Journal of Lie Theory
Volume **17** (2007) 669–684

On the Principal Bundles over a Flag Manifold, II

Hassan Azad and Indranil Biswas

Communicated by E. B. Vinberg

Abstract. Let G be a connected semisimple linear algebraic group defined over an algebraically closed field k and $P \subsetneq G$ a reduced parabolic subgroup that does not contain any simple factor of G. Let $\rho : P \longrightarrow H$ be a homomorphism, where H is a connected reductive linear algebraic group defined over k, with the property that the image $\rho(P)$ is not contained in any proper parabolic subgroup of H. We prove that the principal H–bundle $G \times^P H$ over G/P constructed using ρ is stable with respect to any polarization on G/P. When the characteristic of k is positive, the principal H–bundle $G \times^P H$ is shown to be strongly stable with respect to any polarization on G/P.
Mathematics Subject Classification 2000: 14M15, 14F05.
Keywords and phrases: Homogeneous space, principal bundle, Frobenius, stability.

1. Introduction

Let k be an algebraically closed field. Take any connected semisimple linear algebraic group G defined over k. Let $P \subset G$ be a (reduced) parabolic subgroup such that the image of P in any simple quotient of G is a proper subgroup. In other words, P does not contain any simple factor of P. The subgroup P being parabolic the quotient G/P is a smooth projective variety.

Let H be a connected reductive linear algebraic group defined over the field k. Let

$$\rho : P \longrightarrow H$$

be an irreducible homomorphism. This means that the image $\rho(P)$ is not contained in any proper parabolic subgroup of H. Associated to ρ, we have a principal H–bundle over G/P which can be constructed as follows: Let $G \times^P H$ be the quotient of $G \times H$ for the twisted diagonal action of P whose orbit through any point $(g_0, h_0) \in G \times H$ consists of all $(g_0 g^{-1}, \rho(g)h_0)$, $g \in P$. The composition of the projection $G \times H \longrightarrow G$ with the quotient map $G \longrightarrow G/P$ descends to a projection from $G \times^P H$ to G/P. This descended projection defines a principal H–bundle over G/P. Let E_H denote this principal H–bundle over G/P.

We recall that when the characteristic of k is positive, a principal bundle over a smooth polarized projective variety X defined over k, with a reductive

ISSN 0949–5932 / \$2.50

group as the structure group, is called strongly stable if all the iterated pullbacks of it by the Frobenius morphism of X are stable principal bundles; the details of the definition are given in Section 2. (we assume that a polarization on G/P has been fixed in order to be able to define stable bundles). For our convenience, when the characteristic of k is zero, by a strongly stable principal H–bundle over G/P we will simply mean a stable principal H–bundle over G/P.

The following theorem is the main result proved here (see Theorem 3.4):

Theorem 1.1. *The above principal H–bundle E_H over G/P is strongly stable with respect to any polarization on G/P.*

We note that Theorem 1.1 was proved in [7], [11] under the assumption that $H = \mathrm{GL}(n,k)$ with $\mathrm{char}(k) = 0$. In [2], Theorem 1.1 was proved under the assumption that $k = \mathbb{C}$ using differential geometric methods.

2. Semistability of homogeneous principal bundles

Let k be an algebraically closed field of arbitrary characteristic. Henceforth, the characteristic of k will be denoted by p. Let G be a connected semisimple linear algebraic group defined over the field k. We fix a reduced proper parabolic subgroup

$$P \subsetneq G$$

without any simple factor. This is equivalent to the condition that the image of P in each simple quotient of G is a proper parabolic subgroup.

Fix a very ample line bundle

$$\zeta \in \mathrm{Pic}(G/P) \tag{1}$$

over G/P. Such a line bundle is also called a *polarization* on G/P. It is known that any ample line bundle over G/P is very ample.

Definition 2.1. For a torsionfree coherent sheaf V over G/P, define the degree of V to be the degree of the restriction of V to the general complete intersection curve in G/P obtained by intersecting hyperplanes in G/P from the complete linear system $|\zeta|$. The degree of V will be denoted by $\mathrm{degree}(V)$.

If V is a vector bundle defined over a nonempty Zariski open dense subset $U \subseteq G/P$ such that the complement $(G/P)\setminus U$ is of codimension at least two, then the direct image $\iota_* V$ is a torsionfree coherent sheaf on G/P, where $\iota : U \longrightarrow G/P$ is the inclusion map. For such a vector bundle V, define

$$\mathrm{degree}(V) := \mathrm{degree}(\iota_* V)\,. \tag{2}$$

A torsionfree coherent sheaf E over G/P is called *stable* (respectively, *semistable*) if

$$\frac{\mathrm{degree}(E')}{\mathrm{rank}(E')} < \frac{\mathrm{degree}(E)}{\mathrm{rank}(E)}$$

(respectively, $\frac{\mathrm{degree}(E')}{\mathrm{rank}(E')} \leq \frac{\mathrm{degree}(E)}{\mathrm{rank}(E)}$) for every coherent subsheaf $E' \subset E$ with $0 < \mathrm{rank}(E') < \mathrm{rank}(E)$.

If $p > 0$, where p is the characteristic of the base field k, then

$$F : G/P \longrightarrow G/P \tag{3}$$

will be the Frobenius morphism of the variety G/P. For notational convenience, when $p = 0$, by F we will denote the identity morphism of G/P.

For any $n \geq 1$, let

$$F^n := \overbrace{F \circ \cdots \circ F}^{n\text{-times}} : G/P \longrightarrow G/P$$

be the n–fold composition of the self–map F. By F^0 we will denote the identity map of G/P.

A vector bundle E over G/P is called *strongly stable* (respectively, *strongly semistable*) if for each integer $n \geq 0$, the pullback $(F^n)^*E$ is a stable (respectively, semistable) vector bundle, where F^n is defined above.

Since F^0 is the identity map of G/P, a strongly stable (respectively, strongly semistable) vector bundle is stable (respectively, semistable). We note that in the case where $p = 0$, a strongly stable (respectively, strongly semistable) vector bundle is simply a stable (respectively, semistable) vector bundle.

Let H be a connected reductive linear algebraic group defined over the field k. Let Q be a proper parabolic subgroup of H, and let λ be a character of Q which is trivial on the connected component of the center of H containing the identity element. Such a character λ is called *strictly anti–dominant* if the associated line bundle $L_\lambda = G \times^P k$ over G/P is ample.

A principal H–bundle E_H over G/P is called *stable* (respectively, *semistable*) if for every triple of the form $(Q\,, E_Q\,, \lambda)$, where

- $Q \subsetneq H$ is a reduced parabolic subgroup, and $E_Q \subset E_H$ is a reduction of structure group of E_H to Q over some Zariski open dense subset $U \subset G/P$ such that the codimension of the complement $(G/P) \setminus U$ is at least two, and
- λ is some strictly anti–dominant character of Q,

the inequality

$$\text{degree}(E_Q(\lambda)) > 0$$

(respectively, $\text{degree}(E_Q(\lambda)) \geq 0$) holds, where $E_Q(\lambda)$ is the line bundle over U associated to the principal Q–bundle E_Q for the character λ of Q.

In order to decide whether a given principal H–bundle E_H is semistable (respectively, stable), it suffices to verify the above inequality (respectively, strict inequality) only for those Q which are proper maximal parabolic subgroups of H. More precisely, E_H is semistable (respectively, stable) if and only if for every pair $(Q\,, \sigma)$, where

- $Q \subset H$ is a proper maximal parabolic subgroup, and
- $\sigma : U \longrightarrow E_H/Q$ is a reduction of structure group of E_H to Q over some Zariski open dense subset $U \subset G/P$ such that the codimension of the complement $(G/P) \setminus U$ is at least two,

the inequality

$$\text{degree}(\sigma^* T_{\text{rel}}) > 0 \tag{4}$$

(respectively, $\text{degree}(\sigma^* T_{\text{rel}}) \geq 0$) holds, where T_{rel} is the relative tangent bundle over E_H/Q for the projection $E_H/Q \longrightarrow G/P$. (See [9, page 129, Definition 1.1] and [9, page 131, Lemma 2.1].)

Remark 2.2. We note a couple of points regarding the above definitions.

1. The definition of degree of a torsionfree coherent sheaf on G/P depends on the choice of the polarization ζ in Eqn. (1); see Definition 2.1 and Eqn. (2). Therefore, it is more accurate to call "stable (respectively, semistable) with respect to ζ" instead of calling simply "stable (respectively, semistable)". However, since in all the existing literature the imprecise notation is systematically used, we will stick to it.

2. Let E_{GL_n} be a principal $\text{GL}_n(k)$–bundle over G/P. Let E_V be the vector bundle over G/P of rank n associated to E_{GL_n} for the standard action of $\text{GL}_n(k)$ on $k^{\oplus n}$. The associated vector bundle E_V is stable (respectively, semistable) if and only if the principal $\text{GL}_n(k)$–bundle E_{GL_n} is stable (respectively, semistable). To see this first note that the proper maximal parabolic subgroups of $\text{GL}_n(k)$ are parametrized by the proper nonzero linear subspaces of $k^{\oplus n}$. Giving a reduction of structure group E_Q of E_{GL_n} to a proper maximal parabolic subgroup $Q \subset \text{GL}_n(k)$ is equivalent to giving a subbundle W of E_V whose rank coincides with the dimension of the subspace of $k^{\oplus n}$ which Q preserves. The pullback $\sigma^* T_{\text{rel}}$ in Eqn. (4) coincides with the tensor product $W^* \bigotimes (E_V/W)$. Therefore, we have

$$\text{degree}(\sigma^* T_{\text{rel}}) = \text{degree}(E_V/W) \cdot \text{rank}(W) - \text{degree}(W) \cdot \text{rank}(E_V/W).$$

   Using this equality it follows immediately that the principal $\text{GL}_n(k)$–bundle E_{GL_n} is stable (respectively, semistable) if and only if the associated vector bundle E_V is stable (respectively, semistable).

A principal H–bundle E_H over G/P is called *strongly stable* (respectively, *strongly semistable*) if for each integer $n \geq 0$, the iterated n–fold pullback $(F^n)^* E_H$ is a stable (respectively, semistable) principal H–bundle, where the map F, as before, is the Frobenius morphism in Eqn. (3) when $p > 0$, and it is the identity morphism of G/P when $p = 0$. Also, as before, F^0 is the identity map of G/P.

Let E_H be a principal H–bundle over G/P. A reduction of structure group

$$E_Q \subset E_H$$

to some parabolic subgroup $Q \subset H$ is called *admissible* if for each character λ of Q trivial on the center of H, the associated line bundle $E_Q(\lambda) = E_Q{}^{\lambda}k$ over G/P satisfies the following condition:

$$\text{degree}(E_Q(\lambda)) = 0 \tag{5}$$

[10, page 307, Definition 3.3] (see [3, page 3998–3999] for some explanations of the notion of admissible reduction).

A principal H–bundle E_H over G/P is called *polystable* if either E_H is stable, or there is a proper parabolic subgroup Q and a reduction of structure group $E_{L(Q)} \subset E_H$ to a Levi subgroup $L(Q)$ of Q over G/P such that

- the principal $L(Q)$–bundle $E_{L(Q)}$ is stable, and
- the reduction of structure group of E_H to Q obtained by extending the structure group of $E_{L(Q)}$ using the inclusion of $L(Q)$ in Q is admissible.

A principal H–bundle E_H is called *strongly polystable* if for each integer $n \geq 0$, the iterated n–fold pullback $(F^n)^* E_H$ is polystable.

The quotient map $G \longrightarrow G/P$ defines a principal P–bundle over the projective variety G/P. This tautological principal P–bundle over G/P will be denoted by E_P. The unipotent radical of P will be denoted by $R_u(P)$. The quotient group

$$L(P) := P/R_u(P),$$

which is called the *Levi quotient* of P, is a connected reductive linear algebraic group defined over k. Let

$$q : P \longrightarrow L(P) \tag{6}$$

be the quotient map. Let

$$E_{L(P)} := E_P(L(P)) = (G \times L(P))/P \tag{7}$$

be the principal $L(P)$–bundle over G/P obtained by extending the structure group of the principal P–bundle E_P using the homomorphism q in Eqn. (6). In the construction of the quotient in Eqn. (7), the action of any point $z \in P$ sends any point $(g\,,h) \in G \times L(P)$ to $(gz\,,q(z^{-1})h) \in G \times L(P)$.

Proposition 2.3. *The tautological principal $L(P)$–bundle $E_{L(P)}$ over G/P constructed in Eqn.* (7) *is strongly semistable with respect to any polarization on G/P.*

Proof. The Lie algebra of $L(P)$ will be denoted by $\mathfrak{l}(\mathfrak{p})$. Let $\mathrm{ad}(E_{L(P)})$ be the adjoint bundle for the principal $L(P)$–bundle $E_{L(P)}$. Therefore, $\mathrm{ad}(E_{L(P)})$ is the vector bundle over G/P associated to $E_{L(P)}$ for the adjoint action of $L(P)$ on $\mathfrak{l}(\mathfrak{p})$.

When the characteristic p of the field k is positive, let

$$F_L : L(P) \longrightarrow L(P)$$

be the Frobenius morphism of the group $L(P)$. When $p = 0$, by F_L we will denote the identity map of $L(P)$. For any integer $n \geq 1$, let

$$F_L^n := \overbrace{F_L \circ \cdots \circ F_L}^{n\text{-times}}) : L(P) \longrightarrow L(P) \tag{8}$$

be the n–fold composition of the self–map F_L. By F_L^0 we will denote the identity map of $L(P)$.

For notational convenience, the $L(P)$–module $\mathfrak{l}(\mathfrak{p})$ defined by the adjoint action will be denoted by V.

For any integer $n \geq 0$, let V_n denote the $L(P)$–module given by the composition homomorphism

$$L(P) \xrightarrow{F_L^n} L(P) \longrightarrow \mathrm{Aut}(V)\,,$$

where F_L^n is defined in Eqn. (8), while the above homomorphism $L(P) \to \mathrm{Aut}(V)$ is the adjoint action. The vector bundle over G/P associated to the principal $L(P)$–bundle $E_{L(P)}$ for the $L(P)$–module V_n will be denoted by $E_{L(P)}(V_n)$.

To prove that a principal G'–bundle $E_{G'}$ is semistable, where G' is any connected reductive linear algebraic group over k, it suffices to show that its adjoint vector bundle $\mathrm{ad}(E_{G'})$ is semistable. Indeed, given any reduction of structure group $E_{P'} \subset E_{G'}$ violating the semistability condition for $E_{G'}$, the subbundle $\mathrm{ad}(E_{P'}) \subset \mathrm{ad}(E_{G'})$ violates the semistability condition for $\mathrm{ad}(E_{G'})$. Consequently, to prove that the principal $L(P)$–bundle $E_{L(P)}$ is strongly semistable, it suffices to show that the adjoint vector bundle $\mathrm{ad}(E_{L(P)})$ is strongly semistable.

Consider the Frobenius morphism F in Eqn. (3). The pulled back vector bundle $(F^n)^*\mathrm{ad}(E_{L(P)})$ is identified with the vector bundle $E_{L(P)}(V_n)$. Consequently, to prove the proposition it is enough to show that the above defined associated vector bundle $E_{L(P)}(V_n)$ is semistable for all n.

Let

$$0 = W_n^0 \subset W_n^1 \subset \cdots \subset W_n^{i_n-1} \subset W_n^{i_n} = V_n \tag{9}$$

be a filtration of the $L(P)$-module V_n such that each successive quotient W_n^j/W_n^{j-1}, $j \in [1\,, i_n]$, is an irreducible $L(P)$–module. Let $E_{L(P)}(W_n^j)$, $j \in [0\,, i_n]$, be the vector bundle over G/P associated to the principal $L(P)$–bundle $E_{L(P)}$ for the $L(P)$–module W_n^j. Similarly, let $E_{L(P)}(W_n^j/W_n^{j-1})$, $j \in [1\,, i_n]$, denote the vector bundle associated to $E_{L(P)}$ for the $L(P)$–module W_n^j/W_n^{j-1}. The filtration in Eqn. (9) gives a filtration of subbundles

$$0 = E_{L(P)}(W_n^0) \subset E_{L(P)}(W_n^1) \subset \cdots \subset E_{L(P)}(W_n^{i_n-1}) \subset E_{L(P)}(W_n^{i_n}), \tag{10}$$

where $E_{L(P)}(W_n^{i_n}) = (F^n)^*\mathrm{ad}(E_{L(P)})$. We note that for each $j \in [1\,, i_n]$, the quotient bundle $E_{L(P)}(W_n^j)/E_{L(P)}(W_n^{j-1})$ is canonically identified with

$$E_{L(P)}(W_n^j/W_n^{j-1}).$$

Take any $j \in [1\,, i_n]$. We will show that the vector bundle $E_{L(P)}(W_n^j/W_n^{j-1})$ is semistable. To prove this, assume that $E_{L(P)}(W_n^j/W_n^{j-1})$ is not semistable. Let

$$\mathcal{E}_n^j \subset E_{L(P)}(W_n^j/W_n^{j-1}) \tag{11}$$

be the maximal semistable subsheaf of $E_{L(P)}(W_n^j/W_n^{j-1})$. In other words, $\mathcal{E}_n^j$ is the first term in the Harder–Narasimhan filtration of $E_{L(P)}(W_n^j/W_n^{j-1})$. See [6, page 16, Theorem 1.3.4] for Harder–Narasimhan filtration.

The group G acts on G/P as left translations. The left–translation action of G on itself is a lift of this action of G on G/P to the principal P–bundle E_P which commutes with the principal bundle structure. In other words, the left action of G on E_P and the right action of P on E_P commute. The left–action of G on E_P induces a left–action of G on the principal $L(P)$–bundle $E_{L(P)}$ which commutes with the right–action of $L(P)$ on $E_{L(P)}$. This left–action of G on $E_{L(P)}$

induces a left–action on any bundle associated to $E_{L(P)}$. In particular, the group G acts on the associated vector bundle $E_{L(P)}(W_n^j/W_n^{j-1})$ over G/P that lifts the left–translation action of G on G/P. Since the group G is connected, it preserves any polarization on G/P (the ample line bundles on G/P form a discrete set). Therefore, from the uniqueness of the Harder–Narasimhan filtration it follows that the action of G on $E_{L(P)}(W_n^j/W_n^{j-1})$ preserves the subsheaf $\mathcal{E}_n^j$ in Eqn. (11).

Since the left–translation action of G on G/P is transitive, the fact that the action of G on $E_{L(P)}(W_n^j/W_n^{j-1})$ preserves the subsheaf $\mathcal{E}_n^j$ implies that $\mathcal{E}_n^j$ is in fact a subbundle of $E_{L(P)}(W_n^j/W_n^{j-1})$.

Let $e \in G$ be the identity element. For the action of G on G/P, the isotropy subgroup of the point $eP \in G/P$ is P itself. In other words, the fiber $(E_P)_{eP}$ of E_P over the point eP is identified with P. We note that the fiber $E_{L(P)}(W_n^j/W_n^{j-1})_{eP}$ of $E_{L(P)}(W_n^j/W_n^{j-1})$ over the point $eP \in G/P$ is identified with the vector space underlying the $L(P)$–module W_n^j/W_n^{j-1}. The identification

$$W_n^j/W_n^{j-1} \xrightarrow{\sim} E_{L(P)}(W_n^j/W_n^{j-1})_{eP} \tag{12}$$

is obtained by sending any $v \in W_n^j/W_n^{j-1}$ to the image in $E_{L(P)}(W_n^j/W_n^{j-1})_{eP}$ of the element $(e\,, v) \in G \times W_n^j/W_n^{j-1}$. (There is a natural projection from $E_P = G$ to $E_{L(P)}$ given by the quotient map in Eqn. (6), and also the associated vector bundle $E_{L(P)}(W_n^j/W_n^{j-1})$ is a quotient of $E_{L(P)} \times W_n^j/W_n^{j-1}$; combining these we have a projection from $G \times W_n^j/W_n^{j-1}$ to $E_{L(P)} \times W_n^j/W_n^{j-1}$.)

The fact that the action of G on $E_{L(P)}(W_n^j/W_n^{j-1})$ preserves the subbundle $\mathcal{E}_n^j$ implies that the subspace

$$(\mathcal{E}_n^j)_{eP} \subset E_{L(P)}(W_n^j/W_n^{j-1})_{eP} = W_n^j/W_n^{j-1}$$

is preserved by the action on W_n^j/W_n^{j-1} of the isotropy subgroup P of eP; the group P acts on W_n^j/W_n^{j-1} through the quotient map q in Eqn. (6). Since W_n^j/W_n^{j-1} is an irreducible $L(P)$–module, we conclude that

$$(\mathcal{E}_n^j)_{eP} = W_n^j/W_n^{j-1}\,;$$

note that if $\mathcal{E}_n^j)_{eP} = 0$, then the maximal semistable subsheaf of $E_{L(P)}(W_n^j/W_n^{j-1})$ is zero implying that $E_{L(P)}(W_n^j/W_n^{j-1}) = 0$.
If $(\mathcal{E}_n^j)_{eP} = W_n^j/W_n^{j-1}$, the subbundle $\mathcal{E}_n^j$ actually coincides with

$$E_{L(P)}(W_n^j/W_n^{j-1}).$$

Therefore, we conclude that the vector bundle $E_{L(P)}(W_n^j/W_n^{j-1})$ is semistable.

We will now show that the line bundle

$$\det E_{L(P)}(W_n^j/W_n^{j-1}) = \bigwedge^{\text{top}} E_{L(P)}(W_n^j/W_n^{j-1})$$

over G/P is trivializable.

To prove this first observe that the line bundle $\det E_{L(P)}(W_n^j/W_n^{j-1})$ is associated to the principal $L(P)$–bundle $E_{L(P)}$ for the one–dimensional $L(P)$–module $\bigwedge^{\text{top}}(W_n^j/W_n^{j-1})$. Let $Z(L(P))$ denote the reduced center of $L(P)$. Since $L(P)$ is a reductive group, the quotient $L(P)/Z(L(P))$ is a semisimple group. The restriction to $Z(L(P))$ of the adjoint action of $L(P)$ on its own Lie algebra $\mathfrak{l}(\mathfrak{p})$ clearly coincides with the trivial action. Hence the adjoint action of $L(P)$ on $\mathfrak{l}(\mathfrak{p})$ factors through the semisimple quotient $L(P)/Z(L(P))$. Consequently, the module action of $L(P)$ on $\bigwedge^{\text{top}}(W_n^j/W_n^{j-1})$ factors through $L(P)/Z(L(P))$. In other words, the action of $L(P)$ on $\bigwedge^{\text{top}}(W_n^j/W_n^{j-1})$ is given by a character of $L(P)/Z(L(P))$. The group $L(P)/Z(L(P))$ being semisimple does not admit any nontrivial characters. Hence we conclude that $\bigwedge^{\text{top}}(W_n^j/W_n^{j-1})$ is a trivial $L(P)$–module. This immediately implies that the associated line bundle $\det E_{L(P)}(W_n^j/W_n^{j-1})$ is trivial.

Since $\det E_{L(P)}(W_n^j/W_n^{j-1})$ is a trivial line bundle, we have

$$\text{degree}(\det E_{L(P)}(W_n^j/W_n^{j-1})) = 0$$

with respect to any polarization on G/P. Therefore, the filtration of subbundles of

$$(F^n)^*\text{ad}(E_{L(P)}) = E_{L(P)}(V_n)$$

in Eqn. (10) has the property that each successive quotient is semistable of degree zero. This immediately implies that the vector bundle $E_{L(P)}(V_n)$ is semistable. It was noted earlier that the proposition follows once we have shown that $E_{L(P)}(V_n)$ is semistable for all n. Hence the proof of the proposition is complete. ■

As before, let H be a connected reductive linear algebraic group defined over the field k. Let

$$\rho : P \longrightarrow H \tag{13}$$

be a homomorphism such that the image $\rho(P)$ is not contained in any proper parabolic subgroup of H. We note that such homomorphisms are called *irreducible.* The condition that the homomorphism ρ in Eqn. (13) is irreducible yields that

$$\rho(R_u(P)) = e\,, \tag{14}$$

where $R_u(P) \subset P$ is the unipotent radical, or in other words, ρ factors through the Levi quotient $L(P) := P/R_u(P)$. Indeed, if the image $\rho(R_u(P))$ is nontrivial, then the normalizer, in H, of the nontrivial unipotent subgroup $\rho(R_u(P))$ is contained in some proper parabolic subgroup $\widetilde{P} \subset H$ (see [5, page 185, § 30.3]). Therefore, $R_u(P)$ being a normal subgroup of P, we have $\rho(P) \subset \widetilde{P}$. This contradicts the condition that the homomorphism ρ is irreducible. Hence we conclude that ρ factors through the Levi quotient $P/R_u(P)$.

Let

$$E_H := G \times^P H \tag{15}$$

be the principal H–bundle over G/P obtained by extending the principal P–bundle E_P using the irreducible homomorphism ρ in Eqn. (13). Therefore, E_H is the quotient of $G \times H$ by the twisted diagonal action of P. The twisted diagonal action of any $z \in P$ sends any $(g\,,h) \in G \times H$ to $(gz^{-1}\,,\rho(z)h)$.

Lemma 2.4. *The principal H–bundle E_H over G/P defined in Eqn.* (15) *is strongly semistable with respect to any polarization on G/P.*

Proof. We noted earlier that the irreducible homomorphism ρ factors through the Levi quotient $P/R_u(P)$ (see Eqn. (14)). Let

$$\widetilde{\rho} : L(P) := P/R_u(P) \longrightarrow H \tag{16}$$

be the homomorphism that gives ρ. Since the principal H–bundle E_H is the extension of structure group of E_P using the homomorphism ρ, it follows immediately that E_H is identified with the principal H–bundle over G/P obtained by extending the structure group of the principal $L(P)$–bundle $E_{L(P)}$ using the homomorphism $\widetilde{\rho}$ in Eqn. (16).

Let

$$Z_0 \subset L(P) \tag{17}$$

be the reduced connected component of the center of $L(P)$ containing the identity element. Since $L(P)$ is reductive, the group Z_0 is a torus, i.e., a product of copies of $\mathbb{G}_m$. Let

$$Z_0(H) \subset H$$

be the reduced connected component of the center of H containing the identity element.

We will show that the homomorphism $\widetilde{\rho}$ in Eqn. (16) sends the subgroup Z_0 into $Z_0(H)$.

To prove this, assume that

$$\widetilde{\rho}(Z_0) \not\subset Z_0(H). \tag{18}$$

Since Z_0 is a torus, from Eqn. (18) it follows that the image $\widetilde{\rho}(Z_0)$ is a subtorus of H of positive dimension. The group H being reductive, the centralizer of any subtorus of H not contained in $Z_0(H)$ is a Levi subgroup of some proper parabolic subgroup of H (see [4, page 26, Proposition 1.22]). Therefore, the centralizer of $\widetilde{\rho}(Z_0)$ in H is contained in a Levi subgroup of some proper parabolic subgroup $\widehat{P}$ of H. Since Z_0 lies in the center of $L(P)$, it follows immediately that $\widetilde{\rho}(L(P))$ is contained in a proper parabolic subgroup $\widehat{P}$ of H. In particular, we have

$$\text{image}(\rho) = \text{image}(\widetilde{\rho}) \subset \widehat{P}.$$

But this contradicts the fact that the homomorphism ρ is irreducible. Therefore, we conclude that

$$\widetilde{\rho}(Z_0) \subset Z_0(H). \tag{19}$$

Fix any polarization on G/P. From Proposition 2.3 we know that the principal $L(P)$–bundle $E_{L(P)}$ is strongly semistable. Hence using Eqn. (19) it follows that the principal H–bundle obtained by extending the structure group of the principal $L(P)$–bundle $E_{L(P)}$ by the homomorphism $\widetilde{\rho}$ is also strongly semistable (see [8, page 285, Theorem 3.18] and [8, page 288, Theorem 3.23]). We noted earlier that this principal H–bundle obtained by extending the structure group of $E_{L(P)}$ is identified with E_H. This completes the proof of the lemma. ■

3. Stability of homogeneous principal bundles

We continue with the notation of the previous section.

Proposition 3.1. *The principal H–bundle E_H over G/P, defined in Eqn.* (15)*, is strongly polystable with respect to any polarization on G/P.*

Proof. Fix any polarization on G/P. From Lemma 2.4 we know that E_H is strongly semistable.

To prove the proposition it suffices to show that the principal H–bundle E_H over G/P is polystable. To see this, we recall that E_H is strongly polystable if $(F^n)^*E_H$ is polystable for all $n \geq 0$ (see the definition in Section 2.). If we know that E_H is polystable, to prove that $(F^n)^*E_H$ is polystable, replace ρ by the composition homomorphism

$$L(P) \xrightarrow{F^n_L} L(P) \xrightarrow{\widetilde{\rho}} H\,, \tag{20}$$

where $\widetilde{\rho}$ is the homomorphism in Eqn. (16). The composition homomorphism $L(P) \longrightarrow H$ in Eqn. (20) will be denoted by ρ_n. The condition that ρ is irreducible implies that ρ_n is also irreducible. On the other hand, the pullback $(F^n)^*E_H$ is identified with the principal H–bundle over G/P obtained by extending the structure group of the principal P–bundle E_P using the homomorphism $\rho_n \circ q$, where q is the projection in Eqn. (6). Therefore, to prove the proposition it is enough to show that E_H is polystable.

Assume that the principal H–bundle E_H is not polystable.

Since E_H is semistable but not polystable, it has a unique socle

$$E_Q \subset E_H\,. \tag{21}$$

We recall that the socle is defined as follows:

- $Q \subsetneq H$ is maximal among all the proper parabolic subgroups such that E_H admits an admissible reduction of structure group

  $$E'_Q \subset E_H$$

  for which the associated principal $L(Q')$–bundle $E_{L(Q')} = E'_Q(L(Q'))$ is polystable, where $L(Q')$ is the Levi quotient of Q' (the $L(Q')$–bundle $E_{L(Q')}$ is the extension of structure group of E'_Q using the quotient map $Q \longrightarrow L(Q')$), and
- E_Q in Eqn. (21) is a reduction of structure group of E_H to Q such that the associated principal $L(Q)$–bundle is polystable, where $L(Q)$ is the Levi quotient of Q.

The pair $(Q\,, E_Q)$ is unique in the following sense: for any other pair $(Q_1\,, E_{Q_1})$ satisfying the above conditions, there is some $g \in H$ such that

- $Q_1 = g^{-1}Qg$, and
- $E_{Q_1} = E_Q g$.

(See [6, page 23, Lemma 1.5.5], [1, page 218].) The definition of an admissible reduction is recalled in Eqn. (5).

Let $\mathrm{Ad}(E_H)$ be the group–scheme over G/P associated to the principal H–bundle E_H for the adjoint action of H on itself. Therefore, $\mathrm{Ad}(E_H)$ is the quotient of $E_H \times H$ by the action of H defined by $h \circ (z\,, h') = (zh^{-1}\,, hh'h^{-1})$, where $z \in E_H$, and $h\,, h' \in H$. The group–scheme $\mathrm{Ad}(E_H)$ is also called the *adjoint bundle* of E_H. Let $\mathrm{Ad}(E_Q)$ be the group–scheme over G/P associated to the principal Q–bundle E_Q in Eqn. (21) for the adjoint action of Q on itself. We note that using the inclusion of Q in H, the adjoint bundle $\mathrm{Ad}(E_Q)$ is a subgroup–scheme of $\mathrm{Ad}(E_H)$.

The above uniqueness condition of the pair $(Q\,,E_Q)$ implies that the subgroup–scheme

$$\mathrm{Ad}(E_Q) \subset \mathrm{Ad}(E_H) \tag{22}$$

is independent of the choice of the maximal pair $(Q\,,E_Q)$.

The left–translation action of G on $G\times H$ descends to a lift action of G on E_H. This descended action of G on E_H is a lift of the left–translation action of G on G/P which commutes with the principal bundle structure. As before, commuting with the principal bundle structure means that the left action of G on E_H commutes with the right action of H on E_H.

The action of G on E_H induces an action of G on the adjoint bundle $\mathrm{Ad}(E_H)$. From the uniqueness of the subgroup–scheme $\mathrm{Ad}(E_Q)$ in Eqn. (22) it follows immediately that the action of G on $\mathrm{Ad}(E_H)$ leaves the subgroup–scheme $\mathrm{Ad}(E_Q)$ invariant.

As in the proof of Proposition 2.3, let $e\in G$ be the identity element. The fiber $(E_H)_{eP}$ of E_H over eP is identified with H by sending any $h\in H$ to the image of $(e\,,h)\in G\times H$ in E_H (recall that E_H is a quotient of $G\times H$). The fiber $\mathrm{Ad}(E_H)_{eP}$ of the adjoint bundle $\mathrm{Ad}(E_H)$ over the point $eP\in G/P$ is identified with H by sending any

$$(z\,,h)\in (E_H)_{eP}\times H \;=\; H\times H$$

to $zhz^{-1}\in H$. Let Q' be the proper parabolic subgroup of H given by the image of the inclusion

$$\mathrm{Ad}(E_Q)_{eP}\subset \mathrm{Ad}(E_H)_{eP} \;=\; H$$

in Eqn. (22). From the earlier observation that the action of G on $\mathrm{Ad}(E_H)$ leaves the subgroup–scheme $\mathrm{Ad}(E_Q)$ invariant it follows immediately that the adjoint action of P on H through the homomorphism ρ leaves the subgroup $Q'\subset H$ invariant.

Since Q' is a parabolic subgroup of H, the normalizer of Q' in H coincides with Q' [5, page 143, Corollary B]. Consequently, we have

$$\rho(P)\subset Q'\,.$$

Since Q' is a proper parabolic subgroup of H, this contradicts the assumption on the homomorphism ρ that it is irreducible. Therefore, we conclude that E_H is polystable. This completes the proof of the proposition. ∎

We will need the following proposition to prove that E_H is strongly stable.

Proposition 3.2. *Let V be a finite dimensional left $L(P)$–module on which the action of the subgroup Z_0 in Eqn.* (17) *is trivial. Let E_V be the vector bundle over G/P associated to the tautological principal $L(P)$–bundle $E_{L(P)}$ for the $L(P)$–module V. Assume that the vector bundle E_V is globally generated (i.e., generated by its global sections).*

1. *The vector bundle E_V is trivializable.*

2. *If the $L(P)$–module V is irreducible, then V is a trivial $L(P)$–module of dimension one.*

Proof. Let r be the dimension of V. Fix a point $x_0 \in G/P$. Fix r sections

$$s_1, \cdots, s_r \in H^0(G/P, E_V) \tag{23}$$

such that the fiber $(E_V)_{x_0}$ of E_V at x_0 is spanned by $\{s_i(x_0)\}_{i=1}^r$. Therefore, there is a Zariski open dense subset

$$U_0 \subset G/P$$

containing x_0 such that the restriction $E_V|_{U_0}$ of E_V to U_0 is generated by $\{s_i|_{U_0}\}_{i=1}^r$.

Let $\mathcal{O}_{G/P}$ be the trivial line bundle over G/P. Let

$$\phi : \mathcal{O}_{G/P}^{\oplus r} \longrightarrow E_V \tag{24}$$

be the homomorphism defined by

$$(z; c_1, \cdots, c_r) \longmapsto \sum_{i=1}^{r} c_i \cdot s_i(z),$$

where $z \in G/P$, $c_i \in k$, and s_i are the sections in Eqn. (23).

Consider the one–dimensional $L(P)$–module $\det V := \bigwedge^r V$. Since the subgroup Z_0 acts trivially on V, the action of $L(P)$ on $\det V$ factors through the quotient group $L(P)/Z_0$. In other words, $\det V$ is given by a character of $L(P)/Z_0$. The group $L(P)/Z_0$. is semisimple because $L(P)$ is reductive. Hence $L(P)/Z_0$ does not admit any nontrivial characters. Therefore, $\det V$ is a trivial $L(P)$–module. Consequently, the line bundle

$$E_{L(P)}(\det V) = \det E_V = \bigwedge^r E_V$$

associated to the principal $L(P)$–bundle $E_{L(P)}$ for the $L(P)$–module $\det V$ is trivializable.

Let

$$\det \phi : \mathcal{O}_{G/P} = \bigwedge^r \mathcal{O}_{G/P}^{\oplus r} \longrightarrow \bigwedge^r E_V \cong \mathcal{O}_{G/P}$$

be the homomorphism of line bundles obtained from ϕ constructed in Eqn. (24). The above homomorphism

$$\det \phi : \mathcal{O}_{G/P} \longrightarrow \mathcal{O}_{G/P}$$

is nonzero because it is an isomorphism over the nonempty open subset U_0. This immediately implies that $\det \phi$ is an isomorphism. From this it follows that ϕ is an isomorphism over G/P. In particular, the vector bundle E_V is trivializable. This proves the first statement in the proposition.

Let

$$\widehat{V} := (G/P) \times H^0(G/P, E_V)$$

be the trivial vector bundle over G/P with fiber $H^0(G/P, E_V)$. Since E_V is trivializable, the evaluation of global sections

$$\sigma : \widehat{V} \longrightarrow E_V \tag{25}$$

is an isomorphism of vector bundles. The left–action of G on $E_{L(P)}$ induces an action of G on the associated vector bundle E_V (see the proof of Proposition 2.3 for the action of G on $E_{L(P)}$). This action of G on E_V induces an action of G on $H^0(G/P, E_V)$.

Consider the isomorphism of vector spaces

$$\sigma(eP) : H^0(G/P, E_V) = \widehat{V}_{eP} \longrightarrow (E_V)_{eP} = V , \tag{26}$$

where σ is the isomorphism in Eqn. (25) and $eP \in G/P$ is the point given by the identity element of G; the isomorphism of V with $(E_V)_{eP}$ is constructed as in Eqn. (12). This isomorphism $\sigma(eP)$ in Eqn. (26) clearly commutes with the actions of P (the action of P on $H^0(G/P, E_V)$ is the restriction of the action of G on $H^0(G/P, E_V)$ constructed above, and P acts on V through the homomorphism q in Eqn. (6)).

In particular, the action of P on V extends to an action of G on V. We recall that P is a parabolic subgroup of G that does not contain any simple factor of G. Therefore, the restriction to P of any irreducible representation of G of dimension at least two is reducible.

Consequently, if V is an irreducible $L(P)$–module, then V is of dimension one. Since Z_0 acts trivially on V, and $L(P)/Z_0$ does not admit any nontrivial characters, we conclude that V is a trivial $L(P)$–module of dimension one provided V is irreducible. This completes the proof of the proposition. ∎

Proposition 3.2 can be strengthened, as shown by the following lemma.

Lemma 3.3. *Let V be a finite dimensional left $L(P)$–module on which the action of the subgroup Z_0 in Eqn.* (17) *is trivial. If the associated vector bundle $E_V = E_{L(P)} \times^{L(P)} V$ (see Proposition 3.2) is globally generated, then the $L(P)$–module V is trivializable.*

Proof. Let

$$0 = V_0 \subset V_1 \subset \cdots \subset V_{m-1} \subset V_m = V \tag{27}$$

be a filtration of the $L(P)$–module V such that each successive quotient V_{i+1}/V_i, $i \in [0, m-1]$, is an irreducible $L(P)$–module. For each $i \in [0, m]$, let $E_{L(P)}(V_i)$ be the vector bundle over G/P associated to the principal $L(P)$–bundle $E_{L(P)}$ for the $L(P)$–module V_i. Similarly, for each $i \in [1, m]$, let $E_{L(P)}(V_i/V_{i-1})$ be the vector bundle over G/P associated to $E_{L(P)}$ for V_i/V_{i-1}.

Assume that the associated vector bundle E_V is globally generated.

Since $E_{L(P)}(V_m)$ is globally generated, its quotient $E_{L(P)}(V_m/V_{m-1})$ is also globally generated. From the second part of Proposition 3.2 we know that the $L(P)$–module V_m/V_{m-1} is trivializable. As a consequence, the vector bundle $E_{L(P)}(V_m/V_{m-1})$ is trivializable.

Since both the vector bundles $E_{L(P)}(V_m)$ and $E_{L(P)}(V_m/V_{m-1})$ are trivializable, it can be shown that the vector bundle $E_{L(P)}(V_{m-1})$ is also trivializable. Indeed, the vector bundle $E_{L(P)}(V_m)$ being trivializable, the quotient bundle $E_{L(P)}(V_m/V_{m-1})$ is given by a map from G/P to the Grassmannian $\text{Gr}(r, r')$, where $r := \text{rank}(E_{L(P)}(V_m))$ and $r' := \text{rank}(E_{L(P)}(V_m)/V_{m-1})$. Since $E_{L(P)}(V_m/V_{m-1})$ is trivializable, the pullback of the tautological quotient bundle on $\text{Gr}(r, r')$ to G/P is trivializable. On the other hand, the r'–th exterior power of the tautological quotient bundle on $\text{Gr}(r, r')$ is an ample line bundle. Therefore, the above map from G/P to $\text{Gr}(r, r')$ must be constant. This immediately implies that the vector bundle $E_{L(P)}(V_{m-1})$ is trivializable. In particular, $E_{L(P)}(V_{m-1})$ is globally generated.

Since $E_{L(P)}(V_{m-1})$ is globally generated, from the second part of Proposition 3.2 we know that that the $L(P)$–module V_{m-1}/V_{m-2} is trivializable. Now replacing V by V_{m-1} and using induction we conclude that the $L(P)$–module V_i/V_{i-1} is trivializable for all $i \in [1, m]$.

Consider the homomorphism $L(P) \longrightarrow \text{GL}(V)$ given by the action of $L(P)$ on V. Since the $L(P)$–module V_i/V_{i-1} is trivializable for all $i \in [1, m]$, the image of this homomorphism lies in the unipotent subgroup of $\text{GL}(V)$ associated to the filtration in Eqn. (27). But there are no nonconstant homomorphisms from a reductive group to a unipotent group. Thus V is a trivial $L(P)$–module. This completes the proof of the lemma. ∎

Theorem 3.4. *The principal H–bundle E_H over G/P, defined in Eqn.* (15), *is strongly stable with respect to any polarization on G/P.*

Proof. As in the proof of Proposition 3.4, replacing ρ by the composition homomorphism in Eqn. (20) we conclude that it is enough to show that E_H is stable.

The Lie algebra of H will be denoted by $\mathfrak{h}$. Let

$$\mathfrak{z}(\mathfrak{h}) \subset \mathfrak{h}$$

be the center. Let $\text{ad}(E_H)$ be the vector bundle over G/P associated to the principal H–bundle E_H for the adjoint action of H on $\mathfrak{h}$. Therefore, $\text{ad}(E_H)$ is the adjoint vector bundle for E_H. Since the adjoint action of H on $\mathfrak{h}$ fixes $\mathfrak{z}(\mathfrak{h})$ pointwise, the trivial vector bundle over G/P with fiber $\mathfrak{z}(\mathfrak{h})$ is a subbundle of $\text{ad}(E_H)$. Therefore, we have an inclusion

$$\mathfrak{z}(\mathfrak{h}) \hookrightarrow H^0(G/P, \text{ad}(E_H)). \tag{28}$$

From Proposition 3.1 we know that E_H is polystable. Therefore, to prove that E_H is stable it suffices to show that the homomorphism in Eqn. (28) is surjective. A proof of it can be found in the proof of Proposition 2.3 in [2, page 572].

Let

$$\mathcal{E} \subset \text{ad}(E_H) \tag{29}$$

be the coherent subsheaf generated by its global sections. The action of G on $\text{ad}(E_H)$ induced by the action of G on E_H clearly preserves the subsheaf $\mathcal{E}$ in Eqn. (29) (see the proof of Proposition 3.1 for the action of G on E_H). Since the left–translation action of G on G/P is transitive, and the action of G on $\text{ad}(E_H)$

is a lift of the left–translation action of G on G/P, the subsheaf in Eqn. (29) is actually a subbundle.

Let

$$V = \mathcal{E}_{eP} \subset \mathrm{ad}(E_H)_{eP} = \mathfrak{h} \tag{30}$$

be the submodule of P–module $\mathfrak{h}$ obtained by restricting the homomorphism in Eqn. (29) to the point $eP \in G/P$. We note that the isomorphism of $\mathfrak{h}$ with the fiber $\mathrm{ad}(E_H)_{eP}$ is constructed as in Eqn. (12), the group P acts on $\mathfrak{h}$ through ρ, and P acts on V through q in Eqn. (6). From Eqn. (19) it follows immediately that the subgroup Z_0 acts trivially on $\mathfrak{h}$. Therefore, the action of Z_0 on the $L(P)$–module V in Eqn. (30) is trivial.

The vector bundle $\mathcal{E}$ in Eqn. (29) is clearly globally generated. Hence from Lemma 3.3 we conclude that the $L(P)$–module V in Eqn. (30) is trivial.

We will show that for any element in the complement

$$w \in \mathfrak{h} \setminus \mathfrak{z}(\mathfrak{h}), \tag{31}$$

the reduced isotropy subgroup of H associated to w for the adjoint action of H on $\mathfrak{h}$ is contained in some proper parabolic subgroup. For that, let

$$w = w_s + w_n \tag{32}$$

be the Jordan decomposition of w, where w_s is semisimple and w_n is nilpotent; see [5, page 99, Theorem 15.3] for Jordan decomposition. From the uniqueness of the Jordan decomposition it follows immediately that the reduced isotropy subgroup associated to w for the adjoint action of H is the reduced intersection of the isotropy subgroups associated to w_s and w_n. The centralizer of w_s in H coincides with the centralizer of the torus in H generated by w_s. Therefore, using [4, page 26, Proposition 1.22] we conclude that if

$$w_s \in \mathfrak{h} \setminus \mathfrak{z}(\mathfrak{h}),$$

then the centralizer of w_s in H is contained in some proper parabolic subgroup of H.

If $w_s \in \mathfrak{z}(\mathfrak{h})$, then w_n in Eqn. (32) must be nonzero. If $w_n \neq 0$, from [5, page 185, § 30.3] we know that the centralizer of w_n is contained in some proper parabolic subgroup of H. Therefore, the reduced centralizer in H of the element w in Eqn. (31) is contained in some proper parabolic subgroup of H.

Since the homomorphism $\widetilde{\rho}$ in Eqn. (16) is irreducible, from the above observation we conclude that any trivial submodule of the $L(P)$–module $\mathfrak{h}$ is contained in the center $\mathfrak{z}(\mathfrak{h})$. In particular, the trivial $L(P)$–submodule V in Eqn. (30) is contained in $\mathfrak{z}(\mathfrak{h})$. This immediately implies that the homomorphism in Eqn. (28) is surjective. This completes the proof of the theorem. ∎

References

[1] Anchouche, B., and I. Biswas, *Einstein-Hermitian connections on polystable principal bundles over a compact Kähler manifold*, Amer. Jour. Math. **123** (2001), 207–228.

[2] Azad, H., and I. Biswas, *On the principal bundles over a flag manifold*, J. of Lie Theory **14** (2004), 569–581.

[3] Biswas, I., and S. Subramanian, *Flat holomorphic connections on principal bundles over a projective manifold*, Trans. Amer. Math. Soc. **356** (2004), 3995–4018.

[4] Digne, F., and J. Michel, "Representations of Finite Groups of Lie Type," London Mathematical Society Student Texts **21**, Cambridge University Press, Cambridge, 1991.

[5] Humphreys, J. E., "Linear Algebraic Groups," Graduate Texts in Mathematics **21**, Springer-Verlag, New York, Heidelberg, Berlin, 1987.

[6] Huybrechts, D., and M. Lehn, "The Geometry of Moduli Spaces of Sheaves," Aspects of Mathematics E**31**, Friedr. Vieweg & Sohn, Braunschweig, 1997.

[7] Ramanan, S., *Holomorphic vector bundles on homogeneous spaces*, Topology **5** (1966), 159–177.

[8] Ramanan, S., and A. Ramanathan, *Some remarks on the instability flag*, Tôhoku Math. J. **36** (1984), 269–291.

[9] Ramanathan, A., *Stable principal bundles on a compact Riemann surface*, Math. Ann. **213** (1975), 129–152.

[10] —, *Moduli for principal bundles over algebraic curves: I*, Proc. Ind. Acad. Sci. (Math. Sci.) **106** (1996), 301–328.

[11] Umemura, H., *On a theorem of Ramanan*, Nagoya Math. Jour. **69** (1978), 131–138.

Hassan Azad
Department of Mathematics
Lahore University of Management Sciences, Lahore
Pakistan
hassanaz@kfupm.edu.sa

Indranil Biswas
School of Mathematics
Tata Institute of Fundamental Research, Homi Bhabha Road
Bombay 400005
India
indranil@math.tifr.res.in

Received April 16, 2007
and in final form July 11, 2007