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On Centralizers of Elements in the Lie Algebra of the Special Cremona Group $\mathbf{SA}_2(\mathbf{k})$

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Abstract. We give a description of maximal abelian subalgebras and centralizers of elements in the Lie algebra $sa_2(k) = \{D \in \operatorname{Der} k[x,y] \mid \operatorname{div} D = 0\}$ over an algebraically closed field k of characteristic 0. This description is given in terms of closed polynomials.
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1. Introduction

The special affine Cremona group $SA_n(k)$ over a field k consists of all automorphisms $F = (f_1, \dots, f_n)$ of the polynomial algebra $k[x_1, \dots, x_n]$ with $\det(JF) = 1$, where JF is the Jacobian matrix of F. From [4] it follows that the Lie algebra $sa_n(k)$ of the infinite dimensional algebraic group $SA_n(k)$ consists of all derivations $D = \sum_{i=1}^{n} a_i(x_1, \dots, x_n) \frac{\partial}{\partial x_i}$, $a_i(x_1, \dots, x_n) \in k[x_1, \dots, x_n]$ of the algebra $k[x_1, \dots, x_n]$ with $\operatorname{div} D = \sum_{i=1}^{n} \frac{\partial a_i}{\partial x_i} = 0$.

The aim of this paper is to give a description of centralizers of elements in the Lie algebra $sa_2(k)$ and to describe all maximal abelian subalgebras of this algebra over an algebraically closed field k of characteristic 0. The investigation of the structure of subalgebras in $sa_2(k)$ is of great interest, because many problems (in particular the Jacobian conjecture for $n = 2$) are closely connected with properties of subalgebras in $sa_2(k)$.

To describe centralizers of elements in $sa_2(k)$ we represent this Lie algebra as a quotient algebra of the Lie algebra $P_2(k)$ of all polynomials in two variables with multiplication rule $[f, g] = \det(J(f, g))$, where $\det(J(f, g))$ is the Jacobian of polynomials $f, g \in k[x, y]$. In fact, $P_2(k)$ is a Poisson algebra but we mainly consider it as a Lie algebra. Using results from [3], it is easy to obtain a description of centralizers of elements and of maximal abelian subalgebras in $P_2(k)$ (see also [5]). This description is given in terms of closed polynomials, i. e., polynomials

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$f \in k[x, y]$ for which the subalgebra $k[f]$ is integrally closed in $k[x, y]$. Using some results from [1], one can replace here closed polynomials by irreducible ones.

Notations in the paper are standard. The ground field k is algebraically closed of characteristic 0. The center of a Lie algebra L is denoted by $Z(L)$. It is easy to show that $Z(P_2(k)) = k$, where k is considered as a subalgebra in $P_2(k)$. For a polynomial $f \in k[x, y]$ we denote by $k[f]$ the (associative) subalgebra in $k[x, y]$ generated by f. The one-dimensional vector subspace of $k[x, y]$ spanned on f is denoted by kf. A polynomial $f(x, y) \in k[x, y]$ is called a Jacobian polynomial if there exists a polynomial g such that $[f, g] = \det(J(f, g)) \in k^*$ (see, for example [2], p.245).

2. Closed polynomials

Lemma 2.1. *The Lie algebra $sa_2(k)$ is isomorphic to the quotient algebra of $P_2(k)$ by $Z(P_2(k)) = k$, i. e.,*

$$sa_2(k) \simeq P_2(k)/k.$$

Proof. Any element $f(x, y)$ of the Lie algebra $P_2(k)$ induces the inner derivation $\operatorname{ad} f : P_2(k) \to P_2(k)$, $\operatorname{ad} f(g) = [f, g]$ of the Lie algebra $P_2(k)$. The linear mapping $\operatorname{ad} f$ is also a derivation of the associative algebra $k[x, y]$. It is easy to see that the kernel of the homomorphism of Lie algebras $\operatorname{ad} : P_2(k) \to \operatorname{Der}(k[x, y])$ coincides with k, where k is considered as a subalgebra in $P_2(k)$. Since $\operatorname{ad} f = -\frac{\partial f}{\partial y}\frac{\partial}{\partial x} + \frac{\partial f}{\partial x}\frac{\partial}{\partial y}$, we get $\operatorname{div}(\operatorname{ad} f) = -\frac{\partial^2 f}{\partial y \partial x} + \frac{\partial^2 f}{\partial x \partial y} = 0$. Therefore, $\operatorname{ad} f \in sa_2(k)$. This proves $\operatorname{ad}(P_2(k)) \subseteq sa_2(k)$.

Let us show that ad is a surjective map. Let $D = P(x, y)\frac{\partial}{\partial x} + Q(x, y)\frac{\partial}{\partial y}$ be an element of $sa_2(k)$. Then $\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} = 0$. This condition guarantees the existence of a polynomial $\varphi(x, y)$ (a potential) such that $\frac{\partial \varphi}{\partial x} = Q(x, y)$, $\frac{\partial \varphi}{\partial y} = -P(x, y)$. For φ we obtain $[\varphi, x] = -\frac{\partial \varphi}{\partial y} = P(x, y)$, $[\varphi, y] = \frac{\partial \varphi}{\partial x} = Q(x, y)$, in other words $\operatorname{ad}(\varphi) = D$. This proves the surjectivity of the map ad. Using that $\ker \operatorname{ad} = k$, we obtain $P_2(k)/k \simeq sa_2(k)$. ■

Lemma 2.2. *1) A polynomial $f \in k[x_1, \ldots, x_n] \setminus k$ is closed if and only if $k[f]$ is a maximal element in the partially ordered set (with respect to inclusion)*

$$\mathcal{M} = \{k[h] \mid h \in k[x_1, \ldots, x_n] \setminus k\}.$$

2) Let D be a derivation of $k[x, y]$, $D \neq 0$. Then $\ker D = k[f]$ for some closed polynomial f.

Proof. 1) See [3], Lemma 3.1.

2) From [3], Theorem 2.8, it follows that $\ker D = k[f]$ for some polynomial f. The subalgebra $\ker D = k[f]$ is integrally closed in $k[x, y]$ by the Lemma 2.1 from [3]. Therefore the polynomial f is closed. ■

Let $f, h \in k[x_1, \ldots, x_n]$. We call a polynomial h a generative polynomial of f if h is closed and if $f \in k[h]$, i. e., $f = F(h)$ for some $F(t) \in k[t]$.

Lemma 2.3. *Let $f \in k[x,y] \setminus k$. The polynomial f is closed in the following two cases:*

1) when f is irreducible;

2) when f is a Jacobian polynomial.

Proof. 1) If f is not closed, then $f = F(h)$ for some polynomials $h \in k[x,y]$ and $F(t) \in k[t], \deg F > 1$. Since F is reducible (as a polynomial in one indeterminate) f is reducible as well.

2) Let f be Jacobian but not closed. Then there exists a polynomial $F(t) \in k[t], \deg F \geqslant 2$ such that $f = F(h)$ for some polynomial $h \in k[x,y]$. As f is Jacobian there exists a polynomial $g \in k[x,y]$ with $\det(J(f,g)) = c \in k^*$. Then

$$\det(J(f,g)) = \det(J(F(h),g)) = F^{'}(h)\det(J(h,g)) = c.$$

This is impossible because $\deg F^{'}(h) \geqslant 1$. ∎

Lemma 2.4. *1) If polynomials $f, g \in k[x,y] \setminus k$ are algebraically dependent, there exists a closed polynomial $h \in k[x,y]$ such that $f \in k[h]$ and $g \in k[h]$;*

2) For any polynomial $f \in k[x,y] \setminus k$, there exists a generative polynomial. If h_1, h_2 are two generative polynomials of f, there exist $c_1 \in k^*$, $c_2 \in k$ such that $h_2 = c_1 h_1 + c_2$;

3) In the set of all generative polynomials of a polynomial $f \in k[x,y] \setminus k$ there exists at least one irreducible polynomial.

Proof. 1) If f and g are algebraically dependent, by Corollary 3 from [5] we obtain $[f,g] = 0$. By Lemma 2.2, we get $\ker \operatorname{ad} f = k[h]$ for some closed polynomial $h(x,y)$. Since $f \in \ker \operatorname{ad} f$ and $g \in \ker \operatorname{ad} f$, one concludes $f \in k[h]$ and $g \in k[h]$.

2) Since from the inclusion $k[f] \subsetneq k[g]$ it follows $\deg g < \deg f$, f is contained in some maximal one-generated subalgebra $k[h]$. By Lemma 2.2 h is a generative polynomial of f. Suppose h_1 and h_2 are generative polynomials of f. It means in particular that $f \in k[h_1]$ and $f \in k[h_2]$. Therefore, $f = F_1(h_1)$ and $f = F_2(h_2)$ for some polynomials $F_1(t), F_2(t) \in k[t]$. Then $F_1(h_1) - F_2(h_2) = 0$ and this implies that h_1 and h_2 are algebraically dependent. By 1) we conclude $h_1 \in k[h]$, $h_2 \in k[h]$ for some closed polynomial h. Clearly $k[h_1] = k[h] = k[h_2]$. Therefore $h_2 = c_1 h_1 + c_2$ for some elements $c_1 \in k^*$, $c_2 \in k$.

3) Let h be a generative polynomial of f. Since h is closed it follows from [1] (see Théorème 8) that there exists $c \in k$ such that $h - c$ is an irreducible polynomial. Because $k[h] = k[h-c]$, $h - c$ is also a generative polynomial of f. This proves the Lemma. ∎

Corollary 2.5. *If polynomials $f(x,y)$ and $g(x,y)$ are irreducible and algebraically dependent then $f = c_1 g + c_2$ for some $c_1 \in k^*$, $c_2 \in k$.*

Proof. Since f and g are algebraically dependent, by Lemma 2.4 there exists a closed polynomial h such that $f \in k[h]$, $g \in k[h]$. The irreducible polynomials f and g are closed by Lemma 2.3. Therefore $k[f] = k[h] = k[g]$ and $f = c_1 g + c_2$, for some $c_1 \in k^*$, $c_2 \in k$. ∎

Corollary 2.6. *For any polynomial $f \in k[x,y] \setminus k$ there exist an irreducible polynomial $h(x,y)$ and a polynomial $F(t) \in k[t]$ such that $f = F(h)$.*

Proof. By Lemma 2.4 there exists an irreducible polynomial h such that $f \in k[h]$. This implies the required statement. ■

Lemma 2.7. *1) For any polynomial $f \in P_2(k) \setminus k$ its centralizer $C_{P_2(k)}(f)$ coincides with $k[h]$ for any generative polynomial h of f.*

2) Let A be a maximal abelian subalgebra of the Lie algebra $P_2(k)$. Then $A = k[f]$ for some irreducible polynomial $f \in P_2(k) \setminus k$. Conversely, for any irreducible polynomial $f \in P_2(k) \setminus k$, the subalgebra $k[f]$ is a maximal abelian subalgebra of $P_2(k)$.

Proof. 1) Follows from Lemma 2.2, since $C_{P_2(k)}(f) = \ker \operatorname{ad} f$.

2) Let A be a maximal abelian subalgebra of the Lie algebra $P_2(k)$ and let f be any non-constant polynomial from A. Obviously, $A \subseteq C_{P_2(k)}(f) = k[h]$ for some closed polynomial h. Since $k[h]$ is an abelian subalgebra, $A = k[h]$. By Lemma 2.4, h can be chosen irreducible.

Now let f be an irreducible polynomial. The polynomial f is closed by Lemma 2.3. We shall show that $k[f]$ is a maximal abelian subalgebra. Let g be a polynomial such that $[f,g] = 0$. Then as in the proof of Lemma 2.4 $f \in k[h]$, $g \in k[h]$ for some closed polynomial h. Therefore, using that f is closed, we conclude $g \in k[h] = k[f]$. We proved that all polynomials commuting with f belong to $k[f]$. Therefore $k[f]$ is a maximal abelian subalgebra in $P_2(k)$. ■

3. Main results

Theorem 3.1. *Let $D = P(x,y)\frac{\partial}{\partial x} + Q(x,y)\frac{\partial}{\partial y}$ be a non-zero element of the Lie algebra $sa_2(k)$. Let $f(x,y) \in k[x,y]$ be a polynomial such that $\frac{\partial f}{\partial x} = Q(x,y)$, $\frac{\partial f}{\partial y} = -P(x,y)$ and let $\bar{f}$ be a generative polynomial of f. Then*

1) if $f(x,y)$ is not a Jacobian polynomial,

$$C_{sa_2(k)}(D) = k[\bar{f}]\left(-\frac{\partial \bar{f}}{\partial y}\frac{\partial}{\partial x} + \frac{\partial \bar{f}}{\partial x}\frac{\partial}{\partial y}\right);$$

2) if $f(x,y)$ is a Jacobian polynomial and $g(x,y)$ is a polynomial such that $\det(J(f,g)) \in k^*$,

$$C_{sa_2(k)}(D) = k[f]\left(-\frac{\partial f}{\partial y}\frac{\partial}{\partial x} + \frac{\partial f}{\partial x}\frac{\partial}{\partial y}\right) + k\left(-\frac{\partial g}{\partial y}\frac{\partial}{\partial x} + \frac{\partial g}{\partial x}\frac{\partial}{\partial y}\right)$$

Proof. 1) By Lemma 2.7 $C_{P_2(k)}(f) = k[\bar{f}]$. The homomorphism $\operatorname{ad} : P_2(k) \to sa_2(k)$ takes the polynomial $\bar{f}$ to the derivation $\operatorname{ad} \bar{f} = -\frac{\partial \bar{f}}{\partial y}\frac{\partial}{\partial x} + \frac{\partial \bar{f}}{\partial x}\frac{\partial}{\partial y}$. Let $D_1 = P_1(x,y)\frac{\partial}{\partial x} + Q_1(x,y)\frac{\partial}{\partial y}$ be an arbitrary non-zero element of $C_{sa_2(k)}(D)$. By Lemma 2.1 there exists a polynomial $f_1(x,y)$ such that $\operatorname{ad} f_1 = D_1$. Since $\ker \operatorname{ad} = k$, $[f, f_1]$ lies in k. But f is not a Jacobian polynomial, so we can conclude $[f, f_1] = 0$. Therefore f_1 lies in $C_{P_2(k)}(f) = k[\bar{f}]$. This means that $\mathrm{ad}^{-1}(C_{sa_2(k)}(D)) = k[\bar{f}]$. Using the surjectivity of the homomorphism ad we obtain $C_{sa_2(k)}(D) = \mathrm{ad}(k[\bar{f}]) = k[\bar{f}]\left(-\frac{\partial \bar{f}}{\partial y}\frac{\partial}{\partial x} + \frac{\partial \bar{f}}{\partial x}\frac{\partial}{\partial y}\right)$.

2) Let f be a Jacobian polynomial, i. e., there exists a polynomial g such that $\det(J(f, g)) = c \in k^*$. By Lemma 2.3 the polynomial f is closed, i. e., one can assume $\bar{f} = f$. Since $[\mathrm{ad}\, f, \mathrm{ad}\, g] = \mathrm{ad}\, c = 0$, we have $\mathrm{ad}\, g \in C_{sa_2(k)}(\mathrm{ad}\, f) = C_{sa_2(k)}(D)$. It is easy to see that

$$\mathrm{ad}^{-1}(C_{sa_2(k)}(D)) = \{h \in P_2(k) |\ [f, h] \in k\} = k[\bar{f}] + kg = k[f] + kg.$$

Therefore,

$$C_{sa_2(k)}(D) = \mathrm{ad}(k[f] + kg) = k[f]\left(-\frac{\partial f}{\partial y}\frac{\partial}{\partial x} + \frac{\partial f}{\partial x}\frac{\partial}{\partial y}\right) + k\left(-\frac{\partial g}{\partial y}\frac{\partial}{\partial x} + \frac{\partial g}{\partial x}\frac{\partial}{\partial y}\right).$$

■

Remark 3.2. From Lemma 2.4 it follows that the polynomial $\bar{f}$ in Theorem 3.1 can be chosen irreducible.

Remark 3.3. From the description of centralizers of elements in Theorem 3.1 it follows that the centralizer of a derivation corresponding to a non Jacobian polynomial is an abelian subalgebra, and the centralizer of a derivation corresponding to any Jacobian polynomial is solvable of derived length 2.

Lemma 3.4. *Let $L = k[f] + kg$ be a subalgebra of the Lie algebra $P_2(k)$ with $\det(J(f, g)) = c \in k^*$. If A is a nilpotent subalgebra of L and the nilpotency class of A is at most 2 then either $A \subseteq k[f]$ or A is contained in the subalgebra $k + kf + k(g + p(f))$ for some $p(t) \in k[t]$.*

Proof. Suppose that A is not contained in $k[f]$. As $dimL/k[f] = 1$ the $k-$subspace $A \cap k[f]$ is of codimension 1 in A. Therefore $A = (A \cap k[f]) + k(g + p(f))$ for some $p(t) \in k(t)$. Since $[q(f),\ g + p(f)] = q'(f) \cdot c$ for any polynomial $q(t) \in k[t]$ the subspace $A \cap k[f]$ may not contain polynomials of degree > 1. So the intersection $A \cap k[f]$ is contained in the subalgebra $k + kf$ and therefore $A \subseteq k + kf + k(g + p(f))$. ■

Theorem 3.5. *Let A be a maximal abelian subalgebra of the Lie algebra $sa_2(k)$. Then*

1) if $\dim A = \infty$, then $A = k[f]\left(-\frac{\partial f}{\partial y}\frac{\partial}{\partial x} + \frac{\partial f}{\partial x}\frac{\partial}{\partial y}\right)$, where $f(x, y)$ is an irreducible polynomial. Conversely, for any irreducible polynomial f, the algebra

$$k[f]\left(-\frac{\partial f}{\partial y}\frac{\partial}{\partial x} + \frac{\partial f}{\partial x}\frac{\partial}{\partial y}\right)$$

is a maximal abelian subalgebra in $sa_2(k)$;

2) if $\dim A < \infty$ then $A = kD_1 + kD_2$, where $D_1 = -\frac{\partial f}{\partial y}\frac{\partial}{\partial x} + \frac{\partial f}{\partial x}\frac{\partial}{\partial y}$, $D_2 = -\frac{\partial g}{\partial y}\frac{\partial}{\partial x} + \frac{\partial g}{\partial x}\frac{\partial}{\partial y}$ for some polynomials f, g such that $\det(J(f, g)) \in k^*$. Conversely, for any two polynomials f, g with condition $\det(J(f, g)) \in k^*$ the subalgebra $kD_1 + kD_2$, where D_1 and D_2 are defined as above, is a maximal abelian subalgebra of $sa_2(k)$.

Proof. Let D be an arbitrary non-zero element of A. Then $A \subseteq C_{sa_2(k)}(D)$ and clearly A is a maximal abelian subalgebra of $C_{sa_2(k)}(D)$. By Theorem 3.1 either

$$C_{sa_2(k)}(D) = k[f]\left(-\frac{\partial f}{\partial y}\frac{\partial}{\partial x} + \frac{\partial f}{\partial x}\frac{\partial}{\partial y}\right)$$

or

$$C_{sa_2(k)}(D) = k[f]\left(-\frac{\partial f}{\partial y}\frac{\partial}{\partial x} + \frac{\partial f}{\partial x}\frac{\partial}{\partial y}\right) + k\left(-\frac{\partial g}{\partial y}\frac{\partial}{\partial x} + \frac{\partial g}{\partial x}\frac{\partial}{\partial y}\right).$$

In the first case f is a closed irreducible polynomial, in the second one the polynomials f and g satisfy the condition $\det(J(f,g)) \in k^*$. In the first case $C_{sa_2(k)}(D)$ is an abelian subalgebra. Thus $A = C_{sa_2(k)}(D) = k[f]\left(-\frac{\partial f}{\partial y}\frac{\partial}{\partial x} + \frac{\partial f}{\partial x}\frac{\partial}{\partial y}\right)$.

Consider the second case. Denote $L = \mathrm{ad}^{-1}(C_{sa_2(k)}(D))$ where $\mathrm{ad} : P_2(k) \to sa_2(k)$ is the homomorphism from the Lemma 2.1. Then $\mathrm{ad}^{-1}(A)$ is a subalgebra in L. It is easy to see that $L = k[f] + kg$. Since $\ker \mathrm{ad} = Z(P_2(k)) = k$, we conclude that $\mathrm{ad}^{-1}(A)$ is a nilpotent subalgebra of the nilpotency class $\leqslant 2$. By Lemma 3.4 it holds either $\mathrm{ad}^{-1}(A) \subseteq k[f]$ or $\mathrm{ad}^{-1}(A) \subseteq k + kf + k(g + p(f))$ for some $p(t) \in k[t]$. Since A is a maximal abelian subalgebra of $sa_2(k)$ it follows from inclusion $\mathrm{ad}^{-1}(A) \subseteq k[f]$ that $\mathrm{ad}^{-1}(A) = k[f]$. Then we have $A = k[f]\left(-\frac{\partial f}{\partial y}\frac{\partial}{\partial x} + \frac{\partial f}{\partial x}\frac{\partial}{\partial y}\right)$.

Let now $\mathrm{ad}^{-1}(A) \subseteq k + kf + k(g + p(f))$. Applying the map ad we get the inclusion $A \subseteq \mathrm{ad}(k + kf + k(g + p(f))) = kD_1 + kD_2$, where $D_1 = \mathrm{ad}\, f, D_2 = \mathrm{ad}(g + p(f))$. The subalgebra $kD_1 + kD_2$ is abelian and therefore $A = kD_1 + kD_2$. Denoting $g + p(f)$ by g we have $D_1 = \mathrm{ad}\, f, D_2 = \mathrm{ad}\, g$. So we have proved the necessary conditions for both statements of the Theorem.

Let f be an irreducible polynomial. We will show that $k[f]\left(-\frac{\partial f}{\partial y}\frac{\partial}{\partial x} + \frac{\partial f}{\partial x}\frac{\partial}{\partial y}\right)$ is a maximal abelian subalgebra in $sa_2(k)$. Clearly, since f is an irreducible polynomial, by Lemma 2.7 $k[f]$ is a maximal abelian subalgebra in $P_2(k)$. It is obvious that

$$\mathrm{ad}(k[f]) = k[f]\left(-\frac{\partial f}{\partial y}\frac{\partial}{\partial x} + \frac{\partial f}{\partial x}\frac{\partial}{\partial y}\right)$$

is an abelian subalgebra in $sa_2(k)$. Suppose that $\mathrm{ad}(k[f])$ is not maximal abelian. Then it is properly contained in some maximal abelian subalgebra B of the algebra $sa_2(k)$. Since $\dim B = \infty$, as it was proved above there exists a closed polynomial g such that $B = k[g]\left(-\frac{\partial g}{\partial y}\frac{\partial}{\partial x} + \frac{\partial g}{\partial x}\frac{\partial}{\partial y}\right)$. From this one easily concludes that $k[f]$ is properly contained in $\mathrm{ad}^{-1}(B) = k[g]$. This is impossible by Lemma 2.2, since $k[f]$ is a maximal in the set of subalgebras of the form $k[h]$ in $P_2(k)$. This proves that $k[f]\left(-\frac{\partial f}{\partial y}\frac{\partial}{\partial x} + \frac{\partial f}{\partial x}\frac{\partial}{\partial y}\right)$ is a maximal abelian subalgebra in $sa_2(k)$.

Let now f and g be two polynomials from $k[x,y]$ such that $\det(J(f,g)) \in k^*$. Then the elements $D_1 = \mathrm{ad}\, f$ and $D_2 = \mathrm{ad}\, g$ commute. Therefore $A = kD_1 + kD_2$ is an abelian two-dimensional subalgebra in $sa_2(k)$. Suppose, A is not a maximal abelian subalgebra of the algebra $sa_2(k)$. Then A is contained in some maximal abelian subalgebra B of $sa_2(k)$. If $\dim B = \infty$, by the above proved statement, $B = k[h]\left(-\frac{\partial h}{\partial y}\frac{\partial}{\partial x} + \frac{\partial h}{\partial x}\frac{\partial}{\partial y}\right)$ for some closed polynomial h. Then $\mathrm{ad}^{-1}(B) = k[h]$ is an abelian subalgebra in $P_2(k)$ which contains the non-abelian subalgebra $k + kf + kg$. This is impossible and therefore $\dim B < \infty$. As above one can obtain $\dim B = 2$. This implies $A = B$ which contradicts to our assumption. This contradiction proves that A is a maximal abelian subalgebra in $sa_2(k)$. The sufficient conditions for the both statements of the Theorem are proved. ■

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