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Simplified Proofs for the Pro-Lie Group Theorem and the One-Parameter Subgroup Lifting Lemma

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Abstract. This note is devoted to the theory of projective limits of finite-dimensional Lie groups, as developed in the recent monograph [Hofmann, K. H., and S. A. Morris, "The Lie Theory of Connected Pro-Lie Groups," EMS Publ. House, 2007]. We replace the original, highly non-trivial proof of the One-Parameter Subgroup Lifting Lemma given in the monograph by a shorter and more elementary argument. Furthermore, we shorten (and correct) the proof of the so-called Pro-Lie Group Theorem, which asserts that pro-Lie groups and projective limits of Lie groups coincide.
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By a famous theorem of Yamabe [13], every identity neighbourhood of a connected (or almost connected) locally compact group G contains a closed normal subgroup N such that G/N is a Lie group, and thus is a so-called pro-Lie group. Therefore locally compact pro-Lie groups form a large class of locally compact groups, which has been studied by many authors (see, e.g., [10], [11], [12] as well as [8] and the references therein). Although a small number of papers broached on the topic of non-locally compact pro-Lie groups (like [5] and [4]), a profound structure theory of such groups was only begun recently in [6] and then fully worked out in the monograph [7]. The novel results accomplished in [7] make it clear that the study of general pro-Lie groups is fruitful also for the theory of locally compact groups.

We recall from [7]: For G a Hausdorff topological group, $\mathcal{N}(G)$ denotes the set of all closed normal subgroups N of G such that G/N is a (finite-dimensional) Lie group. If G is complete and $\mathcal{N}(G)$ is a filter basis which converges to 1, then G is called a *pro-Lie group*. It is easy to see that every pro-Lie group is, in particular, a projective limit of Lie groups. Various results which are known in the locally compact case become much more complicated to prove for non-locally compact pro-Lie groups. For example, it is not too hard to see that every locally compact group which is a projective limit of Lie groups is a pro-Lie group (see [3] for an elementary argument; the appeal to the solution of Hilbert's fifth problem in the

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earlier proof in [8] is unnecessary). Also, it has been known for a long time [9] that one-parameter subgroups can be lifted over quotient morphisms $q\colon G \to H$ between locally compact groups, i.e., for each continuous homomorphism $X\colon \mathbb{R} \to \mathbb{H}$ there exists a continuous homomorphism $Y\colon \mathbb{R} \to \mathbb{G}$ such that $X = q \circ Y$. The original proofs for analogues of the preceding two results for general pro-Lie groups as given in [6] and [7] (called the "Pro-Lie Group Theorem" and "One-Parameter Subgroup Lifting Lemma" there) were quite long and complicated. Later, A. A. George Michael gave a short alternative proof of the Pro-Lie Group Theorem, which however was not self-contained but depended on a non-elementary result from outside, the Gleason–Palais Theorem [2, Theorem 7.2]:

If G is a locally arcwise connected topological group in which the compact metrizable subsets are of bounded dimension, then G is a Lie group.

The goal of this note is to record two short and simple arguments, which together with some 10 pages of external reading[1] provide essentially self-contained proofs for both the Pro-Lie Group Theorem and the One-Parameter Subgroup Lifting Lemma (up to well-known facts). In this way, the proof of the latter shrinks from over 3 pages to 8 lines, and the proof of the former by 6 pages. Moreover, the author noticed that the proof of the Pro-Lie Group Theorem in [7] (and [6]) depends on an incorrect assertion,[2] making it all the more important to have a correct self-contained proof available.

It should be stressed that the new proofs do not replace the approach from [7]. To the contrary, results concerning the identity components of projective limits of Lie groups provided in [7, Lemmas 3.20–3.24] (based on a pivotal fact [7, Lemma 3.18] on weakly complete topological vector spaces) form the foundation of our proof of the Pro-Lie Group Theorem. And also our proof of the One-Parameter Subgroup Lifting Lemma is not based on novel techniques, but mainly on a new combination of arguments from [7] (where the bulk of the work is done).

Let us now re-state and prove the theorem and lemma in contention. Notation and terminology from [7] will be used without explanation.

Theorem 0.1. (The Pro-Lie Group Theorem) *Every projective limit of Lie groups is a pro-Lie group.*

Proof. Let G be a projective limit of a projective system $((G_j)_{j\in J}, (f_{jk})_{j\leq k})$ of Lie groups G_j and morphisms $f_{jk}\colon G_k \to G_j$. By [7, Proposition 3.27], G will be a pro-Lie group if we can show that $G/\ker(f_j)$ is a Lie group for each limit map $f_j\colon G \to G_j$. Let H_j be the analytic subgroup of G_j with Lie algebra $\mathcal{L}(f_j)(\mathcal{L}(G))$ (equipped with its Lie group topology). By [7, Lemmas 3.23 and 3.24], f_j restricts and corestricts to a quotient morphism $\phi_j\colon G_0 \to H_j$. Given $g \in G$, write $I^G_g\colon G \to G$, $I^G_g(h) := ghg^{-1}$. Since $\phi_j \circ I^G_g|_{G_0} = I^{G_j}_{f_j(g)} \circ \phi_j$, we see that $I^{G_j}_{f_j(g)}(H_j) \subseteq H_j$ and $I^{G_j}_{f_j(g)}|_{H_j}\colon H_j \to H_j$ is continuous. Hence $Q_j := f_j(G)$ can be

[1] Lemmas 3.17–3.24, Propositions 3.27 and 3.30, Lemma 3.31 and Lemmas 4.16–4.18 in [7].

[2] Parts (iii) and (iv) of the "Closed Subgroup Theorem" [7, Theorem 1.34] are false, as the example $G = \mathbb{R}$, $H = \mathbb{Z}$, $\mathcal{N} = \{\{0\}, \sqrt{2}\,\mathbb{Z}\}$ shows. This invalidates the proof of [7, Lemma 3.29 (iii)]], which is used in [7] to prove the Pro-Lie Group Theorem.

made a Lie group with H_j as an open subgroup. Then the corestriction $q_j\colon G \to Q_j$ of f_j to Q_j is a surjective homomorphism, which is open since so is $f_j|_{G_0}^{H_j} = \phi_j$. If we can show that q_j is continuous, then q_j will be a quotient morphism and thus $G/\ker(f_j) \cong Q_j$ a Lie group. However, by [7, Lemma 3.21], there exists some $k \in I$ such that $k \geq j$ and $f_{jk}((G_k)_0) \subseteq H_j$. Also, it is shown in the proof of [7, Lemma 3.24] that the map $\overline{f}_{jk}\colon (G_k)_0 \to H_j$, $x \mapsto f_{jk}(x)$ is continuous. Since $U := f_k^{-1}((G_k)_0)$ is a neighbourhood of 1 in G and $q_j|_U = \overline{f}_{jk} \circ f_k|_U^{(G_k)_0}$ is continuous, the homomorphism q_j is continuous. ■

Theorem 0.2. (The One-Parameter Subgroup Lifting Lemma) *Let G and H be pro-Lie groups and $f\colon G \to H$ be a quotient morphism of topological groups. Then every one-parameter subgroup X of H lifts to one of G, i.e., there exists a one-parameter subgroup $Y\colon \mathbb{R} \to \mathbb{G}$ such that $X = f \circ Y$.*

Proof. We adapt an argument from [7, p. 193]. By [7, Lemmas 4.16–4.18], we may assume that $H = \mathbb{R}$ and have to show that f is a retraction. If f was not a retraction, we would have $\mathcal{L}(f)(\mathcal{L}(G)) = \{0\}$ and thus $f(G_0) = \{1\}$, since $\exp_G(\mathcal{L}(G))$ generates a dense subgroup of G_0 (by Lemma 3.24 and the proof of Lemma 3.22 in [7]), and $f \circ \exp_G = \exp_H \circ \mathcal{L}(f) = 1$. Hence f factors to a quotient morphism $G/G_0 \to \mathbb{R}$. Since G/G_0 is proto-discrete by [7, Lemma 3.31], it would follow that also its quotient $\mathbb{R}$ is proto-discrete (see [7, Proposition 3.30 (b)]) and hence discrete (as $\mathbb{R}$ has no small subgroups). This is absurd. ■

We mention that the Pro-Lie Group Theorem has no analogue for projective limits of Banach-Lie groups. In fact, consider a Fréchet space E which is not a Banach space but admits a continuous norm $\|.\|$ (e.g., $E = C^\infty([0,1],\mathbb{R})$). Then E is a projective limit of Banach spaces. The $\|.\|$-unit ball U is a 0-neighbourhood in E which does not contain any non-trivial subgroup of E. If there existed a quotient morphism $q\colon E \to G$ to a Banach-Lie group G with kernel in U, then we would have $\ker(q) = \{0\}$. Hence q would be an isomorphism, entailing that the Banach-Lie group G is abelian and simply connected and therefore isomorphic to the additive group of a Banach space. Since E is not a Banach space, we have reached a contradiction.

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