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Central Extensions of the Lie Algebra of Symplectic Vector Fields

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Abstract. For a perfect ideal $\mathfrak{h}$ of the Lie algebra $\mathfrak{g}$, the extendibility of continuous 2-cocycles from $\mathfrak{h}$ to $\mathfrak{g}$ is studied, especially for 2-cocycles of the form $\langle [X,\cdot],\cdot\rangle$ on $\mathfrak{h}$ with $X \in \mathfrak{g}$, when a $\mathfrak{g}$-invariant symmetric bilinear form $\langle\cdot,\cdot\rangle$ on $\mathfrak{h}$ is available. The results are then applied to extend continuous 2-cocycles from the Lie algebra of Hamiltonian vector fields to the Lie algebra of symplectic vector fields on a compact symplectic manifold.
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1. Introduction

Given a compact $2n$-dimensional symplectic manifold (M,ω), we denote by H_f the Hamiltonian vector field associated to the Hamiltonian function f. Then the Lie algebra $\mathfrak{ham}(M,\omega)$ of Hamiltonian vector fields can be identified with $(C_0^\infty(M),\{,\})$, the Lie algebra of zero integral functions with Poisson bracket $\{f,g\} = -\omega(H_f,H_g)$. The following continuous Lie algebra 2-cocycles on the Fréchet Lie algebra $\mathfrak{ham}(M,\omega)$ are considered in [6] Section 9:

$$\sigma_\alpha(H_f,H_g) = \int_M f\alpha(H_g)\omega^n,$$

for α an arbitrary closed 1-form on M.

We study the extendibility of these 2-cocycles to continuous 2-cocycles on the Fréchet Lie algebra $\mathfrak{symp}(M,\omega)$ of symplectic vector fields. It turns out that this property depends only on the de Rham cohomology classes of α and ω. Denoting by $(b_1,b_2) = \int_M b_1 \wedge b_2 \wedge [\omega]^{n-1}$ the symplectic pairing on $H^1_{dR}(M)$, in Theorem 4.2 is shown that σ_α is extendible if and only if

$$(n-1)\int_M [\alpha]\wedge b_1 \wedge b_2 \wedge b_3 \wedge [\omega]^{n-2} = n\sum_{cycl}([\alpha],b_1)(b_2,b_3)$$

for all $b_i \in H^1_{dR}(M)$.

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We observe that if X is the symplectic vector field defined by $i_X\omega = \alpha$, the restriction to $\mathfrak{ham}(M,\omega)$ of the inner derivation $\mathrm{ad}(X)$ is $\mathrm{ad}(X)(f) = -\alpha(H_f)$. It follows that the cocycle σ_α is constructed with the derivation $\mathrm{ad}(X)$ and the $\mathfrak{symp}(M,\omega)$-invariant inner product $\langle H_f, H_g\rangle = \int_M fg\omega^n$ on $\mathfrak{ham}(M,\omega)$, namely $\sigma_\alpha = \langle [X,\cdot],\cdot\rangle$. Moreover $\mathfrak{ham}(M,\omega)$ is a perfect ideal of $\mathfrak{symp}(M,\omega)$ and their quotient is the abelian Lie algebra $H^1_{dR}(M)$.

For this reason in Section 3 we consider for $X \in \mathfrak{g}$ the Lie algebra 2-cocycles $\sigma_X = \langle [X,\cdot],\cdot\rangle$ on the ideal $\mathfrak{h}$ of $\mathfrak{g}$ in the following general setting: a perfect closed ideal $\mathfrak{h}$ of the topological Lie algebra $\mathfrak{g}$, with $\mathfrak{g}/\mathfrak{h}$ abelian and with the canonical projection $p : \mathfrak{g} \to \mathfrak{g}/\mathfrak{h}$ admitting continuous linear sections, and an $\mathrm{ad}(\mathfrak{g})$-invariant bilinear form $\langle\cdot,\cdot\rangle$ on $\mathfrak{h}$. Choosing a continuous linear section $s : \mathfrak{g}/\mathfrak{h} \to \mathfrak{g}$, we define $Q \in \Lambda^4(\mathfrak{g}/\mathfrak{h})^*$ by

$$Q(a,b_1,b_2,b_3) = \frac{1}{3}\sum_{cycl}\langle [sa,sb_1],[sb_2,sb_3]\rangle,$$

where the cyclic sum is taken over the indices 1,2,3. One can view $Q \in H^4(\mathfrak{g}/\mathfrak{h})$ as a characteristic class corresponding to the $\mathfrak{g}$-invariant bilinear form $\langle\cdot,\cdot\rangle$ for the Lie algebra extension

$$0 \to \mathfrak{h} \to \mathfrak{g} \xrightarrow{p} \mathfrak{g}/\mathfrak{h} \to 0.$$

In particular Q does not depend on the choice of the section s. We prove that σ_X is extendible to a continuous 2-cocycle on $\mathfrak{g}$ if and only if $i_{p(X)}Q = 0$. This is shown with the help of the transgression homomorphism t which fits into the exact sequence

$$H^2_c(\mathfrak{g}) \xrightarrow{i^*} H^2_c(\mathfrak{h})^{\mathfrak{g}} \xrightarrow{t} H^3_c(\mathfrak{g}/\mathfrak{h}) \xrightarrow{p^*} H^3_c(\mathfrak{g}).$$

Here $H^*_c(\mathfrak{g})$ denotes the continuous cohomology of $\mathfrak{g}$ and $H^*_c(\mathfrak{h})^{\mathfrak{g}}$ the continuous $\mathfrak{g}$-invariant cohomology of $\mathfrak{h}$.

A result from [6] Section 9 states that $H^2_c(\mathfrak{ham}(M,\omega))$ is isomorphic to $H^1_{dR}(M)$ by $[\alpha] \mapsto [\sigma_\alpha]$. All $[\sigma_\alpha]$ are $\mathfrak{symp}(M,\omega)$-invariant cohomology classes, so $H^2_c(\mathfrak{ham}(M,\omega))^{\mathfrak{symp}(M,\omega)}$ is also isomorphic to $H^1_{dR}(M)$. We show that in this case the transgression map is:

$$t : H^1_{dR}(M) \to \Lambda^3(H^1_{dR}(M))^*$$
$$t(a)(b_1,b_2,b_3) = n(n-1)\int_M a\wedge b_1\wedge b_2\wedge b_3\wedge[\omega]^{n-2} - n^2\sum_{cycl}(a,b_1)(b_2,b_3).$$

The second continuous cohomology space of the Lie algebra of symplectic vector fields turns out to be isomorphic to $\mathrm{Ker}\, t \oplus \Lambda^2 H^1_{dR}(M)^*$.

For the flat $2n$-torus $\mathbb{T}^{2n}$ with canonical symplectic form ω, $\mathfrak{symp}(\mathbb{T}^{2n},\omega)$ is the semidirect product of $\mathfrak{ham}(\mathbb{T}^{2n},\omega)$ with $\mathbb{R}^{2n}$, the abelian Lie algebra of constant vector fields. The transgression map $t : H^2_c(\mathfrak{h})^{\mathfrak{g}} \to H^3_c(\mathfrak{g}/\mathfrak{h})$ is trivial for a Fréchet Lie algebra $\mathfrak{g}$ which is a semidirect product of its perfect ideal $\mathfrak{h}$ with $\mathfrak{g}/\mathfrak{h}$. It follows that all the 2-cocycles σ_α are extendible to the Lie algebra of symplectic vector fields, so $H^2_c(\mathfrak{symp}(\mathbb{T}^2,\omega)) \cong H^1_{dR}(\mathbb{T}^2) \oplus \Lambda^2 H^1_{dR}(\mathbb{T}^2)$.

For any compact symplectic manifold (M,ω) there is a canonical $H^1_{dR}(M)^*$-valued 2-cohomology class λ on $\mathfrak{ham}(M,\omega)$ given by $\lambda([\alpha]) = [\sigma_\alpha]$ for any closed

1-form α. For the 2-torus there is a canonical 2-cocycle Σ representing λ:

$$\Sigma(f,g)=\langle fdg\rangle := \Big(\int_{\mathbb{T}^2} f\partial_x g\omega\Big)[dx]+\Big(\int_{\mathbb{T}^2} f\partial_y g\omega\Big)[dy]\in H^1_{dR}(\mathbb{T}^2)\cong H^1_{dR}(\mathbb{T}^2)^*.$$

We know that Σ is extendible because the transgression map vanishes. An extension of Σ to a continuous 2-cocycle on the Lie algebra of symplectic vector fields is $\Sigma'(X,Y)=\langle f_X df_Y\rangle$ for f_X the unique zero integral function such that $i_X\omega-\langle i_X\omega\rangle = df_X$. The central extension defined by Σ' is Kirillov's 2-dimensional central extension of the Lie algebra of symplectic vector fields on the 2-torus from [2] Section 5.

Surfaces of higher genus $g\geq 2$ have injective transgression maps, so for $[\alpha]\neq 0$ the cocycle σ_α is not extendible to the Lie algebra of symplectic vector fields.

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2. Extending continuous 2-cocycles on a perfect ideal $\mathfrak{h}$ to $\mathfrak{g}$

In this section we prove the exactness of a five term sequence involving the second and third continuous cohomology space associated to a perfect ideal of a Fréchet–Lie algebra.

Let $\mathfrak{g}$ be a Fréchet–Lie algebra and $\mathfrak{z}$ a topological $\mathfrak{g}$-module. On the space $C^p_c(\mathfrak{g},\mathfrak{z})$ of continuous alternating $\mathfrak{z}$-valued maps on $\mathfrak{g}$ we define the differential

$$\begin{aligned} d_{\mathfrak{g}}\sigma(X_1,\dots,X_{p+1}) &= \sum_{i=1}^{p+1}(-1)^{i+1}X_i.\sigma(X_1,\dots,\hat{X}_i,\dots,X_{p+1})\\ &\quad+\sum_{i<j}(-1)^{i+j}\sigma([X_i,X_j],X_1,\dots,\hat{X}_i,\dots,\hat{X}_j,\dots,X_{p+1}). \end{aligned}$$

The cohomology $H^*_c(\mathfrak{g},\mathfrak{z})$ of this chain complex is the *continuous cohomology* of $\mathfrak{g}$ with values in the $\mathfrak{g}$-module $\mathfrak{z}$. We write $H^*_c(\mathfrak{g})$ if $\mathfrak{z}=\mathbb{R}$ is the trivial $\mathfrak{g}$-module. There is a bijection between $H^2_c(\mathfrak{g},\mathfrak{z})$ and topologically split Lie algebra extensions of $\mathfrak{g}$ by the $\mathfrak{g}$-module $\mathfrak{z}$.

Given an ideal $\mathfrak{h}$ of $\mathfrak{g}$, there is a natural action of $\mathfrak{g}$ on $C^p_c(\mathfrak{h})$ by

$$(L_X\sigma)(H_1,\dots,H_p)=-\sum_{i=1}^{p}\sigma(H_1,\dots,[X,H_i],\dots,H_p).$$

It commutes with $d_{\mathfrak{h}}$, hence induces an action on $H^p_c(\mathfrak{h})$. Let $H^*_c(\mathfrak{h})^{\mathfrak{g}}$ denote the $\mathfrak{g}$-invariant continuous cohomology space of $\mathfrak{h}$.

We denote by $\mathfrak{h}^*$ the dual of $\mathfrak{h}$ with its canonical $\mathfrak{g}$-module structure. Let $C^1_c(\mathfrak{g},\mathfrak{h}^*)_T$ be the space of linear maps $\theta:\mathfrak{g}\to\mathfrak{h}^*$ such that the bilinear map $(X,H)\in\mathfrak{g}\times\mathfrak{h}\mapsto\theta(X)(H)$ is continuous and its restriction $\sigma:\mathfrak{h}\times\mathfrak{h}\to\mathbb{R}$ is alternating. Then $B^1(\mathfrak{g},\mathfrak{h}^*)$ is a subset of $C^1_c(\mathfrak{g},\mathfrak{h}^*)_T$ and we define the cohomology space $H^1_c(\mathfrak{g},\mathfrak{h}^*)_T$ to be the quotient $(Z^1(\mathfrak{g},\mathfrak{h}^*)\cap C^1_c(\mathfrak{g},\mathfrak{h}^*)_T)/B^1(\mathfrak{g},\mathfrak{h}^*)$. In the discrete case this space is defined in [5], Remark II.3.

Lemma 2.1 *If $\mathfrak{h}$ is a perfect ideal of the Fréchet–Lie algebra $\mathfrak{g}$, then the restriction map induces an isomorphism in cohomology*

$$\rho : [\theta] \in H^1_c(\mathfrak{g}, \mathfrak{h}^*)_T \mapsto [\sigma] \in H^2_c(\mathfrak{h})^{\mathfrak{g}}$$

defined by $\sigma(H,K) = \theta(H)(K)$. Its inverse is uniquely determined by the relation $L_X\sigma = d_{\mathfrak{h}}\theta(X)$ for all $X \in \mathfrak{g}$, i.e.

$$\theta(X)[H,K] = \sigma([X,H],K) + \sigma(H,[X,K]), \quad H,K \in \mathfrak{h}. \tag{1}$$

Proof. To see that the restriction map ρ is well defined, we check that σ is a 2-cocycle whose cohomology class is $\mathfrak{g}$-invariant. Indeed, by the 1-cocycle condition for θ we get for $X \in \mathfrak{g}$:

$$\begin{aligned} -(L_X\sigma)(H,K) &= \sigma([X,H],K) + \sigma(H,[X,K]) \\ &= \theta([X,H])(K) + \theta(H)([X,K]) = \theta(X)([H,K]). \end{aligned}$$

Specializing to $X \in \mathfrak{h}$ we obtain that σ is a Lie algebra 2-cocycle on $\mathfrak{h}$.

To show that the restriction map ρ is injective, we assume θ is an $\mathfrak{h}^*$-valued 1-cocycle on $\mathfrak{g}$ whose restriction to $\mathfrak{h}$ is a coboundary $\sigma = d_{\mathfrak{h}}\beta$ for $\beta \in \mathfrak{h}^*$. Then we have for $X \in \mathfrak{g}$ and $H,K \in \mathfrak{h}$

$$\begin{aligned} \theta(X)[H,K] &= \theta([X,H])(K) + \theta(H)([X,K]) \\ &= -\beta([[X,H].K]) - \beta([H,[X,K]]) = \beta([[H,K],X]) \end{aligned}$$

and the perfectness of $\mathfrak{h}$ imply $\theta = d_{\mathfrak{g}}\beta$.

Given $[\sigma] \in H^2_c(\mathfrak{h})^{\mathfrak{g}}$, there is a map $\theta \in C^1(\mathfrak{g}, \mathfrak{h}^*)$ such that $L_X\sigma = d_{\mathfrak{h}}\theta(X)$ for all $X \in \mathfrak{g}$. It is uniquely determined since $\mathfrak{h}$ is a perfect Lie algebra, and it extends the 2-cocycle σ. The fact that θ is an $\mathfrak{h}^*$-valued 1-cocycle on $\mathfrak{g}$ follows from

$$d_{\mathfrak{h}}(\theta[X,Y]) = L_{[X,Y]}\sigma = L_X(d_{\mathfrak{h}}\theta(Y)) - L_Y(d_{\mathfrak{h}}\theta(X)) = d_{\mathfrak{h}}(L_X\theta(Y) - L_Y\theta(X))$$

since $[L_X, L_Y] = L_{[X,Y]}$ and $[\mathfrak{h},\mathfrak{h}] = \mathfrak{h}$.

It remains to check that $\theta : \mathfrak{g} \times \mathfrak{h} \to \mathbb{R}$ is continuous. This is the only place where the Fréchet assumption is needed. Since $\mathfrak{h}$ is a perfect Lie algebra, the Lie bracket induces a continuous surjective map on the completion $\Lambda^2(\mathfrak{h})_c$ of $\Lambda^2\mathfrak{h}$ with respect to the projective tensor topology. By the open mapping theorem for Fréchet spaces [7], the linear continuous surjective map $1_{\mathfrak{g}} \times [\cdot,\cdot] : \mathfrak{g} \times \Lambda^2(\mathfrak{h})_c \to \mathfrak{g} \times \mathfrak{h}$ is a quotient map. Because σ is continuous, the fact that the composition $\theta \circ (1_{\mathfrak{g}} \times [\cdot,\cdot])$ is continuous on $\mathfrak{g} \times \Lambda^2(\mathfrak{h})_c$ implies that θ is continuous. Hence $[\theta] \in H^1_c(\mathfrak{g}, \mathfrak{h}^*)_T$ is the preimage of $[\sigma] \in H^2_c(\mathfrak{h})^{\mathfrak{g}}$ and thus ρ is surjective. ∎

Assume that $\mathfrak{h}$ is a perfect ideal of $\mathfrak{g}$ and the exact sequence

$$0 \to \mathfrak{h} \xrightarrow{i} \mathfrak{g} \xrightarrow{p} \mathfrak{g}/\mathfrak{h} \to 0 \tag{2}$$

is topologically split, i.e. p admits a continuous section $s : \mathfrak{g}/\mathfrak{h} \to \mathfrak{g}$. Then we can define a *transgression map*

$$t : H^2_c(\mathfrak{h})^{\mathfrak{g}} \to H^3_c(\mathfrak{g}/\mathfrak{h}), \quad t([\sigma]) = [\overline{d_{\mathfrak{g}}\sigma'}], \tag{3}$$

where $\sigma' \in C_c^2(\mathfrak{g})$ is any continuous alternating extension of the 1-cocycle $\theta : \mathfrak{g} \to \mathfrak{h}^*$ defined by (1). It is well defined since

$$(i_H d_{\mathfrak{g}}\sigma')(X,Y) = -\sigma'([X,Y],H) + \sigma'(X,[Y,H]) - \sigma'(Y,[X,H])$$
$$= -\theta([X,Y])(H) + \theta(X)[Y,H] - \theta(Y)[X,H] = (d_{\mathfrak{g}}\theta)(X,Y)(H) = 0$$

implies that $d_{\mathfrak{g}}\sigma'$ indeed factors through a 3-cocycle on $\mathfrak{g}/\mathfrak{h}$, denoted $\overline{d_{\mathfrak{g}}\sigma'}$. Its cohomology class in $H_c^3(\mathfrak{g}/\mathfrak{h})$ does not depend on the choice of the continuous extension σ': two choices σ' and σ_1' differ by $p^*\beta$ with $\beta \in C_c^2(\mathfrak{g}/\mathfrak{h})$, hence $\overline{d_{\mathfrak{g}}\sigma'}$ and $\overline{d_{\mathfrak{g}}\sigma_1'}$ differ by the coboundary $d_{\mathfrak{g}/\mathfrak{h}}\beta$.

Proposition 2.2 *If the extension $0 \to \mathfrak{h} \xrightarrow{i} \mathfrak{g} \xrightarrow{p} \mathfrak{g}/\mathfrak{h} \to 0$ is a semidirect product and $\mathfrak{h}$ is perfect, then the transgression map vanishes.*

Proof. Denoting by $\cdot$ the Lie algebra action of $\mathfrak{g}/\mathfrak{h}$ on $\mathfrak{h}$, the Lie bracket on the semidirect product is

$$[(H_1,b_1),(H_2,b_2)] = ([H_1,H_2] + b_1 \cdot H_2 - b_2 \cdot H_1, [b_1,b_2])$$

for $H_1, H_2 \in \mathfrak{h}$ and $b_1, b_2 \in \mathfrak{g}/\mathfrak{h}$.

Given $[\sigma] \in H_c^2(\mathfrak{h})^{\mathfrak{g}}$, there is a unique $\mathfrak{h}^*$-valued 1-cocycle θ on $\mathfrak{g}$ determined by $L_X\sigma = d_{\mathfrak{h}}\theta(X)$, $X \in \mathfrak{g}$. In particular $\theta(H,b) = i_H\sigma + \theta(b)$. An alternating extension $\sigma' \in C_c^2(\mathfrak{g})$ of θ is

$$\sigma'((H_1,b_1),(H_2,b_2)) = \sigma(H_1,H_2) + \theta(b_1)(H_2) - \theta(b_2)(H_1).$$

A short computation leads to

$$\begin{aligned} & d_{\mathfrak{g}}\sigma'((H_1,b_1),(H_2,b_2),(H_3,b_3)) \\ = \; & d_{\mathfrak{h}}\sigma(H_1,H_2,H_3) + \sum_{cycl}(d_{\mathfrak{h}}\theta(b_1) - L_{b_1}\sigma)(H_2,H_3) - \sum_{cycl} d_{\mathfrak{g}}\theta(b_1,b_2)(H_3) = 0, \end{aligned}$$

hence $t([\sigma]) = 0$. ■

Theorem 2.3 *If $\mathfrak{h}$ is a perfect ideal of the Fréchet–Lie algebra $\mathfrak{g}$ and (2) is topologically split, then the sequence*

$$0 \to H_c^2(\mathfrak{g}/\mathfrak{h}) \xrightarrow{p^*} H_c^2(\mathfrak{g}) \xrightarrow{i^*} H_c^2(\mathfrak{h})^{\mathfrak{g}} \xrightarrow{t} H_c^3(\mathfrak{g}/\mathfrak{h}) \xrightarrow{p^*} H_c^3(\mathfrak{g})$$

is exact. In particular $H_c^2(\mathfrak{g})/H_c^2(\mathfrak{g}/\mathfrak{h})$ is isomorphic to $\operatorname{Ker} t$.

Proof. There are four things to be shown.

The map $p^* : H_c^2(\mathfrak{g}/\mathfrak{h}) \to H_c^2(\mathfrak{g})$ is injective: Assume $p^*[\beta] = 0$, i.e. β is a continuous 2-cocycle on $\mathfrak{g}/\mathfrak{h}$ such that $p^*\beta = d_{\mathfrak{g}}\alpha$ for a continuous linear map α on $\mathfrak{g}$. Then $[\mathfrak{h},\mathfrak{h}] = \mathfrak{h}$ implies that α vanishes on $\mathfrak{h}$. So $\alpha = p^*\bar{\alpha}$ for a continuous linear map $\bar{\alpha}$ on $\mathfrak{g}/\mathfrak{h}$ and $\beta = d_{\mathfrak{g}/\mathfrak{h}}\bar{\alpha}$.

$\operatorname{Im} p^* = \operatorname{Ker} i^*$: First $i^*p^* = 0$ implies $\operatorname{Im} p^* \subset \operatorname{Ker} i^*$. For the converse let $[\omega] \in \operatorname{Ker} i^*$. Then there is a continuous linear map α on $\mathfrak{h}$ with $i^*\omega = d_{\mathfrak{h}}\alpha$. For any α' extending α continuously to $\mathfrak{g}$, the 2-cocycle $\omega' = \omega - d_{\mathfrak{g}}\alpha'$ vanishes $\mathfrak{h}$.

Then ω' vanishes also on $\mathfrak{g} \times \mathfrak{h}$ because $\mathfrak{h}$ is a perfect ideal and $\omega'(X, [H, K]) = -\omega'(H, [K, X]) - \omega'(K, [X, H]) = 0$. It follows that ω' is of the form $p^*\beta$ for a continuous 2-cocycle β on $\mathfrak{g}/\mathfrak{h}$, hence $[\omega] = p^*[\beta] \in \operatorname{Im} p^*$.

$\operatorname{Im} i^* = \operatorname{Ker} t$: If ω is a 2-cocycle on $\mathfrak{g}$, then $ti^*[\omega] = [\overline{d_\mathfrak{g}\omega}] = 0$, hence $\operatorname{Ker} t \supset \operatorname{Im} i^*$. To show the reverse inclusion, let $[\sigma] \in \operatorname{Ker} t$ and $\theta : \mathfrak{g} \to \mathfrak{h}^*$ the unique 1-cocycle from Lemma 1 extending σ. Because the transgression of $[\sigma]$ is zero, there exists $\gamma \in C^2_c(\mathfrak{g}/\mathfrak{h})$ such that $\overline{d_\mathfrak{g}\sigma'} = d_{\mathfrak{g}/\mathfrak{h}}\gamma$, with σ' a continuous extension of θ. Then $\omega = \sigma' - p^*\gamma$ is a continuous 2-cocycle on $\mathfrak{g}$ extending σ, so $[\sigma] = i^*[\omega]$ and $\operatorname{Ker} t \subset \operatorname{Im} i^*$.

$\operatorname{Im} t = \operatorname{Ker} p^*$: First $p^*t([\sigma]) = p^*[\overline{d_\mathfrak{g}\sigma'}] = [d_\mathfrak{g}\sigma'] = 0$ implies $\operatorname{Im} t \subset \operatorname{Ker} p^*$. For the converse let $[\beta] \in \operatorname{Ker} p^*$, so there is a continuous 2-cochain σ' on $\mathfrak{g}$ such that $p^*\beta = d_\mathfrak{g}\sigma'$. The restriction σ of σ' to $\mathfrak{h}$ satisfies $d_\mathfrak{h}\sigma = 0$ and $L_X\sigma = d_\mathfrak{h} i^* i_X \sigma'$. So the 1-cocycle $\theta : \mathfrak{g} \to \mathfrak{h}^*$ from Lemma 2.1 corresponding to $[\sigma] \in H^2_c(\mathfrak{h})^\mathfrak{g}$ is $\theta(X) = i^* i_X \sigma'$ and σ' is a continuous extension of the map $(X, H) \mapsto \theta(X)(H)$ to $\mathfrak{g} \times \mathfrak{g}$. Hence $t[\sigma] = [\overline{d_\mathfrak{g}\sigma'}] = [\beta]$ and $\operatorname{Ker} p^* \subset \operatorname{Im} t$. ■

Remark 2.4 The 2-cocycle σ on $\mathfrak{h}$ with $[\sigma] \in H^2_c(\mathfrak{h})^\mathfrak{g}$ can be extended to a continuous 2-cocycle on $\mathfrak{g}$ if and only if its transgression is zero. If $\mathfrak{g}/\mathfrak{h}$ is abelian, each extension σ' of the unique $\mathfrak{h}^*$-valued 1-cocycle θ which restricts to σ is a 2-cocycle on $\mathfrak{g}$ extending σ and all the other extensions are obtained by adding elements of the form $p^*\beta$ with $\beta \in \Lambda^2(\mathfrak{g}/\mathfrak{h})^*$.

Any continuous linear section $s : \mathfrak{g}/\mathfrak{h} \to \mathfrak{g}$ defines the continuous retraction $\eta : \mathfrak{g} \to \mathfrak{h}$ by $\eta(X) = X - spX$ for $X \in \mathfrak{g}$. Then there is a unique continuous extension σ' of θ vanishing on the image of s:

$$\begin{aligned}\sigma'(X, Y) &= \theta(X)(\eta(Y)) - \theta(Y)(\eta(X)) - \sigma(\eta(X), \eta(Y)) \\ &= \sigma(\eta(X), \eta(Y)) + \theta(spX)(\eta(Y)) - \theta(spY)(\eta(X)).\end{aligned}$$

The 5-term exact sequence from Theorem 2.3 written for discrete Lie algebra cohomology spaces is the content of Theorem 6 for $m = 2$ in [1]. The Hochschild-Serre spectral sequence for an ideal $\mathfrak{h}$ of $\mathfrak{g}$ is the spectral sequence associated to the filtration $F^pC^{p+q}(\mathfrak{g}) = \{\sigma \in C^{p+q}(\mathfrak{g}) : i_{H_1} \dots i_{H_{q+1}}\sigma = 0, \text{ for all } H_i \in \mathfrak{h}\}$. This means $d_r : E^{p,q}_r \to E^{p+r,q-r+1}_r$ is induced by $d_\mathfrak{g}$, with

$$E^{p,q}_r = \{a \in F^pC^{p+q} : d_\mathfrak{g}a \in F^{p+r}C^{p+q+1}\}/(d_\mathfrak{g}(F^{p-r+1}C^{p+q-1}) + F^{p+1}C^{p+q}).$$

The E_2-term is in this case $E^{p,q}_2 = H^p(\mathfrak{g}/\mathfrak{h}, H^q(\mathfrak{h}))$ and the spectral sequence abuts to $H^*(\mathfrak{g})$. For perfect $\mathfrak{h}$ we obtain that the map $d_3 : E^{0,2}_3 \to E^{3,0}_3$ from the Hochschild-Serre spectral sequence can be identified with the transgression map $t : H^2(\mathfrak{h})^\mathfrak{g} \to H^3(\mathfrak{g}/\mathfrak{h})$.

The restriction map $i^* : H^2(\mathfrak{g}) \to H^2(\mathfrak{h})^\mathfrak{g}$ for an arbitrary ideal $\mathfrak{h}$ of $\mathfrak{g}$ is studied in [5].

3. Special 2-cocycles on $\mathfrak{h}$

In [6] Section 3, a Lie algebra 2-cocycle $\sigma_D(H, K) := \langle D(H), K\rangle$ on $\mathfrak{h}$ is associated to every anti self-dual derivation D of a Lie algebra $\mathfrak{h}$ with respect to an invariant symmetric bilinear form $\langle \cdot, \cdot \rangle$ on $\mathfrak{h}$, i.e. $\langle D(H), K\rangle + \langle H, D(K)\rangle = 0$.

A particular case is the continuous 2-cocycle $\sigma_X = \langle [X,\cdot],\cdot\rangle$ for $X \in \mathfrak{g}$, where $\mathfrak{h}$ is an ideal of the topological Lie algebra $\mathfrak{g}$ with an $\mathrm{ad}(\mathfrak{g})$-invariant continuous symmetric bilinear form $\langle\cdot,\cdot\rangle$ on $\mathfrak{h}$ (here $D = \mathrm{ad}(X)|_{\mathfrak{h}}$). For $H \in \mathfrak{h}$ the cocycle σ_H is the coboundary of $\langle H,\cdot\rangle \in C^1_c(\mathfrak{h})$. Hence we get a linear map $\lambda : \mathfrak{g}/\mathfrak{h} \to H^2_c(\mathfrak{h})$. When $\mathfrak{g}/\mathfrak{h}$ is finite dimensional, we can view λ as a canonical $(\mathfrak{g}/\mathfrak{h})^*$-valued 2-cohomology class on $\mathfrak{h}$.

Proposition 3.1 *Let $\mathfrak{h}$ be a perfect ideal of the topological Lie algebra $\mathfrak{g}$ with an $\mathrm{ad}(\mathfrak{g})$-invariant symmetric bilinear form $\langle\cdot,\cdot\rangle$ on $\mathfrak{h}$ and assume that (2) is topologically split with $\mathfrak{g}/\mathfrak{h}$ abelian. Then the cohomology class of the cocycle $\sigma_X = \langle [X,\cdot],\cdot\rangle$ is $\mathfrak{g}$-invariant and its image $t([\sigma_X]) \in \Lambda^3(\mathfrak{g}/\mathfrak{h})^*$ under the transgression map is $(b_1,b_2,b_3) \mapsto \sum_{cycl}\langle [X,sb_1],[sb_2,sb_3]\rangle$, with $s : \mathfrak{g}/\mathfrak{h} \to \mathfrak{g}$ any continuous section of (2).*

Proof. For $Y \in \mathfrak{g}$ and $H,K \in \mathfrak{h}$ we compute

$$\begin{aligned}(L_Y\sigma_X)(H,K) &= -\sigma_X([Y,H],K) - \sigma_X(H,[Y,K]) \\ &= \langle [X,K],[Y,H]\rangle - \langle [X,H],[Y,K]\rangle \\ &= -\langle H,[Y,[X,K]]\rangle + \langle H,[X,[Y,K]]\rangle \\ &= -\langle H,[K,[X,Y]]\rangle = -\langle [X,Y],[H,K]\rangle.\end{aligned}$$

At the last step we use $[\mathfrak{g},\mathfrak{g}] \subset \mathfrak{h}$ (since $\mathfrak{g}/\mathfrak{h}$ is abelian). We get that $L_Y\sigma_X = d_{\mathfrak{h}}(\theta_X(Y))$ with $\theta_X(Y) = \langle [X,Y],\cdot\rangle$, so the cohomology class $[\sigma_X]$ is $\mathfrak{g}$-invariant and θ_X is the unique $\mathfrak{h}^*$-valued 1-cocycle on $\mathfrak{g}$ extending σ_X.

Let $\sigma'_X \in C^2_c(\mathfrak{g})$ be a continuous extension of the 1-cocycle θ_X. Then $t([\sigma_X]) = [\overline{d_{\mathfrak{g}}\sigma'_X}]$ and

$$\overline{d_{\mathfrak{g}}\sigma'_X}(b_1,b_2,b_3) = \sum_{cycl}\sigma'_X(sb_1,[sb_2,sb_3])$$
$$= \sum_{cycl}\theta_X(sb_1)([sb_2,sb_3]) = \sum_{cycl}\langle [X,sb_1],[sb_2,sb_3]\rangle,$$

for $b_1,b_2,b_3 \in \mathfrak{g}/\mathfrak{h}$. ■

Characteristic classes for Lie algebra extensions: A short exact sequence of Lie algebras:

$$0 \to \mathfrak{h} \xrightarrow{i} \mathfrak{g} \xrightarrow{p} \mathfrak{k} \to 0, \qquad (4)$$

is an extension $\mathfrak{g}$ of $\mathfrak{k}$ by $\mathfrak{h}$. The extension is called abelian if $\mathfrak{h}$ is an abelian Lie algebra. In this case $\mathfrak{h}$ carries a canonical $\mathfrak{k}$-module structure induced by the adjoint action of $\mathfrak{g}$ on $\mathfrak{h}$. An abelian extension of the Lie algebra $\mathfrak{k}$ by the $\mathfrak{k}$-module $\mathfrak{h}$ is described by a cohomology class in $H^2(\mathfrak{k},\mathfrak{h})$.

The *characteristic classes* are the cohomological objects associated to a non-abelian extension of $\mathfrak{k}$ by $\mathfrak{h}$. These are images of the Weil homomorphism defined below and are elements of $H^*(\mathfrak{k},V)$, with V an arbitrary $\mathfrak{k}$-module.

A linear section $s : \mathfrak{k} \to \mathfrak{g}$ for (4) is called a *connection*. The defect of s to be a Lie algebra homomorphism is the *curvature* $\Omega : \mathfrak{k}\times\mathfrak{k} \to \mathfrak{h}$, defined by $\Omega(b_1,b_2) = [sb_1,sb_2] - s[b_1,b_2]$. Denoting by $\eta : \mathfrak{g} \to \mathfrak{h}$ the corresponding retraction, the structure equation and the Bianchi identity hold in the following form: $p^*\Omega = -d\eta + \frac{1}{2}[\eta,\eta]$ and $dp^*\Omega = -[p^*\Omega,\eta]$; see [8] Section 3.

We can recover the notions of connection and curvature of a principal bundle in this way. There is an exact sequence of Lie algebras and $C^\infty(M)$-modules associated to the principal G-bundle $P \to M$,

$$0 \to C^\infty(P,\mathfrak{g})^G \to \mathfrak{X}(P)^G \to \mathfrak{X}(M) \to 0,$$

i.e. the Lie algebra of G-invariant vector fields of P is an extension of the Lie algebra of vector fields on M by the Lie algebra of G-equivariant $\mathfrak{g}$-valued functions on P (vertical G-invariant vector fields on P). A $C^\infty(M)$-linear section s can be identified with the horizontal lift of a principal connection and Ω comes from its curvature 2-form on P.

With the help of the curvature Ω, an analogue of the Weil homomorphism can be constructed like in [3] Section 2. For any $\mathfrak{k}$-module V, let $I^n_V(\mathfrak{h})$ be the set of $\mathfrak{g}$-equivariant V-valued symmetric n-linear mappings on $\mathfrak{h}$, i.e.

$$\sum_{i=1}^n \varphi(H_1,\dots,[X,H_i],\dots,H_n) = p(X)\varphi(H_1,\dots,H_n), \quad \text{for all } X\in\mathfrak{g}.$$

Then the *Weil homomorphism*

$$W : I^n_V(\mathfrak{h}) \to H^{2n}(\mathfrak{k},V), \quad W(\varphi) := [\mathrm{Alt}(\varphi\circ(\Omega\otimes\cdots\otimes\Omega))],$$

does not depend on the chosen connection s. Here Alt denotes anti-symmetrization of multilinear forms.

Remark 3.2 Let $\mathfrak{h}$ be an ideal of the Lie algebra $\mathfrak{g}$ with an $\mathrm{ad}(\mathfrak{g})$-invariant symmetric bilinear form $\langle\cdot,\cdot\rangle$ on $\mathfrak{h}$. Then $\langle\cdot,\cdot\rangle \in I^2_{\mathbb{R}}(\mathfrak{h})$ and the characteristic class $Q = W(\langle\cdot,\cdot\rangle) \in H^4(\mathfrak{g}/\mathfrak{h})$ is the cohomology class of the 4-cocycle

$$(a,b_1,b_2,b_3) \mapsto \frac{1}{3}\sum_{cycl}\langle\Omega(a,b_1),\Omega(b_2,b_3)\rangle, \quad b_i\in\mathfrak{g}/\mathfrak{h}. \tag{5}$$

We can reformulate Proposition 3.1 as:

Corollary 3.3 *If $\mathfrak{h}$ is a perfect ideal of the topological Lie algebra $\mathfrak{g}$ with an $\mathrm{ad}(\mathfrak{g})$-invariant symmetric bilinear form $\langle\cdot,\cdot\rangle$ on $\mathfrak{h}$, and (2) is topologically split with $\mathfrak{g}/\mathfrak{h}$ abelian, then the transgression of $\sigma_X = \langle[X,\cdot],\cdot\rangle$ is $t[\sigma_X] = 3i_{p(X)}Q$ for the characteristic class $Q = W(\langle\cdot,\cdot\rangle)\in\Lambda^4(\mathfrak{g}/\mathfrak{h})^*$.*

Proof. The curvature is in this case $\Omega(a,b) = [sa,sb]$. With Proposition 3.1 we obtain

$$\begin{aligned} t([\sigma_X])(b_1,b_2,b_3) &= \sum_{cycl}\langle[sp(X),sb_1],[sb_2,sb_3]\rangle + \sum_{cycl}\langle[\eta(X),sb_1],[sb_2,sb_3]\rangle \\ &= 3Q(p(X),b_1,b_2,b_3) + \sum_{cycl}\langle\eta(X),[sb_1,[sb_2,sb_3]]\rangle = 3i_{p(X)}Q(b_1,b_2,b_3), \end{aligned}$$

using the relation $X = sp(X)+\eta(X)$ in the first line. ■

Remark 3.4 It follows from Remark 2.4 that σ_X is extendible to $\mathfrak{g}$ if and only if $i_{p(X)}Q = 0$. So the vanishing of the characteristic class Q ensures the extendibility of all σ_X for $X\in\mathfrak{g}$.

4. 2-cocycles on the Lie algebras of Hamiltonian and symplectic vector fields

Let (M,ω) be a compact connected $2n$-dimensional symplectic manifold. A vector field X is called symplectic if $L_X\omega = 0$ and the vector field H_f is called Hamiltonian with Hamiltonian function f if $i_{H_f}\omega = df$. Since M is compact and connected, each Hamiltonian vector field has a unique zero integral Hamiltonian function. The Lie algebra $\mathfrak{ham}(M,\omega)$ of Hamiltonian vector fields on M is an ideal of the Fréchet–Lie algebra $\mathfrak{symp}(M,\omega)$ of symplectic vector fields. It can be identified with the Lie algebra of zero integral functions on M with the Poisson bracket $\{f,g\} = -\omega(H_f,H_g)$. The quotient Lie algebra $\mathfrak{symp}(M,\omega)/\mathfrak{ham}(M,\omega) = H^1_{dR}(M)$ is abelian and the projection of a symplectic vector field X is $p(X) = [i_X\omega]$, so that

$$0 \to \mathfrak{ham}(M,\omega) \to \mathfrak{symp}(M,\omega) \xrightarrow{p} H^1_{dR}(M) \to 0 \tag{6}$$

is a topologically split exact sequence of Lie algebras. We also know that the Lie algebra of Hamiltonian vector fields is perfect [4]. The inner product $\langle H_f, H_g\rangle = \int_M fg\omega^n$ is $\mathfrak{symp}(M,\omega)$-invariant. Indeed, for any symplectic vector field X, the Lie bracket $[X,H_f] = H_{L_Xf}$ and L_Xf is a zero integral Hamiltonian function. So

$$\langle [X,H_f],H_g\rangle + \langle H_f,[X,H_g]\rangle = \int_M (gL_Xf + fL_Xg)\omega^n = \int_M L_X(fg)\omega^n = 0.$$

Hence all the requirements from Section 3 are satisfied.

The special 2-cocycles $\sigma_X = \langle [X,\cdot],\cdot\rangle$ for $X \in \mathfrak{symp}(M,\omega)$ coincide with the cocycles on the Lie algebra of Hamiltonian vector fields considered in [6] Section 9:

$$\sigma_X(H_f,H_g) = \langle [X,H_f],H_g\rangle = -\langle H_f,[X,H_g]\rangle = \langle H_f, H_{\alpha(H_g)}\rangle = \int_M f\alpha(H_g)\omega^n,$$

where $\alpha = i_X\omega$ is a closed 1-form. In the third equality we use $L_Xg = \omega(H_g,X) = -\alpha(H_g)$. It follows from the proof of Proposition 3.1 that the $\mathfrak{ham}(M,\omega)^*$-valued 1-cocycle θ_X on $\mathfrak{symp}(M,\omega)$ extending σ_X is in this case $\theta_X(Y)(H_f) = \langle [X,Y],H_f\rangle = -\int_M f\omega(X,Y)\omega^n$. Note that the Hamiltonian function $-\omega(X,Y)$ for $[X,Y]$ has non-zero integral in general.

To see which of the cocycles σ_X are extendible to the Lie algebra of symplectic vector fields, we compute the characteristic class $Q = \langle\cdot,\cdot\rangle$. Let Ω be the curvature of a connection $s : H^1_{dR}(M) \to \mathfrak{symp}(M,\omega)$, a continuous section of (6). We denote by $(\cdot,\cdot)$ the symplectic pairing on $H^1_{dR}(M)$, i.e. $(b_1,b_2) = \int_M b_1\wedge b_2\wedge[\omega]^{n-1}$. Then $[i_{sb}\omega] = b$ and $\int_M \omega(sb_1,sb_2)\omega^n = n(b_1,b_2)$. We get that $\Omega(b_1,b_2) = [sb_1,sb_2]$ is the Hamiltonian vector field with the zero integral Hamiltonian function $f = n(b_1,b_2) - \omega(sb_1,sb_2)$.

Proposition 4.1 *The characteristic class $Q = W(\langle\cdot,\cdot\rangle) \in \Lambda^4 H^1_{dR}(M)^*$ of the Lie algebra extension (6) is*

$$Q(a,b_1,b_2,b_3) = \frac{1}{3}n(n-1)\int_M a\wedge b_1\wedge b_2\wedge b_3\wedge[\omega]^{n-2} - \frac{1}{3}n^2\sum_{cycl}(a,b_1)(b_2,b_3),$$

with $a, b_i \in H^1_{dR}(M)$ and the cyclic sum taken over the indices 1,2,3.

Proof. Using formula (5) in Section 3 and the fact that $\Omega(b_1,b_2)$ is the Hamiltonian vector field with zero integral Hamiltonian function $n(b_1,b_2)-\omega(sb_1,sb_2)$, we compute:

$$3Q(a,b_1,b_2,b_3)=\sum_{cycl}\langle\Omega(a,b_1),\Omega(b_2,b_3)\rangle$$
$$=\sum_{cycl}\int_M(n(a,b_1)-\omega(sa,sb_1))(n(b_2,b_3)-\omega(sb_2,sb_3))\omega^n$$
$$=\sum_{cycl}\int_M\omega(sa,sb_1)\omega(sb_2,sb_3)\omega^n-n^2\sum_{cycl}(a,b_1)(b_2,b_3).$$

It remains to calculate the first cyclic sum, which we denote by

$$S\overset{not.}{=}\sum_{cycl}\int_M\omega(sa,sb_1)\omega(sb_2,sb_3)\omega^n.$$

Applying the formula $i_X\alpha\wedge\beta=(-1)^{|\alpha|+1}\alpha\wedge i_X\beta$ for $|\alpha|+|\beta|=\dim M+1$, we obtain

$$\begin{aligned}
S &= n\sum_{cycl}\int_M i_{sb_1}i_{sa}\omega\wedge i_{sb_2}\omega\wedge i_{sb_3}\omega\wedge\omega^{n-1}\\
&= n\sum_{cycl}\int_M\omega(sb_2,sb_1)i_{sa}\omega\wedge i_{sb_3}\omega\wedge\omega^{n-1}\\
&\quad -n\sum_{cycl}\int_M\omega(sb_3,sb_1)i_{sa}\omega\wedge i_{sb_2}\omega\wedge\omega^{n-1}\\
&\quad +n(n-1)\sum_{cycl}\int_M i_{sa}\omega\wedge i_{sb_1}\omega\wedge i_{sb_2}\omega\wedge i_{sb_3}\omega\wedge\omega^{n-2}\\
&= \sum_{cycl}\int_M\omega(sa,sb_3)\omega(sb_2,sb_1)\omega^n-\sum_{cycl}\int_M\omega(sa,sb_2)\omega(sb_3,sb_1)\omega^n\\
&\quad +n(n-1)\sum_{cycl}\int_M a\wedge b_1\wedge b_2\wedge b_3\wedge[\omega]^{n-2}\\
&= -2S+3n(n-1)\int_M a\wedge b_1\wedge b_2\wedge b_3\wedge[\omega]^{n-2}.
\end{aligned}$$

We get

$$S=n(n-1)\int_M a\wedge b_1\wedge b_2\wedge b_3\wedge[\omega]^{n-2}$$

and the result follows. ∎

Theorem 4.2 *Given a closed 1-form α on the symplectic manifold (M,ω), the 2-cocycle $\sigma_\alpha(H_f,H_g)=\int_M f\alpha(H_g)\omega^n$ on the Lie algebra of Hamiltonian vector fields is extendible to the Lie algebra of symplectic vector fields if and only if the de Rham cohomology class $a=[\alpha]$ satisfies the relation*

$$(n-1)\int_M a\wedge b_1\wedge b_2\wedge b_3\wedge[\omega]^{n-2}=n\sum_{cycl}(a,b_1)(b_2,b_3)\qquad(7)$$

for all $b_i\in H^1_{dR}(M)$.

Proof. The transgression of $[\sigma_\alpha]$ is

$$t([\sigma_\alpha])(b_1,b_2,b_3) = n(n-1)\int_M a\wedge b_1\wedge b_2\wedge b_3\wedge[\omega]^{n-2} - n^2\sum_{cycl}(a,b_1)(b_2,b_3) \quad (8)$$

by Corollary 3.3 and Proposition 4.1. Then the result follows from Remark 2.4. ■

The following result was announced in [6] Section 9:

Theorem 4.3 *The second continuous cohomology space of the Lie algebra of Hamiltonian vector fields on a compact symplectic manifold M is isomorphic to $H^1_{dR}(M)$, the isomorphism being $[\alpha]\mapsto[\sigma_\alpha]$, with $\sigma_\alpha(H_f,H_g)=\int_M f\alpha(H_g)\omega^n$.*

Each cohomology class $[\sigma_\alpha]$ is $\mathfrak{symp}(M,\omega)$-invariant, hence the second continuous $\mathfrak{symp}(M,\omega)$-invariant cohomology space of $\mathfrak{ham}(M,\omega)$ is again $H^1_{dR}(M)$.

Corollary 4.4 *The second continuous cohomology space of the Lie algebra of symplectic vector fields is isomorphic to $\operatorname{Ker} t\oplus\Lambda^2H^1_{dR}(M)^*$, where $t:H^1_{dR}(M)\to\Lambda^3H^1_{dR}(M)^*$ is the transgression map given by (8).*

Example 4.5 For surfaces the condition (7) becomes $\sum_{cycl}(a,b_1)(b_2,b_3)=0$. On surfaces of genus 1 all σ_α are extendible, since $\dim H^1_{dR}(M)=2$ implies that the transgression map vanishes. On surfaces of genus ≥ 2 none of the σ_α are extendible, i.e. $\operatorname{Ker} t=0$. Indeed, for any non-zero element $a\in H^1_{dR}(M)$, we can find $b_1,b_2,b_3\in H^1_{dR}(M)$ such that $(a,b_1)=(b_2,b_3)=1$ and $(a,b_2)=(a,b_3)=(b_1,b_2)=(b_1,b_3)=0$, hence $t(a)(b_1,b_2,b_3)\neq 0$.

Specializing the previous example to the 2-torus $\mathbb{T}^2=\mathbb{R}^2/\mathbb{Z}^2$ with canonical symplectic form $\omega=dx\wedge dy$, one obtains Kirillov's 2-dimensional central extension of $\mathfrak{symp}(\mathbb{T}^2,\omega)$ from [2] Section 5. We recall that the canonical $H^1_{dR}(M)^*$-valued 2-cohomology class λ on $\mathfrak{ham}(M,\omega)$ mentioned in the beginning of Section 3 is $\lambda([\alpha])=[\sigma_\alpha]$ for any closed 1-form α. The definition is correct since the cohomology class of the cocycle σ_α depends only on the de Rham cohomology class of α.

We identify $H^1_{dR}(\mathbb{T}^2)$ with its dual via $a\mapsto\int_{\mathbb{T}^2}\cdot\wedge a$. There is a canonical 2-cocycle Σ representing λ, namely

$$\Sigma(H_f,H_g)\mapsto[\langle fdg\rangle]\in H^1_{dR}(\mathbb{T}^2)\cong H^1_{dR}(\mathbb{T}^2)^*,$$

with $\langle\cdot\rangle$ denoting the average of a 1-form on the 2-torus

$$\langle adx+bdy\rangle=\Big(\int_{\mathbb{T}^2}a\omega\Big)dx+\Big(\int_{\mathbb{T}^2}b\omega\Big)dy.$$

Indeed,

$$\int_{\mathbb{T}^2}dx\wedge\Sigma(H_f,H_g)=\int_{\mathbb{T}^2}f\partial_y g\,dx\wedge dy=\sigma_{dx}(H_f,H_g)$$

and the same identity holds for dy.

It follows from Example 4.5 above that Σ can be extended to a 2-cocycle on the Lie algebra of symplectic vector fields on the 2-torus. The 1-cocycle extending Σ is

$$\Theta:\mathfrak{symp}(\mathbb{T}^2,\omega)\to\mathfrak{ham}(\mathbb{T}^2,\omega)^*\otimes H^1_{dR}(\mathbb{T}^2),\quad \Theta(X)(H_f)=-[\langle fi_X\omega\rangle].$$

To see this we show that $L_X\Sigma = d_{\mathfrak{h}}(\Theta(X))$. We use $df \wedge dg = -\{f,g\}\omega$ in the following calculation:

$$\begin{aligned}(L_X\Sigma)(H_f,H_g) &= -[\langle L_X f dg\rangle] + [\langle L_X g df\rangle] = -[\langle i_X(df\wedge dg)\rangle]\\ &= [\langle\{f,g\}i_X\omega\rangle] = -\Theta(X)([H_f,H_g]) = d_{\mathfrak{h}}(\Theta(X))(H_f,H_g).\end{aligned}$$

With Remark 2.4 we can find a special extension Σ' of Σ. If α is a closed 1-form on $\mathbb{T}^2$, then $\alpha - \langle\alpha\rangle$ is an exact 1-form. For X a symplectic vector field, $i_X\omega$ is closed and we denote by f_X the unique zero integral function such that $i_X\omega - \langle i_X\omega\rangle = df_X$. Let Σ' be the 2-cocycle extending Σ and vanishing on the image of the connection s defined by $s([dx]) = -\partial_y$, $s([dy]) = \partial_x$. In this case the retraction $\eta : \mathfrak{symp}(\mathbb{T}^2,\omega) \to \mathfrak{ham}(\mathbb{T}^2,\omega)$ is $\eta(X) = X - s[i_X\omega] = X - s[\langle i_X\omega\rangle] = H_{f_X}$. Hence

$$\begin{aligned}\Sigma'(X,Y) &= \Theta(X)(\eta(Y)) - \Theta(Y)(\eta(X)) - \Sigma(\eta(X),\eta(Y))\\ &= -\langle f_Y i_X\omega\rangle + \langle f_X i_Y\omega\rangle - \langle f_X df_Y\rangle\\ &= -\langle f_Y df_X\rangle + \langle f_X\langle i_Y\omega\rangle\rangle - \langle f_Y\langle i_X\omega\rangle\rangle = -\langle f_Y df_X\rangle = \langle f_X df_Y\rangle\end{aligned}$$

is the extension we were looking for.

This means $\Sigma'(\partial_x, H_f) = \Sigma'(\partial_y, H_f) = 0$ and $\Sigma'(H_f,H_g) = \Sigma(H_f,H_g) = [\langle f dg\rangle]$. It defines Kirillov's 2-dimensional central extension of the Lie algebra of symplectic vector fields on the 2-torus

$$0 \to H^1(\mathbb{T}^2) \to \widehat{\mathfrak{symp}(\mathbb{T}^2},\omega) \to \mathfrak{symp}(\mathbb{T}^2,\omega) \to 0.$$

The bracket is given by: $[\partial_x, H_f] = H_{\partial_x f}$, $[\partial_y, H_f] = H_{\partial_y f}$, $[\partial_x,\partial_y] = 0$, $[H_{f_1}, H_{f_2}] = H_{\{f_1,f_2\}} + \langle f_1 df_2\rangle$, the Lie algebra $\mathfrak{symp}(\mathbb{T}^2,\omega)$ being linearly generated by ∂_x, ∂_y and $\mathfrak{ham}(\mathbb{T}^2,\omega)$.

Example 4.6 For the flat $2n$-torus $\mathbb{T}^{2n}$ with canonical symplectic form ω, the Lie algebra of symplectic vector fields is the semidirect product of the Lie algebra of Hamiltonian vector fields with $\mathbb{R}^{2n}$, the abelian Lie algebra of constant vector fields. Proposition 2.2 shows that the transgression map $t : H^2_c(\mathfrak{h})^{\mathfrak{g}} \to H^3_c(\mathfrak{g}/\mathfrak{h})$ is trivial for a Fréchet–Lie algebra $\mathfrak{g}$ which is a semidirect product of its perfect ideal $\mathfrak{h}$ and the quotient Lie algebra $\mathfrak{g}/\mathfrak{h}$. It follows that all the 2-cocycles σ_α are extendible. In particular we recover the result from Example 4.5 for the 2-torus.

Example 4.7 Thurston's symplectic manifold ([9] page 10) is $M = \mathbb{R}^4/\Gamma$ with Γ the discrete group generated by the following symplectic diffeomorphisms of $(\mathbb{R}^4, dx_1\wedge dy_1 + dx_2\wedge dy_2)$:

$$\begin{aligned}(x_1,y_1,x_2,y_2) &\mapsto (x_1,y_1,x_2+1,y_2)\\ (x_1,y_1,x_2,y_2) &\mapsto (x_1,y_1,x_2,y_2+1)\\ (x_1,y_1,x_2,y_2) &\mapsto (x_1+1,y_1,x_2,y_2)\\ (x_1,y_1,x_2,y_2) &\mapsto (x_1,y_1+1,x_2+y_2,y_2).\end{aligned}$$

Since $\Gamma/[\Gamma,\Gamma] = \mathbb{Z}^3$, the first de Rham cohomology group is 3-dimensional. Then the map $t : H^1_{dR}(M) = \mathbb{R}^3 \to \Lambda^3 H^1_{dR}(M)^* = \mathbb{R}$ has a non-trivial kernel, so there are 2-cocycles σ_α on $\mathfrak{ham}(M,\omega)$ which can be extended to $\mathfrak{symp}(M,\omega)$.

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