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Analyticity of Riemannian Exponential Maps on Diff(T)

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Abstract. We study the exponential maps induced by Sobolev type right-invariant (weak) Riemannian metrics of order $k \geq 1$ on the Lie group of smooth, orientation preserving diffeomorphisms of the circle. We prove that each of them defines an *analytic* Fréchet chart of the identity.
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1. Introduction

The aim of this paper is to contribute towards a development of Riemannian geometry for infinite dimensional Lie groups which has attracted a lot of attention since Arnold's seminal paper [1] on hydrodynamics – see [2], [8], [11], [15], [16], [19]. As a case study we consider the Lie group $\mathcal{D} \equiv \mathrm{Diff}(\mathbb{T})$ of orientation preserving C^∞-diffeomorphisms of the circle $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. According to Milnor [18, §9], the group $\mathcal{D}$, endowed with the C^∞-Fréchet differential structure, is *not* analytic i.e., the mapping $\mathcal{D} \times \mathcal{D} \to \mathcal{D}$, $(\varphi, \psi) \mapsto \varphi \circ \psi^{-1}$ is not analytic. Nevertheless, it turns out that for a family of right-invariant weak Riemannian metrics on $\mathcal{D}$, the corresponding Riemannian exponential map defines an *analytic* chart of the identity of $\mathcal{D}$ – see Theorem 1.3 below.

The Lie group $\mathcal{D}$ and its Lie algebra come up in hydrodynamics, playing the role of a configuration space for Burgers equation and the Camassa-Holm equation [16], [19] (see also [15]). The latter equation is a model for (one dimensional) wave propagation in shallow water (cf. [12], and for a derivation based on physical grounds [5]) that has several interesting features which have been intensively studied in recent years.

For any given integer $k \geq 0$, consider the scalar product $\langle \cdot, \cdot \rangle_k : C^\infty(\mathbb{T}) \times$

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$C^\infty(\mathbb{T}) \to \mathbb{R}$,

$$\langle u, v\rangle_k := \sum_{j=0}^{k} \int_{\mathbb{T}} \partial_x^j u \, \partial_x^j v \, dx.$$

It induces a C_F^ω (i.e., Fréchet analytic) weak, right-invariant, Riemannian metric $\nu^{(k)}$ on $\mathcal{D}$:

$$\nu_\varphi^{(k)}(\xi, \eta) := \langle (d_{\mathrm{id}} R_\varphi)^{-1} \xi, (d_{\mathrm{id}} R_\varphi)^{-1} \eta \rangle_k, \quad \forall \varphi \in \mathcal{D}, \quad \text{and} \quad \forall \xi, \eta \in T_\varphi \mathcal{D}$$

where $R_\varphi : \mathcal{D} \to \mathcal{D}$ denotes the right translation $\psi \mapsto \psi \circ \varphi$. The subscript F in C_F^ω refers to the calculus in Fréchet spaces – see Appendix A where we collect some definitions and notions of the calculus in Fréchet spaces. The metric $\nu^{(k)}$ being weak means that the topology induced by $\nu^{(k)}$ on the tangent space $T_\varphi \mathcal{D}$ at an arbitrary point φ in $\mathcal{D}$, is weaker than the Fréchet topology on $C^\infty(\mathbb{T})$.

Definition 1.1. For any given $T > 0$, a C_F^2-smooth curve $\varphi : [0, T] \to \mathcal{D}$, is called a geodesic with respect to $\nu^{(k)}$, or $\nu^{(k)}$-geodesic for short, if it is a critical point of the action functional within the class of C_F^2-smooth variations γ of φ constrained to keep the end points fixed i.e., for any C_F^2-smooth function

$$\gamma : (-\varepsilon, \varepsilon) \times [0, T] \to \mathcal{D}, \, (s, t) \mapsto \gamma(s, t)$$

such that $\gamma(0, t) = \varphi(t)$ and, $\gamma(s, 0) = \varphi(0)$ and $\gamma(s, T) = \varphi(T)$ for any $-\varepsilon < s < \varepsilon$, one has

$$\frac{d}{ds}\Big|_{s=0} \mathcal{E}_k^T(\gamma(s, \cdot)) = 0. \tag{1a}$$

where $\mathcal{E}_k^T$ denotes the action functional

$$\mathcal{E}_k^T(\gamma(s, \cdot)) := \frac{1}{2} \int_0^T \nu_{\gamma(s,t)}^{(k)}(\dot\gamma(s, t), \dot\gamma(s, t)) dt \,, \tag{1b}$$

and $\dot\gamma(s, t) = \partial\gamma(s, t)/\partial t$.

The Euler-Lagrange equations (1a) on $\mathcal{D}$ for critical points of the action functional $\mathcal{E}_k^T$ defined in (1b) are given by

$$\begin{cases} \dot\varphi = v \\ \dot v = F_k(\varphi, v) \end{cases} \tag{2}$$

where

$$F_k(\varphi, v) := R_\varphi \circ A_k^{-1} \circ B_k(v \circ \varphi^{-1}) \tag{3a}$$

in which

$$A_k := \sum_{j=0}^{k} (-1)^j \partial_x^{2j} \tag{3b}$$

and, for any smooth function u in $C^\infty(\mathbb{T})$,

$$\begin{aligned} B_k(u) &:= -2u' A_k u + A_k(uu') - u A_k u' \\ &= -2u' A_k u + Q_k(u) \end{aligned} \tag{3c}$$

where Q_k is a polynomial in $2k+1$ variables, and $Q_k(u)$ is short for the function $Q_k(u, \partial_x u, \ldots, \partial_x^{2k} u)$. In the sequel, we will refer to $Q_k(u)$ as the polynomial in $u, \partial_x u, \cdots, \partial_x^{2k} u$. Here $(\cdot)^\cdot$ stands for d/dt and $(\cdot)'$ for ∂/∂_x. Note that $t \mapsto \varphi(t)$ evolves in $\mathcal{D}$ whereas $t \mapsto \dot{\varphi}(t) = v(t)$ is a vector field along φ i.e., a section of $\varphi(\cdot)^* T\mathcal{D}$. In particular, $v(t)$ belongs to $T_{\varphi(t)}\mathcal{D}$ for any $0 \le t \le T$ for some $T > 0$. We want to study the following initial value problem

$$\begin{cases} (\dot{\varphi}, \dot{v}) = (v, F_k(\varphi, v)) \\ (\varphi(0), v(0)) = (\mathrm{id}, v_0)\,. \end{cases} \tag{4}$$

It is easy to check that (4) is equivalent to

$$\begin{cases} \dot{\varphi} = u \circ \varphi \\ \varphi(0) = \mathrm{id} \end{cases} \tag{5}$$

and

$$\begin{cases} A_k \dot{u} + u A_k u' + 2u' A_k u = 0 \\ u(0) = v_0 \end{cases} \tag{6}$$

where $t \mapsto u(t) = (d_{\mathrm{id}} R_{\varphi(t)})^{-1} \dot{\varphi}(t)$ belongs to $T_{\mathrm{id}}\mathcal{D}$. The initial value problems (4) and (6) are, via (5), two alternative descriptions of the geodesic flow. The first corresponds to the Lagrangian description i.e., tracking the moving point in $\mathcal{D}$ along the section $\varphi(\cdot)^* T\mathcal{D}$ while the latter describes the events in $T_{\mathrm{id}}\mathcal{D}$ from the Eulerian point of view of a fixed observer.

The case $k = 1$ is particularly interesting since the geodesic flow (4) with $v = u \circ \varphi$ is a re-expression of the Camassa-Holm equation (6) – see e.g. [15],[16], and [20]. Indeed, the dynamical system (4) can be used to study the initial value problem for (6) – see e.g. [9], [20].

Theorem 1.2. *Let the integer $k \ge 1$. Then, for any of the right-invariant metrics $\nu^{(k)}$, there exists an open neighborhood $V^{(k)}$ of 0 in $C^\infty(\mathbb{T})$ such that, for any v_0 in $V^{(k)}$, there is a unique $\nu^{(k)}$-geodesic, $(-2,2) \to \mathcal{D}, t \mapsto \varphi(t; v_0)$, issuing from the identity in the direction v_0. Moreover, the map*

$$(-2,2) \times V^{(k)} \to \mathcal{D}, \ (t, v_0) \mapsto \varphi(t; v_0)$$

is Fréchet analytic.

Theorem 1.2 allows to define, for any given $k \ge 1$, the Riemannian exponential map

$$\mathrm{Exp}_k \big|_{V^{(k)}} : V^{(k)} \to \mathcal{D}, \quad v_0 \mapsto \varphi(1; v_0).$$

Theorem 1.3. *For any integer $k \ge 1$, there exists an open neighborhood $\tilde{V}^{(k)} \subseteq V^{(k)}$ of 0 in $C^\infty(\mathbb{T})$ and an open neighborhood $U^{(k)}$ of id in $\mathcal{D}$ so that*

$$\mathrm{Exp}_k \big|_{\tilde{V}^{(k)}} : \tilde{V}^{(k)} \to U^{(k)}, \quad v_0 \mapsto \varphi(1; v_0)$$

is a Fréchet bianalytic diffeomorphism.

Related work: Theorems 1.2 - 1.3 improve the results of Constantin and Kolev in [7, 8] where it was shown that the exponential map is a C^1_F-diffeomorphism.

Method: The approach used to establish Theorems 1.2 - 1.3 is new. It consists in showing that the analyticity of the geodesic flow as described in Theorem 4.3, Theorem 5.2, Proposition 6.1, and Proposition 6.2, can be obtained by an interplay of the structure of the equations describing the geodesic flow from a Lagrangian perspective and the structure of the Euler equations. In [14], we applied our method to define and study exponential maps on the Lie group of orientation preserving diffeomorphisms on the two dimensional torus $\mathbb{R}^2/\mathbb{Z}^2$. Moreover, this new approach improves on the results in [6] for the Virasoro group.

Organization of the paper: Sections 2., 3., and 4. are preliminary. In particular, we show in detail that the vector fields in the equations defining the geodesic flows are analytic in suitable spaces. Theorem 1.2 is proved in section 5.. To show Theorem 1.3, we use Theorem A9 which is a version of the inverse function theorem in a set-up for analytic maps between Fréchet spaces discussed in Appendix A. In section 6., we verify that the assumptions of Theorem A9 are satisfied in our situation.

Case $k = 0$: Our new technique does not apply in the case $k = 0$; the equation (2) with $k = 0$, is no longer a dynamical system (hence, we can not rely on the existence theorem of ODE's to establish existence of geodesics in Sobolev spaces). Indeed, the crucial difference with respect to the case $k \geq 1$, lies in the fact that the inverse of the operator A_k defined in (3b) is the identity operator in the case $k = 0$, hence it is *not* regularizing. More specifically, the expression on the r.h.s. of the system (2) is the map $\mathcal{D}^\ell \times H^\ell \to H^\ell \times H^{\ell-1}$, $(\varphi, v) \mapsto (v, -2vv'/\varphi')$ which is *not* a vector field on $\mathcal{D}^\ell \times H^\ell$. Therefore the proof of Theorem 1.2 in the case $k = 0$ has to be dealt with differently. One way is to notice that the geodesic flow (2) with $k = 0$ is equivalent to the inviscid Burgers equation $\dot{u} + 3uu' = 0$ (cf. (6)) which can be solved implicitly (locally in time) by the method of characteristics. However, in the case $k = 0$, Theorem 1.3 does *not* hold. An explicit counter example is given in [8]. For this reason, in the rest of the paper, we will only concentrate on the case $k \geq 1$.

Notation: The notation we use is standard. In particular, $H^s = H^s(\mathbb{T}) = H^s(\mathbb{T}, \mathbb{R})$ denotes the space of real valued functions on $\mathbb{T}$ of Sobolev class H^s, and for $s \geq 1, W^{s,\infty}$ denotes the Banach space of continuous functions $f : \mathbb{T} \to \mathbb{R}$ for which $\partial_x^j f$ are in $L^\infty = L^\infty(\mathbb{T})$ for every $0 \leq j \leq s$. Further, for $s \geq 2$, $\mathcal{D}^s = \mathcal{D}^s(\mathbb{T})$ denotes the set of orientation preserving C^1-diffeomorphisms $\varphi : \mathbb{T} \to \mathbb{T}$ of class H^s. It is a Hilbert manifold modeled on H^s. The complexification of a real vector space X will be denoted by $X_\mathbb{C}$ i.e., $X_\mathbb{C} := X \otimes \mathbb{C}$.

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2. The group of diffeomorphisms of $\mathbb{T}$

For the convenience of the reader, we collect in this section the results about the group of diffeomorphisms of $\mathbb{T}$ that we need. For a general discussion on this topic see [11] and the references therein. Throughout this section, $s \geq 2$. Denote by $\mathcal{D}^s$

the set of all orientation preserving C^1-diffeomorphisms $\varphi : \mathbb{T} \to \mathbb{T}$ such that φ' belongs to H^{s-1} i.e.,

$$\mathcal{D}^s := \{\varphi : \mathbb{T} \to \mathbb{T},\ C^1 - \text{diffeomorphism} |\quad \varphi' > 0, \quad \text{and} \quad \varphi' \in H^{s-1}\}.$$

$\mathcal{D}^s$ is in a natural way a Hilbert manifold modeled by the Hilbert space H^s. An atlas of $\mathcal{D}^s$ can be described in terms of the lifts of φ in $\mathcal{D}^s$. A lift of φ is of the form

$$\mathbb{R} \to \mathbb{R}, \quad x \mapsto x + f(x)$$

where f is in H^s. The following two Hilbert charts form an atlas of $\mathcal{D}^s$

$$\mathfrak{U}_1^s := \{\varphi = \mathrm{id} + f |\ \ f \in H^s \quad \text{and} \quad |f(0)| < 1/2;\ f' > -1\} \tag{7a}$$

and

$$\mathfrak{U}_2^s := \{\varphi = \mathrm{id} + f |\ \ f \in H^s \quad \text{and} \quad 0 < f(0) < 1;\ f' > -1\}. \tag{7b}$$

(By a slight abuse of notation we denoted a lift of a diffeomorphism φ again by φ.)

Lemma 2.1. *Let $s \geq 2$. Then, for any φ in $\mathcal{D}^s$, the inverse φ^{-1} of φ is again in $\mathcal{D}^s$.*

Proof. Clearly, for any φ in $\mathcal{D}^s$, φ^{-1} is a C^1-diffeomorphism. Using that $(\varphi^{-1})' = (1/\varphi') \circ \varphi^{-1}$ and the fact that H^r is an algebra for any $r \geq 1$, one sees that $(\varphi^{-1})'$ is in H^{s-1}, and hence that φ^{-1} belongs to $\mathcal{D}^s$. ■

Lemma 2.2. *Let $s \geq 2$. Then the following statements hold:*

(i) For any u and φ in H^s and $\mathcal{D}^s$ respectively, $u \circ \varphi$ is in H^s.

(ii) For any φ in $\mathcal{D}^s$, the right-translation $R_\varphi : H^s \to H^s$, $u \mapsto u \circ \varphi$ is uniformly continuous on subsets $\mathcal{W} \subseteq \mathcal{D}^s$ satisfying

$$\sup_{\varphi \in \mathcal{W}} (\|\varphi^{-1}\|_{W^{1,\infty}} + \|\varphi\|_{H^s}) < +\infty. \tag{8}$$

(iii) For any u in H^s, the left-translation $L_u : \mathcal{D}^s \to H^s$, $\varphi \mapsto u \circ \varphi$ is continuous.

Proof. (*i*) First let us prove that for u in L^2 and φ in $\mathcal{D}^2$, the composition $u \circ \varphi$ is in L^2. To see this, note that

$$\|u \circ \varphi\|_{L^2}^2 = \int_{\mathbb{T}} |u(\varphi(x))|^2 dx = \int_{\mathbb{T}} |u(x)|^2 (\varphi^{-1})'(x) dx \leq (\inf_{\mathbb{T}} \varphi')^{-1} \|u\|_{L^2}^2. \tag{9}$$

Moreover, as $s \geq 2$, by the Sobolev embedding theorem, φ' is bounded for any φ in $\mathcal{D}^s$. Hence, the argument above shows that, for any $s \geq 2$, given any u in H^s and any φ in $\mathcal{D}^s$, $u \circ \varphi$ and $(u \circ \varphi)' = (u' \circ \varphi)\varphi'$ are in L^2. For any $2 \leq n \leq s$, we get by the chain and product rules

$$\partial_x^n (u \circ \varphi) = (u' \circ \varphi) \partial_x^n \varphi + \sum_{j=2}^{n} p_{n,j}(\varphi) [(\partial_x^j u) \circ \varphi] \tag{10}$$

where $p_{n,j}(\varphi)$ is a polynomial in $\partial_x\varphi, \dots, \partial_x^{n+1-j}\varphi$. Now, for $s \geq 2$, the fact that u' is in $H^{s-1} \subseteq H^1 \subseteq C^0$ implies that $u' \circ \varphi$ is bounded so that $(u' \circ \varphi)\partial_x^n \varphi$ is in $H^{s-n} \subseteq L^2$. Similarly, for every $2 \leq j \leq n$, $(\partial_x^j u) \circ \varphi$ are seen to be in L^2, and $p_{n,j}(\varphi)$ is bounded. But then, as $\partial_x^n(u \circ \varphi)$ is in L^2 for every $2 \leq n \leq s$, and since $u \circ \varphi$ and $(u \circ \varphi)'$ were shown to be in L^2 too, we conclude that, for any $s \geq 2$, u in H^s and φ in $\mathcal{D}^s$ imply that $u \circ \varphi$ is in H^s as asserted.

(ii) Let $(u_m)_{m\geq 1}$ be a sequence in H^s which converges to u in H^s. By (9), one has

$$\|(u_m - u) \circ \varphi\|_{L^2} \leq \|\varphi^{-1}\|_{W^{1,\infty}}^{1/2} \|u_m - u\|_{L^2},$$

and

$$\|((u_m - u) \circ \varphi)'\|_{L^2} \leq \|\varphi\|_{W^{1,\infty}}^{1/2} \|u_m - u\|_{H^1}.$$

To estimate the derivatives of order $2 \leq n \leq s$, we use formula (10). By the Sobolev embedding theorem,

$$\|[(u_m - u)' \circ \varphi]\partial_x^n \varphi\|_{L^2} \leq \|u_m - u\|_{W^{1,\infty}} \|\varphi\|_{H^n} \leq C\|u_m - u\|_{H^s} \|\varphi\|_{H^s},$$

where $C > 0$ denotes a constant, and for every $2 \leq j \leq n$,

$$\|p_{n,j}(\varphi)[\partial_x^j (u_m - u) \circ \varphi]\|_{L^2} \leq [q_{n,j}(\|\varphi\|_{H^s})]^{1/2} \|\varphi^{-1}\|_{W^{1,\infty}}^{1/2} \|u_m - u\|_{H^s}$$

where $q_{n,j}$ is a polynomial in one variable with positive coefficients. It then follows from (10) and the above estimates that, for any $0 \leq j \leq s$,

$$\|\partial_x^j [R_\varphi (u_m - u)]\|_{L^2} = O(\|u_m - u\|_{H^s})$$

for φ in sets $\mathcal{W} \subseteq \mathcal{D}^s$ satisfying (8). This establishes uniform continuity of the right translation within the set $\mathcal{W}$.

(iii) Let $(\varphi_m)_{m\geq 1} \subseteq \mathcal{D}^s$ be convergent to φ in $\mathcal{D}^s$. First we will assume that u belongs to H^{s+2}. As $s \geq 2$, u is Lipschitz continuous i.e., for any x in $\mathbb{T}$,

$$|u \circ \varphi_m(x) - u \circ \varphi(x)| \leq \|u\|_{W^{1,\infty}} |\varphi_m(x) - \varphi(x)|.$$

This implies

$$\|u \circ \varphi_m - u \circ \varphi\|_{L^2} \leq \|u\|_{W^{1,\infty}} \|\varphi_m - \varphi\|_{L^2}.$$

Similarly, as $(u \circ \varphi_m)' = (u' \circ \varphi_m)\varphi_m'$, one can estimate $\|(u \circ \varphi_m)' - (u \circ \varphi)'\|_{L^2}$ by

$$\begin{aligned}
&\|(u' \circ \varphi_m)(\varphi_m' - \varphi')\|_{L^2} + \|(u' \circ \varphi_m - u' \circ \varphi)\varphi'\|_{L^2} \\
\leq &\|u\|_{W^{1,\infty}} \|(\varphi_m - \varphi)'\|_{L^2} + \|\varphi\|_{W^{1,\infty}} \|u\|_{W^{2,\infty}} \|\varphi_m - \varphi\|_{L^2} \\
\leq &C \max(1, \|\varphi\|_{W^{1,\infty}}) \|u\|_{H^3} \|\varphi_m - \varphi\|_{L^2},
\end{aligned}$$

where in the last step we used the estimate $\|u\|_{W^{1,\infty}} \leq \|u\|_{W^{2,\infty}} \leq C\|u\|_{H^3}$, which follows by the Sobolev embedding theorem.

Next, as in the proof of *(ii)*, we use (10) to estimate the higher order derivatives ($2 \leq n \leq s$). First,

$$\begin{aligned}
\|\partial_x^n (u \circ \varphi_m - u \circ \varphi)\|_{L^2} \leq &\|(u' \circ \varphi_m)\partial_x^n \varphi_m - (u' \circ \varphi)\partial_x^n \varphi\|_{L^2} \\
&+ \textstyle\sum_{j=2}^n \|p_{n,j}(\varphi_m)[(\partial_x^j u) \circ \varphi_m] - p_{n,j}(\varphi)[(\partial_x^j u) \circ \varphi]\|_{L^2}.
\end{aligned}$$

Based on previous estimates, the norm $\|(u'\circ\varphi_m)\partial_x^n\varphi_m-(u'\circ\varphi)\partial_x^n\varphi\|_{L^2}$ is bounded by

$$\|u'\circ\varphi_m-u'\circ\varphi\|_{L^\infty}\|\partial_x^n\varphi_m\|_{L^2}+\|u\|_{W^{1,\infty}}\|\partial_x^n\varphi_m-\partial_x^n\varphi\|_{L^2},$$

while $\|p_{n,j}(\varphi_m)[(\partial_x^j u)\circ\varphi_m]-p_{n,j}(\varphi)[(\partial_x^j u)\circ\varphi]\|_{L^2}$ is bounded by

$$\|(p_{n,j}(\varphi_m)-p_{n,j}(\varphi))[(\partial_x^j u)\circ\varphi_m]\|_{L^2}+\|p_{n,j}(\varphi)[(\partial_x^j u)\circ\varphi_m-(\partial_x^j u)\circ\varphi]\|_{L^2}.$$

One then shows that $\|\partial_x^n(u\circ\varphi_m-u\circ\varphi)\|_{L^2}\to 0$ as $m\to+\infty$ using the estimates

$$\|u'\circ\varphi_m-u'\circ\varphi\|_{L^\infty}\|\partial_x^n\varphi_m\|_{L^2}\le\|u\|_{W^{2,\infty}}\sup_{m\ge1}\{\|\varphi_m\|_{H^n}\}\|\varphi_m-\varphi\|_{H^1},$$
$$\|u'\circ\varphi\|_{L^\infty}\|\partial_x^n\varphi_m-\partial_x^n\varphi\|_{L^2}\le\|u\|_{W^{1,\infty}}\|\varphi_m-\varphi\|_{H^n},$$

and

$$\begin{aligned}&\|(p_{n,j}(\varphi_m)-p_{n,j}(\varphi))[(\partial_x^j u)\circ\varphi_m]\|_{L^2}\le\\&\le\|\partial_x^j u\|_{L^2}\|\varphi_m^{-1}\|_{W^{1,\infty}}^{1/2}\|p_{n,j}(\varphi_m)-p_{n,j}(\varphi)\|_{L^\infty}\\&\le\|u\|_{H^s}\sup_{m\ge1}\{\|\varphi_m^{-1}\|_{W^{1,\infty}}^{1/2}\}r_{n,j}(\|\varphi_m-\varphi\|_{W^{s-1,\infty}})\end{aligned}$$

where $r_{n,j}$ is a polynomial in one variable with positive coefficients, and

$$\begin{aligned}\|p_{n,j}(\varphi)[(\partial_x^j u)\circ\varphi_m-(\partial_x^j u)\circ\varphi]\|_{L^2}&\le\|p_{n,j}(\varphi)\|_{L^\infty}\|\partial_x^j u\|_{W^{1,\infty}}\|\varphi_m-\varphi\|_{L^2}\\&\le q_{n,j}(\|\varphi\|_{W^{n-1,\infty}})\|u\|_{W^{j+1,\infty}}\|\varphi_m-\varphi\|_{L^2}\\&\le Cq_{n,j}(\|\varphi\|_{W^{s-1,\infty}})\|u\|_{H^{s+2}}\|\varphi_m-\varphi\|_{L^2}\end{aligned}$$

where the last inequality results from the fact that, for every $2\le j\le s$, $\|\partial_x^j u\|_{W^{1,\infty}}\le\|u\|_{W^{j+1,\infty}}\le\|u\|_{W^{s+1,\infty}}\le C\|u\|_{H^{s+2}}$ by the Sobolev embedding theorem, and where $q_{n,j}$ is a polynomial in one variable with positive coefficients. Altogether, since u was assumed to be in H^{s+2}, we have shown that $(L_u-L_{u_m})\circ\varphi\to 0$ as $m\to+\infty$ in H^s.

The general case, where u is merely in H^s, is now obtained by a limiting argument. Indeed, as H^{s+2} is densely embedded in H^s, we may choose a sequence $(u_i)_{i\ge1}\subseteq H^{s+2}$ so that $u_i\to u$ in H^s as $i\to+\infty$. Then

$$\|u\circ\varphi_m-u\circ\varphi\|_{H^s}\le\|u\circ\varphi_m-u_i\circ\varphi_m\|_{H^s}+\|u_i\circ\varphi_m-u_i\circ\varphi\|_{H^s}+\|u_i\circ\varphi-u\circ\varphi\|_{H^s}.$$

Note that $\sup_{m\ge1}(\|\varphi_m^{-1}\|_{W^{1,\infty}}+\|\varphi_m^{-1}\|_{H^s})<+\infty$ as $\varphi_m\to\varphi$ in $\mathcal{D}^s$ for $s\ge 2$. Hence, we have by *(ii)* that, for any given $\varepsilon>0$, there exists $i_0=i_0(\varepsilon)\ge 1$ so that, for any $i\ge i_0$, and any $m\ge 1$, the first and third factors on the r.h.s. of the last display are no larger than $\varepsilon/3$. Finally, by the argument above, there exists $m_0(\varepsilon)\ge 1$ such that for every $m\ge m_0$ the middle term is smaller than $\varepsilon/3$, and we are done. ∎

Proposition 2.3. *For any $s\ge 2$ and any r in $\mathbb{Z}_{\ge0}$, the composition $H^{s+r}\times\mathcal{D}^s\to H^s$, $(u,\varphi)\mapsto u\circ\varphi$ is C^r-smooth.*

Proof. Consider first the case $r=0$. It is to prove that, for any sequences $(u_m)_{m\ge1}\subseteq H^s$ and $(\varphi_m)_{m\ge1}\subseteq\mathcal{D}^s$ such that $u_m\to u$ in H^s and $\varphi_m\to\varphi$ in $\mathcal{D}^s$ as $m\to+\infty$, the sequence $(u_m\circ\varphi_m)_{m\ge1}$ converges in H^s to $u\circ\varphi$. Indeed,

$$\|u_m\circ\varphi_m-u\circ\varphi\|_{H^s}\le\|(u_m-u)\circ\varphi_m\|_{H^s}+\|u\circ\varphi_m-u\circ\varphi\|_{H^s}.$$

As φ_m converges to φ in $\mathcal{D}^s$ and $s\ge 2$, one has that $\sup_m\|\varphi_m^{-1}\|_{W^{1,\infty}}<+\infty$. Hence one can apply Lemma 2.2 *(ii)* to conclude that $\|(u_m-u)\circ\varphi_m\|_{H^s}\to 0$. By Lemma 2.2 *(iii)*, $\|u\circ\varphi_m-u\circ\varphi\|_{H^s}\to 0$ as $m\to+\infty$. The proof of the case $r\ge 1$ is similar and is left to the reader. ∎

Proposition 2.4. *For every $s \geq 2$ and any r in $\mathbb{Z}_{\geq 0}$, the map $\mathcal{D}^{s+r} \to \mathcal{D}^s$, $\varphi \mapsto \varphi^{-1}$ is C^r-smooth.*

Proof. As above, we will give the proof of the case $r = 0$, and leave the case $r \geq 1$ to the reader. Let $(\varphi_m)_{m \geq 1}$ be a sequence in $\mathcal{D}^s$ with $\varphi_m \to \varphi$ in $\mathcal{D}^s$. By Lemma 2.1, φ_m^{-1} and φ^{-1} are in $\mathcal{D}^s$. It is to prove that $\varphi_m^{-1} \to \varphi^{-1}$ in $\mathcal{D}^s$. To this end, write

$$\varphi_m^{-1} - \varphi^{-1} = (\mathrm{id} - \varphi^{-1} \circ \varphi_m) \circ \varphi_m^{-1}.$$

By Lemma 2.2 (*iii*), $\varphi^{-1} \circ \varphi_m \to \varphi^{-1} \circ \varphi = \mathrm{id}$ in $\mathcal{D}^s$ as $m \to +\infty$. As $\sup_m(\|\varphi_m^{-1}\|_{H^s} + \|\varphi_m\|_{W^{1,\infty}}) < +\infty$, it then follows from Lemma 2.2 (*ii*) that $(\varphi^{-1} \circ \varphi_m - \mathrm{id}) \circ \varphi_m^{-1} \to 0$ as $m \to +\infty$. ∎

Remark 2.5. From Proposition 2.3 and Proposition 2.4 it follows that the composition and the inverse maps, $\mathcal{D} \times \mathcal{D} \to \mathcal{D}$ respectively $\mathcal{D} \to \mathcal{D}$, are C_F^∞-smooth, hence $\mathcal{D}$ is a Lie group. Its Lie algebra $T_{\mathrm{id}}\mathcal{D}$ can be canonically identified with $C^\infty(\mathbb{T}, \mathbb{R})$, with bracket given by $[u, v] = uv' - u'v$.

3. The vector field $\mathcal{F}_k$

In the present section, let the integer $k \geq 1$, and $\ell \geq \ell_k := 2k + 2$. For any (φ, v) in $\mathcal{D}^\ell \times H^\ell$, consider

$$\mathcal{F}_k(\varphi, v) := (v, F_k(\varphi, v)), \tag{11}$$

where F_k is defined in (3). It follows from Lemma 2.1 and Lemma 2.2 that, for any (φ, v) in $\mathcal{D}^\ell \times H^\ell$, the r.h.s. of (11) is well-defined and belongs to the space $H^\ell \times H^\ell$. In particular, (11) defines a *dynamical system* (ODE) on $\mathcal{D}^\ell \times H^\ell$. Introducing

$$\mathcal{A}_k : \mathcal{D}^\ell \times H^\ell \to \mathcal{D}^\ell \times H^{\ell-2k}, \ \ (\varphi, v) \mapsto (\varphi, R_\varphi \circ A_k \circ R_{\varphi^{-1}} v)$$

and

$$\mathcal{B}_k : \mathcal{D}^\ell \times H^\ell \to \mathcal{D}^\ell \times H^{\ell-2k}, \ \ (\varphi, v) \mapsto (\varphi, R_\varphi \circ B_k \circ R_{\varphi^{-1}} v)$$

with A_k and B_k as in (3b), respectively (3c), we can write

$$F_k = \mathcal{P}roj_2 \circ \mathcal{A}_k^{-1} \circ \mathcal{B}_k$$

where $\mathcal{P}roj_2$ is the projection onto the second component $(\varphi, v) \mapsto v$, and $\mathcal{A}_k^{-1}$ is the inverse of $\mathcal{A}_k$ described in the following proposition.

Proposition 3.1. *Let $k \geq 1$, and $\ell \geq \ell_k = 2k + 2$. Then*

(i) the map

$$\mathcal{A}_k : \mathcal{D}^\ell \times H^\ell \to \mathcal{D}^\ell \times H^{\ell-2k}, \ \ (\varphi, v) \mapsto (\varphi, R_\varphi \circ A_k \circ R_{\varphi^{-1}} v) \tag{12a}$$

is a bianalytic diffeomorphism with inverse given by the map

$$\mathcal{A}_k^{-1} : \mathcal{D}^\ell \times H^{\ell-2k} \to \mathcal{D}^\ell \times H^\ell, \ \ (\varphi, v) \mapsto (\varphi, R_\varphi \circ A_k^{-1} \circ R_{\varphi^{-1}} v). \tag{12b}$$

(ii) The map

$$\mathcal{B}_k : \mathcal{D}^\ell \times H^\ell \to \mathcal{D}^\ell \times H^{\ell-2k}, \ \ (\varphi, v) \mapsto (\varphi, R_\varphi \circ B_k \circ R_{\varphi^{-1}} v) \tag{13}$$

is analytic.

As a consequence,

(iii) the vector field

$$\mathcal{F}_k : \mathcal{D}^\ell \times H^\ell \to H^\ell \times H^\ell, \ \ (\varphi, v) \mapsto \mathcal{A}_k^{-1} \circ \mathcal{B}_k(\varphi, v) \tag{14}$$

is analytic on the Hilbert manifold $\mathcal{D}^\ell \times H^\ell$.

To prove Proposition 3.1 we need two auxiliary lemmas.

Lemma 3.2. *Let $k \geq 1$, and $\ell \geq \ell_k = 2k+2$. Then, for any φ in $\mathcal{D}^\ell$ and v in H^ℓ, and, for any $1 \leq n \leq 2k$, the following statements hold:*

(i) $\partial_x(\varphi^{-1}) \circ \varphi = 1/\varphi'$, and for $n \geq 2$, $(\partial_x^n \varphi^{-1}) \circ \varphi$ is a polynomial in $1/\varphi'$, $\partial_x \varphi, \ldots, \partial_x^n \varphi$ with integer coefficients.

(ii) $\mathtt{D}^n(\varphi, v) := R_\varphi \circ \partial_x^n \circ R_{\varphi^{-1}} v = (\partial_x^n (v \circ \varphi^{-1})) \circ \varphi = \sum_{j=1}^n p_{n,j}(\varphi) \partial_x^j v$ where $p_{n,j}(\varphi)$ is a polynomial in $1/\varphi'$, $\partial_x \varphi, \ldots, \partial_x^{n+1-j}\varphi$ with integer coefficients. In particular, $p_{n,n}(\varphi) = 1/(\varphi')^n$.

Proof. The proof follows by a straightforward application of the chain rule (cf. discussion following (10)). ∎

Recall that $H^s_{\mathbb{C}} := H^s(\mathbb{T}, \mathbb{C})$ is the complexification of H^s.

Lemma 3.3. *For $s \geq 1$, let $W^s_{\mathbb{C}}$ denote the open subset $W^s_{\mathbb{C}} := \{f \in H^s_{\mathbb{C}} : f(x) \neq 0 \ \forall x \in \mathbb{T}\}$. Then, the map $W^s_{\mathbb{C}} \to H^s_{\mathbb{C}}$, $f \mapsto 1/f$ is analytic.*

Proof. Let f in $W^s_{\mathbb{C}}$, and $U_{\epsilon,\mathbb{C}}(f)$ be the neighborhood

$$U_{\epsilon,\mathbb{C}}(f) = \{f - g \mid g \in H^s_{\mathbb{C}}, \ ; \ \|g\|_{H^s_{\mathbb{C}}} < \epsilon\}$$

with $\epsilon > 0$ so small that $\|g/f\|_{H^s_{\mathbb{C}}} < 1$. Such a choice is possible since, $H^s_{\mathbb{C}}$ being a Banach algebra for $s \geq 1$, $\|g/f\|_{H^s_{\mathbb{C}}} \leq C \|g\|_{H^s_{\mathbb{C}}} \|1/f\|_{H^s_{\mathbb{C}}}$ so that it suffices to pick $0 < \epsilon < 1/(C\|1/f\|_{H^s_{\mathbb{C}}})$. Then, $1/(f-g)$ can be written in terms of a series

$$\frac{1}{f-g} = \frac{1}{f}\left(1 + \frac{g}{f} + \left(\frac{g}{f}\right)^2 + \ldots\right)$$

which converges uniformly in $U_{\epsilon,\mathbb{C}}(f)$ to an element in $H^s_{\mathbb{C}}$. ∎

Corollary 3.4. *For any $s \geq 2$, the map $\mathcal{D}^s \to H^{s-1}$, $\varphi \mapsto 1/\varphi'$ is analytic.*

Proof. The map $\mathcal{D}^s \to H^{s-1}$, $\varphi \mapsto 1/\varphi'$ is the composition of the linear map $\mathcal{D}^s \to H^{s-1}$, $\varphi \mapsto \varphi'$, and the analytic map $H^{s-1} \to H^{s-1}$, $\varphi' \mapsto 1/\varphi'$. ∎

Proof of Proposition 3.1. (*i*) By direct computation, one sees that the map defined in (12b) is indeed the inverse of (12a). In particular, this shows that $\mathcal{A}_k$ is bijective. By Lemma 3.2, the definition (3b) of A_k, and that of $\mathtt{D}^n(\varphi, v)$ (cf. Lemma 3.2 (*ii*)), we have that

$$\begin{aligned} R_\varphi \circ A_k \circ R_{\varphi^{-1}} v &= v + \sum_{j=1}^{k} (-1)^j \mathtt{D}^{2j}(\varphi, v) \\ &= v + \sum_{j=1}^{2k} q_{2k,j}(\varphi) \partial_x^j v \end{aligned} \tag{15}$$

where $q_{2k,j}(\varphi)$ is a polynomial in $1/\varphi'$, $\partial_x \varphi, \ldots, \partial_x^{2k+1-j}\varphi$ with integer coefficients. Note that in view of Lemma 3.2 (*ii*), $q_{2k,2k}(\varphi) = (-1)^k/(\varphi')^{2k}$.

By Corollary 3.4, the map

$$\mathcal{D}^\ell \times H^\ell \to (H^{\ell-2k})^{4k+2}, \quad (\varphi, v) \mapsto (1/\varphi', \partial_x\varphi, \ldots, \partial_x^{2k}\varphi, v, \partial_x v, \ldots, \partial_x^{2k} v)$$

is analytic and, since for $\ell \geq \ell_k = 2k+2$, $H^{\ell-2k}$ is a Banach algebra, we conclude that the r.h.s. of (15) is analytic and hence that $\mathcal{A}_k$ is analytic. Moreover, for any (φ_0, v_0) in $\mathcal{D}^\ell \times H^\ell$, the differential $d_{(\varphi_0,v_0)}\mathcal{A}_k : H^\ell \times H^\ell \to H^\ell \times H^{\ell-2k}$ is of the form

$$d_{(\varphi_0,v_0)}\mathcal{A}_k(\delta\varphi, \delta v) = \begin{pmatrix} \delta\varphi & 0 \\ \Lambda(\delta\varphi) & R_{\varphi_0} \circ A_k \circ R_{\varphi_0^{-1}} \delta v \end{pmatrix} \tag{16}$$

where

$$\Lambda : H^\ell \to H^{\ell-2k}, \quad \text{and} \quad R_{\varphi_0} \circ A_k \circ R_{\varphi_0^{-1}} : H^\ell \to H^{\ell-2k}$$

are bounded linear maps. As the latter map is invertible, the open mapping theorem implies that it is a linear isomorphism. Hence, $d_{(\varphi_0,v_0)}\mathcal{A}_k$ is a linear isomorphism and, by the inverse function theorem, the map defined in (12a) is a local C^ω-diffeomorphism and since we have seen that $\mathcal{A}_k$ is bijective, assertion (*i*) follows. The proof of item (*ii*) is similar to the proof of the analyticity of $\mathcal{A}_k$ in part (*i*). ∎

4. The complex analytic extension of $\mathcal{F}_k$

As in the previous section, let $k \geq 1$, and $\ell \geq \ell_k = 2k+2$. Denote by $\mathfrak{U}^\ell_{1,\mathbb{C}}$ the complexification of the Hilbert chart $\mathfrak{U}^\ell_1$ defined in (7a),

$$\mathfrak{U}^\ell_{1,\mathbb{C}} := \{\varphi = \mathrm{id} + f \mid \ f \in H^\ell_\mathbb{C};\ |f(0)| < 1/2;\ \mathtt{Re}\,[(\varphi')^{2k-3}] > 0\}. \tag{17}$$

(The condition $\mathtt{Re}\,[(\varphi')^{2k-3}] > 0$ will be used in the proof of Proposition 6.1.) It follows from Lemma 3.2 and Lemma 3.3 that, for any $1 \leq n \leq 2k$, the map

$$\mathfrak{U}^\ell_1 \times H^\ell \to H^{\ell-n}, \quad (\varphi, v) \mapsto \mathtt{D}^n(\varphi, v) := R_\varphi \circ \partial_x^n \circ R_{\varphi^{-1}} v \tag{18a}$$

can be extended to an analytic map

$$\mathfrak{U}^\ell_{1,\mathbb{C}} \times H^\ell_\mathbb{C} \to H^{\ell-n}_\mathbb{C}, \quad (\varphi, v) \mapsto \mathtt{D}^n_\mathbb{C}(\varphi, v). \tag{18b}$$

As a consequence, the map (cf. (15))

$$\mathfrak{U}_1^\ell \times H^\ell \to H^{\ell-2k}, \quad (\varphi, v) \mapsto \mathtt{A}_k(\varphi, v) := v + \sum_{j=1}^{k} (-1)^j \mathtt{D}^{2j}(\varphi, v) \tag{19a}$$

has an analytic extension

$$\mathfrak{U}_{1,\mathbb{C}}^\ell \times H_{\mathbb{C}}^\ell \to H_{\mathbb{C}}^{\ell-2k}, \quad (\varphi, v) \mapsto \mathtt{A}_{k,\mathbb{C}}(\varphi, v) := v + \sum_{j=1}^{k} (-1)^j \mathtt{D}_{\mathbb{C}}^{2j}(\varphi, v)\,.$$

Note that the latter is of the form

$$\mathtt{A}_{k,\mathbb{C}}(\varphi, v) = v + \sum_{j=1}^{2k} q_{2k,j}(\varphi) \partial_x^j v \tag{19b}$$

where $q_{2k,j}(\varphi)$ is a polynomial in $1/\varphi', \partial_x\varphi, \dots, \partial_x^{2k+1-j}\varphi$. Further, we introduce the analytic map (the complexification of (12a))

$$\mathcal{A}_{k,\mathbb{C}} : \mathfrak{U}_{1,\mathbb{C}}^\ell \times H_{\mathbb{C}}^\ell \to \mathfrak{U}_{1,\mathbb{C}}^\ell \times H_{\mathbb{C}}^{\ell-2k},\ (\varphi, v) \mapsto (\varphi, \mathtt{A}_{k,\mathbb{C}}(\varphi, v))\,. \tag{20}$$

Analogously, Lemma 3.2 and Lemma 3.3 imply that the map (13) can be analytically extended to the map

$$\mathcal{B}_{k,\mathbb{C}} : \mathfrak{U}_{1,\mathbb{C}}^\ell \times H_{\mathbb{C}}^\ell \to \mathfrak{U}_{1,\mathbb{C}}^\ell \times H_{\mathbb{C}}^{\ell-2k}\,. \tag{21}$$

Lemma 4.1. *For any $k \ge 1$ and $\ell \ge \ell_k$, there exists a complex neighborhood $U_{\ell,k;\mathbb{C}}$ of* id *in $\mathfrak{U}_{1,\mathbb{C}}^\ell$ such that* (12b) *can be extended to a bianalytic diffeomorphism*

$$\mathcal{A}_{k,\mathbb{C}}^{-1} : U_{\ell,k;\mathbb{C}} \times H_{\mathbb{C}}^{\ell-2k} \to U_{\ell,k;\mathbb{C}} \times H_{\mathbb{C}}^\ell \tag{22}$$

with inverse $\mathcal{A}_{k,\mathbb{C}}\big|_{U_{\ell,k;\mathbb{C}} \times H_{\mathbb{C}}^\ell}$.

Proof. By Proposition 3.1, for any $\ell \ge \ell_k$, there exists a complex neighborhood $U_{\ell,k;\mathbb{C}} \times W_{\ell,\mathbb{C}}^{(k)}$ of $(\mathrm{id}, 0)$ in $\mathfrak{U}_{1,\mathbb{C}}^\ell \times H_{\mathbb{C}}^\ell$ such that $\mathcal{A}_{k,\mathbb{C}}\big|_{U_{\ell,k;\mathbb{C}} \times W_{\ell,\mathbb{C}}^{(k)}}$ is a bianalytic diffeomorphism onto its image. In particular, cf. (20), for any given φ in $U_{\ell,k;\mathbb{C}}$, the map $\mathtt{A}_{k,\mathbb{C}}(\varphi, \cdot\,) : W_{\ell,\mathbb{C}}^{(k)} \to H_{\mathbb{C}}^{\ell-2k}$ extends, by linearity, to a bounded map

$$\mathtt{A}_{k,\mathbb{C}}(\varphi, \cdot\,) : H_{\mathbb{C}}^\ell \to H_{\mathbb{C}}^{\ell-2k}\,.$$

As $\mathtt{A}_{k,\mathbb{C}}(\varphi, \cdot\,)$ is a bianalytic diffeomorphism near the origin, it is in fact bijective. Hence, by the open mapping theorem, $\mathtt{A}_{k,\mathbb{C}}(\varphi, \cdot\,)$ is a linear isomorphism for any φ in $U_{\ell,k;\mathbb{C}}$ and hence

$$\mathcal{A}_{k,\mathbb{C}}\big|_{U_{\ell,k;\mathbb{C}} \times H_{\mathbb{C}}^\ell} : U_{\ell,k;\mathbb{C}} \times H_{\mathbb{C}}^\ell \to U_{\ell,k;\mathbb{C}} \times H_{\mathbb{C}}^{\ell-2k} \tag{23}$$

is bijective. Moreover, from (20), one sees that, for any (φ_0, v_0) in $U_{\ell,k;\mathbb{C}} \times H_{\mathbb{C}}^\ell$, the differential $d_{(\varphi_0,v_0)}\mathcal{A}_{k,\mathbb{C}}$ of the map in (23) is of the form (16), and hence, a linear isomorphism. Altogether, it follows that (23) is a bianalytic diffeomorphism. ∎

Lemma 4.2. *For any $k \geq 1$ and any $\ell \geq \ell_k$, the neighborhood $U_{\ell,k;\mathbb{C}}$ in Lemma 4.1 can be chosen to be of the form $U^{(k)}_{\ell,\mathbb{C}} := U_{\ell_k,k;\mathbb{C}} \cap \mathfrak{U}^{\ell}_{1,\mathbb{C}}$.*

Proof. For any $\ell \geq \ell_k$, formula (20) defines an analytic map from $U^{(k)}_{\ell,\mathbb{C}} \times H^{\ell}_{\mathbb{C}}$ to $U^{(k)}_{\ell,\mathbb{C}} \times H^{\ell-2k}_{\mathbb{C}}$. By Lemma 4.1, for $\ell = \ell_k$, this map is a bianalytic diffeomorphism. Hence, it is injective for any $\ell \geq \ell_k$. The proof of Lemma 4.1 shows that it suffices to prove that, for any φ in $U^{(k)}_{\ell,\mathbb{C}}$,

$$\mathcal{A}_{k,\mathbb{C}}(\varphi,\cdot\,) : H^{\ell}_{\mathbb{C}} \to H^{\ell-2k}_{\mathbb{C}}, \quad v \mapsto \mathtt{A}_{k,\mathbb{C}}(\varphi, v)\,,$$

is onto. Indeed, for any φ in $U^{(k)}_{\ell,\mathbb{C}}$ and any h in $H^{\ell-2k}_{\mathbb{C}}$, the equation for v, $\mathtt{A}_{k,\mathbb{C}}(\varphi, v) = h$, is by (19b) the ODE

$$v + \sum_{j=1}^{2k} q_{2k,j}(\varphi)\partial_x^j v = h\,.$$

Now, first observe that for φ in $U^{(k)}_{\ell,\mathbb{C}} \subseteq U^{(k)}_{\ell_k,\mathbb{C}}$, the linear operator $\mathtt{A}_{k,\mathbb{C}}(\varphi,\cdot\,)$ maps $H^{\ell_k}_{\mathbb{C}}$ to $H^{\ell_k-2k}_{\mathbb{C}}$, and since the latter is a linear isomorphism, it follows that (for any φ and h as above) this equation has a unique solution v in $H^{\ell_k}_{\mathbb{C}}$. Finally, as $q_{2k,j}(\varphi)$ is in $H^{\ell-2k+j-1}_{\mathbb{C}}$ and $q_{2k,2k}(\varphi) = (-1)^k/(\varphi')^{2k}$ does nowhere vanish by (17), it then follows that v is in $H^{\ell}_{\mathbb{C}}$. ∎

From now on, we choose $U_{\ell,k;\mathbb{C}}$ in Lemma 4.1 to be given by

$$U^{(k)}_{\ell,\mathbb{C}} := U_{\ell_k,k,\mathbb{C}} \cap \mathfrak{U}^{\ell}_{1,\mathbb{C}}\,.$$

Recall that the maps $\mathcal{A}_{k,\mathbb{C}}$ and $\mathcal{B}_{k,\mathbb{C}}$ described in (20) respectively (21) are analytic on $\mathfrak{U}^{\ell}_{1,\mathbb{C}} \times H^{\ell}_{\mathbb{C}}$. By Lemma 4.1, $\mathcal{A}^{-1}_{k,\mathbb{C}}$ is analytic on $U^{(k)}_{\ell,\mathbb{C}} \times H^{\ell-2k}_{\mathbb{C}}$ (cf. Lemma 4.2). Hence, the vector field

$$\mathcal{F}_{k,\mathbb{C}} : U^{(k)}_{\ell,\mathbb{C}} \times H^{\ell}_{\mathbb{C}} \to H^{\ell}_{\mathbb{C}} \times H^{\ell}_{\mathbb{C}}\,, \quad (\varphi, v) \mapsto (v, F_{k,\mathbb{C}}(\varphi, v)) := \mathcal{A}^{-1}_{k,\mathbb{C}} \circ \mathcal{B}_{k,\mathbb{C}}(\varphi, v)\,. \tag{24}$$

is analytic; in fact the analytic extension of the vector field defined in (14). We will study the properties of the dynamical system corresponding to $\mathcal{F}_{k,\mathbb{C}}$ on $U^{(k)}_{\ell,\mathbb{C}}$ i.e.,

$$\begin{cases} \dot{\varphi} = v \\ \dot{v} = F_{k,\mathbb{C}}(\varphi, v)\,. \end{cases} \tag{25}$$

As, for any $k \geq 1$, $(\mathrm{id}, 0)$ is a zero of $\mathcal{F}_{k,\mathbb{C}}$ (and hence an equilibrium solution of (25)), one gets from [10, Theorem 10.8.1] and [10, Theorem 10.8.2] the following result.

Theorem 4.3. *Let $k \geq 1$, and $\ell \geq \ell_k = 2k+2$. Then there exists an open neighborhood $V_{\ell,k;\mathbb{C}}$ of 0 in $H^{\ell}_{\mathbb{C}}$ so that, for any v_0 in $V_{\ell,k;\mathbb{C}}$, the initial value problem for (25) with initial data $(\varphi(0), v(0)) = (\mathrm{id}, v_0)$ has a unique analytic solution*

$$(-2,2) \to U^{(k)}_{\ell,\mathbb{C}} \times H^{\ell}_{\mathbb{C}},\ t \mapsto (\varphi(t; v_0), v(t; v_0)). \tag{26a}$$

Moreover, the flow map,

$$(-2,2) \times V_{\ell,k;\mathbb{C}} \to U^{(k)}_{\ell,\mathbb{C}} \times H^{\ell}_{\mathbb{C}},\ (t, v_0) \mapsto (\varphi(t; v_0), v(t; v_0)) \tag{26b}$$

is analytic.

Remark 4.4. In fact, [10, Theorem 10.8.1] and [10, Theorem 10.8.2] imply that (26b) is a C^1-map over $\mathbb{C}$, and thus that the map (26b) is analytic.

Remark 4.5. Theorem 4.3 does not exclude that $\bigcap_{\ell \geq \ell_k} V_{\ell,k;\mathbb{C}} = \{0\}$. This possibility is ruled out by Theorem 5.2 given in the next section.

5. The exponential map and its analytic extension

As in the previous sections, let $k \geq 1$, and set $\ell_k := 2k+2$. By (26b) of Theorem 4.3, the Riemannian exponential map

$$\mathrm{Exp}_{k,\ell_k} : V_{\ell_k,k;\mathbb{C}} \cap H^{\ell_k} \to \mathcal{D}^{\ell_k},\ v_0 \mapsto \varphi(1; v_0)$$

admits an analytic extension

$$\mathrm{Exp}^{\mathbb{C}}_{k,\ell_k} : V_{\ell_k,k;\mathbb{C}} \to U^{(k)}_{\ell_k,\mathbb{C}},\ v_0 \mapsto \varphi(1; v_0)\,. \tag{27}$$

Set

$$V^{(k)}_{\ell_k,\mathbb{C}} := V_{\ell_k,k;\mathbb{C}}\,.$$

Noting that $d_0 \mathrm{Exp}^{\mathbb{C}}_{k,\ell_k} = \mathrm{Id}_{H^{\ell_k}_{\mathbb{C}}}$, it then follows from the inverse function theorem that, by shrinking the neighborhoods $V^{(k)}_{\ell_k,\mathbb{C}}$ and $U^{(k)}_{\ell_k,\mathbb{C}}$ if necessary, one can ensure that the mapping (27) is a bianalytic diffeomorphism. This will be tacitly assumed in the remaining of the paper. In this section, we study the restriction of $\mathrm{Exp}^{\mathbb{C}}_{k,\ell_k}$ to $V^{(k)}_{\ell_k,\mathbb{C}} \cap C^\infty(\mathbb{T},\mathbb{C})$.

A priori relation: For a while let us study the equation (2) instead of its analytic extension (25). Consider the curve $u = v \circ \varphi^{-1}$ where $t \mapsto (\varphi(t), v(t))$ is an arbitrary C^1-solution of (2) in $(U^{(k)}_{\ell,\mathbb{C}} \cap \mathfrak{U}^\ell_1) \times H^\ell$ on some nontrivial time interval $(-T, T)$. By Proposition 2.3 and Proposition 2.4,

$$(-T,T) \to H^\ell \subseteq H^{\ell-1}, \quad t \mapsto u(t) = v(t) \circ \varphi^{-1}(t)\,,$$

is a C^1-curve in $H^{\ell-1}$. Moreover, it satisfies (6). Our aim is to derive, for any $-T < t < T$, a formula for $(A_k u(t)) \circ \varphi(t)$ which will be used to study regularity properties of the exponential map. By the chain rule,

$$[(A_k u) \circ \varphi]^{\cdot} = (A_k \dot{u}) \circ \varphi + [(A_k u)' \circ \varphi]\dot{\varphi}.$$

As $v = \dot{\varphi} = u \circ \varphi$, the initial value problem (6) then leads to

$$\begin{cases} [(A_k u) \circ \varphi]^{\cdot} + 2(u' \circ \varphi)[(A_k u) \circ \varphi] = 0 \\ (A_k u(0)) \circ \varphi(0) = A_k v_0. \end{cases}$$

By definitions (18a) and (19a), $(A_k u) \circ \varphi = \mathtt{A}_k(\varphi, v)$, and $u' \circ \varphi = \mathtt{D}^1(\varphi, v)$. Hence, the latter can be rewritten as

$$\begin{cases} [\mathtt{A}_k(\varphi, v)]^{\cdot} + 2\mathtt{D}^1(\varphi, v)[\mathtt{A}_k(\varphi, v)] = 0 \\ \mathtt{A}_k(\varphi(0), v(0)) = A_k v_0. \end{cases}$$

where, for $-T < t < T$, $t \mapsto \mathtt{A}_k(\varphi(t), v(t))$ is of class $H^{\ell-2k}$ and $t \mapsto \mathtt{D}^1(\varphi(t), v(t))$ evolves in $H^{\ell-1}$. Both of these curves are C^1-smooth. Solving the latter equation one gets

$$\mathtt{A}_k(\varphi(t), v(t)) = e^{-2\int_0^t \mathtt{D}^1(\varphi(\tau),v(\tau))\,d\tau} A_k v_0. \tag{28}$$

On the other hand, differentiating of (5) with respect to x yields

$$(\varphi')^{\cdot} = \mathtt{D}^1(\varphi, v)\varphi' .$$

Since $\varphi(0) = \mathrm{id}$, one obtains

$$\varphi'(t) = e^{\int_0^t \mathtt{D}^1(\varphi(\tau),v(\tau))\,d\tau} . \tag{29}$$

Hence, (28) can be rewritten as

$$\mathtt{A}_k(\varphi(t), v(t)) = A_k v_0/(\varphi'(t))^2 .$$

Let

$$(U_{\ell,\mathbb{C}}^{(k)} \cap \mathfrak{U}_1^\ell) \times H^\ell \to H^{\ell-2k}, \quad (\varphi, v) \mapsto I_k(\varphi, v) := \mathtt{A}_k(\varphi, v)(\varphi')^2$$

then, the above identity shows that the function

$$I_k(\varphi(t), v(t)) = \mathtt{A}_k(\varphi(t), v(t))(\varphi'(t))^2$$

is independent of t, and is equal to $A_k v_0$. As a consequence, the derivative $\mathcal{L}_{\mathcal{F}_k}(I_k)$ of I_k in the direction $\mathcal{F}_k$ vanishes on the open set $(U_{\ell,\mathbb{C}}^{(k)} \cap \mathfrak{U}_1^\ell) \times H^\ell$.

Now, let us return to the analytic extension (25) of (2). First note that

$$U_{\ell,\mathbb{C}}^{(k)} \times H_{\mathbb{C}}^\ell \to H_{\mathbb{C}}^{\ell-2k}, \quad (\varphi, v) \mapsto I_{k,\mathbb{C}}(\varphi, v) := \mathtt{A}_{k,\mathbb{C}}(\varphi, v)(\varphi')^2 , \tag{30}$$

is an analytic extension of I_k. Further, $\mathcal{F}_{k,\mathbb{C}}$ analytically extends $\mathcal{F}_k$, and the derivative $\mathcal{L}_{\mathcal{F}_{k,\mathbb{C}}}(I_{k,\mathbb{C}})$ of $I_{k,\mathbb{C}}$ in the direction $\mathcal{F}_{k,\mathbb{C}}$, analytically extends $\mathcal{L}_{\mathcal{F}_k}(I_k)$ to $U_{\ell,\mathbb{C}}^{(k)} \times H_{\mathbb{C}}^\ell$ so that

$$\mathcal{L}_{\mathcal{F}_{k,\mathbb{C}}}(I_{k,\mathbb{C}})\big|_{(U_{\ell,\mathbb{C}}^{(k)} \cap \mathfrak{U}_1^\ell) \times H^\ell} = \mathcal{L}_{\mathcal{F}_k}(I_k) .$$

As $\mathcal{L}_{\mathcal{F}_k}(I_k)$ vanishes on $(U_{\ell,\mathbb{C}}^{(k)} \cap \mathfrak{U}_1^\ell) \times H^\ell$, it then follows that $\mathcal{L}_{\mathcal{F}_{k,\mathbb{C}}}(I_{k,\mathbb{C}}) = 0$ everywhere in $U_{\ell,\mathbb{C}}^{(k)} \times H_{\mathbb{C}}^\ell$, – see e.g. [4, Proposition 6.6]. In other words, (30) is a conserved quantity for the solutions of (25) on $U_{\ell,\mathbb{C}}^{(k)} \times H_{\mathbb{C}}^\ell$.

Lemma 5.1. *For any (φ, v) in $U_{\ell,\mathbb{C}}^{(k)} \times H_{\mathbb{C}}^\ell$, and any $1 \le j \le 2k$, the following relation holds*

$$\partial_x^j v = \mathtt{D}_{\mathbb{C}}^j(\varphi, v)(\varphi')^j + \mathtt{D}_{\mathbb{C}}^1(\varphi, v)\partial_x^j \varphi + \dots \tag{31}$$

where $\mathtt{D}_{\mathbb{C}}^j(\varphi, v)$ is the analytic extension of (18a) *to $U_{\ell,\mathbb{C}}^{(k)} \times H_{\mathbb{C}}^\ell$, and $\dots$ stand for a polynomial in the variables $1/\varphi'$, $\partial_x \varphi, \dots, \partial_x^{j-1}\varphi$ and $\partial_x v, \dots, \partial_x^{j-1} v$.*

Proof. First let us consider (φ, v) in $(U^{(k)}_{\ell,\mathbb{C}} \cap \mathfrak{U}^\ell_1) \times H^\ell$. Then φ is in $\mathcal{D}^\ell$ and $u := v \circ \varphi^{-1}$ is well-defined in H^ℓ, and thus we can write $v = u \circ \varphi$. By differentiation we get

$$v' = (u' \circ \varphi)\varphi' . \tag{32a}$$

As $u' \circ \varphi = R_\varphi \circ \partial_x \circ R_{\varphi^{-1}} v$, one has by definition (18a) that

$$v' = \mathtt{D}^1(\varphi, v)\varphi' . \tag{32b}$$

By the analyticity of $\mathtt{D}^1_{\mathbb{C}}$, the identity (32b) continues to hold for (φ, v) in $U^{(k)}_{\ell,\mathbb{C}} \times H^\ell_{\mathbb{C}}$ i.e.,

$$v' = \mathtt{D}^1_{\mathbb{C}}(\varphi, v)\varphi' .$$

For $j \geq 2$, we argue similarly i.e., given (φ, v) in $(U^{(k)}_{\ell,\mathbb{C}} \cap \mathfrak{U}^\ell_1) \times H^\ell$, we differentiate (32a) $(j-1)$ times to get

$$\partial^j_x v = \mathtt{D}^j(\varphi, v)(\varphi')^j + \mathtt{D}^1(\varphi, v)\partial^j_x \varphi + \dots \tag{32c}$$

where $\dots$ stand for a polynomial in $1/\varphi'$, $\partial_x \varphi, \dots, \partial^{j-1}_x \varphi$ and $\partial_x v, \dots, \partial^{j-1}_x v$. Finally, (32c) extends by analyticity to $U^{(k)}_{\ell,\mathbb{C}} \times H^\ell_{\mathbb{C}}$ leading to (31). ■

Now, let $t \mapsto (\varphi(t), v(t))$ be a solution of (25) in $C^1((-2,2), U^{(k)}_{\ell,\mathbb{C}} \times H^\ell_{\mathbb{C}})$. Then, by Lemma 5.1, the curve $t \mapsto (A_k - 1)\varphi(t) = \sum_{j=1}^k (-1)^j \partial^{2j}_x \varphi(t)$ satisfies the inhomogeneous transport equation

$$\begin{cases} ((A_k - 1)\varphi)^{\cdot} - \mathtt{D}^1_{\mathbb{C}}(\varphi, v)(A_k - 1)\varphi = \mathtt{A}_{k,\mathbb{C}}(\varphi, v)(\varphi')^{2k} + g_{2k-1}(\varphi, v) \\ (A_k - 1)\varphi(0) = 0 \end{cases} \tag{33}$$

where $g_{2k-1}(\varphi, v)$ is a polynomial (with constant coefficients) in $1/\varphi'$, $\partial_x \varphi, \dots, \partial^{2k-1}_x \varphi$ and v, $\partial_x v, \dots, \partial^{2k-1}_x v$. (For convenience, we consider $(A_k - 1)\varphi$ instead of $A_k \varphi$ so that the initial value problem (33) involves periodic functions only.) Integrating (33) by the method of variation of parameters, and using (29) to write the final expression in compact form, we get that, for any $-2 < t < 2$,

$$(A_k - 1)\varphi(t) = \varphi'(t) \int_0^t \left[\mathtt{A}_{k,\mathbb{C}}(\varphi(\tau), v(\tau))(\varphi'(\tau))^{2k-1} + \frac{g_{2k-1}(\varphi(\tau), v(\tau))}{\varphi'(\tau)} \right] d\tau.$$

By (30) which when evaluated at the solution of (25) is equal to $A_k v_0$, we then have that, for any $-2 < t < 2$,

$$(A_k - 1)\varphi(t) - \varphi'(t) \left(\int_0^t (\varphi'(\tau))^{2k-3} d\tau \right) A_k v_0 = \varphi'(t) \int_0^t \rho_{2k-1}(\varphi(\tau), v(\tau))\, d\tau \tag{34}$$

where

$$U^{(k)}_{\ell,\mathbb{C}} \times H^\ell_{\mathbb{C}} \to H^{\ell-2k+1}_{\mathbb{C}}, \quad (\varphi, v) \mapsto \rho_{2k-1}(\varphi, v) := g_{2k-1}(\varphi, v)/\varphi' \tag{35}$$

is analytic by Lemma 3.3. As we will explain in detail later on, the a priori relation (34) plays a fundamental role in the proofs of Theorem 5.2 and Theorem 1.3 below.

Theorem 5.2. *Let $k \geq 1$, and $\ell \geq \ell_k = 2k+2$. Then, for any v_0 in $V^{(k)}_{\ell,\mathbb{C}} := V^{(k)}_{\ell_k,\mathbb{C}} \cap H^\ell_\mathbb{C}$, there exists a unique solution of* (25) *in* $C^1((-2,2), U^{(k)}_{\ell,\mathbb{C}} \times H^\ell_\mathbb{C})$ *with initial data* (id, v_0). *Moreover, the flow map*

$$(-2,2) \times V^{(k)}_{\ell,\mathbb{C}} \to U^{(k)}_{\ell,\mathbb{C}} \times H^\ell_\mathbb{C},\ (t, v_0) \mapsto (\varphi(t; v_0), v(t; v_0))$$

is analytic.

Proof. We argue by induction with respect to $\ell \geq \ell_k$. For $\ell = \ell_k$ the statement follows from Theorem 4.3 since, by definition, $V^{(k)}_{\ell_k,\mathbb{C}} = V_{\ell_k,k;\mathbb{C}}$. Assume that the statement is true for any given $\ell > \ell_k$ i.e., that given any v_0 in $V^{(k)}_{\ell,\mathbb{C}}$, there exists a unique solution of (25) in $C^1((-2,2), U^{(k)}_{\ell,\mathbb{C}} \times H^\ell_\mathbb{C})$ with initial data (id, v_0) and, in addition, that the flow map $(-2,2) \times V^{(k)}_{\ell,\mathbb{C}} \to U^{(k)}_{\ell,\mathbb{C}} \times H^\ell_\mathbb{C}$ is analytic. Then, by (35), the r.h.s. of (34) is in $H^{\ell-2k+1}_\mathbb{C}$. Now, let v_0 be in $V^{(k)}_{\ell+1,\mathbb{C}}$ and, for $-2 < t < 2$, let

$$t \mapsto \zeta(t) := (\varphi(t), v(t)) \in U^{(k)}_{\ell,\mathbb{C}} \times H^\ell_\mathbb{C}$$

be the corresponding solution of (25) issuing from (id, v_0). In particular, $t \mapsto \zeta(t)$ satisfies the integral equation

$$\zeta(t) = (\mathrm{id}, v_0) + \int_0^t \mathcal{F}_{k,\mathbb{C}}(\varphi(\tau), v(\tau))\, d\tau\,. \tag{36}$$

As v_0 belongs to $V^{(k)}_{\ell+1,\mathbb{C}}$ and $k \geq 1$, $A_k v_0$ and hence the second term on the l.h.s. of (34) are in $H^{\ell-2k+1}_\mathbb{C}$. Altogether, it then follows from (34) that $t \mapsto (A_k - 1)\varphi(t)$ is a C^1-curve evolving in $H^{\ell-2k+1}_\mathbb{C}$ for $-2 < t < 2$. Hence, as $v = \dot\varphi$, $t \mapsto \zeta(t) = (\varphi(t), v(t))$ is a continuous curve in $U^{(k)}_{\ell+1,\mathbb{C}} \times H^{\ell+1}_\mathbb{C}$. By the analyticity of the map (24), it then follows that the integrand in (36) is a continuous function of τ with values in $H^{\ell+1}_\mathbb{C} \times H^{\ell+1}_\mathbb{C}$. Finally, the integral equation (36) implies that ζ is a solution of (25) in $C^1((-2,2), U^{(k)}_{\ell+1,\mathbb{C}} \times H^{\ell+1}_\mathbb{C})$ with initial data (id, v_0). The second statement of the theorem follows by combining [10, Theorem 10.8.1] and [10, Theorem 10.8.2] (cf. Remark 4.4). ∎

Proof of Theorem 1.2. Theorem 1.2 is an immediate consequence of Theorem 5.2. Indeed, for any $k \geq 1$,

$$V^{(k)} := V^{(k)}_\mathbb{C} \cap C^\infty(\mathbb{T}, \mathbb{R}) \tag{37}$$

satifies the properties stated in Theorem 1.2. ∎

Theorem 5.2 allows to define the exponential map. Recall that, for any $\ell \geq \ell_k$,

$$V^{(k)}_{\ell,\mathbb{C}} := V^{(k)}_{\ell_k,\mathbb{C}} \cap H^\ell_\mathbb{C}\,. \tag{38a}$$

where $V^{(k)}_{\ell_k,\mathbb{C}} = V_{\ell_k,k;\mathbb{C}}$. (Note that for $\ell \geq \ell_k + 1$, $V^{(k)}_{\ell,\mathbb{C}}$ might not coincide with the neighborhood $V_{\ell,k;\mathbb{C}}$ introduced in Theorem 4.3.) Introduce

$$V^{(k)}_\mathbb{C} := V^{(k)}_{\ell_k,\mathbb{C}} \cap C^\infty(\mathbb{T}, \mathbb{C}) = \bigcap_{\ell \geq \ell_k} V^{(k)}_{\ell,\mathbb{C}} \tag{38b}$$

$$U^{(k)}_\mathbb{C} := U^{(k)}_{\ell_k,\mathbb{C}} \cap C^\infty(\mathbb{T}, \mathbb{C}) = \bigcap_{\ell \geq \ell_k} U^{(k)}_{\ell,\mathbb{C}}\,.$$

By Theorem 5.2, for any $\ell \geq \ell_k$, the restriction $\mathrm{Exp}^{\mathbb{C}}_{k;\ell}$ of $\mathrm{Exp}^{\mathbb{C}}_{k,\ell_k}$ to $V^{(k)}_{\ell,\mathbb{C}}$ takes values in $U^{(k)}_{\ell,\mathbb{C}}$. Moreover, $\mathrm{Exp}^{\mathbb{C}}_{k,\ell} : V^{(k)}_{\ell,\mathbb{C}} \to U^{(k)}_{\ell,\mathbb{C}}$ is analytic. Hence, the restriction $\mathrm{Exp}^{\mathbb{C}}_{k}$ of $\mathrm{Exp}^{\mathbb{C}}_{k,\ell_k}$ to $V^{(k)}_{\mathbb{C}}$ takes values in $U^{(k)}_{\ell_k,\mathbb{C}} \cap C^\infty(\mathbb{T},\mathbb{C})$ i.e.,

$$\mathrm{Exp}^{\mathbb{C}}_{k} : V^{(k)}_{\mathbb{C}} \to U^{(k)}_{\mathbb{C}}, \quad v_0 \mapsto \varphi(1; v_0).$$

6. A Fréchet analytic chart of id in $\mathcal{D}(\mathbb{T})$

Theorem 1.3 states that the exponential map $\mathrm{Exp}_k \big|_{V^{(k)}} : V^{(k)} \to \mathcal{D}$ that is defined using Theorem 1.2, can be used to define an analytic chart of the identity in $\mathcal{D}$.

Proof of Theorem 1.3. By definitions (37) and (38b), $V^{(k)} = V^{(k)}_{\mathbb{C}} \cap C^\infty(\mathbb{T},\mathbb{R})$, and the map $\mathrm{Exp}^{\mathbb{C}}_{k}$, defined at the end of the previous section, is the analytic extension of Exp_k. We want to apply to $\mathrm{Exp}^{\mathbb{C}}_{k}$ the inverse function theorem in Fréchet spaces, Theorem A9. Fix $k \geq 1$. To match the notation of this theorem, we write, for any integer $n \geq 0$, $\ell := \ell_k + n$ ($\ell_k = 2k+2$), and define

$$X_n := H^\ell_{\mathbb{C}}; \quad Y_n := H^\ell_{\mathbb{C}}, \qquad \text{and} \qquad V_n := V^{(k)}_{\ell,\mathbb{C}}; \quad U_n := \mathrm{Exp}^{\mathbb{C}}_{k,\ell}(V^{(k)}_{\ell,\mathbb{C}})$$

where $V^{(k)}_{\ell,\mathbb{C}}$ is defined in (38a), and $\mathrm{Exp}^{\mathbb{C}}_{k,\ell}$ is the exponential map introduced in section 5.. Further, let $f := \mathrm{Exp}^{\mathbb{C}}_{k,\ell_k} : V_0 \to U_0$. It follows from our construction that f is a C^1-diffeomorphism and hence item (a) of Theorem A9 is verified. Assumption (b) holds in view of Theorem 5.2, whereas items (c) and (d) hold, respectively, by Proposition 6.1, and Proposition 6.2 below. Hence, Theorem 1.3 follows from Theorem A9. ■

It remains to show the two propositions used in the proof above.

Proposition 6.1. *Let $k \geq 1$, and $\ell_k = 2k+2$. Then, for any $\ell = \ell_k + n$, $n \geq 0$, and any v_0 in $V^{(k)}_{\ell_k,\mathbb{C}}$,*

$$\mathrm{Exp}^{\mathbb{C}}_{k,\ell_k}(v_0) \in U^{(k)}_{\ell,\mathbb{C}} \quad \Longrightarrow \quad v_0 \in V^{(k)}_{\ell,\mathbb{C}}.$$

Proof. Let $(\varphi(\cdot; v_0), v(\cdot; v_0))$ denote the solution of (25) with initial data (id, v_0) with v_0 in $V^{(k)}_{\ell_k,\mathbb{C}}$, and suppose that $(A_k - 1)\varphi(1; v_0)$ belongs to $H^{\ell-2k}_{\mathbb{C}}$. We will show that v_0 has to be in $H^\ell_{\mathbb{C}}$. For $\ell = \ell_k$, the result holds by construction. Now, inductively, assume that v_0 is in $V^{(k)}_{\ell,\mathbb{C}}$ for any given $\ell > \ell_k$. By Theorem 5.2, this solution actually lies in $C^1((-2,2), U^{(k)}_{\ell,\mathbb{C}} \times H^\ell_{\mathbb{C}})$. This implies that the r.h.s. of (34), when evaluated at $t = 1$, is in $H^{\ell-2k+1}_{\mathbb{C}}$. Moreover, as $U^{(k)}_{\ell,\mathbb{C}} \subseteq \mathfrak{U}^\ell_{1,\mathbb{C}}$, we have by definition (17) that the factor $\varphi'(1) \int_0^1 (\varphi'(\tau))^{2k-3}\, d\tau$ does not vanish and is in $H^{\ell-1}_{\mathbb{C}} \subseteq H^{\ell-2k+1}_{\mathbb{C}}$ since $k \geq 1$. Altogether, it follows from (34) that $A_k v_0$ lies in $H^{\ell-2k+1}_{\mathbb{C}}$, and hence that v_0 is in $H^{\ell+1}_{\mathbb{C}}$. ■

By Theorem 5.2, $(-2,2) \times V^{(k)}_{\ell,\mathbb{C}} \to U^{(k)}_{\ell,\mathbb{C}} \times H^\ell_{\mathbb{C}}$, $(t, v_0) \mapsto (\varphi(t; v_0), v(t; v_0))$ is analytic. Then, the variation δv_0 in $H^\ell_{\mathbb{C}}$ of the initial data v_0 in $V^{(k)}_{\ell,\mathbb{C}}$ induces the variation of $t \mapsto (\varphi(t; v_0), v(t; v_0))$

$$t \mapsto (\delta\varphi(t), \delta v(t)) := \frac{d}{d\epsilon}\Big|_{\epsilon=0} (\varphi(t; v_0 + \epsilon\delta v_0), v(t; v_0 + \epsilon\delta v_0))$$

which is a continuous curve in $H^\ell_\mathbb{C} \times H^\ell_\mathbb{C}$. Differentiating (34) in direction δv_0 in $H^\ell_\mathbb{C}$ at v_0 in $V^{(k)}_{\ell,\mathbb{C}}$ yields

$$(A_k - 1)(\delta\varphi(t)) - \varphi'(t)\left(\int_0^t (\varphi'(\tau))^{2k-3}\, d\tau\right) A_k(\delta v_0) = \\ = P_{2k-1}(\varphi(t), v(t); \delta\varphi(t), \delta v(t)) \qquad (39)$$

where $P_{2k-1} : U^{(k)}_{\ell,\mathbb{C}} \times H^\ell_\mathbb{C} \times H^\ell_\mathbb{C} \times H^\ell_\mathbb{C} \to H^{\ell-2k+1}_\mathbb{C}$ is analytic.

Proposition 6.2. *Let $k \geq 1$, $\ell_k = 2k+2$, and $\ell = \ell_k + n$, $n \geq 0$. Assume that v_0 is in $V^{(k)}_{\ell,\mathbb{C}}$. Then*

$$(d_{v_0}\mathrm{Exp}^\mathbb{C}_{k,\ell})(H^\ell_\mathbb{C} \backslash H^{\ell+1}_\mathbb{C}) \subseteq H^\ell_\mathbb{C} \backslash H^{\ell+1}_\mathbb{C}\,.$$

Proof. Assume $v_0 \in V^{(k)}_{\ell,\mathbb{C}}$, and let $(\varphi(\cdot; v_0), v(\cdot; v_0))$ in $C^1((-2,2), U^{(k)}_\ell \times H^\ell_\mathbb{C})$ be the unique solution of (25) issuing from (id, v_0) as guaranteed by Theorem 5.2. As before, it follows from (17) that the factor in front of $A_k\delta v_0$ in formula (39), evaluated at $t = 1$, is a non-zero function in $H^{\ell-1}_\mathbb{C} \subseteq H^{\ell-2k+1}_\mathbb{C}$ whereas the term on the r.h.s. of this identity is in $H^{\ell-2k+1}_\mathbb{C}$. Hence, just as in the proof of Proposition 6.1 which followed from analyzing (34), the statement of Proposition 6.2 can be obtained from (39), evaluated at $t = 1$. ∎

A Analytic maps between Fréchet spaces

For the convenience of the reader we collect in this appendix some definitions and notions from the calculus in Fréchet spaces and present an inverse function theorem valid in a set-up for Fréchet spaces which is suitable for our purposes. For more details on the theory of smooth functions in Fréchet spaces we refer the reader to [13]. For the theory of analytic functions in Fréchet spaces, we follow the approach developped in [3, 4] (cf. also [17]). In the sequel $\mathbb{K}$ denotes either the field $\mathbb{C}$ of complex numbers or the field $\mathbb{R}$ of real numbers.

Fréchet spaces: Consider the pair $(X, \{||\cdot||_n\}_{n\geq 0})$ where X is a vector space over $\mathbb{K}$ and $\{||\cdot||_n\}_{n\geq 0}$ is a countable collection of seminorms. A topology on X is defined in the usual way as follows: A basis of open neighborhoods of 0 in X is given by the sets

$$U_{\epsilon,k_1,\ldots,k_s} := \{x \in X \;:\; ||x||_{k_j} < \epsilon \quad \forall 1 \leq j \leq s\}$$

where $s, k_1, \ldots, k_s$ are nonnegative integers and $\epsilon > 0$. Then the topology on X is defined as the collection of open sets generated by the sets $x + U_{\epsilon,k_1,\ldots,k_s}$, for arbitrary x in X and arbitrary $s, k_1, \ldots, k_s$ in $\mathbb{Z}_{\geq 0}$ and $\epsilon > 0$. In this way, X becomes a topological vector space. Note that a sequence $(x_k)_{k\geq 0}$ converges to x in X iff, for any $n \geq 0$, $||x_k - x||_n \to 0$ as $k \to +\infty$.

Moreover, the topological vector space X described above is *Hausdorff* iff, for any x in X, $||x||_n = 0$ for every n in $\mathbb{Z}_{\geq 0}$ implies $x = 0$. A sequence $(x_k)_{k\in\mathbb{N}}$ is called *Cauchy* iff it is a Cauchy sequence with respect to any of the seminorms. By definition, X is complete iff every Cauchy sequence converges in X.

Definition A1. A pair $(X, \{||\cdot||_n\}_{n\geq 0})$ consisting of a topological vector space X and a countable system of seminorms $\{||\cdot||_n\}_{n\geq 0}$ is called a *Fréchet space* [3] iff the topology of X is the one induced by $\{||\cdot||_n\}_{n\in\mathbb{Z}_{\geq 0}}$, and X is Hausdorff and complete.

The space of continuous maps $f: U \to Y$ from an open subset $U \subseteq X$ into the Fréchet space Y is denoted by $C^0(U, Y)$.

C^1_F-differentiability: Let $f: U \subseteq X \to Y$ be a map from an open set U of a Fréchet space X to a Fréchet space Y.

Definition A2. If the limit

$$\lim_{\epsilon\in\mathbb{K},\epsilon\to 0} \frac{1}{\epsilon}(f(x+\epsilon h) - f(x))$$

in Y exists with respect to the Fréchet topology of Y, we say that f *is differentiable at x in the direction h*. The limit is declared to be the *directional derivative* of f at the point x in U in the direction h in X. Following [3, 4], we denote it by $\delta_x f(h)$.

Definition A3. If the directional derivative $\delta_x f(h)$ exists for any x in U and any h in X, and the map

$$(x, h) \mapsto \delta_x f(h),\ U \times X \to Y$$

is continuous with respect to the Fréchet topology on $U \times X$ and Y, then f is called *continuously differentiable on U*. The space of all such maps is denoted by $C^1_F(U, Y)$.[4] A map $f: U \to V$ from an open set $U \subseteq X$ onto an open set $V \subseteq Y$ is called *a C^1_F-diffeomorphism* if f is a homeomorphism and f as well as f^{-1} are C^1_F-smooth.

Lemma A4. *Let $U \subseteq X$ be an open subset. Then*

(i) $C^1_F(U, Y) \subseteq C^0(U, Y)$

(ii) Assume that a map f in $C^1(U, Y)$ is a C^1_F-diffeomorphism onto an open subset $V \subseteq Y$. Then, for any x in U, $\delta_x f: X \to Y$ is a linear isomorphism.

Proof. Statement (i) follows from [13, Theorem 3.2.2.] whereas statement (ii) is a consequence of [13, Theorem 3.3.4.]. ■

Analytic functions in Fréchet spaces: Let X and Y be Fréchet spaces over $\mathbb{K}$ and let $f: U \subseteq X \to Y$ be a map from an open set $U \subseteq X$ into Y. A map $f_s: X \to Y$ is called a *homogeneous polynomial* of degree $s \in \mathbb{Z}_{\geq 0}$ if there exists a s-linear symmetric map $\mathtt{f}_s: X^s \to Y$ such that $f_s(x) = \mathtt{f}_s(x, \ldots, x)$ for any x in X (cf. [3, Definition 2]).

[3] Unlike for the standard notion of a Fréchet space, here the countable system of seminorms defining the topology of X is part of the structure of the space.

[4] Note that even in the case where X and Y are Banach spaces this definition of continuous differentiability is weaker than the usual one (cf. [13]). In order to distinguish it from the classical one we write C^1_F instead of C^1. We refer to [13] for a discussion of the reasons to introduce the notion of C^1_F-differentiability.

Definition A5. Following [4, Definition 5.6], a continuous function $f : U \to Y$ is called *analytic* if, for any x in U, there exist an open neighborhood V of 0 in X and a sequence of continuous homogeneous polynomials $(f_s)_{s\geq 0}$, $\deg f_s = s$, such that $x + V \subseteq U$ and, for any h in V, $f(x+h) = \sum_{s=0}^{\infty} f_s(h)$ converges in Y[5].

We will need the following lemma.

Lemma A6. *Assume that X and Y are Fréchet spaces over $\mathbb{C}$, $U \subseteq X$ is an open subset of X, and $f : U \to Y$ is in $C_F^1(U, Y)$. Then f is analytic.*

Remark A7. The converse of Lemma A6 is true as well. More precisely, assume that X and Y are $\mathbb{K}$-Fréchet spaces and $f : U \to Y$ is analytic. Then, by the definition of an analytic map, f is in $C_F^n(U, Y)$ for any $n \geq 0$.

Proof of Lemma A6. According to [4, Theorem 6.2], it suffices to prove that f is continuous and that it is analytic on affine lines. By Lemma A4 (i), f is continuous. By Definition A3, it follows that, for any x in U, and any h in X, the map $f_{x,h} : z \mapsto f(x + zh)$ with values in Y, defined on the open set $\{z \in \mathbb{C} \ : \ x + zh \in U\} \subseteq \mathbb{C}$ is (complex) differentiable. In particular, by [4, Theorem 3.1], $f_{x,h}(z)$ is analytic. Hence, h in X being arbitrary, f is analytic on affine lines. ■

Analytic functions in Fréchet spaces over $\mathbb{R}$: Now, assume that X and Y are Fréchet spaces over $\mathbb{R}$. Denote by $X_{\mathbb{C}} = X \otimes \mathbb{C}$ the complexification of X. The following theorem follows directly from [3, Theorem 3] and [4, Theorem 7.1].

Theorem A8. *Let U be an open subset of X. A function $f : U \to Y$ is analytic iff there exists a complex neighborhood $\tilde{U} \supseteq U$ in $X_{\mathbb{C}}$ and an analytic function $\tilde{f} : \tilde{U} \to Y_{\mathbb{C}}$ such that $\tilde{f}|_U = f$.*

In this paper we consider mainly the following spaces:

Fréchet space $C^\infty(\mathbb{T})$: The space $C^\infty(\mathbb{T}) \equiv C^\infty(\mathbb{T}, \mathbb{R})$ denotes the real vector space of real-valued C^∞-smooth, 1-periodic functions $u : \mathbb{R} \to \mathbb{R}$. The topology on $C^\infty(\mathbb{T})$ is induced by the countable system of Sobolev norms:

$$\|u\|_n := \|u\|_{H^n} = \Big(\sum_{j=0}^{n} \int_0^1 [\partial_x^j u(x)]^2 \, dx \Big)^{1/2}, \quad n \geq 0.$$

Fréchet manifold $\mathcal{D}$: By definition, $\mathcal{D}$ denotes the group of C^∞-smooth positively oriented diffeomorphisms of the torus $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. A Fréchet manifold structure on $\mathcal{D}$ can be introduced as follows: Passing to the universal cover $\mathbb{R} \to \mathbb{T}$, any element φ of $\mathcal{D}$ gives rise to a smooth diffeomorphism of $\mathbb{R}$ in $C^\infty(\mathbb{R}, \mathbb{R})$, again denoted by φ, satisfying the *normalization condition*

$$-1/2 < \varphi(0) < 1/2 \qquad \text{or} \qquad 0 < \varphi(0) < 1.$$

[5] In case $\mathbb{K} = \mathbb{R}$, an analytic function $f : U \to Y$ is sometimes called real analytic.

The function $f := \varphi - \mathrm{id}$ is 1-periodic and hence lies in $C^\infty(\mathbb{T})$. Moreover $f'(x) > -1$ for any x in $\mathbb{R}$. The above normalizations give rise, respectively, to two charts $\mathfrak{U}_1$, $\mathfrak{U}_2$ of $\mathcal{D}$ with $\mathfrak{U}_1 \cup \mathfrak{U}_2 = \mathcal{D}$, defined by

$$\mathfrak{F}_j : \mathfrak{V}_j \to \mathfrak{U}_j, \quad f \mapsto \varphi := \mathrm{id} + f$$

where $j = 1, 2$, and

$$\begin{aligned} \mathfrak{V}_1 &:= \{f \in C^\infty(\mathbb{T}) \mid -1/2 < f(0) < 1/2 \text{ and } f' > -1\} \\ \mathfrak{V}_2 &:= \{f \in C^\infty(\mathbb{T}) \mid 0 < f(0) < 1 \text{ and } f' > -1\}. \end{aligned}$$

As $\mathfrak{V}_1, \mathfrak{V}_2$ are both open subsets in the Fréchet space $C^\infty(\mathbb{T})$, the construction above gives an atlas of Fréchet charts of $\mathcal{D}$. In this way, $\mathcal{D}$ is a Fréchet manifold modeled on $C^\infty(\mathbb{T})$.

Hilbert manifold $\mathcal{D}^s(\mathbb{T})$ ($s \geq 2$): $\mathcal{D}^s = \mathcal{D}^s(\mathbb{T})$ denotes the group of positively oriented bijective transformations of $\mathbb{T}$ of class H^s. By definition, a bijective transformation φ of $\mathbb{T}$ is of class H^s iff the lift $\tilde{\varphi} : \mathbb{R} \to \mathbb{R}$ of φ, determined by the normalization, $0 \leq \tilde{\varphi}(0) < 1$, and its inverse $\tilde{\varphi}^{-1}$ both lie in the Sobolev space $H^s_{loc}(\mathbb{R}, \mathbb{R})$. As for $\mathcal{D}$ one can introduce an atlas for $\mathcal{D}^s$ with two charts in H^s, making $\mathcal{D}^s$ a Hilbert manifold modeled on H^s.

Hilbert approximations: Assume that for a given Fréchet space X over $\mathbb{K}$ there is a sequence of $\mathbb{K}$-Hilbert spaces $(X_n, ||\cdot||_n)_{n\geq 0}$ such that

$$X_0 \supseteq X_1 \supseteq X_2 \supseteq \cdots \supseteq X \quad \text{and} \quad X = \bigcap_{n=0}^{\infty} X_n$$

where $\{||\cdot||_n\}_{n\geq 0}$ is a sequence of norms inducing the topology on X so that $||x||_0 \leq ||x||_1 \leq ||x||_2 \leq \dots$ for any x in X. Such a sequence of Hilbert spaces $(X_n, ||\cdot||_n)_{n\geq 0}$ is called a *Hilbert approximation* of the Fréchet space X. For Fréchet spaces admitting Hilbert approximations one can prove the following version of the inverse function theorem.

Theorem A9. *Let X and Y be Fréchet spaces over $\mathbb{K} = \mathbb{C}$ or $\mathbb{R}$ with Hilbert approximations $(X_n, \|\cdot\|_n)_{n\geq 0}$, and respectively $(Y_n, |\cdot|_n)_{n\geq 0}$. Let $f : V_0 \to U_0$ be a map between the open subsets $V_0 \subseteq X_0$ and $U_0 \subseteq Y_0$ of the Hilbert spaces X_0, respectively Y_0. Define, for any $n \geq 0$,*

$$V_n := V_0 \cap X_n, \qquad U_n := U_0 \cap Y_n.$$

Assume that, for any $n \geq 0$, the following properties are satisfied:

(a) *$f : V_0 \to U_0$ is a bijective C^1-map, and, for any x in $V := V_0 \cap X$, $d_x f : X_0 \to Y_0$ is a linear isomorphism;*

(b) *$f(V_n) \subseteq Y_n$, and the restriction $f\big|_{V_n} : V_n \to Y_n$ is a C^1-map;*

(c) *$f(V_n) \supseteq U_n$;*

(d) *for any x in V, $d_x f(X_n \backslash X_{n+1}) \subseteq Y_n \backslash Y_{n+1}$.*

Then for the open subsets $V := V_0 \cap X \subseteq X$ *and* $U := U_0 \cap Y \subseteq Y$, *one has* $f(V) \subseteq U$ *and the map* $f_\infty := f\big|_V : V \to U$ *is a* C^1_F*-diffeomorphism.*

Proof. By properties (a) and (b), $f_n := f\big|_{V_n} : V_n \to U_n$ is a well-defined, injective C^1-map. By (c), f_n is onto. Hence, $f_\infty := f\big|_V : V \to U$ is bijective. In order to prove that $f_\infty : V \to U$ is a C^1_F-diffeomorphism, consider, for any $n \geq 0$, and any x in V, the differential $d_x f_n : X_n \to Y_n$. As $(d_x f)\big|_{X_n} = d_x f_n$ and, by (a), $d_x f : X_0 \to Y_0$ is bijective, one concludes that $d_x f_n$ is one-to-one. We prove by induction (with respect to n) that, for any x in V, $d_x f_n : X_n \to Y_n$ is onto. For $n = 0$ ($V \subseteq V_0$), the statement is true by property (a). Next, assume that for arbitrary positive integer n, and arbitrary x in V, $d_x f_{n-1} : X_{n-1} \to Y_{n-1}$ is onto. Then, for any x in V, and η in $Y_n \subseteq Y_{n-1}$, there exists a (unique) ξ in X_{n-1} verifying $d_x f_{n-1}(\xi) = \eta$. By property (d), it follows that ξ belongs to X_n. In other words, for any given $n \geq 0$, and any x in V, we have that the map $d_x f_n : X_n \to Y_n$ is bijective, and thus, by Banach's theorem, the inverse $(d_x f_n)^{-1} : Y_n \to X_n$ is a bounded linear operator. As, for any $n \geq 0$, f_n is C^1-smooth, the map

$$V'_n \times X_n \to Y_n,\ (x, \xi) \mapsto d_x f_n(\xi) \tag{40}$$

is continuous and, by the inverse function theorem it follows that

$$U'_n \times Y_n \to X_n,\ (y, \eta) \mapsto d_y(f_n^{-1})(\eta) \tag{41}$$

is continuous as well. Here V'_n (U'_n) denotes the subset V (U) with the topology induced by $|\cdot|_n$ ($\|\cdot\|_n$). As for any x in V, and $n \geq 0$,

$$\delta_x f_\infty = d_x f_n\big|_X$$

one gets from (40) - (41) that

$$V \times X \to Y,\ (x, \xi) \to \delta_x f_\infty(\xi)$$

and

$$U \times Y \to X,\ (x, \eta) \mapsto \delta_y f_\infty^{-1}(\eta)$$

are continuous. In particular, one concludes (cf. Definition A3) that

$$f_\infty : V \to U$$

is a C^1_F-diffeomorphism. ■

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