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Birational Isomorphisms between Twisted Group Actions

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Abstract. Let X be an algebraic variety with a generically free action of a connected algebraic group G. Given an automorphism $\phi\colon G \to G$, we will denote by X^ϕ the same variety X with the G-action given by $g\colon x \to \phi(g)\cdot x$.

We construct examples of G-varieties X such that X and X^ϕ are not G-equivariantly isomorphic. The problem of whether or not such examples can exist in the case where X is a vector space with a generically free linear action, remains open. On the other hand, we prove that X and X^ϕ are always stably birationally isomorphic, i.e., $X \times \mathbb{A}^m$ and $X^\phi \times \mathbb{A}^m$ are G-equivariantly birationally isomorphic for a suitable $m \geq 0$.
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1. Introduction

Throughout this note all algebraic varieties, algebraic groups, group actions and maps between them will be defined over a fixed base field k. By a G-variety X we shall mean an algebraic variety with a (regular) action of a linear algebraic group G. A morphism (respectively, rational map, birational isomorphism) of G-varieties is a G-equivariant morphism (respectively, rational map, birational isomorphism).

Given an automorphism ϕ of G, we can "twist" a group action $\alpha\colon G \times X \longrightarrow X$ by ϕ to obtain a new G-action α^ϕ on X as follows:

$$\alpha^\phi\colon G \times X \xrightarrow{(\phi, id)} G \times X \xrightarrow{\alpha} X\,.$$

Note that the new action has the same orbits as the old one. If X is a G-variety (via α) then we will denote the "twisted" G-variety (i.e., X with the G-action via α^ϕ) by X^ϕ.

Now suppose that the action α of G on X is generically free, i.e., that there exists a G-invariant open dense subset U of X such that the stabilizer of every geometric point of U is trivial. In this paper we will address the following question:

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Problem 1.1. Are X and X^ϕ birationally isomorphic as G-varieties over k?

Katsylo [6] conjectured that when $k = \mathbb{C}$ generically free linear G-representations V and W are (G-equivariantly) birationally isomorphic if and only if $\dim(V) = \dim(W)$. This conjecture is known to be false for some finite groups G; in particular, there are counterexamples where $W = V^\phi$ for an automorphism ϕ of G, see [16, Section 7]. V. L. Popov has suggested that counterexamples of this form should also exist in the case where G is connected. (In this case Katsylo's conjecture is still open.) The present paper was largely motivated by this suggestion.

Before proceeding to state our main results, we will make two simple observations. First, the answer to Problem 1.1 depends only on the class of ϕ in the group of outer automorphisms of G. Indeed, suppose $\phi' = \phi \circ \mathrm{inn}_h$, where $\mathrm{inn}_h \colon G \longrightarrow G$ is conjugation by $h \in G$, i.e., $\phi(g) = hgh^{-1}$. Then X^ϕ and $X^{\phi'}$ are isomorphic via $x \mapsto h \cdot x$. In particular, X and X^ϕ are always isomorphic if G has no outer automorphisms, e.g., if G is the full symmetric group S_n $(n \neq 6)$ or if G is a semisimple algebraic group whose Dynkin diagram has no non-trivial automorphisms; cf. [10, Theorem 25.16]. The latter class of groups includes every split (almost) simple algebraic group, other than those of type A_n, D_n and E_6; cf. [10, Sections 24A and 25B].

Secondly, the G-invariant rational functions for X and X^ϕ are exactly the same, i.e.,

$$k(X)^G = k(X^\phi)^G \subset k(X)\,. \tag{1}$$

Recall that the inclusion $k(X) \subset k(X)^G$ induces a dominant rational map $\pi \colon X \dashrightarrow X/G$, which is called *the rational quotient map.* The k-variety X/G is defined (up to birational isomorphism) by the condition $k(X/G) = k(X)^G$; cf. [13, Section 2.4]. Thus (1) can be rephrased by saying that a rational quotient map π for X is also a rational quotient map for X^ϕ.

The main results of this note are Theorems 1.2 and 1.3 below.

Theorem 1.2. *Let X be a generically free G-variety and $\phi \colon G \longrightarrow G$ be an automorphism of G. Then the G-varieties X and X^ϕ are stably birationally isomorphic. More precisely there exists an integer $m \geq 0$ and a G-equivariant birational isomorphism*

$$f \colon X \times \mathbb{A}^m \dashrightarrow X^\phi \times \mathbb{A}^m$$

such that the diagram

$$\begin{tikzcd} X \times \mathbb{A}^m \arrow[r, dashed, "f"] \arrow[d, dashed] & X^\phi \times \mathbb{A}^m \arrow[d, dashed] \\ X/G \arrow[r, equal] & X^\phi/G, \end{tikzcd}$$

commutes.

Here $\mathbb{A}^m$ is the m-dimensional affine space with trivial G-action, and the vertical map on the left is the composition of the projection $X \times \mathbb{A}^m \longrightarrow X$ with the rational quotient map $X \dashrightarrow X/G$ (similarly for the vertical map on the right).

Theorem 1.3. *Let $n \geq 3$ and let ϕ be the (outer) automorphism of $G = \mathrm{PGL}_n$ given by $g \longrightarrow (g^{-1})^{\mathrm{transpose}}$. Assume the base field k contains a primitive nth root of unity. Then there exists a generically free PGL_n-variety X such that X and X^{ϕ} are not birationally isomorphic over k.*

This gives a negative answer to Problem 1.1. The question of whether or not similar examples can exist in the case where X is a vector space with a generically free linear action of a connected linear algebraic group G, remains open.

2. The no-name lemma

Recall that a G-bundle $\pi\colon E \longrightarrow X$ is an algebraic vector bundle with a G-action on E and X such that π is G-equivariant and the action of every $g \in G$ restricts to a linear map $\pi^{-1}(x) \longrightarrow \pi^{-1}(gx)$ for every $x \in X$.

Our proof of Theorem 1.2 in the next section will heavily rely the following result.

Lemma 2.1. (No-name Lemma) *Let $\pi\colon E \longrightarrow X$ be a G-bundle of rank r. Assume that the G-action on X is generically free. Then there exists a birational isomorphism $\pi\colon E \overset{\simeq}{\dashrightarrow} X \times \mathbb{A}^r$ of G-varieties such that the following diagram commutes*

$$\begin{tikzcd} E \arrow[r, dashed, "\phi"] \arrow[d, "\pi"'] & X \times \mathbb{A}^r \arrow[dl, "\mathrm{pr}_1"] \\ X & \end{tikzcd} \tag{2}$$

Here G is assumed to act trivially on $\mathbb{A}^r$, and pr_1 denotes the projection to the first factor.

The term "no-name lemma" is due to Dolgachev [4]. In the case where G is a finite group, it is a variant of Hilbert's Theorem 90 classically known as Speiser's Lemma [21]; for a modern treatment, see [5, Proposition 1.1], [9, Proposition 1.3] or [20, Appendix 3]. In the case where the base field k is algebraically closed, $\mathrm{char}(k) = 0$, and G is an arbitrary linear algebraic group, to the best of our knowledge the no-name lemma first appeared in [1], [6]. For a detailed proof in this setting, see [2, Section 4].

In the sequel we would like to use Lemma 2.1 in the case where k is not necessarily algebraically closed. With this in mind, we will prove a more general variant of this result (Proposition 2.3 below). For the rest of this section we will work over an arbitrary base field k.

Remark 2.2. Suppose G is a group scheme of finite type over k, X is an arbitrary quasi-separated scheme (or algebraic space) over k on which G acts (quasi-separated means that the diagonal embedding $X \hookrightarrow X \times_{\operatorname{Spec} k} X$ is quasi-compact; this is automatically satisfied when X is of finite type over k). Assume further that the G-action on X is *free*, i.e., that the stabilizers of all geometric points of X are trivial (as group schemes). Equivalently, the morphism $G \times_{\operatorname{Spec} k} X \longrightarrow X \times_{\operatorname{Spec} k} X$ defined in functorial terms by $(g, x) \mapsto (gx, x)$ is categorically injective (i.e., it is injective on geometric points and unramified). Then, by a result of Artin [8, Corollaire 10.4], the quotient sheaf X/G in the fppf topology is an algebraic space, and $G \times_{\operatorname{Spec} k} X = X \times_{X/G} X$. There is a Zariski open dense subspace $V \subseteq X/G$ that is a scheme ([7, Proposition 6.7]); if U is the inverse image of V in X, then the restriction $U \to U/G = V$ is a G-torsor (i.e. a principal G-bundle in the fppf topology, cf. [3]). In the case where X is a k-variety, this is a the rational quotient map, i.e., $k(U/G) = k(X)^G$; cf. [13, Lemma 2.1].

Proposition 2.3. *Assume that G is a group scheme of finite type over k, acting on a quasi-separated k-scheme X, with a non-empty invariant open subscheme on which the action is free. Let $\mathcal{E}$ be a G-equivariant locally free sheaf of rank r on X. Then there exists a non-empty open G-invariant subscheme U of X, such that the restriction $\mathcal{E}\mid_U$ is isomorphic to the trivial G-equivariant sheaf $\mathcal{O}_U^r$.*

To see that Lemma 2.1 (over an arbitrary base field k) follows from Proposition 2.3, recall the the well-known equivalence between the category of G-equivariant vector bundles on X and the category of G-equivariant locally free sheaves on X. One passes from a G-bundle $V \to X$ to the G-equivariant locally free sheaf of sections of V; conversely, to each G-equivariant locally free sheaf $\mathcal{E}$ on X one associates the spectrum of the sheaf of symmetric algebras of the dual $\mathcal{E}^\vee$ over X.

Note also that in the course of proving Lemma 2.1, we may assume without loss of generality that X is *primitive*, i.e., G transitively permutes the irreducible components of X (equivalently, $k(X)^G$ is a field). Indeed, an arbitrary G-variety X is easily seen to be birationally isomorphic to a disjoint union of primitive G-varieties $X_1, \dots, X_r$, and it suffices to prove Lemma 2.1 for each X_i. On the other hand, if X is primitive, then every non-empty G-invariant open subset is dense in X. This shows that Lemma 2.1 follows from Proposition 2.3, as claimed.

Proof of Proposition 2.3. After replacing X by a non-empty open subscheme we may assume that the action of G on X is free. By passing to a dense invariant subscheme of X, we may assume that X/G is a scheme, and $X \to X/G$ is a G-torsor; see Remark 2.2. By descent theory, the G-equivariant sheaf $\mathcal{E}$ comes from a locally free sheaf $\mathcal{F}$ on X/G; see, for example, [22, Theorem 4.46]. By restricting to a non-empty open subscheme of X/G once again, we may assume that $\mathcal{F}$ is isomorphic to $\mathcal{O}_{X/G}^r$. Then $\mathcal{E}$ is G-equivariantly isomorphic to $\mathcal{O}_X^r$, as claimed. ∎

Remark 2.4. The same argument goes through if the base field k (or, equivalently, the base scheme $\operatorname{Spec}(k)$) is replaced by a algebraic space B, so that X is defined over B, and the group scheme G is assumed to be flat and finitely presented over B.

3. Proof of Theorem 1.2

We will prove Theorem 1.2 in two steps: first in the case where $X = V$ is a generically free linear representation of G, then for arbitrary X.

Step 1: Suppose $X = V$ is a generically free linear representation of G.
Let $m = \dim(V)$. By the no-name lemma, there exist G-equivariant

birational isomorphisms α and β such that the diagram

$$\begin{tikzcd} V \times \mathbb{A}^m \arrow[r, dashed, "\alpha"] \arrow[d, "pr_1"] & V \times V^\phi \arrow[d, "pr_1"] & V \times V^\phi \arrow[r, dashed, "\beta"] \arrow[d, "pr_2"] & V^\phi \times \mathbb{A}^m \arrow[d, "pr_1"] \\ V \arrow[r, equal] \arrow[d, dashed, "\pi_V"] & V \arrow[d, dashed, "\pi_V"] & V^\phi \arrow[r, equal] \arrow[d, dashed, "\pi_{V^\phi}"] & V^\phi \arrow[d, dashed, "\pi_{V^\phi}"] \\ V/G \arrow[r, equal] & V/G \arrow[r, equal] & V^\phi/G \arrow[r, equal] & V^\phi/G \end{tikzcd}$$

commutes. Now we can take $f = \beta \circ \alpha \colon V \times \mathbb{A}^m \dashrightarrow V^\phi \times \mathbb{A}^m$.

Step 2: Suppose X is an arbitrary generically free G-variety.

Let V be a generically free linear representation of G and $p \colon X \times V \longrightarrow V$ be the projection onto the second factor. By the no-name lemma, $X \times V$ is birationally isomorphic to $X \times \mathbb{A}^m$; this yields a dominant rational map of G-varieties $X \times \mathbb{A}^m \dashrightarrow V$, which we will continue to denote by p. After replacing X by $X \times \mathbb{A}^m$, we may assume that there exists a dominant rational map $p \colon X \dashrightarrow V$. We now consider the commutative diagram

$$\begin{tikzcd} X \arrow[r, dashed, "p"] \arrow[d, dashed] & V \arrow[d, dashed] \\ X/G \arrow[r, dashed, "p/G"] & V/G, \end{tikzcd}$$

where the vertical arrows are rational quotient maps. We claim that X is birationally isomorphic to the fiber product $X/G \times_{V/G} V$, where the G-action on this fiber product is induced from the G-action on V (in other words, G acts trivially on X/G and on V/G). In the case where k is an algebraically closed field of characteristic zero, this is proved in [14, Lemma 2.16]. For general k, choose an open invariant subscheme U of V such that G acts freely over U, the quotient U/G is a scheme, and the projection $U \to U/G$ is a G-torsor; see Remark 2.2. By restricting X, we may assume that the morphism $X \to X/G$ is also a G-torsor, and that X maps into U. Then we get a commutative diagram

$$\begin{tikzcd} X \arrow[r, "p"] \arrow[d] & U \arrow[d] \\ X/G \arrow[r, "p/G"] & U/G \end{tikzcd}$$

where the columns are G-torsors and the top row is G-equivariant. Any such diagram is well known to be cartesian; this proves our claim.

Similarly, $X^\phi \simeq X^\phi/G \times_{V^\phi/G} V^\phi$. By Step 1 there is a G-equivariant birational isomorphism $f \colon V \times \mathbb{A}^m \dashrightarrow V^\phi \times \mathbb{A}^m$ which makes the diagram

$$\begin{tikzcd} & V \times \mathbb{A}^m \arrow[d, dashed, "\pi_V"] \arrow[drr, dashed, "f"] & & \\ X/G \arrow[r, dashed, "p/G"] \arrow[dr, dashed, "id"] & V/G \arrow[drr, dashed, "id"] & & V^\phi \times \mathbb{A}^m \arrow[d, dashed, "\pi_{V^\phi}"] \\ & X^\phi/G \arrow[rr, dashed, "p/G"] & & V^\phi/G \end{tikzcd}$$

commute. Consequently, f induces a G-equivariant birational isomorphism between the fiber products $X/G \times_{V/G} (V \times \mathbb{A}^m)$ and $X^\phi/G \times_{V/G} (V^\phi \times \mathbb{A}^m)$ i.e., between $X \times \mathbb{A}^m$ and $X^\phi \times \mathbb{A}^m$. This completes the proof of Theorem 1.2.

Remark 3.1. Our proof shows that the integer m in the statement of Theorem 1.2 can be taken to be the minimal value of $2\dim(W)$, as W ranges over the generically free linear representations of G.

4. Examples where X and X^ϕ are birationally isomorphic

The question of whether or not X and X^ϕ are birationally isomorphic over k is delicate in general. Birational isomorphism over $K = k(X)^G$ is more accessible because it has a natural interpretation in terms of Galois cohomology. In this section we will to show that in many cases X and X^ϕ are, indeed, birationally isomorphic over K (and thus over k).

Let X be a primitive G-variety. Recall that X is called primitive if G transitively permutes the irreducible components of X; see Section 2. That is, X is primitive if $K = k(X)^G$ is a field or equivalently, if the rational quotient variety X/G is irreducible. As we saw in Remark 2.2, after replacing X by an open G-invariant subvariety we may assume that the rational quotient map $\pi\colon X \dashrightarrow X/G$ is regular and is a G-torsor. Hence, the G-action on X gives rise to a Galois cohomology class in $H^1(K, G)$, which we shall denote by $[X]$; cf., [18], [12].

An automorphism ϕ of G induces an automorphism ϕ_* of the (pointed) cohomology set $H^1(K, G)$, where $[X^\phi] = \phi_*([X])$. The G-varieties X and X^ϕ are birationally isomorphic over K if and only if $\phi_*([X]) = [X]$ in $H^1(K, G)$.

Example 4.1. If $[X] = 1$ then $\phi_*([X]) = 1$; hence, X and X^ϕ are birationally isomorphic. Explicitly, in this case X is birationally isomorphic to the "split" G-variety $Y \times G$, with G acting on the second component by left translations, and a birational isomorphism between $Y \times G$ and $(Y \times G)^\phi$ is given by $(y, g) \longrightarrow (y, \phi(g))$. In particular, if G is a special group, i.e., $H^1(K, G) = \{1\}$ for every K/k, then X and X^ϕ are birationally isomorphic for every generically free G-variety X. Examples of special groups are $G = \mathrm{GL}_n$, SL_n, Sp_{2n}; see [19, Chapter X].

The following lemma extends this simple argument a bit further.

Lemma 4.2. *Let X be a primitive generically free G-variety and $K = k(X)^G$. Suppose $[X]$ lies in the image of the natural map $H^1(K, G_0) \longrightarrow H^1(K, G)$, where G_0 is a closed subgroup of G such that $\phi_{|G_0} = \mathrm{id}\colon G_0 \longrightarrow G_0$. Then X and X^ϕ are birationally isomorphic as K-varieties.*

Proof. Let $i\colon G_0 \hookrightarrow G$ be the inclusion map. The commutative diagram

$$\begin{tikzcd} G_0 \arrow[r, "i"] \arrow[d, "id"] & G \arrow[d, "\phi"] \\ G_0 \arrow[r, "i"] & G \end{tikzcd}$$

of groups induces a commutative diagram of cohomology sets

$$\begin{array}{ccc} H^1(K, G_0) & \xrightarrow{i_*} & H^1(K, G) \\ \Big\downarrow{\scriptstyle id} & & \Big\downarrow{\scriptstyle \phi_*} \\ H^1(K, G_0) & \xrightarrow{i_*} & H^1(K, G). \end{array}$$

Since $[X]$ is in the image of i_*, this diagram shows that $\phi_*([X]) = [X]$. ∎

Note that if $[X] = 1$ then $[X]$ is the image of the trivial element of $H^1(K, G_0)$, where $G_0 = \{1\}$. Example 4.1 is thus a special case of Lemma 4.2. We now turn to a more sophisticated application of Lemma 4.2, with non-trivial G_0.

Lemma 4.3. *Let k be a field of characteristic $\neq 2$, n be a positive integer, $a_1, \ldots, a_n \in k^*$,*

$$q(x_1, \ldots, x_n) = a_1 x_1^2 + \cdots + a x_n^2$$

be a non-degenerate quadratic form and ϕ be the automorphism of the special orthogonal group $\mathrm{SO}(q)$ given by

$$\phi(g) = hgh^{-1}, \text{ where } h = \mathrm{diag}(-1, 1, \ldots, 1) \in \mathrm{O}(q). \tag{3}$$

Then X and X^ϕ are birationally isomorphic over $K = k(X)^{\mathrm{SO}(q)}$ (and hence, over k) for any irreducible generically free $\mathrm{SO}(q)$-variety X.

We remark that if n is odd then ϕ is an inner automorphism; indeed, $\phi(g) = (-h)g(-h)^{-1}$ and $-h \in \mathrm{SO}(q)$. In this case a K-isomorphism between X and X^ϕ is given by $x \mapsto (-h) \cdot (x)$. Thus the above lemma is only of interest in the case where n is even; however, the argument below is valid for every $n \geq 1$.

Proof. Let $D_0 \simeq (\mathbb{Z}/2\mathbb{Z})^{n-1}$, $D \simeq (\mathbb{Z}/2\mathbb{Z})^n$ be the subgroups of diagonal matrices in $\mathrm{SO}(q)$, $\mathrm{O}(q)$ respectively, and $i \colon D_0 \hookrightarrow \mathrm{SO}(q)$, $j \colon D \hookrightarrow \mathrm{SO}(q)$ be the natural inclusion maps. Note that ϕ restricts to a trivial automorphism of D_0. Thus, in view of Lemma 4.2 it suffices to show that $i_* \colon H^1(K, D_0) \longrightarrow H^1(K, \mathrm{SO}(q))$ is surjective.

Consider the commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & D_0 & \longrightarrow & D & \xrightarrow{\det} & \mathbb{Z}/2\mathbb{Z} & \longrightarrow & 1 \\ & & \Big\downarrow{\scriptstyle i} & & \Big\downarrow{\scriptstyle j} & & & & \\ & & \mathrm{SO}(q) & \longrightarrow & \mathrm{O}(q) & & & & \end{array}$$

of algebraic groups and the induced commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & H^1(K, D_0) & \longrightarrow & (K^*/(K^*)^2)^n & \xrightarrow{p} & K^*/(K^*)^2 & \longrightarrow & 1 \\ & & \Big\downarrow{\scriptstyle i_*} & & \Big\downarrow{\scriptstyle j_*} & & & & \\ & & H^1(K, \mathrm{SO}(q)) & \longrightarrow & H^1(K, \mathrm{O}(q)) & & & & \end{array}$$

in cohomology. The top row is an exact sequence of abelian groups; $H^1(K, D_0)$ can thus be identified with the kernel of the product map $p\colon (K^*/(K^*)^2)^n \longrightarrow K^*/(K^*)^2$, where $p(b_1, \ldots, b_n) = b_1 \ldots b_n \pmod{(K^*)^2}$.

Now recall that $H^1(K, \mathrm{O}(q))$ is in a natural 1-1 correspondence with isometry classes of n-dimensional quadratic forms q' and that j_* takes $(b_1, \ldots, b_n) \in (K^*/K^{*2})^n$ to the quadratic form $q' = a_1b_1x_1^2 + \cdots + a_nb_nx_n^2$. Similarly, $H^1(K, \mathrm{SO}(q))$ is in a natural 1-1 correspondence with isometry classes of n-dimensional quadratic forms q' such that q' has the same discriminant as q, and i_* takes $(b_1, \ldots, b_n) \in (K^*/K^{*2})^n$, with $b_1 \ldots b_n = 1$ in K^*/K^{*2}, to $q' = a_1b_1x_1^2 + \cdots + a_nb_nx_n^2$; cf., [10, Section 29.E] or [18, III, Appendix 2, §2]. It is clear from this description that both i_* and j_* are surjective. ∎

We will now assume that the base field k is algebraically closed (and still of characteristic $\neq 2$). In this case, up to isomorphism, there is only one non-degenerate quadratic form q of each dimension $n \geq 1$, so we will set $q(x_1, \ldots, x_n) = x_1^2 + \cdots + x_n^2$ and write SO_n in place of $\mathrm{SO}(q)$.

Proposition 4.4. *Let k be an algebraically closed base field of characteristic $\neq 2$ and let X be a generically free SO_n-variety defined over k. Then X and X^ϕ are birationally isomorphic over $K = k(X)^{\mathrm{SO}_n}$ for every automorphism ϕ of SO_n.*

Proof. If $n = 1$ then the group SO_n is trivial, and there is nothing to prove. If $n = 2$ then SO_n is isomorphic to the 1-dimensional torus $\mathbb{G}_m$. Since this group is special (see e.g., [18, Proposition X.2]), in this case the proposition follows from Example 4.1.

We may thus assume that $n \geq 3$. As we remarked in the introduction, we are allowed to replace ϕ by $\phi' = \phi \circ \mathrm{inn}_h$, where inn_h denotes conjugation by $h \in \mathrm{SO}(q)$; indeed, X^ϕ and $X^{\phi'}$ are isomorphic (over K) via $x \mapsto h \cdot x$. For this reason we will be interested in the group $\mathrm{Out}(\mathrm{SO}_n)$ of outer automorphisms of SO_n.

Since SO_n is semisimple for every $n \geq 3$ (in fact, it is even simple, unless $n = 4$), we have

$$|\operatorname{Out}(\mathrm{SO}_n)| \leq |\operatorname{Aut}(\operatorname{Dyn}(\mathrm{SO}_n))|,$$

where $\operatorname{Dyn}(\mathrm{SO}_n)$ is the Dynkin diagram of SO_n; see [10, Theorem 25.16].

If $n = 2m + 1 \geq 3$ is odd, then SO_n is a simple group of type B_m. In this case $\operatorname{Dyn}(\mathrm{SO}_n)$ has no non-trivial automorphisms and thus $\operatorname{Out}(\mathrm{SO}_n) = \{1\}$. In other words, ϕ is an inner automorphism, and the proposition follows.

Now suppose $n = 2m \geq 4$ is even. Recall that SO_4 is a semisimple group of type $A_1 + A_1$, and SO_n is a simple group of type D_m for any $n \geq 6$. In both cases $|\operatorname{Aut}(\operatorname{Dyn}(\mathrm{SO}_n))| = 2$ and thus $|\operatorname{Out}(\mathrm{SO}_n)| \leq 2$. On the other hand, it is easy to see that the automorphism ϕ defined in (3) is not inner. Thus $|\operatorname{Out}(\mathrm{SO}_n)| = 2$, and after composing ϕ with an inner automorphism, we may assume that either $\phi = \mathrm{id}$ or ϕ is as in (3). In the former case the assertion of the proposition is trivial, in the latter if follows from Lemma 4.3. ∎

5. Proof of Theorem 1.3

Recall that elements of $H^1(K, \mathrm{PGL}_n)$ are in a natural 1-1 correspondence with

(i) generically free PGL_n-varieties X, with $k(X)^{\mathrm{PGL}_n} = K$, up to birational isomorphism over K, or alternatively, with

(ii) central simple algebras A/K of degree n, up to K-isomorphism;

see [18], [12], [15]. We will denote the central simple algebra corresponding to an irreducible generically free PGL_n-variety X (respectively, to an element $\alpha \in H^1(K, \mathrm{PGL}_n)$) by A_X (respectively, by A_α). If $\phi\colon \mathrm{PGL}_n \longrightarrow \mathrm{PGL}_n$ is the automorphism given by $g \longrightarrow (g^{-1})^{\text{transpose}}$ then $A_{\phi_*(\alpha)}$ is the opposite algebra A_α^{op}; cf. e.g., [19, pp. 152-153]. In other words, $A_{X^\phi} = A_X^{op}$. The following lemma gives a necessary and sufficient conditions for X and X^ϕ to be birationally isomorphic over K.

Lemma 5.1. *Let X be an irreducible generically free PGL_n-variety. Then the following conditions are equivalent.*

(a) X and X^ϕ are birationally isomorphic over $K = k(X)^{\mathrm{PGL}_n}$,

(b) A_X is K-isomorphic to A_X^{op},

(c) A_X has order 1 or 2 in the Brauer group $\mathrm{Br}(K)$.

Proof. (a) $\iff$ (b): Let $\alpha = [X] \in H^1(K, \mathrm{PGL}_n)$. Then as we observed in Section 4, $[X^\phi] = \phi_*(\alpha)$. Thus $A_{X^\phi} = A_{\phi_*(\alpha)} = A_\alpha^{op} = A_X^{op}$, so that X and X^ϕ are birationally isomorphic over K if and only if A_X is K-isomorphic to A_X^{op}.

The equivalence of (b) and (c) is obvious, since A^{op} is the inverse of A in $\mathrm{Br}(K)$. ∎

Lemma 5.1 does not directly address Problem 1.1, i.e., it does not tell us under what conditions X and X^ϕ are birationally isomorphic over the base field k. Note however, that a birational isomorphism $\alpha\colon X \longrightarrow X^\phi$ defined over k, restricts to a k-automorphism of the field of invariants $K = k(X)^{\mathrm{PGL}_n} = k(X^\phi)^{\mathrm{PGL}_n}$. Our proof of Theorem 1.3 is based on the observation that if $\mathrm{Aut}_k(K) = \{1\}$, then

X and X^ϕ are isomorphic over $k \iff$

X and X^ϕ are isomorphic over $K \iff$

A_X has order 1 or 2 in the Brauer group of K.

Thus in order to prove Theorem 1.3 it suffices to construct (i) a finitely generated field extension K/k such that $\mathrm{Aut}_k(K) = \{1\}$ and (ii) a central simple algebra A/K of degree n and exponent n. These constructions are carried out in Lemmas 5.2 and 5.3 below.

Lemma 5.2. *For any field k there exists a smooth projective geometrically connected algebraic surface S/k such that $S(k) \neq \emptyset$ which admits no non-trivial birational automorphisms. In other words,* $\mathrm{Aut}_k\, k(S) = \{1\}$.

Proof. According to [11], there exist smooth projective geometrically connected curves C/k, of arbitrary genus $g \geq 3$, such that the automorphism group of the extension $C_{\overline{k}}$ to the algebraic closure of k is trivial. It is easy to see that all the examples constructed in [11] have rational points over k (for example, each has a rational point over the origin in $\mathbb{P}^1_k$, and another one at infinity). Choose two such curves C_1 and C_2, of genus g_1 and g_2 respectively, with $g_1 < g_2$, and set

$S = C_1 \times C_2$. By the Hurwitz formula every map from C_1 to C_2 is constant and every non-constant separable map from C_i to itself is the identity.

We claim that every birational automorphism $f\colon S \dashrightarrow S$ is trivial. To prove this, we may extend the base field to $\overline{k}$, and assume that k is algebraically closed. Note that there are no rational curves in S, since there are no non-constant maps from $\mathbb{P}^1$ to C_1 or to C_2. By the standard result on the resolution of indeterminacies of rational maps by sequences of blow-ups, we see that every rational map $S \dashrightarrow S$ is in fact regular. Hence a birational isomorphism of S is a biregular isomorphism.

Since every morphism $C_1 \to C_2$ is constant, we see that the composite $S \xrightarrow{f} S \xrightarrow{\mathrm{pr}_2} C_2$ factors through the second projection $\mathrm{pr}_2\colon S \to C_2$ to give a commutative diagram

$$\begin{tikzcd} S \arrow[r, "f"] \arrow[d, "\mathrm{pr}_2"] & S \arrow[d, "\mathrm{pr}_2"] \\ C_2 \arrow[r, "g"] & C_2. \end{tikzcd}$$

Since $\mathrm{pr}_2 \circ f$ is smooth, g must be separable, so it is the identity. Then f sends each fiber $\mathrm{pr}_2^{-1}(y) \simeq C_1$ isomorphically to itself; and each of these restrictions must be the identity. This implies that f is the identity, as claimed. ∎

Lemma 5.3. *Let X be an algebraic variety over a field k of dimension $d \geq 2$ with a smooth rational k-point, K its fraction field, n a positive integer not divisible by* $\mathrm{char}(k)$ *such that k contains all nth roots of 1. Then there exists a division algebra D/K of degree n and exponent n.*

Proof. Let $x \in X(k)$ be a smooth rational point; consider a system of local parameters $t_1, \ldots, t_d$ in the local ring $\mathcal{O}_x(X)_{X,x}$; here $d = \dim(X) = \mathrm{trdeg}_k(K) \geq 2$. We claim that the symbol algebra $D = (t_1, t_2)_n$, i.e., the K-algebra given by generators x_1, x_2 and relations $x_1^n = t_1$, $x_2^n = t_2$, $x_1 x_2 = \zeta x_2 x_1$ has exponent n in $\mathrm{Br}(K)$. (Here ζ_n is a primitive nth root of unity in k.)

To prove this, consider the completion $\widehat{\mathcal{O}}_x(X) = k[[t_1, \ldots, t_d]]$ of the local ring $\mathcal{O}_x(X)$, where $k[[t_1, \ldots, t_d]]$ denotes the ring of formal power series in the variables $t_1, \ldots, t_d$. Note that $\mathcal{O}_x(X) \subset \widehat{\mathcal{O}}_x(X)$ and thus, after passing to the fields of fractions, $K \subset k((t_1, \ldots, t_d))$. The image of D under the restriction map $\mathrm{Br}(K) \longrightarrow \mathrm{Br}\big(k((t_1, \ldots, t_d))\big)$ is the symbol algebra $D' = (t_1, t_2)_n$ over $k((t_1, \ldots, t_d))$. A simple valuation-theoretic argument shows that D' has exponent n; cf. [17, Proposition 3.3.26]. Hence, so does D. This completes the proof of Lemma 5.3 and thus of Theorem 1.3. ∎

Remark 5.4. Theorem 1.3 remains valid if the condition that k contains a primitive nth root of unity is replaced by the (weaker) condition that k contains a primitive mth root of unity for some divisor m of n such that $m \geq 3$. The proof is the same, except that instead of the symbol algebra $D = (t_1, t_2)_n$ of degree n and exponent n, we use the algebra $\mathrm{M}_{n/m}(E)$ of degree n and exponent m, where $E = (t_1, t_2)_m$.

Remark 5.5. Theorem 1.3 fails for $n = 2$; indeed, A and A^{op} are isomorphic over K for any central simple algebra A/K of degree 2. Alternatively, $\mathrm{PGL}_2 \simeq \mathrm{SO}_3$, so if Theorem 1.3 were true for $n = 2$, it would contradict Proposition 4.4.

6. Further examples

In this section we will assume that G is a finite group and k is an algebraically closed field of characteristic zero.

Proposition 6.1. *(a) For every finitely generated field extension K/k with $K \neq k$ and every finite group G there exists a G-Galois extension L/K. Equivalently, there exists an irreducible G-variety X such that $k(X)^G = K$.*

(b) Suppose $\mathrm{Aut}_k(K) = \{1\}$ and $\phi\colon G \longrightarrow G$ is an outer automorphism of a finite group G. Then for every X, as in (a), the G-varieties X and X^ϕ are not birationally isomorphic.

Proof. (a) By the Riemann existence theorem there exists a G-Galois extension $L_0/k(t)$, where t is an independent variable. Hence, there exists a G-Galois extension $L_1/K(t)$, where $L_1 = L \otimes_{k(t)} K(t)$. The Hilbert irreducibility theorem now allows us to construct a G-Galois extension L/K by suitably specializing t in K.

(b) Irreducible G-varieties X (up to birational isomorphism) such that $k(X)^G = K$, are in 1-1 correspondence with G-Galois field extensions L/K. A birational isomorphism $\alpha\colon X \dashrightarrow X^\phi$ of G-varieties induces an isomorphism

$$\begin{tikzcd} L = k(X) \arrow[r, "\alpha"] \arrow[d, dash] & k(X^\phi) = L \arrow[d, dash] \\ K \arrow[r, equal] & K. \end{tikzcd}$$

Then $\alpha \in \mathrm{Gal}(L/K) = G$, and since the above diagram commutes, we have $\alpha g(l) = \phi(g)\alpha(l)$ for every $g \in G$ and $l \in L$. In other words, $\phi(g) = \alpha g \alpha^{-1}$, contradicting our assumption that ϕ is an outer automorphism. ■

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