$, $\mathfrak{d}_{4,1}$ and $\mathfrak{d}_{4,2}$. If (4) does not splits then one gets $\mathbb{R} \times \mathfrak{h}_3$. ■

Notice that according to the classifications of complex structures in [22] [20] and symplectic structures [21] the non completely solvable Lie algebras which could admit compatible pairs (J, ω) are $\mathbb{R} \times \mathfrak{e}(2)$, $\mathfrak{r}'_{4,0,\delta}$, $\delta \neq 0$, and $\mathfrak{d}'_{4,\delta}$ with $\delta \neq 0$. These Lie algebras admit Kähler structures (see Table (3.3)) and moreover the Lie algebras $\mathbb{R} \times \mathfrak{e}(2)$ and $\mathfrak{r}'_{4,0,\delta}$ satisfy the following splitting short exact sequence of Lie algebras

$$0 \longrightarrow \mathfrak{h} = J\mathfrak{h} \longrightarrow \mathfrak{g} \longrightarrow \mathfrak{h}^{\perp} \longrightarrow 0,$$

where $\mathfrak{h}$ is an abelian ideal but not a ω-lagrangian ideal of $(\mathfrak{g}, \omega)$.

Let $\mathfrak{g}$ be a Lie algebra admitting a complex structure J and let us denote by $\mathcal{S}_c(\mathfrak{g}, J)$ the set of all symplectic forms ω that are compatible with J. Our goal now is to parametrize the elements of $\mathcal{S}_c(\mathfrak{g}, J)$. In the previous paragraphs we found the Lie algebras $\mathfrak{g}$ for which $\mathcal{S}_c(\mathfrak{g}, J) \neq \emptyset$ for some complex structure J.

Let $\{e^i\}$ denotes the dual basis on $\mathfrak{g}^*$ of the basis $\{e_i\}$ in $\mathfrak{g}$ (as in (2.3)). Then we use $e^{ijk\cdots}$ to denote $e^i \wedge e^j \wedge e^k \wedge \ldots$.

Proposition 3.3. *Let $\mathfrak{g}$ be a Kähler Lie algebra, then $\mathfrak{g}$ is isomorphic to one of the following Lie algebras endowed with complex and compatible symplectic structures listed in* Table 3.3 *below.*

Proof. The complete proof follows a case by case study. Making use of the classifications of complex structures we found in [22] and [20], then for a fixed complex structure J on a given Lie algebra $\mathfrak{g}$ we verify the compatibility condition with the symplectic forms given in [21].

We shall give the details in the case $\mathfrak{r}'_2$, the Lie algebra corresponding to $\mathfrak{aff}(\mathbb{C})$. The other cases should be handled in a similar way.

$\mathfrak{g}$	Complex structure	Compatible symplectic 2-forms
$\mathfrak{rh}_3$:	$Je_1 = e_2,\ Je_3 = e_4$	$a_{13+24}(e^{13}+e^{24}) + a_{14-23}(e^{14}-e^{23}) + {}$ ${}+a_{12}e^{12},\ a_{13}^2 + a_{14}^2 \neq 0$
$\mathfrak{rr}_{3,0}$:	$Je_1 = e_2,\ Je_3 = e_4$	$a_{12}e^{12} + a_{34}e^{34},\ a_{12}a_{34} \neq 0$
$\mathfrak{rr}'_{3,0}$:	$Je_1 = e_4,\ Je_2 = e_3$	$a_{14}e^{14} + a_{23}e^{23},\ a_{14}a_{23} \neq 0$
$\mathfrak{r}_2\mathfrak{r}_2$:	$Je_1 = e_2,\ Je_3 = e_4$	$a_{12}e^{12} + a_{34}e^{34},\ a_{12}a_{34} \neq 0$
$\mathfrak{r}'_2$:	$J_1e_1 = e_3,\ J_1e_2 = e_4$	$a_{13-24}(e^{13}-e^{24}) + a_{14+23}(e^{14}+e^{23}),$ $a_{13-24}^2 + a_{14+23}^2 \neq 0$
	$J_2e_1 = -e_2,\ J_2e_3 = e_4$	$a_{13-24}(e^{13}-e^{24}) + a_{14+23}(e^{14}+e^{23}) + {}$ ${}+a_{12}e^{12},\ a_{13-24}^2 + a_{14+23}^2 \neq 0$
$\mathfrak{r}_{4,-1,-1}$:	$Je_4 = e_1,\ Je_2 = e_3$	$a_{12+34}(e^{12}+e^{34}) + a_{13-24}(e^{13}-e^{24}) + {}$ ${}+a_{14}e^{14},\ a_{12+34}^2 + a_{13-24}^2 \neq 0$
$\mathfrak{r}'_{4,0,\delta}$:	$J_1e_4 = e_1,\ J_1e_2 = e_3,$ $J_2e_4 = e_1,\ J_2e_2 = -e_3$	$a_{14}e^{14} + a_{23}e^{23},\ a_{14}a_{23} \neq 0$
$\mathfrak{d}_{4,1}$:	$Je_1 = e_4,\ Je_2 = e_3$	$a_{12-34}(e^{12}-e^{34}) + e_{14}e^{14},\ a_{12-34} \neq 0$
$\mathfrak{d}_{4,2}$:	$J_1e_4 = -e_2,\ J_1e_1 = e_3$	$a_{14+23}(e^{14}+e^{23}) + a_{24}e^2 \wedge e^4,\ a_{14+23} \neq 0$
	$J_2e_4 = -2e_1,\ J_2e_2 = e_3$	$a_{14}e^{14} + a_{23}e^{23},\ a_{14}a_{23} \neq 0$
$\mathfrak{d}_{4,1/2}$:	$J_1e_4 = e_3,\ J_1e_1 = e_2$ $J_2e_4 = e_3,\ J_2e_1 = -e_2$	$a_{12-34}(e^{12}-e^{34}),\ a_{12-34} \neq 0$
$\mathfrak{d}'_{4,\delta}$:	$J_1e_4 = e_3,\ J_1e_1 = e_2,$ $J_2e_4 = -e_3,\ J_2e_1 = e_2,$ $J_3e_4 = -e_3,\ J_3e_1 = -e_2,$ $J_4e_4 = e_3,\ J_4e_1 = e_2,$	$a_{12-\delta 34}(e^{12}-\delta e^{34}),\ a_{12-34} \neq 0$

Table 3.3

As we can see in the classification of Snow [22] the complex structures on $\mathfrak{r}'_2$ are given by: $J_1e_1 = e_3$, $J_1e_2 = e_4$; and for the other type of complex structures, denoting $a_1 \in \mathbb{C}$ by $a_1 = \mu + i\nu$, with $\nu \neq 0$; we have $J_{\mu,\nu}e_1 = \frac{\mu}{\nu}e_1 + (\frac{\nu^2+\mu^2}{\nu})e_2$, $J_{\mu,\nu}e_3 = e_4$. On the other hand any sympletic structure has the form: $\omega = a_{12}(e^1 \wedge e^2) + a_{13-24}(e^1 \wedge e^3 - e^2 \wedge e^4) + a_{14+23}(e^1 \wedge e^4 + e^2 \wedge e^3)$, with $a_{14+23}^2 + a_{13-24}^2 \neq 0$. Assuming that there exists a Kähler structure it holds $\omega(JX, JY) = \omega(X, Y)$ for all $X, Y \in \mathfrak{g}$ and this condition produces equations on the coefficients of ω which should be verified in each case.

For J_1 we need to compute only the following:

$$\omega(e_1, e_2) = a_{12} = \omega(e_3, e_4) \qquad \omega(e_1, e_4) = a_{14+23} = \omega(e_3, -e_2)$$

Thus these equalities impose the condition $a_{12} = 0$. And so any Kähler structure concerning J_1 has the form $\omega = a_{13-24}(e^1 \wedge e^3 - e^2 \wedge e^4) + a_{14+23}(e^1 \wedge e^4 + e^2 \wedge e^3)$ with $a_{13-24}^2 + a_{14+23}^2 \neq 0$.

For the second case corresponding to $J_{\mu,\nu}$, by computing $\omega(e_2, e_4)$, $\omega(e_1, e_3)$, we get respectively:

$$\text{i)}\quad (1 + \tfrac{1}{\nu})a_{24-13} \;=\; \tfrac{\mu}{\nu}a_{14+23} \qquad \text{ii)}\quad (1 + \tfrac{\mu^2+\nu^2}{\nu})a_{24-13} \;=\; -\tfrac{\mu}{\nu}a_{14+23}$$

By comparing i) and ii) we get:

$$(1 + \tfrac{1}{\nu})a_{24-13} \;=\; -(1 + \tfrac{\mu^2+\nu^2}{\nu})a_{24-13}$$

and this equality implies either iii)$a_{24-13} = 0$ or iv) $1 + \frac{1}{\nu} + 1 + \frac{\mu^2+\nu^2}{\nu} = 0$. As $a_{24-13} \neq 0$ (since in this case we would also get $a_{14+23} = 0$ and this would be a contradiction) then it must hold iv), that is $1 + \mu^2 + \nu^2 = -2\nu$ and that implies $\mu^2 = -2\nu - 1 - \nu^2 = -(\nu + 1)^2$ and that is possible only if $\mu = 0$ and $\nu = -1$. For this complex structure J, given by $Je_1 = -e_2$ $Je_3 = e_4$, it is not difficult to prove that for any symplectic structure ω it always holds $\omega(JX, JY) = \omega(X, Y)$, that is, any symplectic structure on $\mathfrak{g}$ is compatible with J. In this way we have completed the proof of the assertion. ■

In the following we shall simplify the notation: parameters with four subindices will be denoted only with the first two ones, hence for instance $a_{14+23} \to a_{14}$. By the computations of the pseudo-Kähler metrics the parameters are those satisfying conditions of Table (3.3).

Remark 3.4. The set $\mathcal{S}_c(\mathfrak{g}, J)$ can be parametrized by $\mathbb{R}^*$, $\mathbb{R} \times \mathbb{R}^*$, $\mathbb{R}^* \times \mathbb{R}^*$ or $\mathbb{R}^2 - \{0\}$ and by $\mathbb{R} \times (\mathbb{R}^2 - \{0\})$.

Remark 3.5. The complex structure which endowes the Lie algebra $\mathfrak{r}_2'$ with a complex Lie bracket is given by $Je_1 = e_2$ and $Je_3 = e_4$, which does not admit a compatible symplectic structure. In fact, assume that Ω is a 2-form compatible with J, then $\Omega = \alpha e^{12} + \beta(e^{13} + e^{24}) + \gamma(e^{14} - e^{23})$. Hence $d\Omega = 0$ if and only if $\beta = 0 = \gamma$. Thus there is no symplectic structure compatible with J. In [10] it is proved that any closed 2-form is always degenerate when it is compatible with a complex structure J which gives $\mathfrak{g}$ a structure of complex Lie algebra.

Remark 3.6. Among the four dimensional Lie algebras we find many examples of Lie algebras, such that the set of complex structures $\mathcal{C}$ and the set of symplectic structures $\mathcal{S}$ are both nonempty and however there is no compatible pair (J, ω). This situation occurs for instance on the Lie algebras $\mathfrak{h}_4$ or the family $\mathfrak{d}_{4,\lambda}$ for $\lambda \neq 1/2, 1, 2$ (Compare results in [20] [21] and [22]).

Reading the previous list of Proposition 3.3 by looking at the structure of the Lie algebras we get the following Corollary.

Corollary 3.7. *Let $\mathfrak{g}$ be a Kähler four dimensional Lie algebra. If $\mathfrak{g}$ is unimodular then it is isomorphic either to $\mathbb{R} \times \mathfrak{h}_3$ or $\mathbb{R} \times \mathfrak{e}(2)$.*

If $\mathfrak{g}$ is not unimodular then either:

i) $\dim \mathfrak{g}' = 1$ *and it is isomorphic to* $\mathbb{R}^2 \times \mathfrak{aff}(\mathbb{R})$, *ii)* $\dim \mathfrak{g}' = 2$ *and* $\mathfrak{g}$ *is a non trivial extension of* $\mathfrak{e}(1,1)$, $\mathfrak{aff}(\mathbb{R}) \times \mathfrak{aff}(\mathbb{R})$, *or an extension of* $\mathfrak{e}(2)$ *or*

iii) $\mathfrak{g}' \simeq \mathbb{R}^3$ *and* $\mathfrak{g} \simeq \mathfrak{r}_{4,-1,-1}$ *or* $\mathfrak{r}'_{4,0,\delta}$ *or*

iv) $\mathfrak{g}' \simeq \mathfrak{h}_3$ *and the action of* $e_4 \notin \mathfrak{g}'$ *diagonalizes with set of eigenvalues one of the following ones* $\{1,1,0\}$, $\{1,2,-1\}$, $\{1,\frac{1}{2},\frac{1}{2}\}$, $\{1,\frac{1}{2}+i\delta,\frac{1}{2}-i\delta\}$, *with* $\delta > 0$.

Proof. According to [2], if $\dim \mathfrak{g}' = 1$ then $\mathfrak{g}$ is a trivial extension of $\mathfrak{h}_3$ or $\mathfrak{aff}(\mathbb{R})$; the non trivial extension of $\mathfrak{e}(1,1)$ is $\mathfrak{r}_2\mathfrak{r}_2$ and the extensions of $\mathfrak{e}(2)$ are isomorphic either to $\mathfrak{aff}(\mathbb{C})$ or $\mathbb{R}\times\mathfrak{e}(2)$. The rest of the proof follows by looking at the adjoint actions on any Kähler Lie algebra with three dimensional commutator. ■

Corollary 3.8. *Let* $\mathfrak{g}$ *be a nilpotent (non abelian) four dimensional Kähler Lie algebra, then it is isomorphic to* $\mathbb{R} \times \mathfrak{h}_3$ *and any complex structure is abelian.*

Proof. Among the four dimensional Lie algebras the non abelian nilpotent ones are $\mathbb{R} \times \mathfrak{h}_3$ and $\mathfrak{n}_4$. Only $\mathbb{R} \times \mathfrak{h}_3$ admits a compatible pair (J,ω) and in fact the previous table parametrizes elements of $\mathcal{S}_c(\mathbb{R}\times\mathfrak{h}_3, J)$ for a fixed complex structure J. ■

Remark 3.9. $\mathbb{R} \times \mathfrak{h}_3$ is the Lie algebra underlying the Kodaira Thurston nilmanifold [23] for which actually any complex structure J admits a compatible symplectic form ω.

Corollary 3.10. *Let* $\mathfrak{g}$ *be a four dimensional Lie algebra for which any complex structure gives rise to a Kähler structure on* $\mathfrak{g}$. *Then* $\mathfrak{g}$ *is isomorphic either to* $\mathbb{R} \times \mathfrak{h}_3$, $\mathbb{R}^2 \times \mathfrak{aff}(\mathbb{R})$, $\mathbb{R} \times \mathfrak{e}(2)$, $\mathfrak{r}_{4,-1,-1}$, $\mathfrak{r}'_{4,0,\delta}$, $\mathfrak{d}_{4,1}$ $\mathfrak{d}_{4,2}$.

Corollary 3.11. *Let* $\mathfrak{g}$ *be a four dimensional Lie algebra admitting abelian complex structures. Then* $(\mathfrak{g}, J)$ *is Kähler if and only if* $\mathfrak{g}$ *is symplectic and* J *is abelian.*

Proof. According to (2.5) and the results of [22], the four dimensional Lie algebras which are Kähler and admit abelian complex structures are $\mathbb{R} \times \mathfrak{h}_3$, $\mathbb{R}^2 \times \mathfrak{aff}(\mathbb{R})$, $\mathfrak{aff}(\mathbb{R}) \times \mathfrak{aff}(\mathbb{R})$, $\mathfrak{aff}(\mathbb{C})$ and $\mathfrak{d}_{4,1}$. Among these Lie algebras only $\mathfrak{aff}(\mathbb{C})$ admits complex structures which are not abelian. On $\mathfrak{aff}(\mathbb{C})$ there is a curve of non equivalent complex structures. Among the points of this curve there is one which belongs to the abelian class. The class which represent this point and one class more corresponding to an abelian structure admit a compatible symplectic structure and the complex structure which are not abelian do not admit a compatible symplectic structure. ■

In dimension four a pseudo-Riemannian Kähler metric must be definite or neutral. Notice that the set of pseudo-Riemannian Kähler metrics on each Kähler Lie algebra $(\mathfrak{g}, J, g)$ can be identified with the set $\mathcal{S}_c(\mathfrak{g}, J)$.

We use the following notation to describe the pseudo-Riemannian metrics. If $\{e_i\}$ is the basis of Proposition (2.3) then $\{e^i\}$ is its dual basis on $\mathfrak{g}^*$ and symmetric two tensors are of the form $e^i \cdot e^j$ where $\cdot$ denotes the symmetric product of 1-forms. We denote by z_i the coordinates of $z \in \mathfrak{g}$ with respect to the basis $\{e_i\}$.

Corollary 3.12. *Let $(\mathfrak{g}, J)$ be a non abelian four dimensional Kähler Lie algebra with complex structure J admitting only definite Kähler metrics then $(\mathfrak{g}, J)$ is isomorphic either to the Lie algebra $(\mathfrak{d}_{4,1/2}, J_1)$, or to $(\mathfrak{d}'_{4,\delta}, J_1, J_3)$.*

The Kähler Lie algebras $(\mathbb{R}\times\mathfrak{h}_3, J)$, $(\mathfrak{aff}(\mathbb{C}), J_1, J_2)$, $(\mathfrak{r}_{4,-1,-1}, J)$, $(\mathfrak{d}_{4,1}, J)$ and $(\mathfrak{d}'_{4,\delta}, J_2, J_4)$ admit only neutral pseudo-Riemannian metrics.

Proof. In the case of the completely solvable Kähler Lie algebras or $\mathfrak{aff}(\mathbb{C})$ the assertions follow from the proof of Propositions (3.1) and (3.2). In fact these Kähler Lie algebras can be constructed in terms of splitting exact sequences of Lie algebras, verifying some extra conditions. We need to study the assertions in the cases $\mathbb{R} \times \mathfrak{e}(2)$, $\mathfrak{r}'_{4,0,\delta}$ and $\mathfrak{d}'_{4,\delta}$ with $\delta \neq 0$. Looking at the pseudo-Kähler metrics on $\mathbb{R} \times \mathfrak{e}(2)$, $\mathfrak{r}'_{4,0,\delta}$ (see Propositions (4.4) and (4.9)) it is possible to verify that both cases admit definite and neutral metrics. In the case of $\mathfrak{d}'_{4,\delta}$ the complex structures J_1 and J_3 admit only definite compatible pseudometrics and the complex structures J_2 and J_4 admit only neutral compatible pseudometrics. ∎

The following propositions offer an alternative model for four dimensional Kähler Lie algebras since the existence of a lagrangian ideal is a strong condition. The next constructions are based on the existence of an abelian ideal which does not need to be lagrangian.

Proposition 3.13. *The following Kähler four dimensional Lie algebras: $(\mathbb{R}^2 \times \mathfrak{aff}(\mathbb{R}), J)$, $(\mathbb{R} \times \mathfrak{e}(2), J)$, $(\mathfrak{aff}(\mathbb{R}) \times \mathfrak{aff}(\mathbb{R}), J)$, $(\mathfrak{r}'_{4,0,\delta}, J_1, J_2)$ endowed with a pseudo-Kähler metric, satisfy the following splitting short exact sequence of Lie algebras:*

$$0 \longrightarrow \mathfrak{h} = J\mathfrak{h} \longrightarrow \mathfrak{g} \longrightarrow \mathfrak{h}^{\perp} \longrightarrow 0$$

where the sum is orthogonal.

Proof. For the Lie algebras of the proposition, with a given pseudo-Kähler metric, we exhibit a abelian ideal satisfying $J\mathfrak{h} = \mathfrak{h}$:

$\mathbb{R}^2 \times \mathfrak{aff}(\mathbb{R}), J$	$g = a_{12}(e^1 \cdot e^1 + e^2 \cdot e^2) + a_{34}(e^3 \cdot e^3 + e^4 \cdot e^4)$	$\mathfrak{h} = \mathrm{span}\{e_1, e_2\}$
$\mathbb{R} \times \mathfrak{e}(2), J$	$g = a_{14}(e^1 \cdot e^1 + e^4 \cdot e^4) + a_{23}(e^2 \cdot e^2 + e^3 \cdot e^3)$	$\mathfrak{h} = \mathrm{span}\{e_2, e_3\}$
$\mathfrak{aff}(\mathbb{R})^2, J$	$g = a_{12}(e^1 \cdot e^1 + e^2 \cdot e^2) + a_{34}(e^3 \cdot e^3 + e^4 \cdot e^4)$	$\mathfrak{h} = \mathrm{span}\{e_1, e_2\}$
$\mathfrak{r}'_{4,0,\delta}, J_1, J_2$	$g = a_{14}(e^1 \cdot e^1 + e^4 \cdot e^4) + a_{23}(e^2 \cdot e^2 + e^3 \cdot e^3)$	$\mathfrak{h} = \mathrm{span}\{e_2, e_3\}$

∎

Proposition 3.14. *The following Kähler four dimensional Lie algebras: $(\mathbb{R} \times \mathfrak{h}_3, J)$, $(\mathfrak{r}_{4,-1,-1}, J)$, $(\mathfrak{d}_{4,2}, J_1)$, endowed with a pseudo-Kähler metric, satisfy the following splitting exact sequence of Lie algebras:*

$$0 \longrightarrow \mathfrak{h} = \mathfrak{h}^{\perp} \longrightarrow \mathfrak{g} \longrightarrow J\mathfrak{h} \longrightarrow 0.$$

Proof. For the Lie algebras of the proposition, with a fixed pseudo-Kähler metric g, we exhibit an ideal satisfying $\mathfrak{h} = \mathfrak{h}^\perp$ and $\mathfrak{h} \cap J\mathfrak{h} = 0$:

$$\begin{array}{lll} \mathbb{R} \times \mathfrak{h}_3, J & g = e^1 \cdot e^3 - e^2 \cdot e^4 & \mathfrak{h} = \text{span}\{e_2, e_3\} \\ \mathfrak{r}_{4,-1,-1}, J & g = a_{13}(e^1 \cdot e^2 - e^3 \cdot e^4) & \mathfrak{h} = \text{span}\{e_1, e_3\} \\ \mathfrak{d}_{4,2}, J_1 & g = a_{14}(e^1 \cdot e^2 + e^3 \cdot e^4) & \mathfrak{h} = \text{span}\{e_2, e_3\} \end{array}$$

■

Remark 3.15. The Lie algebras of Proposition (3.14) are those admitting a hypersymplectic structure [1].

4. On the geometry of left invariant pseudo-Kähler metrics in four dimensional Lie algebras

In this section we study the geometry of the Lie group G whose Lie algebra $\mathfrak{g}$ is endowed with a Kähler structure. Because of the left invariant property all results in this sections are presented at the level of the Lie algebra. We make use of the models (3.1) and (3.2) to find totally geodesic submanifolds. We determine Ricci flat and Einstein Kähler metrics. In the definite case Ricci flat metrics are flat [3]. In the non definite case this is not true in general. However in dimension four if $\mathfrak{g}$ is unimodular and the Kähler metric is Ricci flat, then it is flat. Some proofs in this section follow a case by case study. In those situations, we shall explain the computations and give them just only in one case to exemplify the work should be done.[2]

Let ∇ be the Levi Civita connection corresponding to the pseudo-Riemannian metric g. This is determined by the Koszul formula

$$2g(\nabla_x y, z) = g([x, y], z) - g([y, z], x) + g([z, x], y)$$

It is known that the completeness of the left invariant connection ∇ on G can be studied by considering the corresponding connection on the Lie algebra $\mathfrak{g}$. Indeed the connection ∇ on G will be (geodesically) complete if and only if the differential equation on $\mathfrak{g}$

$$\dot{x}(t) = -\nabla_{x(t)} x(t)$$

admits solutions $x(t) \subset \mathfrak{g}$ defined for all $t \in \mathbb{R}$ (see for instance [14]).

A submanifold N on a Riemannian manifold (M, g) is totally geodesic if $\nabla_x y \in TN$ for $x, y \in TN$. At the level of the Lie algebra we have totally geodesic subspaces, subalgebras, etc. which are in correspondence with totally geodesic submanifolds, subgroups, etc on the corresponding Lie group G with left invariant pseudometric g.

Proposition 4.1. *Let $(\mathfrak{g}, J, g)$ be a Kähler Lie algebra and assume that $\mathfrak{h}$ is an ideal satisfying $J\mathfrak{h} = \mathfrak{h}^\perp$ and $\mathfrak{h} \cap J\mathfrak{h} = 0$ (that is $\mathfrak{h}$ is ω-lagrangian as in (3.1)) then for $x, y \in \mathfrak{h}$ it holds*

$$\nabla_x y \in J\mathfrak{h}; \qquad \nabla_{Jx} Jy \in J\mathfrak{h}; \qquad \nabla_x Jy \in \mathfrak{h}; \qquad \nabla_{Jx} y \in \mathfrak{h}.$$

Thus the subgroup corresponding to $J\mathfrak{h}$ on the Lie group G is totally geodesic.

[2] More details can be found in a previous version of this paper available in arxiv.

Proposition 4.2. *Let $(\mathfrak{g}, J, g)$ be a Kähler Lie algebra and assume that $\mathfrak{h}$ is a abelian ideal satisfying $J\mathfrak{h} = \mathfrak{h} = \mathfrak{h}^{\perp}$. Thus $\mathfrak{g} = \mathfrak{h} \rtimes \mathfrak{k}$ with $J\mathfrak{k} \subset \mathfrak{k}$. Then it holds:*

- $\nabla_z y \in \mathfrak{h}$ *for all* $x, y \in \mathfrak{h}$, *and* $z \in \mathfrak{g}$;
- *for* $z \in \mathfrak{k}$ *it holds* $g([x, y], z) = 2g(\nabla_x y, z) = 2g(\nabla_x z, y) = -2g(\nabla_z x, y)$;

Therefore the normal subgroup H corresponding to the ideal $\mathfrak{h}$ on the Lie group G is totally geodesic.

The proofs of the previous two propositions follow from the Koszul formula for the Levi Civita connection and the features announced in Propositions (3.1) and (3.2).

Recall that a pseudometric on a Lie algebra $\mathfrak{g}$ is called a *Walker* metric if there exists a null and parallel subspace $W \subset \mathfrak{g}$, i.e. there is a subspace W satisfying $g(W, W) = 0$ and $\nabla_y W \subset W$ for all y (see [24]). The previous proposition show examples of Walker metrics in dimension four (compare with [18]).

Corollary 4.3. *The neutral metrics on the Kähler Lie algebras of Proposition (3.2) are Walker.*

The curvature tensor $R(x, y)$ and the Ricci tensor $ric(x, y)$ are respectively defined by:

$$R(x, y) = [\nabla_x, \nabla_y] - \nabla_{[x,y]} \qquad ric(x, y) = -\sum_i \varepsilon_i g(R(x, v_i)y, v_i)$$

where $\{v_i\}$ is a frame field on $\mathfrak{g}$ and ε_i equals $g(v_i, v_i)$. The left invariant property allows to speak in the following setting. We say that the metric is flat if $R \equiv 0$ and Ricci flat if $ric \equiv 0$.

It is clear that the existence of flat or non flat pseudo-Kähler metrics is a property which is invariant under complex isomorphisms, i.e. if J and J' are equivalent complex structures then there exists a flat (resp. non flat) pseudo-Kähler metric for J if and only there exists such a metric for J'.

Theorem 4.4. *Let $\mathfrak{g}$ be a unimodular four dimensional Kähler Lie algebra with pseudo-Kähler metric g. Then g is flat and its Levi Civita connection is complete.*

Proof. Among the Kähler Lie algebras of (3.3) the unimodular ones are $\mathbb{R} \times \mathfrak{h}_3$ and $\mathbb{R} \times \mathfrak{e}(2)$.

Let us work in detail the case $\mathbb{R} \times \mathfrak{e}(2)$ (the other one can be done in a similar way).

Any pseudo-Kähler metric is $g = a_{14}(e^1 \cdot e^1 - e^4 \cdot e^4) + a_{23}(e^2 \cdot e^2 + e^3 \cdot e^3)$ and the corresponding Levi Civita connection is

$$\nabla_z y = z_1 y_3 e_2 - z_1 y_2 e_3.$$

Then g is complete; in fact looking at the geodesic equation in coordinates we have:

$$x_1' = 0, \quad x_2' = x_1 x_3, \quad x_3' = -x_1 x_2, \quad x_4' = 0$$

whose solution for a given initial condition is defined in $\mathbb{R}$. It also holds $\nabla_{[x,y]} \equiv 0$ and since $\nabla_x \nabla_y = \nabla_y \nabla_x$, the curvature tensor vanishes which implies that g is flat. ∎

In the non definite case Ricci flat metrics do not need to be flat. Known counterexamples for this in the case of neutral metrics are provided by hypersymplectic structures [15] [1].

Hypersymplectic four dimensional Lie algebras were classified in [1]. Aside from the abelian Lie algebra there are only three Lie algebras which admit a hypersymplectic structure: $\mathbb{R} \times \mathfrak{h}_3$, $\mathfrak{r}_{4,-1,-1}$ and $\mathfrak{d}_{4,2}$. In theorem (4.4) the Lie algebra $\mathbb{R} \times \mathfrak{e}(2)$ is flat and do not admit hypersymplectic structures [1]. In the following theorem we shall complete the list of Kähler Lie algebras $(\mathfrak{g}, J, g)$ whose pseudometric is Ricci flat.

Remark 4.5. It is known that for a given complex product structure on a four dimensional Lie algebra there is only one compatible metric, up to a non zero constant (see for instance [1]).

Theorem 4.6. *Let $(\mathfrak{g}, J)$ be a non unimodular four dimensional Kähler Lie algebra with Kähler metric g which is Ricci flat. Then $(\mathfrak{g}, J)$ is isomorphic either to $(\mathfrak{r}_{4,-1,-1}, J)$, $(\mathfrak{d}_{4,2}, J_2)$, $(\mathfrak{aff}(\mathbb{C}), J_1)$. Moreover these Lie algebras have flat metrics and also Ricci flat but non flat metrics.*

Proof. For each one of these Lie algebras the pseudo-Kähler metrics can be obtain from Table (3.3). With these we do the computations to prove the assertions. We shall show the Levi Civita connection, curvature and Ricci curvature tensors only on $\mathfrak{aff}(\mathbb{C})$.

$\mathfrak{aff}(\mathbb{C})$:
$J_2 e_2 = e_1,\ J_2 e_3 = e_4$

$$\begin{pmatrix} -s & 0 & a_{14} & -a_{13} \\ 0 & -s & -a_{13} & -a_{14} \\ a_{14} & -a_{13} & 0 & 0 \\ -a_{13} & -a_{14} & 0 & 0 \end{pmatrix}$$

$$\begin{aligned} \nabla_Z Y &= (-z_1 y_1 + z_2 y_2) e_1 - (z_2 y_1 + z_1 y_2) e_2 \\ &\quad + (\tfrac{s}{\varepsilon}\alpha y_1 + \tfrac{s}{\varepsilon}\beta y_2 + z_1 y_3 - z_2 y_4) e_3 + \\ &\quad + (\tfrac{s}{\varepsilon}\beta y_1 - \tfrac{s}{\varepsilon}\alpha y_2 + z_2 y_3 + z_1 y_4) e_4 \\ \varepsilon &= a_{13}^2 + a_{14}^2 \\ \alpha &= -a_{14} z_1 + a_{13} z_2 \\ \beta &= a_{13} z_1 + a_{14} z_2 \\ R(X,Y)Z &= 2\tfrac{(x_1 y_2 - x_2 y_1)}{\varepsilon}[(a_{13} z_1 + \\ &\qquad a_{14} z_2) e_3 + (a_{14} z_1 - a_{13} z_2) e_4] \\ ric(X,Y) &= 0 \\ g(R(v,w)w,v) &= -s(v_1 w_2 - v_2 w_1)^2 \end{aligned}$$

The other Kähler Lie algebras do not admit Ricci flat metrics (see results of Proposition (4.8) and Theorem (4.9)).

Notice that in all cases the commutator is a totally geodesic submanifold. Moreover in $\mathfrak{aff}(\mathbb{C})$ we have $\nabla_{\mathfrak{g}'}\mathfrak{g}' = 0$, and in the other cases $\nabla_{\mathfrak{g}'}\mathfrak{g}' \subset \text{span}\{e_3\}$ for any s. If $s = 0$ then in $\mathfrak{r}_{4,-1,-1}$ we get that the Levi Civita connection restricted to the commutator is always zero. ∎

Remark 4.7. Among these Ricci flat metrics there are examples of complete and non complete metrics [1].

An *Einstein metric* g is proportional to its corresponding Ricci tensor, i.e. $g(x,y) = \nu\, ric(x,y)$ for all $x, y \in \mathfrak{g}$ and ν be a real constant. We shall determine Einstein Kähler metrics in the four dimensional case.

Proposition 4.8. *Let* $(\mathfrak{g}, J, g)$ *be a Kähler Lie algebra with Einstein metric* g. *Then if* g *is non Ricci flat,* g *is a Kähler metric corresponding to one of the following Lie algebras:*

$$\begin{array}{lll}
\mathfrak{aff}(\mathbb{R}) \times \mathfrak{aff}(\mathbb{R}) & J & g = \alpha(e^1.e^1 + e^2.e^2 + e^3.e^3 + e^4.e^4) \\
\mathfrak{aff}(\mathbb{C}) & J_1 & g = \alpha(e^1.e^1 - e^2.e^2 + e^3.e^3 - e^4 e^4) \\
\mathfrak{d}_{4,1/2} & \begin{array}{l} J_1, \\ J_2 \end{array} & \begin{array}{l} g = \alpha(e^1.e^1 + e^2.e^2 + e^3.e^3 + e^4.e^4) \\ g = \alpha(e^1.e^1 + e^2.e^2 - e^3.e^3 - e^4.e^4) \end{array} \\
\mathfrak{d}'_{4,\delta} & \begin{array}{l} J_1, J_3 \\ J_2, J_4 \end{array} & \begin{array}{l} g = \alpha(e^1.e^1 + e^2.e^2 + \delta(e^3.e^3 + e^4.e^4)) \\ g = \alpha(e^1.e^1 + e^2.e^2 - \delta(e^3.e^3 + e^4.e^4)) \end{array}
\end{array}$$

In all cases $\alpha \neq 0$.

Proof. Let us work with the following Lie algebra and give the details in this case. The other cases should be handled in a similar way. One must write the pseudo-Kähler metric, compute the Levi Civita connection, the curvature and Ricci curvature tensors and finally one compares the Ricci tensor with the metric tensor.

$$\begin{array}{ll}
\mathfrak{aff}(\mathbb{C}): & \nabla_Z Y = (z_3 y_3 - z_4 y_4)e_1 + (z_4 y_3 + z_3 y_4)e_2 - \\
J_1 e_1 = e_3,\ J_1 e_2 = e_4 & \quad -(z_3 y_1 - z_4 y_2)e_3 - (z_4 y_1 + z_3 y_2)e_4 \\
 & R(X,Y) = -\nabla_{[X,Y]} \\
\begin{pmatrix} a_{13} & a_{14} & 0 & 0 \\ a_{14} & -a_{13} & 0 & 0 \\ 0 & 0 & a_{13} & a_{14} \\ 0 & 0 & a_{14} & -a_{13} \end{pmatrix} & \begin{array}{l} ric(X,Y) = 2(-x_1 y_1 + x_2 y_2 - x_3 y_3 + x_4 y_4) \\ g(R(v,w)w,v) = -a_{13}(\alpha^2 - \beta^2) - 2a_{14}\alpha\beta \\ \alpha = v_1 w_3 - v_3 w_1 + v_4 w_2 - v_2 w_4 \\ \beta = w_4 v_1 - v_4 w_1 + v_2 w_3 - v_3 w_2 \end{array}
\end{array}$$

Therefore when $a_{14} = 0$ and $a_{13} \neq 0$ the corresponding metric is Einstein.

The proof will be completed with the results of the Theorem (4.9), proving that there are no more Einstein metrics. ∎

We shall finish this geometric study with the characterization of four dimensional Kähler Lie algebras which are not Einstein.

Theorem 4.9. *Let* $(\mathfrak{g}, J, g)$ *be a Kähler Lie algebra. If* $\mathfrak{g}$ *does not admit an Einstein Kähler metric then* $\mathfrak{g}$ *is isomorphic to* $\mathbb{R}^2 \times \mathfrak{aff}(\mathbb{R})$, $\mathfrak{r}'_{4,0,\delta}$, $\mathfrak{d}_{4,1}$.

Proof. The previous propositions show all examples of Lie algebras admitting Einstein Kähler pseudometrics. The Lie algebras $\mathbb{R}^2 \times \mathfrak{aff}(\mathbb{R})$, $\mathfrak{r}'_{4,0,\delta}$, $\mathfrak{d}_{4,1}$ do not admit Einstein Kähler metrics and this follows from a case by case study. We shall show computations only in case $\mathfrak{d}_{4,1}$.

$$\begin{array}{ll}
\mathfrak{d}_{4,1}: \nabla_Z Y = -z_1 y_4 e_1 - (z_3 y_1 + z_1 y_3)e_2 + & \\
\qquad\qquad +(z_1 y_2 - z_3 y_4)e_3 + z_1 y_1 e_4 & J e_1 = e_4,\ J e_2 = e_3 \\
\begin{array}{l} R(X,Y) = -\nabla_{[X,Y]} \\ ric(X,Y) = -2(x_1 y_1 + x_4 y_4) \\ g(R(v,w)w,v) = -\alpha(a_{14}\alpha - 2\beta a_{12}) \\ \alpha = v_4 w_1 - v_1 w_4 \\ \beta = v_1 w_2 - w_1 v_2 + v_4 w_3 - w_4 v_3 \end{array} & \begin{pmatrix} a_{14} & 0 & -a_{12} & 0 \\ 0 & 0 & 0 & a_{12} \\ -a_{12} & 0 & 0 & 0 \\ 0 & a_{12} & 0 & a_{14} \end{pmatrix}
\end{array}$$

Finally let us remark that the Lie algebra $\mathfrak{d}_{4,2}$ admits two non equivalent complex structures, one of them admits a compatible Einstein pseudometric. But for the other one J_2 this is not the case as similar computations as done above show. ■

Corollary 4.10. *Let $(\mathfrak{g}, J)$ be a four dimensional Kähler Lie algebra. Then the commutator is totally geodesic.*

Proof. It follows from the Levi Civita connection computed at the corresponding elements in the commutator. ■

5. A picture in global coordinates

In this section we shall write the pseudo-Kähler metrics in global complex coordinates (the real expression can also be done with the information we present in the following paragraphs). The following table summarizes the results. In the first column we write the corresponding Lie algebra, the invariant complex structure and the homogeneous complex manifold according to [22] and [20]. In the second column we present left invariant 1-forms and the metric in terms of complex coordinates.

$\mathbb{R}\times\mathfrak{h}_3$ $Jv_1 = v_2,\ Jv_3 = v_4$ $\mathbb{C}^2$	$v^1 = dx,\ v^2 = dy,\ v^3 = dz + \frac{y}{2}dx - \frac{x}{2}dy,\ v^4 = dt$ with $u = v_1 + iv_2,\ w = v_3 + iv_4$ $g = a_{12}dud\overline{u} + (a_{14} - ia_{13})dud\overline{w} + (a_{14} + ia_{13})d\overline{u}dw$ flat (4.4)
$\mathbb{R}^2\times\mathfrak{aff}(\mathbb{R})$ $Jv_1 = v_2,\ Jv_3 = v_4$ $\mathbb{C}\times\mathbb{H}$	$v^1 = dt,\ v^2 = e^{-t}dx,\ v^3 = dy,\ v^4 = dz$ with $u = v_1 + iv_2,\ w = v_3 + iv_4$ $g = a_{12}dud\overline{u} + a_{34}dwd\overline{w}$
$\mathbb{R}\times\mathfrak{e}(2)$ $Jv_1 = v_4,\ Jv_2 = v_3$ $\mathbb{C}^2$	$v^1 = dt,\ v^2 = \cos t dx + \sin t dy,\ v^3 = \sin t dx+$ $+\cos t dy,\ v^4 = dz$ with $u = v_1 + iv_4,\ w = v_2 + iv_4$ $g = a_{14}dud\overline{u} + a_{23}dwd\overline{w}$
$\mathfrak{aff}(\mathbb{R})\times\mathfrak{aff}(\mathbb{R})$ $Jv_1 = v_2,\ Jv_3 = v_4$ $\mathbb{H}\times\mathbb{H}$	$v^1 = dx,\ v^2 = e^{-x}dy,\ v^3 = dz,\ v^4 = e^{-z}dt$ with $u = v_1 + iv_2,\ w = v_3 + iv_4$ $g = a_{12}dud\overline{u} + a_{34}dwd\overline{w}$ Einstein if $a_{12} = a_{34} \neq 0$ (4.8)
$\mathfrak{aff}(\mathbb{C})$ $J_1v_1 = v_3,\ J_1v_2 = v_4$ $\mathbb{C}^2$	$v^1 = dt,\ v^3 = e^{-t}(\cos z\, dx + \sin z\, dy),$ $v^2 = dz,\ v^4 = e^{-t}(-\sin z\, dx + \cos z\, dy),$ with $u = v_1 + iv_3,\ w = v_2 + iv_4$ $g = a_{13}(du^2 + d\overline{u}^2 + dw^2 + d\overline{w}^2)+$ $+a_{14}i(du^2 - d\overline{u}^2 + dw^2 - d\overline{w}^2)$ Einstein if $a_{14} = 0$ (4.8)

$J_2v_1 = -v_2,\ J_2v_3 = v_4$ $\mathbb{C}^2$	with $u = v_1 + iv_2,\ w = v_3 + iv_4$ $g = sdud\overline{u} + a_{14}(dudw + d\overline{u}d\overline{w}) -$ $-ia_{13}(dudw - d\overline{u}d\overline{w})$ Ricci flat always & flat if $s = 0$ (4.6)
$\mathfrak{r}_{4,-1,-1}$ $Jv_4 = v_1,\ Jv_2 = v_3$ $\mathbb{C}\times\mathbb{H}$	$v^1 = e^{-t}dx,\ v^2 = e^t dy,\ v^3 = e^t dz,\ v^4 = dt$ with $u = v_4 + iv_1,\ w = v_2 + iv_3$ $g = -sdud\overline{u} - (a_{12} + ia_{13})dud\overline{w} - (a_{12} -$ $-ia_{13})d\overline{u}dw$, Ricci flat always & flat if $s = 0$ (4.6)
$\mathfrak{r}'_{4,0,\delta}$ $J_1v_4 = v_1,\ J_1v_2 = v_3$ $\mathbb{C}\times\mathbb{H}$	$v^1 = e^{-t}dx,\ v^2 = (\cos t dy + \sin t dz),\ v^4 = dt$ $v^3 = (-\sin t dy + \cos t dz)$, with $u = v_4 + iv_1$, $w = v_2 + iv_3 g_1 = -a_{14}dud\overline{u} + a_{23}dwd\overline{w}$
$J_2v_4 = v_1,\ J_2v_2 = -v_3$ $\mathbb{C}\times\mathbb{H}$	with $u = v_4 + iv_1,\ w = v_2 + iv_3$ $g_2 = a_{14}dud\overline{u} + a_{23}dwd\overline{w}$
$\mathfrak{d}_{4,1}$ $Jv_1 = v_4,\ Jv_2 = v_3$ $\mathbb{C}\times\mathbb{H}$	$v^1 = e^{-t}dx,\ v^2 = dy,\ v^3 = e^{-t}dz - \frac{x}{2}e^{-t}dy,\ v^4 = dt$ with $u = v_1 + iv_4,\ w = v_2 + iv_3$ $g_1 = a_{14}dud\overline{u} - ia_{12}(dud\overline{w} - d\overline{u}dw)$
$\mathfrak{d}_{4,1/2}$ $J_1v_1 = v_2,\ J_1v_2 = v_3$ $\mathbb{D}^2$	$v^1 = e^{-t/2}dx,\ v^2 = e^{-t/2}dy,\ v^3 = e^{-t}dz - \frac{x}{2}e^{-t}dy$, $v^4 = dt$ with $u = v_1 + iv_2,\ w = v_4 + iv_3$ $g_1 = a_{12}(dud\overline{u} + dwd\overline{w})$, Einstein (4.8)
$J_2v_1 = -v_2,\ J_2v_2 = v_3$ $(\mathbb{D}^{2^c})^0$	with $u = v_1 + iv_2,\ w = v_4 + iv_3$ $g_1 = a_{12}(-dud\overline{u} + dwd\overline{w})$, Einstein (4.8)
$\mathfrak{d}_{4,2}$ $J_1v_2 = v_4,\ J_1v_1 = v_3$ $\mathbb{C}\times\mathbb{H}$	$v^1 = e^{-2t}dx,\ v^2 = e^t dy,\ v^3 = e^{-t}dz - \frac{x}{2}e^{-t}dy$, $v^4 = dt$ with $u = v_2 + iv_4,\ w = v_1 + iv_3$ $g_1 = sdud\overline{u} + a_{14}(dud\overline{w} + d\overline{u}dw)$ Ricci flat always & flat if $s = 0$ (4.6)
$J_2v_1 = 1/2v_4,\ J_2v_2 = v_3$ $\mathbb{C}\times\mathbb{H}$	with $u = \sqrt{2}/2v_1 + i\sqrt{2}v_4,\ w = v_2 + iv_3$ $g_1 = a_{14}dud\overline{u} + a_{23}dwd\overline{w}$
$\mathfrak{d}'_{4,\delta}$ $J_1v_1 = v_2,\ J_1v_4 = v_3$ $\mathbb{D}^2$	$v^1 = e^{-\delta t/2}(\cos t dx - \sin t dy),\ v^4 = dt$ $v^2 = e^{-\delta t/2}(\sin t dx + \cos t dy)$, $v^3 = e^{-t}dz + xe^{-t\delta/2}(\sin 2t dx - \cos 2t dy)$, with $u = v_1 + iv_2,\ w = v_4 + iv_3$, $g_1 = a_{12}(dud\overline{u} + \delta dwd\overline{w})$, Einstein (4.8)
$J_2v_1 = v_2,\ J_2v_4 = -v_3$ $(\mathbb{D}^{2^c})^0$	with $u = v_1 + iv_2,\ w = v_4 + iv_3$ $g_2 = a_{12}(dud\overline{u} - \delta dwd\overline{w})$

6. Some generalizations

In this section we show some geometric features in higher dimensions. On the one hand we shall give constructions of Kähler Lie algebras in higher dimensions. We obtain examples of Ricci flat metrics by generalizing a Kähler structure on $\mathfrak{aff}(\mathbb{C})$. On the other hand we shall compare hypersymplectic and Walker metrics.

Let A be an associative Lie algebra. Then $\mathfrak{aff}(A)$ is the Lie algebra $A \oplus A$ with Lie bracket given by:

$$[(a,b)(c,d)] = (ac - ca, ad - cb)$$

An almost complex structure on $\mathfrak{aff}(A)$ is defined by $K(a,b) = (b, -a)$ which is integrable and parallel for the torsion free connection $\nabla_{(a,b)}(c,d) = (ac, ad)$.

Affine Lie algebras play an important role in the characterization of the solvable Lie algebras admitting an abelian complex structure [5]. In dimension four the list of affine Lie algebras consists of $\mathbb{R} \times \mathfrak{h}_3$, $\mathbb{R}^2 \times \mathfrak{aff}(\mathbb{R})$, $\mathfrak{aff}(\mathbb{R}) \times \mathfrak{aff}(\mathbb{R})$, $\mathfrak{d}_{4,1}$ and $\mathfrak{aff}(\mathbb{C})$.

Assume that A is commutative and that e_i i= 1... n is a basis of A. Let $v_i = (e_i, 0)$ and $w_i = (0, e_i)$ be a basis of $\mathfrak{aff}(A)$. Consider the dual basis v^i w^i of $\mathfrak{aff}(A)^*$ and define a non degenerate two form by $\omega = \sum v^i \wedge w^i$. Indeed ω is K invariant and by computing $d\omega$ one gets that it is closed.

Proposition 6.1. *The Lie algebras $\mathfrak{aff}(A)$ carry a Kähler structure for any commutative algebra A.*

This Kähler structure does not necessarily induces a Ricci flat metric. See for example $\mathbb{R}^2 \times \mathfrak{aff}(\mathbb{R})$.

Assume now that A is a commutative complex algebra and consider J to be the almost complex structure on $\mathfrak{aff}(A)$ given by $J(a,b) = (-ia, ib)$. Let ∇ be the connection on $\mathfrak{aff}(A)$ given by

$$\nabla_{(a,b)}(c,d) = (-ac, ad).$$

Then since A is commutative ∇ is torsion free. Furthermore the connection is flat. Indeed

$$R((a,b),(c,d)) = \nabla_{[(a,b),(c,d)]} = 0$$

and J is parallel, that is $\nabla J = 0$. We shall prove that ∇ is a metric connection.

Take coordinates u^i on $A \oplus 0$ and w^i on $0 \oplus A$. Let g be the (pseudo-) metric on $\mathfrak{aff}(A)$ defined by:

$$g((a,b),(c,d)) = \sum_i (du^i dw^i + d\overline{u}^i d\overline{w}^i) = \sum_i Re\,(ad + bc)^i$$

then ∇ is the Levi Civita connection of g. It is easy to verify that $\nabla_{(a,b)}$ is skew symmetric with respect to g.

Proposition 6.2. *The Lie algebras $\mathfrak{aff}(A)$ are endowed with a neutral Ricci flat Kähler metric for a commutative complex algebra A.*

For a curve $(a(t), b(t))$ on $\mathfrak{aff}(A)$ the geodesic equation related to the previous pseudo-Kähler metric: $-\nabla_{(a,b)}(a,b) = (a', b')$ gives rise the following equations $a' = a^2$, $b' = -ab$, which have the non trivial solutions $a = (\kappa_1 - t)^{-1}$, $b = \kappa_2(t - \kappa_1)$ for κ_1, κ_2 constants, showing that the metric is not complete except when $a = 0$ and $b = \kappa$ is also constant.

A Walker metric g on a Lie algebra $\mathfrak{g}$ is characterized by the existence of a subspace $W \subset \mathfrak{g}$ satisfying: $(*)$ $g(W,W) = 0$ and $\nabla_y W \subset W$ for all $y \in \mathfrak{g}$ where ∇ denotes the Levi Civita connection for g.

Since $g([x,y],z) = g(\nabla_x y, z) + g(\nabla_y z, x) = 0$ for all $x, y, z \in \mathfrak{g}$ then $W \subset W^{\perp}$. Thus if the dimension of W is a half of the dimension of W then W must be a subalgebra.

A hypersymplectic metric on a Lie algebra $\mathfrak{g}$ is an example of a Walker metric (see section 4). The following result explains how to construct hypersymmplectic metrics from Walker Kähler metrics. The proof follows from the previous observation and features of hypersymplectic Lie algebras (see [1] for instance).

Proposition 6.3. *Let g be a Walker Kähler metric on a Lie algebra $\mathfrak{g}$ for which W satisfies conditions $(*)$ and assume that $\mathfrak{g} = W \oplus JW$. Then g is a hypersymplectic metric on $\mathfrak{g}$.*

The condition $W \oplus JW$ is necesary as proved by $(\mathfrak{aff}(\mathbb{C}), J_2, g_2)$. In fact g_2 is a Walker Kähler metric but not hypersymplectic. The condition for g of being Kähler is necessary as we see in the following example.

Example 6.4. Consider on $\mathfrak{aff}(\mathbb{C})$ the complex structure given by $J(a,b) = (ia, ib)$ and consider the metric g defined by $g((a,b),(c,d)) = Re(a\overline{d} + b\overline{c})$. Then g is compatible with J and the Levi Civita connection for g is

$$\nabla_{(a,b)}(c,d) = (-\frac{1}{2}(a\overline{c} + c\overline{a}), a\frac{(d+\overline{d})}{2} + c\frac{(\overline{b} - b)}{2})$$

It is proved that J is not parallel (see (3.5)), hence this metric is not pseudo-Kähler. However the metric is Walker (compare with [18])

References

[1] Andrada, A., *Hypersymplectic four dimensional Lie algebras*, available at arXiv: math.DG/0310462.

[2] Andrada, A., M. L. Barberis, I. Dotti, and G. Ovando, *Product structures on four dimensional solvable Lie algebras*, Homol. Homot. and Appl. **7** (2005), 9–37.

[3] Alekseevskii, D. V., and B. N. Kimel'fel'd, *Structure of homogenous Riemannian spaces with zero Ricci curvature*, Funktional Anal. i Prolozen **9** (1975), 97–102.

[4] Barberis, M. L., and I. Dotti, *Hypercomplex structures on a class of solvable Lie groups*, Quart. J. Math. Oxford, **47** (1996), 389–404.

[5] —, *Abelian complex structures on solvable Lie algebras*, J. of Lie Theory, **14**, (2004), 25–34.

[6] Benson, C., and C. Gordon, *Kähler and symplectic structures on nilmanifolds*, Proc. Amer. Math. Soc. **108** (1990), 971–980.

[7] —, *Kähler structures on compact solvmanifolds*, Topology **27** (1988), 513–518.

[8] Boyom, N., *Models for solvable symplectic Lie groups*, Indiana Univ. J. **42** (1993), 1149–1167.

[9] Chu, B. Y., *Symplectic homogeneous spaces*, Trans. Amer. Math. Soc. **197** (1974), 145–159.

[10] Cordero, L., M. Fernández, and L. Ugarte, *Lefschetz complex conditions for complex manifolds*, Ann. Glob. Anal. Geom. **22** (2002), 355–373.

[11] —, *Pseudo-Kähler metrics on six dimensional nilpotent Lie algebras*, J. of Geom. and Phys. **50** (2004), 115–137.

[12] Dardié, J. M., and A. Medina, *Algèbres de Lie kählériennes et double extension*, J. Algebra **185** (1996), 774–795.

[13] Dorfmeister, J., and K. Nakajima, *The fundamental conjecture for homogeneous Kähler manifolds*, Acta Math. **161**, (1986), 189–208.

[14] Guediri, M., *Sur la complétude des pseudo-métriques invariantes à gauche sur les groupes de Lie nilpotentes*, Rend. Sem. Mat. Univ. Pol. Torino **52** (1994), 371–376.

[15] Kamada, H., *Neutral hyper-Kähler structures on primary Kodaira surfaces*, Tsukuba J. Math. **23** (1999), 321–332.

[16] Kobayashi, S., and K. Nomizu, "Foundations of Differential Geometry," Interscience Publishers, New York, 1969.

[17] Lichnerowicz, A., and A. Medina, *On Lie groups with left invariant symplectic or Kählerian structures*, Lett. Math. Phys. **16** (1988), 225–235.

[18] Matsushita, Y., *Four dimensional Walker metrics and symplectic structures*, J. of Geom. and Phys. **52** (2004), 89–99.

[19] Oprea, J., and A. Tralle, "Symplectic manifolds with no Kähler structure," Lecture Notes in Math. **1661**, Springer-Verlag, 1997.

[20] Ovando, G., *Invariant complex structures on solvable real Lie groups*, Man. Math. **103** (2000), 19–30.

[21] —, *Four dimensional symplectic Lie algebras*, available at arXiv: math.DG/0407501.

[22] Snow, J. E., *Invariant Complex Structures on Four Dimensional Solvable Real Lie Groups*, Man. Math, **66** (1990), 397–412.

[23] Thurston, W. P., *Some simple examples of symplectic manifolds*, Proc. Amer. Math. Soc. **55** (1976), 467–468.

[24] Walker, A. G., *Canonical form for a Riemannian space with a parallel field of null planes*, Quart. J. of Math. **1** (1950), 69–79.

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