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Finite-dimensional Lie Subalgebras of the Weyl Algebra

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Abstract. We classify up to isomorphism all finite-dimensional Lie algebras that can be realised as Lie subalgebras of the complex Weyl algebra A_1. The list we obtain turns out to be countable and, for example, the only non-solvable Lie algebras with this property are: $\mathfrak{sl}(2)$, $\mathfrak{sl}(2) \times \mathbf{C}$ and $\mathfrak{sl}(2) \ltimes \mathcal{H}_3$. We then give several different characterisations, normal forms and isotropy groups for the action of $\mathrm{Aut}(A_1) \times \mathrm{Aut}(\mathfrak{sl}(2))$ on a class of realisations of $\mathfrak{sl}(2)$ in A_1.
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1. Introduction

The Weyl algebra A_1 is the complex associative algebra generated by elements p and q satisfying the relation $pq - qp = 1$. It is well known that the Lie algebras $\mathfrak{sl}(2)$, $\mathfrak{sl}(2) \times \mathbb{C}$ and $\mathfrak{sl}(2) \ltimes \mathcal{H}_3$ (where $\mathcal{H}_3$ denotes the three-dimensional Heisenberg algebra) can be realised as Lie subalgebras of A_1. In [11], A. Simoni and F. Zaccaria proved that the only complex semi-simple Lie algebra that can be realised in A_1 is $\mathfrak{sl}(2)$ and a remarkable property of realisations of $\mathfrak{sl}(2)$ in A_1 was proved by A. Joseph in [9], where he showed that the spectrum of the realisation in A_1 of suitably normalised semi-simple elements of $\mathfrak{sl}(2)$ is either $\mathbb{Z}$ or $2\mathbb{Z}$. In [7], J. Igusa gave a necessary condition for two elements of A_1 to generate an infinite-dimensional Lie subalgebra.

In this article we find all complex finite-dimensional Lie algebras which can be realised in A_1. If $\mathfrak{g}$ is a complex finite-dimensional Lie algebra and $A_1^{\mathfrak{g}}$ denotes the set of injective Lie algebra homomorphisms from $\mathfrak{g}$ to A_1, we prove (Theorems 4.10, 4.11, 4.15 and 4.17)

Theorem 1.1. *Let $\mathfrak{g}$ be a complex finite-dimensional non-abelian Lie algebra. Then $A_1^{\mathfrak{g}} \neq \emptyset$ iff $\mathfrak{g}$ is isomorphic to one of the following:*

1) $\mathfrak{sl}(2)$,
2) $\mathfrak{sl}(2) \times \mathbb{C}$,
3) $\mathfrak{sl}(2) \ltimes \mathcal{H}_3$,
4) $\mathcal{L}_n$ $(n \geq 2)$,
5) $\tilde{\mathcal{L}}_n$ $(n \geq 2)$,
6) $\mathfrak{r}(i_1, \ldots, i_n)$ *($i_1 < \ldots < i_n$ are positive integers).*

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Here, $\mathcal{L}_n$ is a nilpotent, in fact filiform, Lie algebra, $\tilde{\mathcal{L}}_n$ is isomorphic to a semi-direct product $\mathbb{C} \ltimes \mathcal{L}_n$ and $\mathfrak{r}(i_1, \dots i_n)$ is isomorphic to a semi-direct product $\mathbb{C} \ltimes \mathbb{C}^n$. For the precise definitions of these Lie algebras see section 3. Note that only a finite number of non-solvable Lie algebras and only a countable family of solvable Lie algebras appear in the list of Theorem 1.1. Since all derivations of A_1 are inner, this theorem also leads to the classification of all finite-dimensional Lie algebras which can be realised in $\mathrm{Der}(A_1)$ (see Theorem 4.20) and we give explicit examples of subgroups of $\mathrm{Aut}(A_1)$ which exponentiate them (see section 4.4).

In the second part of the paper we study a particular family of realisations of $\mathfrak{sl}(2)$ in A_1. The group $\mathrm{Aut}(A_1) \times \mathrm{Aut}(\mathfrak{sl}(2))$ acts naturally on $A_1^{\mathfrak{sl}(2)}$ and we give several characterisations of the orbit of $\mathcal{N} = \{f_I,\ f_{II}^b\ :\ b \in \mathbb{C}\} \subseteq A_1^{\mathfrak{sl}(2)}$ where

$$\begin{array}{ll} f_I(e_+) = -\frac{1}{2}q^2 & f_{II}^b(e_+) = (b + pq)q \\ f_I(e_-) = \frac{1}{2}p^2 & f_{II}^b(e_-) = -p \\ f_I(e_0) = \frac{1}{2}(pq + qp) & f_{II}^b(e_0) = 2pq + b \end{array} \tag{1.1}$$

(e_+, e_-, e_0 is the standard basis of $\mathfrak{sl}(2)$). The realisations f_{II}^b were first introduced in this context by A. Joseph in [9] and f_I is the natural embedding of $\mathfrak{sl}(2)$ in A_1. We prove that no two elements of $\mathcal{N}$ are in the same orbit and the second main result of this paper is (Theorem 6.2)

Theorem 1.2. *Let $f \in A_1^{\mathfrak{sl}(2)}$. Then the following statements are equivalent:*

(i) There exists $\gamma \in \mathrm{Aut}(A_1) \times \mathrm{Aut}(\mathfrak{sl}(2))$ such that $\gamma.f \in \mathcal{N}$;

(ii) There exists $z \in \mathfrak{sl}(2) \setminus \{0\}$ such that $\mathrm{ad}(f(z))$ has a strictly semi-simple eigenvector;

(iii) There exists $z \in \mathfrak{sl}(2) \setminus \{0\}$ such that $f(z)$ is strictly semi-simple;

(iv) There exists $z \in \mathfrak{sl}(2) \setminus \{0\}$ such that $\mathrm{ad}(f(z))$ has a strictly nilpotent eigenvector;

(v) There exists $z \in \mathfrak{sl}(2) \setminus \{0\}$ such that $f(z)$ is strictly nilpotent;

(vi) There exists $z \in \mathfrak{sl}(2) \setminus \{0\}$ such that $\mathrm{ad}(f(z))$ can be exponentiated in $\mathrm{Aut}(A_1)$.

The terms strictly nilpotent and strictly semi-simple for non-zero elements in A_1 were defined by J. Dixmier in [4] (see also subsection 2.2 of this paper) and for the precise definition of exponentiation in this context, see subsection 2.3. We further show that the isotropy of f_I is isomorphic to $SL(2, \mathbb{C})$ and that the isotropy of f_{II}^b is isomorphic to a Borel subgroup of $SO(3, \mathbb{C})$. Finally, for the sake of completeness, we give explicit formulae for a realisation of $\mathfrak{sl}(2)$ in A_1 which does not satisfy any of the criteria of Theorem 1.2. (see also [9]).

The paper is organised as follows. In section 2 we recall the basic properties of the Weyl algebra and in particular the Dixmier partition which is essential to this article. In section 3 we give some examples of Lie algebras which can be realised as Lie subalgebras of A_1 and in section 4 we obtain the classification of all finite-dimensional Lie algebras with this property. Sections 5 and 6 are devoted to the study of an explicit family of realisations of $\mathfrak{sl}(2)$ and its orbit under the action of the group $\mathrm{Aut}(A_1) \times \mathrm{Aut}(\mathfrak{sl}(2))$.

2. Properties of the Weyl algebra and the Dixmier partition

In this section we give the basic properties of the Weyl algebra A_1 and its Dixmier partition. In particular we give a simple characterisation of the set of elements $Z \in A_1$ such that $\mathrm{ad}(Z)$ can be exponentiated.

2.1. Basic properties.

Definition 2.1. The *Weyl algebra* A_1 is the complex associative algebra generated by elements p and q subject to the relation $pq - qp = 1$.

There is a natural action of A_1 on $\mathbb{C}[x]$ defined by

$$p \cdot P(x) = P'(x), \quad q \cdot P(x) = xP(x), \ \forall P \in \mathbb{C}[x] \tag{2.1}$$

and this establishes an isomorphism of A_1 with the algebra of polynomial coefficient differential operators acting on $\mathbb{C}[x]$. We will refer to this representation as the *standard representation of* A_1.

Properties of A_1 (see [4], for example):

P1 The elements $\{p^i q^j : i, j \in \mathbb{N}\}$ constitute a basis of A_1 and the centre of A_1 is $\mathbb{C}$.

P2 The linear subspace $W_1 =< p, q >$ spanned by p and q has a unique symplectic structure ω such that $\omega(p, q) = 1$ and the group $SL(W_1)$ of symplectic transformations of (W_1, ω) is isomorphic to $SL(2, \mathbb{C})$. It is well known that the inclusion of W_1 in A_1 extends to an algebra isomorphism from the quotient of the tensor algebra $T(W_1)$ by the two-sided ideal generated by $v_1 \otimes v_2 - v_2 \otimes v_1 - \omega(v_1, v_2)1$ to A_1 and, since this ideal is stable under its action, $SL(W_1)$ acts naturally on A_1. The map $\delta : S(W_1) \to A_1$ given on $S^n(W_1)$ by

$$\delta(v_1 \odot \ldots \odot v_n) = \sum_{\sigma \in S_n} \frac{1}{n!} v_{\sigma(1)} \ldots v_{\sigma(n)}$$

is an $SL(W_1)$-equivariant *linear* isomorphism. If we set $W_n = \delta(S_n(W_1))$ one can show that $A_1 = \bigoplus_{n\in\mathbb{N}} W_n$ and that $[W_i, W_j] \subseteq W_{i+j-2}$. In particular,

$$[W_2, W_2] \subseteq W_2, \ [W_2, W_i] \subseteq W_i \tag{2.2}$$

so that W_2 is a Lie algebra and W_i is a representation of W_2. This action of W_2 on W_1 establishes an $SL(W_1)$-equivariant Lie algebra isomorphism $W_2 \cong \mathfrak{sl}(W_1)$.

P3 The algebra A_1 satifies the commutative centraliser condition (*ccc*): the centraliser $C(x)$ of any element $x \in A_1 \setminus \mathbb{C}$ is a commutative subalgebra (see [1] and [4]).

P3′ If $x, y \in A_1 \setminus \mathbb{C}$ then

$$C(x) \cap C(y) = \begin{cases} \mathbb{C} & \text{if } xy - yx \neq 0 \\ C(x) & \text{if } xy - yx = 0, \end{cases}$$

(see Corollary 4.7 of [4]).

P4 Two elements $p', q' \in A_1$ satisfying $[p', q'] = 1$ uniquely define an algebra homomorphism from A_1 to itself and conversely, given an homomorphism $\alpha : A_1 \to A_1$ we have $[\alpha(p), \alpha(q)] = 1$. J. Dixmier conjectured in 1968 that an algebra homomorphism of A_1 is invertible and thus in fact an automorphism (see [4]). This conjecture is still undecided.

P5 For $n \in \mathbb{N}$, the map $\mathrm{ad}(p^{n+1}) : A_1 \to A_1$ given by $\mathrm{ad}(p^{n+1})(a) = [p^{n+1}, a]$ is locally nilpotent, *i.e.*, for each $a \in A_1$, there exists an $N \in \mathbb{N}$ (depending on a) such that $\mathrm{ad}^N(p^{n+1})(a) = 0$. For $\lambda \in \mathbb{C}$ one can then define $\Phi_{n,\lambda} : A_1 \to A_1$ by $\Phi_{n,\lambda}(a) = \sum_{k=0}^{N} \frac{(\frac{\lambda}{n+1} ad(p^{n+1}))^k}{k!}(a) = \exp(\frac{\lambda}{n+1}\mathrm{ad}(p^{n+1}))(a)$ and this is the unique automorphism of A_1 such that

$$\Phi_{n,\lambda}(p) = p, \quad \Phi_{n,\lambda}(q) = q + \lambda p^n. \tag{2.3}$$

One defines $\Phi'_{n,\lambda} = \exp(-\frac{\lambda}{n+1}\mathrm{ad}(q^{n+1}))$ similarly and shows that it is the unique automorphism of A_1 such that

$$\Phi'_{n,\lambda}(q) = q, \quad \Phi'_{n,\lambda}(p) = p + \lambda q^n. \tag{2.4}$$

The group of automorphisms of A_1 is generated by the $\Phi_{n,\lambda}$ and the $\Phi'_{n,\lambda}$ (see [4]).

2.2. The Dixmier partition.

Let $x \in A_1$. Set

$$\begin{aligned} N(x) &= \big\{y \in A_1 :\ \mathrm{ad}^m(x)(y) = 0, \text{ for some positive integer } m\big\} \\ C(x) &= \big\{y \in A_1 :\ \mathrm{ad}(x)(y) = 0\big\} \\ D(x) &= \Big\langle y \in A_1 : \mathrm{ad}(x)(y) = \lambda y \text{ for some } \lambda \in \mathbb{C}\Big\rangle. \end{aligned}$$

It is immediate that $N(x) \cap D(x) = C(x)$ and Dixmier showed that for all $x \in A_1$, either $C(x) = N(x)$ or $C(x) = D(x)$. As a consequence he proved (see [4])

Theorem 2.2. *(Dixmier partition) The set $A_1 \setminus \mathbb{C}$ is a disjoint union of the following non-empty subsets.*

$$\begin{aligned} \Delta_1 &= \big\{x \in A_1 \setminus \mathbb{C} :\ D(x) = C(x),\ N(x) \neq C(x),\ N(x) = A_1\big\} \\ \Delta_2 &= \big\{x \in A_1 \setminus \mathbb{C} :\ D(x) = C(x),\ N(x) \neq C(x),\ N(x) \neq A_1\big\} \\ \Delta_3 &= \big\{x \in A_1 \setminus \mathbb{C} :\ D(x) \neq C(x),\ N(x) = C(x),\ D(x) = A_1\big\} \\ \Delta_4 &= \big\{x \in A_1 \setminus \mathbb{C} :\ D(x) \neq C(x),\ N(x) = C(x),\ D(x) \neq A_1\big\} \\ \Delta_5 &= \big\{x \in A_1 \setminus \mathbb{C} :\ D(x) = C(x),\ N(x) = C(x),\ C(x) \neq A_1\big\} \end{aligned}$$

Note that this partition is stable under the action of $\mathrm{Aut}(A_1)$ and multiplication by a non-zero scalar.

Elements of $\Delta_1 \cup \Delta_2$ (resp. of Δ_1) are said to be nilpotent (resp. strictly nilpotent) and elements of $\Delta_3 \cup \Delta_4$ (resp. of Δ_3) are said to be semi-simple (resp. strictly semi-simple). In fact, $x \in \Delta_1$ iff there exists an automorphism α of A_1 such that $\alpha(x)$ is a polynomial in p (Theorem 9.1 of [4]) and $x \in \Delta_3$ iff there exists an automorphism α of A_1 such that $\alpha(x) = \mu pq + \nu$, for some $\mu \in \mathbb{C}^*$ and $\nu \in \mathbb{C}$ (Theorem 9.2 of [4]).

Recall (see **P2**) that W_2 is a Lie algebra isomorphic to $\mathfrak{sl}(2)$ and therefore elements of W_2 are either semi-simple or nilpotent in the Lie algebra sense.

Proposition 2.3. *([4], Lemme 8.6). Let $x \in A_1 \backslash \mathbb{C}$ be of the form $\alpha + w_1 + w_2$ where $\alpha \in \mathbb{C}$, $w_1 \in W_1$ and $w_2 \in W_2$.*
(i) If w_2 is nilpotent in the Lie algebra sense, then $x \in \Delta_1$.
(ii) If w_2 is semi-simple in the Lie algebra sense, then $x \in \Delta_3$.

2.3. Characterisation of $\Delta_1 \cup \Delta_3$ in terms of exponentiation.

Definition 2.4. Let $Z \in A_1$. One says that $\mathrm{ad}(Z)$ can be exponentiated if there exists a group homomorphism $\Phi : \mathbb{C} \to \mathrm{Aut}(A_1)$ such that

1. for all $a \in A_1$, the vector space $V_a =< \Phi(t)(a) : \ t \in \mathbb{C} >$ is finite-dimensional;
2. $\Phi_a : \mathbb{C} \to V_a$ is holomorphic and $\frac{d}{dt}\big|_0 \Phi_a(t) = [Z, a]$ where $\Phi_a(t) = \Phi(t)(a)$. (Since A_1 is infinite-dimensional, we impose 1 so that 2 makes sense).

Example 2.5. If $Z \in A_1$ is such that $\mathrm{ad}(Z)$ is locally nilpotent then $\mathrm{ad}(Z)$ can be exponentiated in this sense (cf. **P5** when Z is a polynomial in p). If $Z = pq + \nu$, $\mathrm{ad}(Z)$ can be exponentiated by the group homomorphism $\Psi : \mathbb{C} \to \mathrm{Aut}(A_1)$ given on the canonical basis $< p^i q^j : \ i, j \in \mathbb{N} >$ by $\Psi(t)(p^i q^j) = e^{t(j-i)} p^i q^j$.

Lemma 2.6. *Suppose $\mathrm{ad}(Z)$ can be exponentiated in the above sense.*

1. *For all $a \in A_1$, the finite-dimensional vector space V_a is stable under the action of $\mathrm{ad}(Z)$.*
2. *$\Phi(t)(a) = e^{t\mathrm{ad}(Z)|_{V_a}}(a)$ for all $a \in A_1$ ($e^{t\mathrm{ad}(Z)|_{V_a}}$ is well defined by 1).*

Proof. Fix $a \in A_1$ and $t_0 \in \mathbb{C}$. Then

$$\Phi(t + t_0)(a) = \Phi(t)\Phi(t_0)(a)$$

and both sides are in the finite-dimensional vector space V_a so we can differentiate with respect to t. This gives

$$\frac{d}{dt}\bigg|_0 \Phi(t + t_0)(a) = \frac{d}{dt}\bigg|_0 \Phi(t)\Phi(t_0)(a) = [Z, \Phi(t_0)(a)]. \tag{2.5}$$

The LHS is in V_a and hence $[Z, \Phi(t_0)(a)]$ also. This proves part 1.

Part 2 follows from the fact that the curves $t \mapsto \Phi(t)(a)$ and $t \mapsto e^{t\mathrm{ad}(Z)|_{V_a}}(a)$ are contained in the finite-dimensional vector space V_a and are solutions of the same first order differential equation

$$\frac{d}{dt}\gamma(t) = [Z, \gamma(t)]$$

with the same initial condition $\gamma(0) = a$. ■

Proposition 2.7. *Let $Z \in A_1 \setminus \{0\}$. Then $\mathrm{ad}(Z)$ can be exponentiated iff $Z \in \Delta_1 \cup \Delta_3$.*

Proof. $(\Rightarrow)$: Suppose that $Z \notin \Delta_1 \cup \Delta_3$. Set

$$F(Z) = \Big\{ a \in A_1 : \dim(< \mathrm{ad}^n(Z)(a), n \in \mathbb{N} >) < \infty \Big\}.$$

Then by Corollary 6.6 of [4], $F(Z) = D(Z) \cup N(Z)$ and so by Theorem 2.2, $F(Z) \neq A_1$. Let $a \in A_1 \setminus F(Z)$. By hypothesis, V_a is finite-dimensional, stable under $\mathrm{ad}(Z)$ and contains a. Hence $< \mathrm{ad}^n(Z)(a) : n \in \mathbb{N} > \subseteq V_a$ is also finite-dimensional, which is a contradiction.

$(\Leftarrow)$: If $Z \in \Delta_1 \cup \Delta_3$ then up to an automorphism of A_1, Z is equal to $pq + \alpha$ or to a polynomial in p. The result follows from Example 2.5. ∎

3. Examples of Lie subalgebras of the Weyl algebra

In this section we give examples of Lie algebras which can be realised as Lie subalgebras of A_1 (see also [4], [7], [9] and [11]).

Definition 3.1. Let $\mathfrak{g}$ be a complex Lie algebra.

$$A_1^{\mathfrak{g}} = \{ f \in \mathrm{Hom}(\mathfrak{g}, A_1) : f \text{ is injective}, f([a,b]) = f(a)f(b) - f(b)f(a)\, \forall a, b \in A_1 \}.$$

If $A_1^{\mathfrak{g}} \neq \emptyset$ we will say that the Lie algebra $\mathfrak{g}$ can be realised as a Lie subalgebra of A_1 and an element of $A_1^{\mathfrak{g}}$ will be called a realisation of $\mathfrak{g}$ in A_1.

Remark 3.2. Let $f : \mathfrak{g} \to A_1$ be a realisation of $\mathfrak{g}$ and $x \in \mathfrak{g} \setminus Z_{\mathfrak{g}}$. Then $f(x)$ is semi-simple (resp. nilpotent) in the A_1 sense if x is semi-simple (resp. nilpotent) in the Lie algebra sense. For example, when $\mathrm{ad}(x)$ is diagonalisable, there exist $y \in \mathfrak{g}$ and $\lambda \in \mathbb{C}^*$ such that $[x, y] = \lambda y$; hence $f(y) \in D(f(x))$, $f(y) \notin C(f(x))$ and $f(x)$ is semi-simple by Theorem 2.2.

Let us now give some examples of Lie algebras $\mathfrak{g}$ for which $A_1^{\mathfrak{g}} \neq \emptyset$.

E1 We saw in **P2** that W_2 is a Lie subalgebra of A_1 isomorphic to $\mathfrak{sl}(2)$. The standard basis

$$X = -\frac{1}{2}q^2,\ Y = \frac{1}{2}p^2,\ H = \frac{1}{2}(pq + qp) = pq - \frac{1}{2}, \tag{3.1}$$

satisfies the commutation relations $[X, Y] = H$, $[H, X] = 2X$ and $[H, Y] = -2Y$.

E2 The elements $1, X, Y, H$ span $W_0 \oplus W_2$ which is a Lie subalgebra of A_1 isomorphic to the direct product $\mathfrak{sl}(2) \times \mathbb{C}$.

E3 The elements $1, p, q$ span $W_0 \oplus W_1$ which is a Lie subalgebra isomorphic to the three dimensional Heisenberg algebra $\mathcal{H}_3$.

E4 The elements $1, p, q, X, Y, H$ span $W_0 \oplus W_1 \oplus W_2$ which is a Lie subalgebra of A_1 isomorphic to a semi-direct product $\mathfrak{sl}(2) \ltimes \mathcal{H}_3$.

E5 The associative subalgebra of A_1 generated by p is infinite-dimensional abelian and therefore any finite-dimensional abelian Lie algebra can be realised in A_1.

E6 The $(n+1)$ elements $q, 1, p, \ldots, p^{n-1}$ span a non-abelian nilpotent Lie subalgebra isomorphic to the filiform Lie algebra $\mathcal{L}_n$ (see [12]). If we set $X_0 = -q, X_k = \frac{1}{(n-k)!}p^{n-k}$ for $k = 1, \ldots, n$, then the only non-zero commutation relations are: $[X_0, X_k] = X_{k+1}$ for $k = 1, \ldots, n-1$.

E7 The $(n+2)$ elements $pq, q, 1, p, \ldots, p^{n-1}$ span a non-nilpotent solvable Lie subalgebra whose derived algebra is isomorphic to $\mathcal{L}_n$. If we set $h = pq, X_0 = -q$ and $X_k = \frac{1}{(n-k)!}p^{n-k}$ for $k = 1, \ldots, n$, then the only non-zero commutation relations are: $[h, X_0] = X_0, [h, X_k] = -(n-k)X_k$ and $[X_0, X_k] = X_{k+1}$ for $k = 1, \ldots, n-1$. We denote this Lie algebra by $\tilde{\mathcal{L}}_n$.

E8 The $(n+1)$ elements $pq, p^{i_1}, \ldots, p^{i_n}$, where $i_1, \ldots, i_n$ are distinct positive integers not all zero, span a non-nilpotent solvable Lie subalgebra whose derived algebra is $n-$dimensional and abelian. We denote this Lie algebra by $\mathfrak{r}(i_1, \ldots, i_n)$. It is clear that $\mathfrak{r}(i_{\sigma(1)}, \ldots, i_{\sigma(n)}) \cong \mathfrak{r}(i_1, \ldots, i_n)$ for any permutation $\sigma \in S_n$, that $\mathfrak{r}(i_1, \ldots, i_n)$ has a non-trivial centre iff one of the indices is zero and that $\mathfrak{r}(0, i_2, \ldots, i_n) \cong \mathfrak{r}_{n-1}(i_2, \ldots, i_n) \times \mathbb{C}$. Note also that $\mathfrak{r}(mi_1, \ldots mi_n) \cong \mathfrak{r}(i_1, \ldots i_n)$ if $m \in \mathbb{N}^*$. If we set $h = pq, X_k = p^{i_k}$ for $k = 1, \ldots, n$ the only non-zero commutation relations are: $[h, X_k] = -i_k X_k$.

4. Classification of finite-dimensional Lie algebras which can be realised in the Weyl algebra

In this section we obtain all finite-dimensional Lie algebras that can be realised as subalgebras of A_1. The only such non-solvable Lie algebras are isomorphic to either $\mathfrak{sl}(2)$, $\mathfrak{sl}(2) \times \mathbb{C}$ or $\mathfrak{sl}(2) \ltimes \mathcal{H}_3$. This is basically a consequence of the fact that A_1 satisfies **P3**, the "commutative centraliser condition". We then show that a non-abelian nilpotent Lie algebra which can be realised as a subalgebra of A_1 is isomorphic to an $\mathcal{L}_n$ (cf. **E6**) and this is consequence of the properties **P3** and **P3′**. Finally, we show that a solvable non-nilpotent Lie algebra which can be realised as a subalgebra of A_1 is isomorphic to either an $\tilde{\mathcal{L}}_n$ (cf. **E7**) or to an $\mathfrak{r}(i_1, \ldots, i_n)$ (cf. **E8**). This result is more difficult to prove and follows from special properties of the spectrum of semi-simple elements of A_1.

4.1. Non-solvable Lie algebras.

Proposition 4.1. *(Theorem 3 of [11]) If $\mathfrak{g}$ is a semi-simple complex Lie algebra of rank > 1, then $A_1^{\mathfrak{g}} = \emptyset$.*

Proof. Suppose for contradiction that there exists $f \in A_1^{\mathfrak{g}}$. If $\mathfrak{g} = \mathfrak{g}_1 \times \mathfrak{g}_2$ is a direct product of two non-abelian Lie subalgebras then $f(\mathfrak{g}_1)$, which is not abelian, is contained in the commutant of $f(\mathfrak{g}_2)$. By the *ccc* this means that $f(\mathfrak{g}_2) \subseteq \mathbb{C}$ which is impossible since $\mathfrak{g}_2$ is not abelian. If $\mathfrak{g}$ is simple, let $\mathfrak{h} \subset \mathfrak{g}$ be a Cartan subalgebra and let

$$\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$$

be the corresponding triangular decomposition for some choice of simple roots. Let $x \in \mathfrak{n}_+$ be a non-zero highest root vector. The commutant of x in $\mathfrak{g}$, $Z_{\mathfrak{g}}(x)$, contains $\mathfrak{n}_+$ which is not abelian since $\mathrm{rank}(\mathfrak{g}) > 1$. Hence $C(f(x))$ contains $f(\mathfrak{n}_+)$ which is not abelian and so by the *ccc*, $f(x) \in \mathbb{C}$ and $x \in Z_{\mathfrak{g}}$. But $\mathfrak{g}$ is semi-simple and hence $x = 0$ which is a contradiction. ∎

Proposition 4.2. *Let $\mathfrak{g} = \mathfrak{g}_1 \times \mathfrak{z}$ be a reductive complex Lie algebra where $\mathfrak{g}_1$ is semi-simple and $\mathfrak{z}$ is the centre. If $A_1^{\mathfrak{g}} \neq \emptyset$, then*

1. $\mathfrak{g}_1 \cong \mathfrak{sl}(2)$.
2. *For any $f \in A_1^{\mathfrak{g}}$, $f(\mathfrak{z}) \subseteq \mathbb{C}$.*

Proof. By Proposition 4.1 one has $\mathfrak{g}_1 \cong \mathfrak{sl}(2)$. Let z be an element of $\mathfrak{z}$. Then $C(f(z))$ contains $f(\mathfrak{g}_1)$ which is not abelian and hence, by the *ccc*, $f(z)$ is a scalar. ∎

Remark 4.3. A noncommutative algebra A is said to satisfy the commutative centraliser condition if the centraliser of any element not in the centre Z_A is a commutative subalgebra. Proposition 4.2 then remains true if we replace A_1 by A and $\mathbb{C}$ by Z_A. One example of such an algebra is the universal enveloping algebra $\mathcal{U}(\mathfrak{sl}(2))$ (for other examples see [3]).

Remark 4.4. Using the classification theorem of Kac, this result implies that a classical complex simple Lie superalgebra contained in A_1 is isomorphic to $\mathfrak{osp}(1|2)$.

By the theorem of Levi-Malcev, any finite-dimensional non-solvable Lie algebra $\mathfrak{g}$ is isomorphic to a semi-direct product of its radical $\mathfrak{r}$ and a semi-simple subalgebra $\mathfrak{s}$. Suppose $\mathfrak{g}$ is realisable in A_1. Then $\mathfrak{s}$ is isomorphic to $\mathfrak{sl}(2)$ by Proposition 4.1 and by analysing the action of this $\mathfrak{sl}(2)$ on $\mathfrak{r}$ we will show that there are in fact only three possibilities for $\mathfrak{r}$.

Definition 4.5. (i) Three non-zero elements X, Y, H of A_1 are called an $\mathfrak{sl}(2)$ *triplet* if they satisfy the relations:

$$[H, X] = 2X, \quad [H, Y] = -2Y, \quad [X, Y] = H. \tag{4.1}$$

(ii) An element $v \in A_1$ is of weight $\lambda \in \mathbb{C}$ if $[H, v] = \lambda v$.
(iii) The set of elements of weight λ in a linear subspace $E \subseteq A_1$ will be denoted by E_λ. (Note that E_0 is abelian by the *ccc*.)

Lemma 4.6. *Let X, Y, H be an $\mathfrak{sl}(2)$ triplet and let $\mathfrak{l} \subseteq A_1$ be a Lie subalgebra stable under $\mathrm{ad}(X), \mathrm{ad}(Y)$ and $\mathrm{ad}(H)$. Suppose there exists $v \neq 0$ in $\mathfrak{l}$ of weight $\lambda \in \mathbb{C}^*$ such that $[X, v] = 0$. Then $[v, [Y, v]] \neq 0$ and is of weight $2\lambda - 2$.*

Proof. We suppose for contradiction that $[v, [Y, v]] = 0$. Then $C(v)$ contains X and $[Y, v]$. But $[X, [Y, v]] = [H, v] + [Y, [X, v]] = \lambda v \neq 0$ and therefore $C(v)$ is non-abelian. By the *ccc*, $v \in \mathbb{C}$ and so $[H, v] = 0$ which is a contradiction. Hence, $[v, [Y, v]] \neq 0$ and is obviously of weight $2\lambda - 2$. ∎

Proposition 4.7. *Let X, Y, H be an $\mathfrak{sl}(2)$ triplet and let $\mathfrak{l} \subseteq A_1$ be a finite-dimensional Lie subalgebra stable under $\mathrm{ad}(X), \mathrm{ad}(Y)$ and $\mathrm{ad}(H)$. Then $\mathfrak{l}_\lambda = \{0\}$ for $|\lambda| > 2$.*

Proof. Recall first that since $\mathfrak{l}$ is a *finite*-dimensional representation of $\mathfrak{sl}(2)$, $\mathfrak{l}_\lambda = \{0\}$ iff $\mathfrak{l}_{-\lambda} = \{0\}$. Let λ_{max} be the largest eigenvalue of $\mathrm{ad}(H)$ restricted to $\mathfrak{l}$ and suppose that $\lambda_{max} > 2$. Then $\lambda_{max} \geq 2\lambda_{max} - 2$ by Lemma 4.6 which is a contradiction. ■

Proposition 4.8. *Let X, Y, H be an $\mathfrak{sl}(2)$ triplet and let $\mathfrak{r} \subseteq A_1$ be a finite-dimensional solvable Lie subalgebra stable under $\mathrm{ad}(X), \mathrm{ad}(Y)$ and $\mathrm{ad}(H)$. Then $\mathfrak{r}_\lambda = \{0\}$ for $|\lambda| > 1$ and $\mathfrak{r}_0 \subseteq \mathbb{C}$.*

Proof. Recall that if $\mathfrak{g}$ is a Lie algebra the upper central series $(\mathfrak{g}^{(i)})_{i\in\mathbb{N}}$ is defined inductively by: $\mathfrak{g}^{(0)} = \mathfrak{g}$, $\mathfrak{g}^{(i+1)} = [\mathfrak{g}^{(i)}, \mathfrak{g}^{(i)}]$. Suppose for contradiction that $\mathfrak{r}_2 \neq \{0\}$. Then, by repeated application of Lemma 4.6, it follows that $\mathfrak{r}_2^{(i)} \neq \{0\}$ for all positive i. But $\mathfrak{r}$ is solvable so that by definition $\mathfrak{r}^{(m)} = \{0\}$ for some $m \in \mathbb{N}$ and this is a contradiction. Now, since $\mathfrak{r} = \mathfrak{r}_0 \oplus \mathfrak{r}_{-1} \oplus \mathfrak{r}_1$ it follows that $[X, \mathfrak{r}_0] = [Y, \mathfrak{r}_0] = \{0\}$ and therefore, by the *ccc*, that $\mathfrak{r}_0 \subseteq \mathbb{C}$. ■

Proposition 4.9. *Let X, Y, H be an $\mathfrak{sl}(2)$ triplet and let $\mathfrak{r} \subseteq A_1$ be a finite-dimensional solvable Lie subalgebra stable under $\mathrm{ad}(X), \mathrm{ad}(Y)$ and $\mathrm{ad}(H)$. Then $\mathfrak{r}$ is isomorphic to either $\{0\}$, $\mathbb{C}$ or $\mathcal{H}_3$ (the three-dimensional Heisenberg algebra).*

Proof. We have $\mathfrak{r} = \mathfrak{r}_0 \oplus \mathfrak{r}_{-1} \oplus \mathfrak{r}_1$ (by Proposition 4.8), $\mathfrak{r}_0 \subseteq \mathbb{C}$ (by Proposition 4.8) and $[\mathfrak{r}_{-1}, \mathfrak{r}_1] \subseteq \mathfrak{r}_0$ since $\mathrm{ad}(H)$ is a derivation.

Suppose first that $\dim \mathfrak{r}_1 \geq 2$ and let $v \in \mathfrak{r}_1 \setminus \{0\}$. The kernel, $Z_{\mathfrak{r}_{-1}}(v)$, of the linear map $\mathrm{ad}(v) : \mathfrak{r}_{-1} \to \mathfrak{r}_0$ is of dimension ≥ 1 and therefore contains a non-zero vector w. Hence $C(v)$ contains X and w. But $[X, w] \neq 0$ since $\mathrm{ad}(X) : \mathfrak{r}_{-1} \to \mathfrak{r}_1$ is an isomorphism and so $C(v)$ is not abelian. By the *ccc*, v is a scalar which is a contradiction since $[H, v] = v$. Therefore $\dim \mathfrak{r}_{-1} = \dim \mathfrak{r}_1 \leq 1$.

If $\dim \mathfrak{r}_1 = 1$, then $\mathfrak{r}$ is not abelian by Lemma 4.6 and so $\mathfrak{r}_0 = \mathbb{C}$ by Proposition 4.8. Hence $\mathfrak{r}_0$ is the centre of $\mathfrak{r}$ and it is now obvious that $\mathfrak{r}$ is isomorphic to the three dimensional Heisenberg algebra. If $\dim \mathfrak{r}_1 = 0$, then $\mathfrak{r}_0 = \{0\}$ or $\mathbb{C}$ by Proposition 4.8. ■

We can now conclude this subsection with the following theorem:

Theorem 4.10. *Let $\mathfrak{g}$ be a finite-dimensional non-solvable Lie algebra. Then $A_1^{\mathfrak{g}} \neq \emptyset$ iff $\mathfrak{g}$ is isomorphic to one of the following:*

1. $\mathfrak{sl}(2)$,
2. $\mathfrak{sl}(2) \times \mathbb{C}$,
3. $\mathfrak{sl}(2) \ltimes \mathcal{H}_3$.

Proof. Immediate from the Levi-Malcev theorem, Propositions 4.1 and 4.9. Note that by the *ccc*, $\mathfrak{sl}(2) \times \mathcal{H}_3$ cannot be realised as a Lie subalgebra of A_1. ■

4.2. Nilpotent non-abelian Lie algebras.

Theorem 4.11. *Let $\mathfrak{n} \subset A_1$ be a nilpotent, non-abelian Lie subalgebra of dimension n. Then $\mathfrak{n} \cong \mathcal{L}_{n-1}$.*

Proof. Let $(\mathfrak{n}_{(i)})_{i\in\mathbb{N}}$ defined by $\mathfrak{n}_{(0)} = \mathfrak{n}$ and $\mathfrak{n}_{(i+1)} = [\mathfrak{n}, \mathfrak{n}_{(i)}]$ be the lower central series. Let $k \neq 0$ be the unique positive integer such that $\mathfrak{n}_{(k)} \neq \{0\}$ and $\mathfrak{n}_{(k+1)} = \{0\}$. The theorem will essentially be a consequence of the following lemma:

Lemma 4.12. *(a)* $Z_{\mathfrak{n}} = \mathfrak{n}_{(k)} = \mathbb{C}$.

(b) There exist $P, Q \in \mathfrak{n}$ such that $[P,Q] = 1$ and $\mathfrak{n} =< P > \oplus Z_{\mathfrak{n}}(Q)$.

(c) $Z_{\mathfrak{n}}(Q)$ is abelian and $\mathfrak{n}_{(1)} \subset Z_{\mathfrak{n}}(Q)$.

(d) $\dim \mathfrak{n}_{(i)} = n - i - 1$ *for* $1 \leq i \leq k$.

(e) $k = n - 2$.

Proof. (a): Let $z \in Z_{\mathfrak{n}}$. Then $C(z)$ contains $\mathfrak{n}$ which is not commutative. By the *ccc*, z is a scalar and therefore $Z_{\mathfrak{n}} \subseteq \mathbb{C}$. But Z_n contains $\mathfrak{n}_{(k)}$ and so $Z_{\mathfrak{n}} = \mathfrak{n}_{(k)} = \mathbb{C}$.

(b): Since $\mathfrak{n}_{(k)} = [\mathfrak{n}, \mathfrak{n}_{(k-1)}] = \mathbb{C}$, there exist $P \in \mathfrak{n}$, $Q \in \mathfrak{n}_{(k-1)}$ such that $[P,Q] = 1$. It is clear that $< P > \cap Z_{\mathfrak{n}}(Q) = \{0\}$ since P and Q do not commute. We now show that $\mathfrak{n} =< P > + Z_{\mathfrak{n}}(Q)$. Let $v \in \mathfrak{n}$. There exists $\lambda \in \mathbb{C}$ such that $[v, Q] = \lambda$ since $[\mathfrak{n}, \mathfrak{n}_{(k-1)}] = \mathbb{C}$. Then $v - \lambda P \in Z_{\mathfrak{n}}(Q)$ and $v = \lambda P + (v - \lambda P) \in < P > + Z_{\mathfrak{n}}(Q)$.

(c): By the *ccc*, $Z_{\mathfrak{n}}(Q)$ is abelian. Let $z \in Z_{\mathfrak{n}}(Q)$. Then

$$[Q, [P, z]] = [[Q, P], z] + [P, [Q, z]] = 0$$

because $[Q, P] = -1$ and $[Q, z] = 0$. Hence $[P, Z_{\mathfrak{n}}(Q)] \subseteq Z_{\mathfrak{n}}(Q)$ and since $[Z_{\mathfrak{n}}(Q), Z_{\mathfrak{n}}(Q)] = \{0\}$, this means using (b) that $\mathfrak{n}_{(1)} = [\mathfrak{n}, \mathfrak{n}]$ is contained in $Z_{\mathfrak{n}}(Q)$.

(d): Since $\mathfrak{n} =< P > \oplus Z_{\mathfrak{n}}(Q)$ and since $Z_{\mathfrak{n}}(Q)$ is abelian, we have

$$\mathfrak{n}_{(i)} = \mathrm{ad}^i(P)(Z_{\mathfrak{n}}(Q)) \ \forall i \in \mathbb{N}^*. \tag{4.2}$$

If $z \in Z_{\mathfrak{n}}(Q)$ is such that $\mathrm{ad}(P)(z) = 0$ then $z \in C(P) \cap C(Q)$. By property **P3′**, $C(P) \cap C(Q) = \mathbb{C}$ since P and Q do not commute. Hence $z \in \mathbb{C}$ and

$$\mathrm{Ker}\ \mathrm{ad}(P)|_{\mathfrak{n}} =< P > \oplus \mathbb{C}, \quad \mathrm{Ker}\ \mathrm{ad}(P)|_{Z_{\mathfrak{n}}(Q)} = \mathbb{C}. \tag{4.3}$$

This implies (d).

(e): It follows immediately from (d) that $k = n - 2$. ∎

We will now prove that $\mathfrak{n} \cong \mathcal{L}_{n-1}$. Since $\mathfrak{n}_{(k-1)} = \mathrm{ad}^{k-1}(P)(Z_{\mathfrak{n}}(Q))$ and since $Q \in \mathfrak{n}_{(k-1)}$, there exists $w \in Z_{\mathfrak{n}}(Q)$ such that $\mathrm{ad}^{k-1}(P)(w) = Q$. It is clear that $X_1 = w, X_2 = \mathrm{ad}(P)(w), X_3 = \mathrm{ad}^2(P)(w), \ldots, X_{k+1} = \mathrm{ad}^k(P)(w)$ are linearly independent (since $\mathrm{ad}^k(P)(w) \neq 0$ and $\mathrm{ad}^{k+1}(P)(w) = 0$) and by a dimension count, these vectors form a basis of $Z_{\mathfrak{n}}(Q)$. The only non-zero commutation relations of $\mathfrak{n}$ in the basis $X_0 = P, X_1, \ldots, X_{n-1}$ are

$$[X_0, X_i] = X_{i+1} \text{ for } 1 \leq i \leq n-2,$$

which are the standard commutation relations for $\mathcal{L}_{n-1}$ (cf. **E6**). ■

4.3. Solvable non-nilpotent Lie algebras.

Lemma 4.13. *Let $\mathfrak{g} \subset A_1$ be a finite-dimensional Lie subalgebra and let $\mathfrak{g}'$ be its derived algebra. Suppose there exists $h \in \mathfrak{g}$ be such that $\mathrm{ad}(h)|_{\mathfrak{g}}$ is not nilpotent. Let $0, \lambda_1, \ldots, \lambda_k$ be the distinct eigenvalues of $\mathrm{ad}(h)|_{\mathfrak{g}}$ and let $E_0, E_{\lambda_1}, \ldots, E_{\lambda_k}$ be the corresponding eigenspaces.*

1. $\mathfrak{g} = E_0 \oplus E_{\lambda_1} \oplus \ldots \oplus E_{\lambda_k}$
2. $\mathfrak{g}' = (\mathfrak{g}' \cap E_0) \oplus E_{\lambda_1} \oplus \ldots \oplus E_{\lambda_k}$
3. $E_0 =< h >$ *or* $E_0 =< h > \oplus \mathbb{C}$.

Proof. By the theory of endomorphisms,

$$\mathfrak{g} = \mathrm{Ker}\ \mathrm{ad}^{m_0}(h)|_{\mathfrak{g}} \oplus \mathrm{Ker}(\mathrm{ad}(h)|_{\mathfrak{g}} - \lambda_1)^{m_1} \oplus \cdots \oplus \mathrm{Ker}(\mathrm{ad}(h)|_{\mathfrak{g}} - \lambda_k)^{m_k},$$

where the characteristic polynomial of $\mathrm{ad}(h)|_{\mathfrak{g}}$ is $x^{m_0}(x-\lambda_1)^{m_1} \ldots (x-\lambda_k)^{m_k}$. By Proposition 6.5 of [4],

$$\mathrm{Ker}(\mathrm{ad}(h) - \lambda_j)^{m_j} = \mathrm{Ker}(\mathrm{ad}(h) - \lambda_j).$$

Since $\mathrm{ad}(h)$ has a non-zero eigenvalue, $C(h) \neq D(h)$ (cf. subsection 2.2) and therefore, by Theorem 2.2, $C(h) = N(h)$ so that

$$\mathrm{Ker}\ \mathrm{ad}^{m_0}(h) = \mathrm{Ker}\ \mathrm{ad}(h).$$

This proves part 1. To prove part 2 it is sufficient to note that $E_{\lambda_i} \subseteq \mathfrak{g}'$ since $v_i = \frac{1}{\lambda_i}[h, v_i]$ for all $v_i \in E_{\lambda_i}$.

To prove part 3, let $h' \in E_0$. Then $[h', E_{\lambda_i}] \subseteq E_{\lambda_i}$ since $[h, h'] = 0$. Thus there exist $\alpha \in \mathbb{C}$, $v \in E_{\lambda_1} \setminus \{0\}$ such that $[h', v] = \alpha v$. Hence $[h' - \frac{\alpha}{\lambda_1}h, v] = 0$ which means that h and v commute with $h' - \frac{\alpha}{\lambda_1}h$. But $[h, v] \neq 0$ and so by the *ccc*, $h' - \frac{\alpha}{\lambda_1}h \in \mathbb{C}$. ■

Remark 4.14. Taking $\mathfrak{g}$ semi-simple, this lemma provides an alternative proof of Proposition 4.1. However the proof we gave for Proposition 4.1 works for any algebra satisfying the *ccc*, whereas the proof of the theorem above depends on the existence of the Dixmier partition (Theorem 2.2) and other special properties not in general available for an algebra satisfying the *ccc*.

If $\mathfrak{g} \subset A_1$ is a finite-dimensional solvable non-nilpotent Lie subalgebra then, by Theorem 4.11, the derived algebra $\mathfrak{g}'$ is isomorphic to either an $\mathcal{L}_n$ or is abelian. We first treat the case where $\mathfrak{g}' \cong \mathcal{L}_n$.

Theorem 4.15. *Let $\mathfrak{r} \subset A_1$ be a finite-dimensional solvable non-nilpotent Lie subalgebra whose derived algebra $\mathfrak{r}'$ is isomorphic to $\mathcal{L}_n$. Then $\mathfrak{r}$ is isomorphic to $\tilde{\mathcal{L}}_n$.*

Proof. Since $\mathfrak{r} \subset A_1$ is a finite-dimensional non-nilpotent Lie algebra, there exists $h \in \mathfrak{r}$ such that $\mathrm{ad}(h)|_{\mathfrak{r}}$ is not nilpotent by Engel's theorem. By Lemma 4.13, $\mathrm{ad}(h)|_{\mathfrak{r}'}$ is diagonalisable and therefore by Theorem 1 of [6] there exists a basis $X_0, \ldots, X_n$ of $\mathfrak{r}' \cong \mathcal{L}_n$ of eigenvectors of $\mathrm{ad}(h)$ such that the only non-zero commutation relations are

$$\begin{aligned} [X_0, X_i] &= X_{i+1},\ i = 1, .., n-1 \\ [h, X_j] &= \alpha_j X_j,\ j = 0, .., n. \end{aligned} \tag{4.4}$$

Hence X_n is in the centre of $\mathfrak{r}$ and by the *ccc* we must have $X_n \in \mathbb{C}$, $[h, X_n] = 0$ and $\alpha_n = 0$. But $\mathrm{ad}(h)|_{\mathfrak{r}'}$ is a (non-zero) derivation so its eigenvalues satisfy

$$\alpha_{i+1} = \alpha_0 + \alpha_i,\ i = 1, ..., n-1. \tag{4.5}$$

From this it follows that the eigenvalues of $\mathrm{ad}(h)|_{\mathfrak{r}'}$ are $\alpha_0, (1-n)\alpha_0, (2-n)\alpha_0, \ldots, 0$ and hence $\alpha_0 \neq 0$ and the eigenspaces are of dimension 1. The only non-zero commutation relations of $\mathfrak{r}$ are then

$$\begin{aligned} [X_0, X_i] &= X_{i+1},\ i = 1, .., n-1 \\ \left[\frac{1}{\alpha_0} h, X_0\right] &= X_0 \\ \left[\frac{1}{\alpha_0} h, X_i\right] &= -(n-i) X_i\ i = 1, .., n-1 \end{aligned} \tag{4.6}$$

and this shows that $\mathfrak{r}$ is isomorphic to $\tilde{\mathcal{L}}_n$ (cf. **E7**). ∎

It now remains to treat the case where $\mathfrak{g} \subset A_1$ is a finite-dimensional solvable non-nilpotent Lie subalgebra whose derived algebra is abelian. We will need the following lemma (also see Theorem 3.2 of [8]):

Lemma 4.16. *Let $h, X_1, X_2 \in A_1 \setminus \{0\}$ and $(\lambda_1, \lambda_2) \in \mathbb{Z}^2 \setminus \{(0,0)\}$ be such that*

$$[h, X_1] = \lambda_1 X_1,\ [h, X_2] = \lambda_2 X_2,\ \textit{and}\ [X_1, X_2] = 0. \tag{4.7}$$

Then $\lambda_1 \lambda_2 > 0$ and there exists $a \in \mathbb{C}^*$ such that $X_1^{|\lambda_2|} = a X_2^{|\lambda_1|}$.

Proof. Since $[X_1, X_2] = 0$, by Theorem 3.1 of [8] there exist $m, n \in \mathbb{N}^*$ and $\alpha_{ij} \in \mathbb{C}$ not all equal to zero such that

$$\sum_{i=0}^{m} \sum_{j=0}^{n} \alpha_{ij} X_1^i X_2^j = 0 \tag{4.8}$$

This can be rewritten as

$$\sum_{u \in U} \left(\sum_{(i,j): i\lambda_1 + j\lambda_2 = u} \alpha_{ij} X_1^i X_2^j \right) = 0, \tag{4.9}$$

where $U = \{i\lambda_1 + j\lambda_2 \in \mathbb{Z} : \ 0 \leq i \leq m, 0 \leq j \leq n\}$. Since eigenvectors corresponding to distinct eigenvalues of h are linearly independent we deduce that for all $u \in U$,

$$\sum_{(i,j):i\lambda_1+j\lambda_2=u} \alpha_{ij} X_1^i X_2^j = 0.$$

Choose $(i_0, j_0) \in \mathbb{Z}^2$ such that $\alpha_{i_0 j_0} \neq 0$. We set $u_0 = i_0\lambda_1 + j_0\lambda_2$ and

$$S = \Big\{(i,j) \in \mathbb{Z}^2 : \ i\lambda_1 + j\lambda_2 = u_0\Big\} \cap \Big([0,m] \times [0,n]\Big).$$

In $\mathbb{R}^2$ the solutions (x, y) of the equation $x\lambda_1 + y\lambda_2 = u_0$ define an affine line and S is a discrete subset of this line. It is then easy to see that there exist $(i', j') \in \mathbb{Z}^2$ with $j' \geq 0$ and $(i_m, j_m) \in S$ such that

$$i'\lambda_1 + j'\lambda_2 = 0 \tag{4.10}$$

and every element (i,j) of S is of the form $(i,j) = (i_m, j_m) + k_{ij}(i', j')$ for some $k_{ij} \in \mathbb{N}$. One can then write (in the field of fractions of A_1, see page 210 of [4])

$$\sum_{(i,j):i\lambda_1+j\lambda_2=u_0} \alpha_{ij} X_1^i X_2^j = X_1^{i_m} X_2^{j_m} \left(\sum_{(i,j):i\lambda_1+j\lambda_2=u_0} \alpha_{ij} (X_1^{i'} X_2^{j'})^{k_{ij}} \right) = 0.$$

Since by hypothesis $X_1^{i_m} X_2^{j_m} \neq 0$, this means that $X_1^{i'} X_2^{j'}$ satisfies a polynomial equation. Factorising (in the field of fractions of A_1) we deduce that there exists $c \in \mathbb{C}^*$ such that $X_1^{i'} X_2^{j'} = c$. Hence $X_1^{i'\lambda_1} X_2^{j'\lambda_1} = c^{\lambda_1}$, $(X_1^{-\lambda_2} X_2^{\lambda_1})^{j'} = c^{\lambda_1}$ (by (4.10)) and therefore $X_1^{-\lambda_2} X_2^{\lambda_1} \in \mathbb{C}^*$.

If $\lambda_1\lambda_2 \leq 0$ this means that there exists $b \in \mathbb{C}^*$ such that $X_1^{|\lambda_2|} X_2^{|\lambda_1|} = b$ which is clearly impossible since in A_1 the only invertible elements are the scalars (see page 210 of [4]). Hence $\lambda_1\lambda_2 > 0$ and there exists $a \in \mathbb{C}$ such that $X_1^{|\lambda_2|} = a X_2^{|\lambda_1|}$. ■

We can now show that a finite-dimensional solvable non-nilpotent Lie subalgebra of A_1 whose derived algebra is abelian is isomorphic to an $\mathfrak{r}(i_1, \dots i_n)$ (cf. **E8**).

Theorem 4.17. *Let $\mathfrak{r} \subset A_1$ be a finite-dimensional solvable non-nilpotent Lie subalgebra whose derived algebra $\mathfrak{r}'$ is abelian. Then there exist distinct positive integers $i_1, \dots, i_n$ not all zero such that $\mathfrak{r}$ is isomorphic to $\mathfrak{r}(i_1, \dots i_n)$.*

Proof. Let $h \in \mathfrak{r}$ be such that $\mathrm{ad}(h)|_{\mathfrak{r}}$ is not nilpotent, let $0, \lambda_1, \dots, \lambda_k$ be its distinct eigenvalues and let $E_0, E_{\lambda_1}, \dots, E_{\lambda_k}$ be the corresponding eigenspaces. By Lemma 4.13,

$$\mathfrak{r} = E_0 \oplus E_{\lambda_1} \oplus E_{\lambda_2} \oplus \cdots \oplus E_{\lambda_k}$$

and since $E_{\lambda_1} \oplus E_{\lambda_2} \oplus \dots E_{\lambda_k} \subseteq \mathfrak{r}'$ is abelian we deduce that

$$\mathfrak{r}' = E_{\lambda_1} \oplus E_{\lambda_2} \oplus \cdots \oplus E_{\lambda_k}.$$

It is clear that if the centre of $\mathfrak{r}$ is $\mathbb{C}$, then by Lemma 4.13, $\mathfrak{r} \cong \tilde{\mathfrak{r}} \times \mathbb{C}$ where $\tilde{\mathfrak{r}} = \langle h \rangle \oplus E_{\lambda_1} \oplus E_{\lambda_2} \oplus \dots E_{\lambda_k}$ satisfies the hypothesis of the theorem and has trivial centre. Therefore to prove the theorem it is enough to consider the case where $\mathfrak{r}$ has trivial centre (see **E8**) .

First, note that by Lemma 4.13(3) we have $E_0 =< h >$. Next, since $h \in \Delta_3 \cup \Delta_4$ (by Remark 3.2), there exists $\rho \in \mathbb{C}^*$ such that the eigenvalues of $\mathrm{ad}(h)$ are integer multiples of ρ (Corollary 9.3 of [4] if $h \in \Delta_3$ and Theorem 1.3 of [10] if $h \in \Delta_4$). Hence $\frac{1}{\rho}\mathrm{ad}(h)|_{\mathfrak{r}}$ has integer eigenvalues. Choose $X_1 \in E_{\lambda_i}$ and $X_2 \in E_{\lambda_j}$. Since we know that $[E_{\lambda_i}, E_{\lambda_j}] = 0$, applying Lemma 4.16 to $\frac{1}{\rho}h, X_1, X_2$ we deduce that $\frac{\lambda_i}{\rho}$ and $\frac{\lambda_j}{\rho}$, hence λ_i and λ_j, are of the same sign. Similarly, if $X_1, X_2 \in E_{\lambda_i}$, then applying Lemma 4.16 to $\frac{1}{\lambda_1}h, X_1, X_2$ it follows (since the eigenvalue is one) that $X_1 = aX_2$ for some $a \in \mathbb{C}^*$ and hence that E_{λ_i} is of dimension one. It is now clear that $\mathfrak{r}$ is isomorphic to $\mathfrak{r}(|\frac{\lambda_1}{\rho}|, \ldots, |\frac{\lambda_k}{\rho}|)$. ■

Corollary 4.18. *Let $\mathfrak{g}$ be a complex Lie algebra which contains a finite-dimensional subalgebra not isomorphic to one of the following: $\mathfrak{sl}(2)$, $\mathfrak{sl}(2)\times\mathbb{C}$, $\mathfrak{sl}(2)\ltimes\mathcal{H}_3$, $\mathcal{L}_n$, $\tilde{\mathcal{L}}_n$ or $\mathfrak{r}(i_1, \ldots, i_n)$. Then $A_1^{\mathfrak{g}} = \emptyset$.*

Proof. Immediate from Theorems 4.10, 4.11, 4.15 and 4.17. ■

Corollary 4.19. *Let $\tilde{\mathfrak{g}}$ be an affine Kac-Moody algebra associated to the simple complex Lie algebra $\mathfrak{g}$. Then $A_1^{\tilde{\mathfrak{g}}} = \emptyset$.*

Proof. Since $\tilde{\mathfrak{g}}$ contains a reductive Lie algebra isomorphic to $\mathfrak{g}\times\mathbb{C}^2$ (generated by $\mathfrak{g}$, the derivation and the central element), the result follows immediately from Corollary 4.18. ■

4.4. Finite-dimensional Lie subalgebras of Der(A_1).

In this subsection we find all finite-dimensional Lie algebras that can be realised as subalgebras of $\mathrm{Der}(A_1)$ and we give some examples of Lie subgroups of $\mathrm{Aut}(A_1)$ which exponentiate them.

Theorem 4.20. *Let $\mathfrak{g} \subseteq \mathrm{Der}(A_1)$ be a finite-dimensional non-abelian Lie subalgebra. Then $\mathfrak{g}$ is isomorphic to one of the following:*

1) $\mathfrak{sl}(2)$,
2) $\mathfrak{sl}(2) \ltimes \mathbb{C}^2$,
3) $\mathcal{L}_n$ $(n \geq 2)$,
4) $\tilde{\mathcal{L}}_n/\mathbb{C}$ $(n \geq 2)$,
5) $\mathfrak{r}(i_1, \ldots, i_n)$ $(0 < i_1 < \cdots < i_n)$.

Proof. Consider the commutative diagram:

$$\begin{array}{ccccc} A_1 & \xrightarrow{\pi} & A_1/\mathbb{C} & \xrightarrow{\mathrm{ad}} & \mathrm{Der}(A_1) \\ \uparrow & & \uparrow & & \uparrow \\ \pi^{-1}(\mathrm{ad}^{-1}(\mathfrak{g})) & \xrightarrow{\pi} & \mathrm{ad}^{-1}(\mathfrak{g}) & \xrightarrow{\mathrm{ad}} & \mathfrak{g} \end{array}$$

In this diagram, $\pi : A_1 \to A_1/\mathbb{C}$ is a surjective Lie algebra homomorphism and $\mathrm{ad} : A_1/\mathbb{C} \to \mathrm{Der}(A_1)$ is a Lie algebra isomorphism (see page 210 of [4]).

If $\mathfrak{g}$ is not abelian then $\pi^{-1}(\mathrm{ad}^{-1}(\mathfrak{g}))$ is a finite-dimensional non-abelian subalgebra of A_1 containing $\mathbb{C}$ and $\mathfrak{g} \cong \pi^{-1}(\mathrm{ad}^{-1}(\mathfrak{g}))/\mathbb{C}$. By the results of the previous subsections $\pi^{-1}(\mathrm{ad}^{-1}(\mathfrak{g}))$ is isomorphic to one of the following: $\mathfrak{sl}(2)\times\mathbb{C}$, $\mathfrak{sl}(2) \ltimes \mathcal{H}_3$, an $\mathcal{L}_n$, an $\tilde{\mathcal{L}}_n$ or an $\mathfrak{r}(0, i_2, \ldots, i_n)$. Taking the quotient by $\mathbb{C}$ this shows that $\mathfrak{g}$ is isomorphic to one of: $\mathfrak{sl}(2)$, $\mathfrak{sl}(2) \ltimes \mathbb{C}^2$, an $\mathcal{L}_n/\mathbb{C}$, an $\tilde{\mathcal{L}}_n/\mathbb{C}$ or an $\mathfrak{r}(i_2, \ldots, i_n)$ (cf. **E8**). Since $\mathcal{L}_n/\mathbb{C} \cong \mathcal{L}_{n-1}$ this proves the theorem. ■

Remark 4.21. The derived Lie algebra of $\tilde{\mathcal{L}}_n/\mathbb{C}$ is isomorphic to $\mathcal{L}_{n-1}$ but $\tilde{\mathcal{L}}_n/\mathbb{C}$ is isomorphic to neither $\tilde{\mathcal{L}}_{n-1}$ nor $\mathcal{L}_n$, both of whose derived algebras are also isomorphic to $\mathcal{L}_{n-1}$. A basis for $\tilde{\mathcal{L}}_n/\mathbb{C}$ is $\bar{h}, \bar{X}_0, \ldots, \bar{X}_{n-1}$ (see **E7** for notations) and the only non-zero commutation relations are $[\bar{h}, \bar{X}_0] = \bar{X}_0$, $[\bar{h}, \bar{X}_k] = -(n-k)\bar{X}_k$ for $k = 1, \ldots, n-1$ and $[\bar{X}_0, \bar{X}_k] = \bar{X}_{k+1}$ for $k = 1, \ldots, n-2$. In particular, $\tilde{\mathcal{L}}_n/\mathbb{C}$ is not nilpotent and its centre is trivial.

This theorem implies that if G is a finite-dimensional, connected and in some sense Lie subgroup of $\mathrm{Aut}(A_1)$, then G is either abelian or a discrete quotient of $SL(2,\mathbb{C})$, $SL(2,\mathbb{C}) \ltimes \mathbb{C}^2$ or of the simply-connected Lie groups L_n, $\tilde{L}_n/\mathbb{C}$ and $R(i_1, \ldots, i_n)$ corresponding respectively to the Lie algebras $\mathcal{L}_n$, $\tilde{\mathcal{L}}_n/\mathbb{C}$ and $\mathfrak{r}(i_1, \ldots, i_n)$. We now give explicit constructions of $R(i_1, \ldots, i_n)$, $\tilde{L}_n$ and L_n and show that the groups $SL(2,\mathbb{C})$, $SL(2,\mathbb{C}) \ltimes \mathbb{C}^2$, $R(i_1, \ldots, i_n)/\mathbb{Z}$, $\tilde{L}_n/(\mathbb{C}\times\mathbb{Z})$ and L_n can be holomorphically embedded in $\mathrm{Aut}(A_1)$.

E9 Define $\hat{\alpha}_1 : SL(2,\mathbb{C}) \to \mathrm{Aut}(A_1)$ by

$$\hat{\alpha}_1(\begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix})(p) = a_2 q + a_4 p, \quad \hat{\alpha}_1(\begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix})(q) = a_1 q + a_3 p$$

and $\hat{\alpha}_2 : \mathbb{C}^2 \to \mathrm{Aut}(A_1)$ by

$$\hat{\alpha}_2(\begin{pmatrix} b_1 \\ b_2 \end{pmatrix})(p) = p - b_1, \quad \hat{\alpha}_2(\begin{pmatrix} b_1 \\ b_2 \end{pmatrix})(p) = q + b_2.$$

Then one checks that $\hat{\alpha}_1$ and $\hat{\alpha}_2$ are injective group homomorphisms such that

$$\hat{\alpha}_2(\begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}\begin{pmatrix} b_1 \\ b_2 \end{pmatrix}) = \hat{\alpha}_1(\begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix})\ \hat{\alpha}_2(\begin{pmatrix} b_1 \\ b_2 \end{pmatrix})\ \hat{\alpha}_1(\begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix})^{-1}$$

and so there exists a unique injective group homomorphism $\hat{\alpha} : SL(2,\mathbb{C}) \ltimes \mathbb{C}^2 \to \mathrm{Aut}(A_1)$ extending $\hat{\alpha}_1$ and $\hat{\alpha}_2$.

Remark 4.22. The derivative of $\hat{\alpha}_1 : SL(2,\mathbb{C}) \to \mathrm{Aut}(A_1)$ is $\mathrm{ad} \circ f_I$ (cf. Example 5.3) in the following sense: if $z \in \mathfrak{sl}(2)$, then $\Phi_z : \mathbb{C} \to \mathrm{Aut}(A_1)$ defined by $\Phi_z(t) = \hat{\alpha}_1(e^{t\,z})$ is a group homomorphism which exponentiates $\mathrm{ad}(f_I(z))$ in the sense of Definition 2.4. Similarly one sees that $\hat{\alpha}_2(\begin{pmatrix} b_1 \\ b_2 \end{pmatrix}) = e^{\mathrm{ad}(b_1 q + b_2 p)}$.

E10 For $i_1, \ldots, i_n \in \mathbb{N}^*$ we define $R(i_1, \ldots, i_n)$ to be the (simply-connected) Lie group whose underlying manifold is $\mathbb{C}^{n+1}$ and whose group law is

$$(a_1, \ldots, a_n, v).(a'_1, \ldots, a'_n, v') = (a_1 + a'_1 e^{-vi_1}, \ldots, a_n + a'_n e^{-vi_n}, v + v')$$

One checks that the Lie algebra of $R(i_1, \ldots, i_n)$ is isomorphic to $\mathfrak{r}(i_1, \ldots, i_n)$ and that $\Phi : R(i_1, \ldots, i_n) \to \mathrm{Aut}(A_1)$ defined by

$$\begin{aligned}\Phi((a_1, \ldots, a_n, v))(p) &= e^{-v} p, \\ \Phi((a_1, \ldots, a_n, v))(q) &= e^{v}(q + \sum_{k=1}^{n} \frac{a_k}{(i_k - 1)!} p^{i_k - 1}).\end{aligned}$$

is a group homomorphism with discrete kernel isomorphic to $\mathbb{Z}$. Note that $\Phi((a_1, \ldots, a_n, v)) = e^{\mathrm{ad}(\frac{a_1}{i_1!} p^{i_1} + \cdots + \frac{a_n}{i_n!} p^{i_n})} e^{\mathrm{ad}(vpq)}$ which means that Φ exponentiates the formulae of example **E8**.

E11 We define $\tilde{L}_n$ to be the (simply-connected) Lie group whose underlying manifold is $\mathbb{C}^{n+2}$ and whose group law is

$$(a_1, \ldots, a_n, t, v).(a'_1, \ldots, a'_n, t', v') = (a''_1, \ldots, a''_n, t'', v'') \tag{4.11}$$

where

$$\begin{aligned}a''_k &= a_k e^{(n-k)v'} + \sum_{j=1}^{k-1} \frac{t^{k-j}}{(k-j)!} a'_j e^{-(k-j)v'} + a'_k, \\ t'' &= t' + t e^{-v'}, \\ v'' &= v + v'.\end{aligned}$$

One checks that the Lie algebra of $\tilde{L}_n$ is isomorphic to $\tilde{\mathcal{L}}_n$, that $R(n-1, n-2, \ldots, 0)$ is a subgroup of $\tilde{L}_n$ by the inclusion:

$$(a_1, a_2, \ldots, a_n, v) \mapsto (a_1 e^{(n-1)v}, a_2 e^{(n-2)v}, \ldots, a_n, 0, v)$$

and that $\tilde{L}_n / \{(0, \ldots, 0, a_n, 0, 0) : \ a_n \in \mathbb{C}\}$ is a simply-connected Lie group whose Lie algebra is isomorphic to $\tilde{\mathcal{L}}_n / \mathbb{C}$. In fact one can extend the map Φ of **E10** to $\tilde{\Phi} : \tilde{L}_n \to \mathrm{Aut}(A_1)$ by setting $\tilde{\Phi}((a_1, \ldots, a_n, t, v)) = \Phi((a_1 e^{(-n+1)v}, a_2 e^{(-n+2)v}, \ldots, a_n, v)) e^{-t\mathrm{ad}(q)}$. Explicitly, this gives

$$\begin{aligned}\tilde{\Phi}((a_1, \ldots, a_n, t, v))(p) &= e^{-v} p + t, \\ \tilde{\Phi}((a_1, \ldots, a_n, t, v))(q) &= e^{v} \left(q + \sum_{k=1}^{n-1} \frac{a_k e^{(-n+k)v}}{(n-k-1)!} p^{n-k-1} \right).\end{aligned} \tag{4.12}$$

Using (4.11) and (4.12) one checks that $\tilde{\Phi}$ is a group homomorphism whose kernel is the subgroup $\{(0, \ldots, 0, a_n, 0, 2\pi i k) : \ k \in \mathbb{Z}, a_n \in \mathbb{C}\}$, isomorphic to the direct product $\mathbb{C} \times \mathbb{Z}$. Finally, note that the subgroup L_n of $\tilde{L}_n$ defined by $v = 0$ is simply-connected and that its Lie algebra is isomorphic to $\mathcal{L}_n$. The restriction of $\tilde{\Phi}$ to L_n factors to give an injection of $L_n / \mathbb{C}$ in $\mathrm{Aut}(A_1)$ and $L_n / \mathbb{C}$ is a simply-connected Lie group whose Lie algebra is isomorphic to $\mathcal{L}_{n-1}$.

5. A family $\mathcal{N}$ of $\mathfrak{sl}(2)$ realisations in the Weyl algebra

In this section we study some explicit examples of realisations of $\mathfrak{sl}(2)$ in A_1 first given by Joseph in [9]. We show that distinct members of this family are inequivalent under the action of the group $\mathrm{Aut}(A_1) \times \mathrm{Aut}(\mathfrak{sl}(2))$.

We denote by $\mathcal{U}(\mathfrak{sl}(2)))$ the universal enveloping algebra of $\mathfrak{sl}(2)$ and by $\hat{f} : \mathcal{U}(\mathfrak{sl}(2))) \to A_1$ the natural extension of $f \in A_1^{\mathfrak{sl}(2)}$ to $\mathcal{U}(\mathfrak{sl}(2)))$. We also set

$$e_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, e_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, e_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and the images of e_+, e_-, e_0 under the natural inclusion $\mathfrak{sl}(2) \subset \mathcal{U}(\mathfrak{sl}(2))$ will be respectively denoted by x, y, h.

Definition 5.1. The group $\mathrm{Aut}(A_1) \times \mathrm{Aut}(\mathcal{U}(\mathfrak{sl}(2)))$ acts on $A_1^{\mathfrak{sl}(2)}$ by:

$$(\alpha, w).f = \alpha \circ \hat{f} \circ w^{-1}|_{\mathfrak{sl}(2)} \tag{5.1}$$

where $\alpha \in \mathrm{Aut}(A_1)$, $w \in \mathrm{Aut}(\mathcal{U}(\mathfrak{sl}(2)))$ and $f \in A_1^{\mathfrak{sl}(2)}$.

Remark 5.2. The natural inclusion $\mathrm{Aut}(\mathfrak{sl}(2)) \hookrightarrow \mathrm{Aut}(\mathcal{U}(\mathfrak{sl}(2)))$ is strict, see for example equations (6.6).

The set $A_1^{sl(2)}$ is in bijection with the set of $\mathfrak{sl}(2)$ triplets. If X, Y, H is an $sl(2)$ triplet, $f : \mathfrak{sl}(2) \to A_1$ given by $f(e_+) = X, f(e_-) = Y, f(e_0) = H$ is a Lie algebra homomorphism and conversely, $f(e_+), f(e_-), f(e_0)$ is an $\mathfrak{sl}(2)$ triplet if $f : \mathfrak{sl}(2) \to A_1$ is a Lie algebra homomorphism.

Example 5.3. From **E1** we know that

$$X = -\frac{1}{2}q^2,\ Y = \frac{1}{2}p^2,\ H = \frac{1}{2}(pq + qp),$$

form an $\mathfrak{sl}(2)$ triplet. We denote by $f_I : \mathfrak{sl}(2) \to A_1$ the corresponding Lie algebra homomorphism.

This realisation of $\mathfrak{sl}(2)$ has the following properties:

(i) the set of eigenvalues of $\mathrm{ad}(f_I(e_0))$ is $\mathbb{Z}$;

(ii) $f_I(n) \in \Delta_1$ for any nilpotent $n \in \mathfrak{sl}(2) \setminus \{0\}$;

(iii) $f_I(s) \in \Delta_3$ for any semi-simple $s \in \mathfrak{sl}(2) \setminus \{0\}$.

Property (i) follows from the equation $\mathrm{ad}(f_I(e_0))(p^i q^j) = (j - i)p^i q^j$; (ii) and (iii) follow from Proposition 2.3 and Remark 3.2 since $f_I(\mathfrak{sl}(2)) = W_2$.

Remark 5.4. In the standard representation of A_1, the operators X, Y, H of Example 5.3 are represented by the differential operators:

$$X = -\frac{1}{2}x^2,\ Y = \frac{1}{2}\frac{d^2}{dx^2},\ H = x\frac{d}{dx} + \frac{1}{2}.$$

It is interesting to notice that one can obtain $\mathfrak{sl}(2)$ triplets represented by differential operators of arbitrary order by applying appropriate automorphisms of A_1 to this example.

Example 5.5. (See [9].) For $b \in \mathbb{C}$, the three elements of A_1

$$X = (b + pq)q, \ Y = -p, \ H = 2pq + b,$$

form an $\mathfrak{sl}(2)$ triplet. We denote by $f_{II}^b : \mathfrak{sl}(2) \to A_1$ the corresponding Lie algebra homomorphism.

This realisation of $\mathfrak{sl}(2)$ is fundamentally different from Example 5.3 with respect to each of the three properties above:

(i) the set of eigenvalues of $\mathrm{ad}(f_{II}^b(e_0))$ is $2\mathbb{Z}$;

(ii) there exists a nilpotent $n \in \mathfrak{sl}(2)$ such that $f_{II}^b(n) \in \Delta_2$;

(iii) there exists a semi-simple $s \in \mathfrak{sl}(2)$ such that $f_{II}^b(s) \in \Delta_4$.

Property (i) is obvious and properties (ii) and (iii) are consequences of the

Lemma 5.6. $\lambda X + \mu Y + \nu H \in \Delta_1 \cup \Delta_3$ *iff* $\lambda = 0$.

Proof. ($\Rightarrow$): One shows by induction that

$$\mathrm{ad}^n(\lambda X + \mu Y + \nu H)(q) = n!\lambda^n a^n q^{n+1} + h_n(q)$$

where $h_n(q)$ is a polynomial in q of degree at most n. It then follows that if $\lambda \neq 0$, $\big(\mathrm{ad}^n(\lambda X + \mu Y + \nu H)(q)\big)_{n \in \mathbb{N}}$ spans an infinite-dimensional vector space and hence $\lambda X + \mu Y + \nu H \notin \Delta_1 \cup \Delta_3$ by Corollary 6.6 of [4].

($\Leftarrow$): If $\lambda = 0$ the result follows from Proposition 2.3. ■

Remark 5.7. (See [9].) For $b \in \mathbb{C}$, the three elements of A_1

$$X = -q, \ Y = p(b + pq), \ H = 2pq + b$$

form an $\mathfrak{sl}(2)$ triplet. The corresponding Lie homomorphism is easily seen to be $(\alpha, \tau) \cdot f_{II}^{-(b+2)}$ where $\tau \in \mathrm{Aut}(\mathfrak{sl}(2))$ and $\alpha \in \mathrm{Aut}(A_1)$ are given by $\tau(e_+) = e_-$, $\tau(e_-) = e_+$ and $\alpha(p) = q$, $\alpha(q) = -p$. However, note that there does not exist $\beta \in \mathrm{Aut}(A_1)$ and $b' \in \mathbb{C}$ such that $(\beta, Id) \cdot f_{II}^{b'} = (\alpha, \tau) \cdot f_{II}^{-(b+2)}$ since $(\alpha, \tau) \cdot f_{II}^{-(b+2)}(e_-) \in \Delta_2$ and $f_{II}^{b'}(e_-) \in \Delta_1$. This shows that the $\mathfrak{sl}(2)$ triplets (1) and (2) of Lemma 2.4 in [9] are equivalent under the group $\mathrm{Aut}(A_1) \times \mathrm{Aut}(\mathfrak{sl}(2))$ but not equivalent under $\mathrm{Aut}(A_1)$.

Definition 5.8. Define $\mathcal{N} \subseteq A_1^{\mathfrak{sl}(2)}$ by $\mathcal{N} = \{f_I, f_{II}^b : b \in \mathbb{C}\}$.

The following lemma, which is a variant of Lemma 2.4 of Joseph [9], shows that the $\mathfrak{sl}(2)$ triplets in $\mathcal{N}$ are essentially the only $\mathfrak{sl}(2)$ triplets whose semi-simple element is of the form $\mu pq + \nu$.

Lemma 5.9. *An $\mathfrak{sl}(2)$ triplet X, Y, H in A_1 such that $H = \mu pq + \nu$ with $\mu, \nu \in \mathbb{C}$ is equivalent to an $\mathfrak{sl}(2)$ triplet in $\mathcal{N}$ under the action* $\mathrm{Aut}(A_1) \times \mathrm{Aut}(\mathfrak{sl}(2))$.

Proof. It is easy to see that the $\mathfrak{sl}(2)$ commutations relations imply that $\mu = \pm 1, \pm 2$. The lemma now follows from Joseph [9] and Remark 5.7. ∎

5.1. Inequivalence of elements of $\mathcal{N}$ under $\mathrm{Aut}(A_1) \times \mathrm{Aut}(\mathfrak{sl}(2))$.

In [9] Joseph showed that $A_1^{\mathfrak{sl}(2)}$ is a disjoint union $S_1 \cup S_2$ where $f \in S_1$ (resp. $f \in S_2$) iff the set of eigenvalues of $\mathrm{ad}(f(e_0))$ is $2\mathbb{Z}$ (resp. $\mathbb{Z}$). Furthermore, he subdivided S_1 (resp. S_2) into a disjoint union $S_{11} \cup S_{12} \cup \cdots \cup S_{1\infty}$ (resp. $S_{21} \cup S_{22} \cup \dots$), showed that for $1 < r < \infty$ (resp. $1 \le r < \infty$) the S_{1r} (resp. S_{2r}) are stable under the action of $\mathrm{Aut}(A_1) \times \mathrm{Aut}(\mathcal{U}(\mathfrak{sl}(2)))$ and that $f_I \in S_{21}$ and $f_{II}^b \in S_{11}$. This means in particular that f_I and f_{II}^b are inequivalent under $\mathrm{Aut}(A_1) \times \mathrm{Aut}(\mathfrak{sl}(2))$. We will prove that f_{II}^b and $f_{II}^{b'}$ are inequivalent under $\mathrm{Aut}(A_1) \times \mathrm{Aut}(\mathfrak{sl}(2))$ if $b \neq b'$ so that distinct elements of $\mathcal{N}$ are inequivalent under the action of this group. It is not known whether distinct elements of $\mathcal{N}$ are inequivalent under the action of $\mathrm{Aut}(A_1) \times \mathrm{Aut}(\mathcal{U}(\mathfrak{sl}(2)))$.

Definition 5.10. Let $f \in A_1^{\mathfrak{sl}(2)}$, let $\hat{f} : \mathcal{U}(\mathfrak{sl}(2)) \to A_1$ be the natural extension of f to the universal enveloping algebra and let $Q_f = \hat{f}(Q)$ be the image by $\hat{f}$ of the Casimir operator $Q = \frac{1}{2}h^2 + xy + yx$ of $\mathfrak{sl}(2)$.

It is easy to see that if $f, g \in A_1^{\mathfrak{sl}(2)}$ are equivalent under $\mathrm{Aut}(A_1) \times \mathrm{Aut}(\mathcal{U}(\mathfrak{sl}(2)))$ then $\hat{f}(Q) = \hat{g}(Q)$ (cf. [2] or [5]). Calculation shows that $Q_{f_{II}^b} = b(\frac{1}{2}b + 1)$ and thus:

Proposition 5.11. *Let $b, b' \in \mathbb{C}$ be such that $b' \neq b$ and $b' \neq -b-2$. Then $f_{II}^{b'}$ is not equivalent to f_{II}^b under* $\mathrm{Aut}(A_1) \times \mathrm{Aut}(\mathcal{U}(\mathfrak{sl}(2)))$.

This means that the only elements of $\mathcal{N}$ which can be equivalent under the action of $\mathrm{Aut}(A_1) \times \mathrm{Aut}(\mathcal{U}(\mathfrak{sl}(2)))$, *a fortiori* under the action $\mathrm{Aut}(A_1) \times \mathrm{Aut}(\mathfrak{sl}(2))$, are f_{II}^b and f_{II}^{-b-2}.

Proposition 5.12. *If $b \neq -1$, f_{II}^b is not equivalent to f_{II}^{-b-2} under* $\mathrm{Aut}(A_1) \times \mathrm{Aut}(\mathfrak{sl}(2))$.

Proof: The $\mathfrak{sl}(2)$ triplets corresponding to f_{II}^b and f_{II}^{-b-2} are

$$\begin{array}{ll} X = (b+pq)q & X' = (-b-2+pq)q \\ Y = -p & Y' = -p \\ H = 2pq + b & H' = 2pq - b - 2. \end{array} \tag{5.2}$$

Suppose there exists $(\alpha, w) \in \mathrm{Aut}(A_1) \times \mathrm{Aut}(\mathfrak{sl}(2))$ such that $f_{II}^b = (\alpha, w) \cdot f_{II}^{-b-2}$, i.e., such that $\alpha \circ \cdot f_{II}^{-b-2} = f_{II}^b \circ w$. Then writing $p' = \alpha(p), q' = \alpha(q)$ and $w = \mathrm{Ad}(g)$, this gives (see the Appendix)

$$\begin{array}{rl} (-b-2+p'q')q' & = a_1^2(b+pq)q - a_3^2(-p) - a_1a_3(2pq+b) \\ -p' & = -a_2^2(b+pq)q + a_4^2(-p) + a_2a_4(2pq+b) \\ 2p'q' - b - 2 & = -2a_1a_2(b+pq)q + 2a_3a_4(-p) + (a_1a_4 + a_2a_3)(2pq+b). \end{array} \tag{5.3}$$

Substituting the second equation in the third equation, we obtain

$$\begin{aligned} & -2[-a_2^2(b+pq)q + a_4^2(-p) + a_2a_4(2pq+b)]q' - b - 2 \\ &= -2a_1a_2(b+pq)q + 2a_3a_4(-p) + (a_1a_4 + a_2a_3)(2pq+b). \end{aligned} \tag{5.4}$$

If $a_2 \neq 0$, the expansion of q' in the standard basis can have only a constant term, otherwise the term $2a_2^2(pq)qq'$ on the LHS contains terms which are not present in the RHS; but then $[p', q'] = 0$ which is a contradiction and hence $a_2 = 0$. Since $a_1a_4 - a_2a_3 = 1$ this implies that $a_1a_4 = 1$. Equation 5.4 now reduces to

$$-2[a_4^2(-p)]q' - b - 2 = 2a_3a_4(-p) + (2pq + b) \tag{5.5}$$

and equating the constant term on both sides of (5.5) gives $b = -1$, a contradiction.

Proposition 5.13. *Distinct elements of $\mathcal{N}$ are not equivalent under the action of* $\mathrm{Aut}(A_1) \times \mathrm{Aut}(\mathfrak{sl}(2))$.

Proof. Immediate from Proposition 5.11 and Proposition 5.12. ∎

6. The orbit of $\mathcal{N}$ under $\mathrm{Aut}(A_1) \times \mathrm{Aut}(\mathfrak{sl}(2))$

In this section we give various characterisations of $\mathcal{D}$, the orbit of $\mathcal{N}$ under $\mathrm{Aut}(A_1) \times \mathrm{Aut}(\mathfrak{sl}(2))$, in terms of the Dixmier partition and also in terms of exponentiation. We then calculate the isotropy groups of each of the elements $\mathcal{N}$ and finally, for the sake of completeness, we give an explicit example of an element in $A_1^{\mathfrak{sl}(2)}$ which is not in the orbit of $\mathcal{N}$.

6.1. Characterisations in terms of the Dixmier partition.

Definition 6.1.

$$\begin{array}{rcl}
\mathcal{D} &=& \{f \in A_1^{\mathfrak{sl}(2)} : \exists(\alpha, w) \in \mathrm{Aut}(A_1) \times \mathrm{Aut}(\mathfrak{sl}(2)) \text{ s.t } (\alpha, w)\cdot f \in \mathcal{N}\} \\
\mathcal{D}_3 &=& \{f \in A_1^{\mathfrak{sl}(2)} : \exists z \in \mathfrak{sl}(2) \setminus \{0\} \text{ s.t } f(z) \in \Delta_3\} \\
\mathcal{D}'_3 &=& \{f \in A_1^{\mathfrak{sl}(2)} : \exists z \in \mathfrak{sl}(2) \setminus \{0\} \text{ s.t } \mathrm{ad}(f(z)) \text{ has an eigenvector in } \Delta_3\} \\
\mathcal{D}_1 &=& \{f \in A_1^{\mathfrak{sl}(2)} : \exists z \in \mathfrak{sl}(2) \setminus \{0\} \text{ s.t } f(z) \in \Delta_1\} \\
\mathcal{D}'_1 &=& \{f \in A_1^{\mathfrak{sl}(2)} : \exists z \in \mathfrak{sl}(2) \setminus \{0\} \text{ s.t } \mathrm{ad}(f(z)) \text{ has an eigenvector in } \Delta_1\} \\
\mathcal{E} &=& \{f \in A_1^{\mathfrak{sl}(2)} : \exists z \in \mathfrak{sl}(2) \setminus \{0\} \text{ s.t } \mathrm{ad}f(z) \text{ can be exponentiated }\}
\end{array}$$

We now show that the above sets are the same. This means in particular that $\mathcal{N}$ is a set of canonical forms for the action of $\mathrm{Aut}(A_1) \times \mathrm{Aut}(\mathfrak{sl}(2))$ on $\mathcal{D}_3, \mathcal{D}'_3, \mathcal{D}_1, \mathcal{D}'_1$ and $\mathcal{E}$.

Theorem 6.2. $\mathcal{D} = \mathcal{D}_3 = \mathcal{D}'_3 = \mathcal{D}_1 = \mathcal{D}'_1 = \mathcal{E}$.

Proof. First, note that $\mathcal{D}_3, \mathcal{D}'_3, \mathcal{D}_1, \mathcal{D}'_1$ and $\mathcal{E}$ are stable under the action of $\mathrm{Aut}(A_1) \times \mathrm{Aut}(\mathfrak{sl}(2))$ and that the inclusions $\mathcal{D}_3 \subseteq \mathcal{D}'_3$ and $\mathcal{D}_1 \subseteq \mathcal{D}'_1$ are obvious. We have already seen that $\mathcal{N} \subseteq \mathcal{D}_3$, $\mathcal{N} \subseteq \mathcal{D}_1$ and $\mathcal{N} \subseteq \mathcal{E}$ (cf. Examples 5.3, 5.5 and Proposition 2.7) and hence, to prove the theorem it will be sufficient to show that $\mathcal{D}_3 \subseteq \mathcal{D}$, $\mathcal{D}'_3 \subseteq \mathcal{D}_3$, $\mathcal{D}'_1 \subseteq \mathcal{D}_3$ and $\mathcal{E} \subseteq \mathcal{D}_1$.

$\boldsymbol{\mathcal{D}_3 \subseteq \mathcal{D}}$: Let $f \in \mathcal{D}_3$. By hypothesis there exists $z \in \mathfrak{sl}(2)$ such that $f(z) \in \Delta_3$ and z must be semi-simple by Remark 3.2. By rescaling we can always suppose that the eigenvalues (in $\mathfrak{sl}(2)$) of $\mathrm{ad}(z)$ are $-2, 0$ and 2 and then there exists $w \in \mathrm{Aut}(\mathfrak{sl}(2))$ such that $w^{-1}(e_0) = z$. By Theorem 9.2 of [4], there exist $\alpha \in \mathrm{Aut}(A_1), \mu \in \mathbb{C}^*$ and $\nu \in \mathbb{C}$ such that $(\alpha, w) \cdot f(e_0) = \alpha \circ f \circ w^{-1}(e_0) = \mu pq + \nu$. By Lemma 5.9, there exists $(\alpha', w') \in \mathrm{Aut}(A_1) \times \mathrm{Aut}(\mathfrak{sl}(2))$ such $(\alpha', w') \cdot ((\alpha, w) \cdot f) \in \mathcal{N}$ and hence $f \in \mathcal{D}$.

$\mathcal{D}'_3 \subseteq \mathcal{D}_3$: We need the following lemma

Lemma 6.3. *Let $S \in \Delta_3$. Then*

$$\begin{array}{llcl} (i) & C(S) & \subset & \mathbb{C} \cup \Delta_3 \cup \Delta_5; \\ (ii) & C(S) \cap \Delta_3 & = & \{\mu S + \nu : \mu \in \mathbb{C}^*, \nu \in \mathbb{C}\}. \end{array} \tag{6.1}$$

Proof. Without loss of generality we can suppose that $S = pq$ since any element of Δ_3 is equivalent under $\mathrm{Aut}(A_1)$ to $\mu' pq + \nu'$ (Theorem 9.2 of [4]) and since $C(\mu' pq + \nu') = C(pq)$. First note that $C(pq) = \mathbb{C}[pq]$ (see Proposition 5.3 of [4]). Let $Z = a_k (pq)^k + \cdots + a_0$ be a polynomial of degree k in pq. Then a simple induction shows that

$$[Z, p^m q^n] = k(n-m) a_k p^{m+k-1} q^{n+k-1} + \text{ terms of lower degree in } p.$$

From this it follows that if $k > 1$ the only eigenvalue of $\mathrm{ad}(Z)$ is 0 and so $D(Z) = C(Z)$. By iteration of this formula it also follows that that if $k > 1$, $\mathrm{Ker}\,\mathrm{ad}^\ell(Z) = \mathrm{Ker}\,\mathrm{ad}(Z)$ and so $N(Z) = C(Z)$. This means that if $k > 1$, $Z \in \Delta_5$ (see Theorem 2.2). It is clear if $k = 1$ then $Z = a_1 pq + a_0$ is in Δ_3 and the lemma is proved. ∎

Let $f \in \mathcal{D}'_3$. There exists $z \in \mathfrak{sl}(2) \setminus \{0\}$, $S \in \Delta_3$ and $\lambda \in \mathbb{C}$ such that $[f(z), S] = \lambda S$. This implies that $\mathrm{ad}^2(S) f(z) = 0$, that is $f(z) \in N(S)$. Since $S \in \Delta_3$, $N(S) = C(S)$ (by Theorem 2.2), and thus $f(z) \in C(S)$. However, z is either semi-simple or nilpotent so that $f(z) \in \Delta_1 \cup \Delta_2 \cup \Delta_3 \cup \Delta_4$ by remark 3.2. Hence, by the previous Lemma, this means that $f(z) \in \Delta_3$ and so $f \in \mathcal{D}_3$.

$\mathcal{D}'_1 \subseteq \mathcal{D}_3$: We need the following two lemmas

Lemma 6.4. *Let $a \in A_1$, $\mu \in \mathbb{C}^*$ be such that $[a, p] = -\mu p$. There exists $\alpha_3 \in \mathrm{Aut}(A_1)$ and $a_0 \in \mathbb{C}$ such that $\alpha_3^{-1}(a) = \mu pq + a_0$.*

Proof. Let $a = \sum_{ij} h_{ij} p^i q^j$. Since

$$[a, p] = -\mu p \;\Leftrightarrow\; [a - \mu pq, p] = 0, \tag{6.2}$$

and $[p, p^i q^j] = j p^i q^{j-1}$, one has

$$a = \mu pq + \sum_{i=0}^{N} a_i p^i, \tag{6.3}$$

where $a_i \in \mathbb{C}$ and $N \in \mathbb{N}$. One can then write

$$a = \mu p(q + \sum_{i=1}^{N} \frac{a_i}{\mu} p^{i-1}) + a_0.$$

But $[p, q'] = 1$ where $q' = q + \sum_{i=1}^{N} \frac{a_i}{\mu} p^{i-1}$ and it is easy to see that the homomorphism $\alpha_3 : A_1 \to A_1$ given by $\alpha_3(p) = p$, $\alpha_3(q) = q'$ is invertible. Hence

$$\alpha_3^{-1}(a) = \mu pq + a_0.$$

∎

Lemma 6.5. *Let $a \in A_1$, $\lambda \in \mathbb{C}^*$ and let $g(p) = \sum_{k=0}^n b_k p^k$ be a polynomial of degree n in p. If $[a, g(p)] = \lambda g(p)$ then there exists $\alpha_2 \in \mathrm{Aut}(A_1)$ such that $[\alpha_2^{-1}(a), p] = \frac{\lambda}{n} p$.*

Proof. Since $< p^i q^j >$ is a basis of A_1, we can write

$$[a, p^k] = \sum_{i=0}^{N_k} f_{i,k}(p)\; q^i,$$

where $k \in \mathbb{N}^*, N_k \in \mathbb{N}$ and $f_{N_k,k}$ is a non-zero polynomial of degree M_k in p. Let $c_k \neq 0$ be the coefficient of $p^{M_k} q^{N_k}$ in $[a, p^k]$. A straightforward induction argument shows that $N_k = N_1$, $M_k = M_1 + k - 1$ and $c_k = k c_1$.

Thus, one has

$$[a, \sum_{k=0}^n b_k p^k] = \sum_{k=0}^n \left(\sum_{i=0}^{N_1} b_k f_{i,k}(p) q^i \right).$$

But since $[a, g(p)] = \lambda g(p)$ by hypothesis, this means that $b_n c_n = b_n n c_1 = b_n \lambda$, $N_1 = 0$, $M_n = n$ and so $c_1 = \frac{\lambda}{n}$ and $M_1 = 1$. Therefore

$$[a, p] = \frac{\lambda}{n} p + \nu$$

for some constant $\nu \in \mathbb{C}$. But $[\frac{\lambda}{n} p + \nu, \frac{n}{\lambda} q] = 1$ and so there exists an unique automorphism α_2 of A_1 such that $\alpha_2(p) = \frac{\lambda}{n} p + \nu$ and $\alpha_2(q) = \frac{n}{\lambda} q$. Hence

$$[\alpha_2^{-1}(a), p)] = \frac{\lambda}{n} p. \qquad \blacksquare$$

We now prove that $\mathcal{D}_1' \subseteq \mathcal{D}_3$. Let $f \in \mathcal{D}_1'$. By hypothesis there exist $z \in \mathfrak{sl}(2)$, $N \in \Delta_1$ and $\lambda \in \mathbb{C}$ such that

$$[f(z), N] = \lambda N. \tag{6.4}$$

By Theorem 9.1 of [4] there exist $\alpha_1 \in \mathrm{Aut}(A_1)$ such that $\alpha_1(N) \in \mathbb{C}[p]$. If $\lambda \neq 0$, $\alpha_1 \circ f(z)$ satisfies the hypothesis of Lemma 6.5. and, by Lemmas 6.5 and 6.4, there exist $\alpha_2, \alpha_3 \in \mathrm{Aut}(A_1)$, $\mu \in \mathbb{C}^*$ and $a_0 \in \mathbb{C}$ such that $\alpha_3^{-1} \circ \alpha_2^{-1} \circ \alpha_1 \circ f(z) = \mu pq + a_0$. Since $\mu pq + a_0$ is in Δ_3, this means that $\alpha_3^{-1} \circ \alpha_2^{-1} \circ \alpha_1 \circ f \in \mathcal{D}_3$ and hence $f \in \mathcal{D}_3$. Il $\lambda = 0$, we can reduce to the $\lambda \neq 0$ case as follows. If $\lambda = 0$, then $f(z) \in C(N)$. But $C(N) \subseteq \Delta_1$ (by Theorem 9.1 of [4]) so that $f(z) \in \Delta_1$ and, by Remark 3.2, z is nilpotent. There exists $s \in \mathfrak{sl}(2)$ semi-simple such that $[s, z] = 2z$ and then $[f(s), f(z)] = 2f(z)$. This is exactly the equation (6.4) with $f(z)$ replaced by $f(s)$, N replaced by $f(z)$ and λ replaced by 2. The argument above in the $\lambda \neq 0$ case implies that $f \in \mathcal{D}_3$.

$\mathcal{E} \subseteq \mathcal{D}_1$: If $f \in \mathcal{E}$ then by the definition of $\mathcal{E}$ and Proposition 2.7, $f \in \mathcal{D}_3 \cup \mathcal{D}_1 = \mathcal{D}_1$. $\blacksquare$

Corollary 6.6. *Let $f \in A_1^{\mathfrak{sl}(2)}$. The following are equivalent:*

(1) $f(\mathfrak{sl}(2)) \subseteq \Delta_1 \cup \Delta_3$.

(2) For all $z \in \mathfrak{sl}(2)$, $f(z)$ *can be exponentiated.*

(3) There exists $(\alpha, w) \in \mathrm{Aut}(A_1) \times \mathrm{Aut}(\mathfrak{sl}(2))$ *such that* $(\alpha, w).f = f_I$.

Proof. Parts (1) and (2) are equivalent by Proposition 2.7.

(1) $\Rightarrow$ (3): The condition (1) implies that $f \in \mathcal{D}_3$ which in turn, by Theorem 6.2, implies that $f \in \mathcal{D}$. Hence there exists $(\alpha, w) \in \mathrm{Aut}(A_1) \times \mathrm{Aut}(\mathcal{U}(\mathfrak{sl}(2)))$ such that $f' = (\alpha, w).f \in \mathcal{N}$ and then by (1) f' satisfies $f'(\mathfrak{sl}(2)) \subseteq \Delta_1 \cup \Delta_3$. By Lemma 5.6 this means that $f' = f_I$. The implication (3) $\Rightarrow$ (1) is obvious.

∎

6.2. Isotropy groups.

Recall that if a group G acts on a set X, the isotropy of $x \in X$ in G is by definition the subgroup $\{g \in G : g.x = x\}$. In this subsection we calculate the isotropy of f_I and f_{II}^b under the action of the group $\mathrm{Aut}(A_1) \times \mathrm{Aut}(\mathfrak{sl}(2))$.

In order to calculate the isotropy of f_I we need the following lemma. As usual, the homomorphism $\mathrm{Ad} : SL(2,\mathbb{C}) \to \mathrm{Aut}(\mathfrak{sl}(2))$ is defined by $\mathrm{Ad}(g)(x) = gxg^{-1}$, where $g \in SL(2,\mathbb{C})$ and $x \in \mathfrak{sl}(2)$. Recall that Ad is a double covering map and that $\mathrm{Aut}(\mathfrak{sl}(2)) \cong SO(3,\mathbb{C})$.

Lemma 6.7. *There exists a group homomorphism* $\hat{\alpha}_1 : SL(2,\mathbb{C}) \to \mathrm{Aut}(A_1)$ *such that* $f_I \circ \mathrm{Ad}(g) = \hat{\alpha}_1(g) \circ f_I$ *for all* $g \in SL(2,\mathbb{C})$. *Furthermore,* $\hat{\alpha}_1$ *does not factor through* $\mathrm{Ad} : SL(2,\mathbb{C}) \to \mathrm{Aut}(\mathfrak{sl}(2))$.

Proof. Define $\hat{\alpha}_1 : SL(2,\mathbb{C}) \to \mathrm{Aut}(A_1)$ by (see **E9**, subsection 4.4)

$$\hat{\alpha}_1(\begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix})(p) = a_2 q + a_4 p, \ \hat{\alpha}_1(\begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix})(q) = a_1 q + a_3 p.$$

The formula $f_I \circ \mathrm{Ad}(g) = \hat{\alpha}_1(g) \circ f_I$ follows by a straightforward calculation and $\hat{\alpha}_1$ cannot factor through $\mathrm{Ad} : SL(2,\mathbb{C}) \to \mathrm{Aut}(\mathfrak{sl}(2))$ since $\hat{\alpha}_1(-Id) \neq \hat{\alpha}_1(Id)$. ∎

Proposition 6.8. *Let $\mathcal{I}_{f_I}$ be the isotropy of f_I in* $\mathrm{Aut}(A_1) \times \mathrm{Aut}(\mathfrak{sl}(2))$. *Then*

$$\mathcal{I}_{f_I} = \big\{(\hat{\alpha}_1(g), \mathrm{Ad}(g)) \ : \ g \in SL(2,\mathbb{C})\big\}.$$

In particular, $\mathcal{I}_{f_I}$ *is isomorphic to* $SL(2,\mathbb{C})$.

Proof. The inclusion $\big\{(\hat{\alpha}_1(g), \mathrm{Ad}(g)) \ : \ g \in SL(2,\mathbb{C})\big\} \subseteq \mathcal{I}_{f_I}$ follows immediately from the lemma. To prove inclusion in the opposite sense, let $(\alpha, w) \in \mathcal{I}_{f_I}$ and choose $g \in SL(2,\mathbb{C})$ such that $w = \mathrm{Ad}(g)$. By definition

$$f_I \circ \omega = \alpha \circ f_I$$

and, since by the previous lemma $f_I \circ \omega = \hat{\alpha}_1(g) \circ f_I$, we get $\alpha^{-1} \circ \hat{\alpha}_1(g) \circ f_I = f_I$. Using the explicit formulae for f_I (cf. Example 5.3) this implies that $q'^2 = q^2$, $p'^2 = p^2$ where we have written $p' = \alpha^{-1} \circ \hat{\alpha}_1(g)(p)$ and $q' = \alpha^{-1} \circ \hat{\alpha}_1(g)(q)$. Hence $q' \in C(q^2) = \mathbb{C}[q]$ and $p' \in C(p^2) = \mathbb{C}[p]$. It is then easy to see that either $q' = q$ and $p' = p$, or $q' = -q$ and $p' = -p$. Thus either $\alpha^{-1} \circ \hat{\alpha}_1(g) = 1$ or $\alpha^{-1} \circ \hat{\alpha}_1(g) = \hat{\alpha}_1(-Id)$ which means either $\alpha = \hat{\alpha}_1(g)$ or $\alpha = \hat{\alpha}_1(-g)$. Since $w = \mathrm{Ad}(g) = \mathrm{Ad}(-g)$, this completes the proof of the proposition. ∎

To calculate the isotropy of f_{II}^b we need the following definition and lemma:

Definition 6.9. Let $\hat{B} = \left\{ \begin{pmatrix} a_1 & 0 \\ a_3 & \frac{1}{a_1} \end{pmatrix} \in SL(2,\mathbb{C}) : a_1 \in \mathbb{C}^*,\ a_3 \in \mathbb{C} \right\}$ and let B be the subgroup $\mathrm{Ad}(\hat{B})$ of $\mathrm{Aut}(\mathfrak{sl}(2))$.

Lemma 6.10. *There exists a group homomorphism* $\hat{\beta} : \hat{B} \to \mathrm{Aut}(A_1)$ *such that for all* $g \in \hat{B}$, $f_{II}^b \circ \mathrm{Ad}(g) = \hat{\beta}(g) \circ f_{II}^b$. *Furthermore,* $\hat{\beta}(g) = \hat{\beta}(-g)$ *and* $\hat{\beta}$ *factors through* $\mathrm{Ad} : \hat{B} \to B$ *to a group homomorphism* $\beta : B \to \mathrm{Aut}(A_1)$.

Remark 6.11. The derivative of $\hat{\beta} : \hat{B} \to \mathrm{Aut}(A_1)$ is the restriction of the Lie homomorphism $\mathrm{ad} \circ f_{II}^b : \mathfrak{sl}(2) \to \mathrm{Der}(A_1)$ to the Lie algebra $\mathfrak{b} \subseteq \mathfrak{sl}(2)$ of $\hat{B}$. This makes sense since every element of $\mathrm{ad} \circ f_{II}^b(\mathfrak{b})$ is exponentiable. Note that the elements of $\mathrm{ad} \circ f_{II}^b\big(\mathfrak{sl}(2) \setminus \mathfrak{b}\big)$ are not exponentiable by Lemmas 5.6 and 2.7.

Proof. Define $\hat{\beta} : \hat{B} \to \mathrm{Aut}(A_1)$ by

$$\hat{\beta}(\begin{pmatrix} a_1 & 0 \\ a_3 & \frac{1}{a_1} \end{pmatrix})(p) = \frac{1}{a_1^2} p,\ \hat{\beta}(\begin{pmatrix} a_1 & 0 \\ a_3 & \frac{1}{a_1} \end{pmatrix})(q) = a_1^2 (q - \frac{a_3}{a_1}).$$

The result then follows by a straightforward calculation. ∎

Proposition 6.12. *Let* $\mathcal{I}_{f_{II}^b}$ *be the isotropy of* f_{II}^b *in* $\mathrm{Aut}(A_1) \times \mathrm{Aut}(\mathfrak{sl}(2))$. *Then*

$$\mathcal{I}_{f_{II}^b} = \{(\beta(w), w)\ :\ w \in B\}.$$

In particular, $\mathcal{I}_{f_{II}^b}$ *is isomorphic to a Borel subgroup of* $\mathrm{Aut}(\mathfrak{sl}(2))$.

Proof. The inclusion $\{(\beta(w), w)\ :\ w \in B\} \subseteq \mathcal{I}_{f_{II}^b}$ follows immediately from the lemma. To prove inclusion in the opposite sense, let $(\alpha, w) \in \mathcal{I}_{f_{II}^b}$ and choose $g \in SL(2,\mathbb{C})$ such that $w = \mathrm{Ad}(g)$. By definition, we have

$$\alpha \circ f_{II}^b = f_{II}^b \circ w, \tag{6.5}$$

which is equivalent to

$$\alpha \circ f_{II}^b(x) = f_{II}^b(gxg^{-1}) \qquad \forall x \in \mathfrak{sl}(2).$$

If $x \in \mathfrak{b} =< e_0, e_- >$ then $f_{II}^b(x) \in \Delta_1 \cup \Delta_3$ (by Lemma 5.6), $\alpha \circ f_{II}^b(x) \in \Delta_1 \cup \Delta_3$ (since the Dixmier partition is invariant under $\mathrm{Aut}(A_1)$) and hence $f_{II}^b(gxg^{-1}) \in \Delta_1 \cup \Delta_3$. By Lemma 5.6, this means that $gxg^{-1} \in \mathfrak{b}$ and we have shown that

$g\mathfrak{b}g^{-1} = \mathfrak{b}$. Since $\mathfrak{b}$ is a Borel subalgebra of $\mathfrak{sl}(2)$ and is the Lie algebra of $\hat{B}$, this implies that $g \in \hat{B}$.

By the previous lemma and equation (6.5), we have

$$\alpha \circ f^b_{II} = f^b_{II} \circ w = f^b_{II} \circ \mathrm{Ad}(g) = \hat{\beta}(g) \circ f^b_{II} = \beta(w) \circ f^b_{II}$$

and hence

$$\beta(w)^{-1} \circ \alpha \circ f^b_{II} = f^b_{II}.$$

Writing $p' = \beta(w)^{-1} \circ \alpha(p)$, $q' = \beta(w)^{-1} \circ \alpha(q)$ and using the explicit formulae for f^b_{II} (cf. Example 5.5), this implies that $p' = p$ and $2p'q' + b = 2pq + b$. Hence $0 = 2(pq' - pq) = 2p(q' - q)$, which implies that $q' = q$ and $\alpha = \beta(w)$. ■

6.3. Other examples.

One can construct elements of $A_1^{\mathfrak{sl}(2)}$ which are not in the orbit of $\mathcal{N}$ under $\mathrm{Aut}(A_1) \times \mathrm{Aut}(\mathfrak{sl}(2))$ by letting $\mathrm{Aut}(\mathcal{U}(\mathfrak{sl}(2))) \setminus \mathrm{Aut}(\mathfrak{sl}(2))$ act on $\mathcal{N}$. In this section, for the sake of completeness, we give an explicit example (see also page 127 of [9]).

Define $g \in A_1^{\mathfrak{sl}(2)}$ by $g = f^1_{II} \circ w$, where $w = \exp(\mathrm{ad}(x^2)) \in \mathrm{Aut}(\mathcal{U}(\mathfrak{sl}(2)))$ is given by:

$$\begin{aligned} w(x) &= x, \\ w(y) &= y + hx + xh - 4x^3, \\ w(h) &= h - 4x^2. \end{aligned} \tag{6.6}$$

Explicitly, this gives

$$\begin{aligned} g(e_+) &= (1 + pq)q \\ g(e_-) &= -p + 4p^2q^3 - 4p^3q^6 + 12p^2q^5 \\ g(e_0) &= 2pq - 4p^2q^4 + 1. \end{aligned} \tag{6.7}$$

Proposition 6.13. $g \notin \mathcal{D}$.

Proof. It is enough to prove that there does not exist $(\alpha, w) \in \mathrm{Aut}(A_1) \times \mathrm{Aut}(\mathfrak{sl}(2))$ and $f \in \mathcal{N}$ such that $(\alpha, w).f = g$. First note that if $(\alpha, w).f = g$ then either $f = f^1_{II}$ or $f = f^{-3}_{II}$ by Proposition 5.10 and the formula $Q_{f^b_{II}} = \frac{1}{2}b(b+1)$. Suppose there exists $(\alpha, w) \in \mathrm{Aut}(A_1) \times \mathrm{Aut}(\mathfrak{sl}(2))$ such that $(\alpha, w).f^b_{II} = g$ with $b \in \{1, -3\}$.

The $\mathfrak{sl}(2)$ triplets corresponding to g and f^b_{II} are

$$\begin{array}{ll} X = (1 + pq)q & X' = (b + pq)q \\ Y = -p + 4p^2q^3 - 4p^3q^6 + 12p^2q^5 & Y' = -p \\ H = 2pq + 1 - 4p^4q^2 & H' = 2pq + b. \end{array} \tag{6.8}$$

Writing $p' = \alpha(p),\ q' = \alpha(q)$ and $w = \mathrm{Ad}(\begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix})$, this gives (see Appendix)

$$\begin{aligned}
(b + p'q')q' &= a_1^2 X - a_3^2 Y - a_1 a_3 H \\
&= a_1^2((1+pq)q) - a_3^2(-p + 4p^2q^3 - 4p^3q^6 + 12p^2q^5) \\
&\quad -a_1a_3(2pq + 1 - 4p^4q^2); \\
-p' &= -a_2^2 X + a_4^2 Y + a_2 a_4 H \\
&= -a_2^2(1+pq)q + a_4^2(-p + 4p^2q^3 - 4p^3q^6 + 12p^2q^5) \\
&\quad +a_2a_4(2pq + 1 - 4p^4q^2); \\
2p'q' + b &= -2a_1a_2 X + 2a_3a_4 Y + (a_1a_4 + a_2a_3)H \\
&= -2a_1a_2(1+pq)q + 2a_3a_4(-p + 4p^2q^3 - 4p^3q^6 + 12p^2q^5) \\
&\quad +(a_1a_4 + a_2a_3)(2pq + 1 - 4p^4q^2).
\end{aligned}$$

Substituting the second equation in the third equation, we obtain

$$\begin{aligned}
&-2\Big(-a_2^2(1+pq)q + a_4^2(-p + 4p^2q^3 - 4p^3q^6 + 12p^2q^5) \\
&\qquad +a_2a_4(2pq + 1 - 4p^4q^2)\Big)q' + b \\
&= -2a_1a_2(1+pq)q + 2a_3a_4(-p + 4p^2q^3 - 4p^3q^6 + 12p^2q^5) \\
&\qquad +(a_1a_4 + a_2a_3)(2pq + 1 - 4p^4q^2).
\end{aligned}$$

If $a_4 \neq 0$ then the expansion of q' in the standard basis would consist of only a scalar term otherwise $-2a_4^2p^3q^6q'$ on the LHS contains terms which are not present on the RHS; but then $[p', q'] = 0$ which is a contradiction and hence $a_4 = 0$.

The equation above now reduces to

$$2a_2^2(q + pq^2)q' + b = -2a_1a_2(q + pq^2) - (2pq + 1 - 4p^4q^2). \tag{6.9}$$

Let k be the highest power of q appearing in the expansion of q' in the standard basis. Then the highest power appearing in the expansion of the LHS in the standard basis is $k+2$. Comparing with the RHS gives $k = 0$ and $q' = f(p)$ where f is a polynomial in p of degree at most 3. This polynomial must in fact be of degree 3 to provide the term $4p^4q^2$ on the RHS but then this introduces a term in p^3q on the LHS which is not present on the RHS. In conclusion (6.9) has no solutions and thus $g \notin \mathcal{D}$. ■

Corollary 6.14. $g(\mathfrak{sl}(2)) \subseteq \Delta_2 \cup \Delta_4$.

Proof. Immediate from Theorem 6.2. ■

With respect to the Joseph decompositions $S_1 = S_{11} \cup S_{12} \cup \cdots \cup S_{1\infty}$ and $S_2 = S_{21} \cup S_{22} \cup \ldots$, one can show that $g \in S_{1\infty}$ (see [9] page 127 and the Appendix). It is not known whether S_{ij} is non-empty when $j > 1$ and so it seems reasonable to conjecture that in fact, all elements of $A_1^{\mathfrak{sl}(2)}$ are obtained from $\mathcal{N}$ by the action of the group $\mathrm{Aut}(A_1) \times \mathrm{Aut}(\mathcal{U}(\mathfrak{sl}(2)))$.

A Appendix

Proposition A1. *Let* $g = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \in SL(2,\mathbb{C})$. *Define* $w \in \mathrm{Aut}(\mathfrak{sl}(2))$ *by* $w(z) = gzg^{-1}$ *for any* $z \in \mathfrak{sl}(2)$. *Then*

$$\begin{aligned} w(e_+) &= \begin{pmatrix} -a_1a_3 & a_1^2 \\ -a_3^2 & a_1a_3 \end{pmatrix} \\ w(e_-) &= \begin{pmatrix} a_2a_4 & -a_2^2 \\ a_4^2 & -a_2a_4 \end{pmatrix} \\ w(e_0) &= \begin{pmatrix} a_1a_4 + a_2a_3 & -2a_1a_2 \\ 2a_3a_4 & -(a_1a_4 + a_2a_3) \end{pmatrix}, \end{aligned}$$

where e_+, e_-, e_0 *is the standard basis of* $\mathfrak{sl}(2)$.

Proposition A2. *Let* $g \in A_1^{\mathfrak{sl}(2)}$ *be defined by equations* (6.7). *Then* $g \in S_{1\infty}$.

Proof. By [9], $S_1 = S_{11} \cup S_{12} \cup \cdots \cup S_{1\infty}$ and S_{1r} $(1 < r < \infty)$ are stable under $\mathrm{Aut}(A_1) \times \mathrm{Aut}(\mathcal{U}(\mathfrak{sl}(2)))$. Hence to prove that $g \in S_{1\infty}$ it is enough to show that $g \notin S_{11}$. Set $H = g(e_0)$, $X = g(e_+)$, $Y = g(e_-)$ and, for $m \in 2\mathbb{Z}$, define $D\big(H, m\big)$ by

$$D\big(H, m\big) = \{z \in A_1 :\ [H, z] = mz\}.$$

Then, by definition (see [9]), $g \notin S_{11}$ iff $D\big(H, 2\big) = X\mathbb{C}[H]$ and $D\big(H, -2\big) = Y\mathbb{C}[H]$. By Lemma 3.1 of [9], there exists $y_2 \in D\big(H, 2\big)$ such that $D\big(H, 2\big) = y_2\mathbb{C}[H]$. Since $X \in D\big(H, 2\big)$ there exists a polynomial $a_nH^n + \cdots + a_0$ such that

$$X = y_2(a_nH^n + \cdots + a_0),$$

which gives

$$q + pq^2 = y_2\left(a_n(2pq - 4p^2q^4 + 1)^n + \cdots + a_0\right).$$

Comparing the highest power of p, one has $n = 0$, $y_2 = \frac{1}{a_0}X$ and thus $D\big(H, 2\big) = X\mathbb{C}[H]$. A similar but slightly more complicated argument shows that $D\big(H, -2\big) = Y\mathbb{C}[H]$. ■

References

[1] Amitsur, S. A., *Commutative Linear Differential Operators*, Pacific J. of Math. **88** (1958), 1–10.

[2] Arnal, D., and G. Pinczon, *On Algebraically Irreducible Representations of the Lie Algebra* $\mathfrak{sl}(2)$, J. Math. Phys. **15** (1974), 350–359.

[3] Bavula, V. V., *Dixmier's Problem* 6 *for the Weyl Algebra (the Generic Type Problem)*, math.RA/0402244.

[4] Dixmier, J., *Sur les Algèbres de Weyl*, Bull. Soc. Math. France **96** (1968), 209–242.

[5] —, *Quotients Simples de l'Algèbre Enveloppante de* $\mathfrak{sl}(2)$, J. of Algebra **24** (1973), 151–164.

[6] Goze, M.. and Yu. Hakimjanov, *Sur les Algèbres de Lie Nilpotentes Admettant un Tore de Dérivations*, Manuscripta Math. **84** (1994), 115–124.

[7] Igusa, J., *On Lie Algebras Generated by Two Differential Operators*, Prog. Math. **14** (1981), 187–195.

[8] Joseph, A., *Commuting Polynomials in Quantum Canonical Operators and Realizations of Lie Algebras*, J. Math. Phys. **13**, (1972) 351–357.

[9] —, *A Characterization Theorem for Realizations of* $\mathfrak{sl}(2)$, Proc. Camb. Phil. Soc. **75** (1974), 119–131.

[10] —, *The Weyl Algebra—Semisimple and Nilpotent Elements*, Amer. J. Math. **97** (1975), 597–615.

[11] Simoni, A., and F. Zaccaria, *On the Realization of Semi-Simple Lie Algebras with Quantum Canonical Variables*, Nuovo Cimento A (10) **59** (1969), 280–292.

[12] Vergne, M., *Cohomologie des Algèbres de Lie Nilpotentes. Applications à l'Étude de la Variété des Algèbres de Lie Nilpotentes*, Bull. Soc. Math. France **98** (1970), 81–116.

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