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Nilpotent Metric Lie Algebras of Small Dimension

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Abstract. In [12] we developed a general classification scheme for metric Lie algebras, i.e. for finite-dimensional Lie algebras equipped with a non-degenerate invariant inner product. Here we determine all nilpotent Lie algebras $\mathfrak{l}$ with $\dim[\mathfrak{l},\mathfrak{l}] = 2$ which are used in this scheme. Furthermore, we use the scheme to classify all nilpotent metric Lie algebras of dimension at most 10.
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1. Introduction

All of what we know of nilpotent Lie algebras suggests that it is hopeless to try and classify them. Even under suitable reasonable conditions the classification of nilpotent Lie algebras is a difficult problem, see [7, 8] for a summary of what is currently known. Here we are mainly interested in real finite-dimensional nilpotent Lie algebras which have a non-degenerate invariant inner product. We call a Lie algebra together with such an inner product a metric Lie algebra. Metric Lie algebras are not only interesting from an algebraic but also from a geometric point of view since they correspond to Lie groups with a bi-invariant pseudo-Riemannian metric, i.e. to special pseudo-Riemannian symmetric spaces.

There are several constructions of metric Lie algebras. The best-known one is the method of double extensions developed independently by Medina and Revoy [13] and Favre and Santharoubane [6]. Each indecomposable non-simple metric Lie algebra can be obtained by such an extension from a lower-dimensional metric Lie algebra. This allows an inductive construction of all metric Lie algebras. In [6] nilpotent metric Lie algebras of dimension at most 7 are classified using this method. Unfortunately, in general a given metric Lie algebra can be obtained in many different ways by this construction. This fact causes trouble if one wants to apply double extensions to classification problems for metric Lie algebras of higher dimension with arbitrary index of the metric.

A second method by Medina and Revoy [14] produces examples of metric Lie algebras from an arbitrary Lie algebra $\mathfrak{l}$, an orthogonal $\mathfrak{l}$-module $\mathfrak{a}$ and a 3-cocycle on $\mathfrak{l}$ with values in the one-dimensional trivial representation. For further results on the structure of metric Lie algebras see e.g. [1, 4, 10].

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In [12] we present a construction which goes back to an idea of Bérard Bergery used in his unpublished work on pseudo-Riemannian symmetric spaces [2, 3]. We call it quadratic extension. It looks similar to the second method by Medina and Revoy, but the data describing the extension contain an additional 2-cocycle. The quadratic extension has the following advantages compared to the previously mentioned constructions. Any metric Lie algebra without simple ideals has the structure of a quadratic extension (of course we are here interested only in metric Lie algebras without simple ideals since semi-simple metric Lie algebras are known). Though also here different extensions can give isomorphic metric Lie algebras, we can use the method for classification since we can always distinguish one of these extensions (up to equivalence). We call this extension the balanced one. Hence any metric Lie algebra can be obtained as a balanced quadratic extension in a unique way. However, balanced quadratic extensions use only semi-simple orthogonal modules. Consequently, compared to the construction in [14] our method has the advantage that it is sufficient to use semi-simple orthogonal modules. In particular, in the case of nilpotent metric Lie algebras we may restrict ourselves to trivial modules.

Similar to ordinary extensions quadratic extensions of Lie algebras are described by a second cohomology, too. But now we have to use a sort of non-linear cohomology. More exactly, there is a one-to-one correspondence between equivalence classes of quadratic extensions of a Lie algebra $\mathfrak{l}$ by an orthogonal module $\mathfrak{a}$ and elements of the second quadratic cohomology set $\mathcal{H}^2_Q(\mathfrak{l},\mathfrak{a})$. Quadratic cohomology sets are a special case of cohomology sets associated by Grishkov [9] to a cochain complex with a cup product taking values in a second complex, see also [12] for a self-contained presentation. In order to classify metric Lie algebras the main problem is to find those cohomology classes that correspond to balanced quadratic extensions. We will call such cohomology classes admissible and denote the subset of admissible cohomology classes in $\mathcal{H}^2_Q(\mathfrak{l},\mathfrak{a})$ by $\mathcal{H}^2_Q(\mathfrak{l},\mathfrak{a})_\sharp$. Then the moduli space of all isomorphism classes of metric Lie algebras can be written as

$$\coprod_{(\mathfrak{l},\mathfrak{a})} \mathcal{H}^2_Q(\mathfrak{l},\mathfrak{a})_\sharp / G_{(\mathfrak{l},\mathfrak{a})}, \tag{1}$$

where the union is taken over all isomorphism classes of pairs $(\mathfrak{l},\mathfrak{a})$ of Lie algebras $\mathfrak{l}$ and semi-simple orthogonal $\mathfrak{l}$-modules $\mathfrak{a}$ and $G_{(\mathfrak{l},\mathfrak{a})}$ is the automorphism group of the pair $(\mathfrak{l},\mathfrak{a})$. See [12] for detailed explanations, in particular for a general characterisation of the subset $\mathcal{H}^2_Q(\mathfrak{l},\mathfrak{a})_\sharp \subset \mathcal{H}^2_Q(\mathfrak{l},\mathfrak{a})$. Note that it is not necessary to consider all Lie algebras $\mathfrak{l}$ in this union, since in most cases $\mathcal{H}^2_Q(\mathfrak{l},\mathfrak{a})_\sharp = \emptyset$.

The description (1) of the moduli space of isomorphism classes of metric Lie algebras allows a systematic approach to the construction and classification of metric Lie algebras. Of course it is far from being an explicit classification (e.g. a list). A full classification would require that we can determine all Lie algebras $\mathfrak{l}$ for which $\mathcal{H}^2_Q(\mathfrak{l},\mathfrak{a})_\sharp$ is not empty for some orthogonal $\mathfrak{l}$-module $\mathfrak{a}$. These Lie algebras are called admissible. Although admissibility is a very strong condition it seems to be hard to give a classification of these Lie algebras. Another difficulty is the explicit computation of the cohomology sets.

However, (1) yields a general classification scheme, i.e. a systematic method which can be used to obtain a full classification for metric Lie algebras satisfying suitable additional assumptions. Such assumptions can be, e.g., restrictions on the index of the inner product or on the structure of the Lie algebra. These restrictions give additional conditions for the Lie algebras $\mathfrak{l}$ occuring in (1). Hence, in order to get a classification from (1) one has first to determine all admissible Lie algebras $\mathfrak{l}$ which satisfy these additional conditions and afterwards one has to determine orbit sets of cohomology classes of these Lie algebras. For example, the classification of metric Lie algebras with index p leads to the classification problem for admissible Lie algebras of dimension $\dim \mathfrak{l} \leq p$. In [11] and [12] we show how one can solve this problem for small p. In particular, we give a classification of all metric Lie algebras whose invariant inner product is of index two or three.

The aim of this paper is to give a further example which shows that (1) is really a useful mean for concrete classification problems concerning metric Lie algebras. We have seen that the classification of admissible Lie algebras $\mathfrak{l}$ is a main step in solving such problems. The first part of the paper deals with the determination of a certain class of admissible Lie algebras. We will classify all nilpotent admissible Lie algebras $\mathfrak{l}$ whose derived algebra $\mathfrak{l}'$ is two-dimensional. In particular, we will see how restrictive the admissibility condition is. We can prove that such Lie algebras are direct sums $\mathfrak{g} \oplus \mathbb{R}^k$, where $\mathfrak{g}$ is nilpotent and admissible of dimension at most 6. The precise classification result is stated in Theorem 3.1. The easier case of solvable non-nilpotent admissible Lie algebras with two-dimensional nilpotent radical was already treated in [12]. We will prove Theorem 3.1 exploiting only the admissibility condition, i.e. without using other classification results on nilpotent Lie algebras.

In the second part we show how to apply the general classification scheme for metric Lie algebras to low-dimensional nilpotent metric Lie algebras. For an upper bound of the dimension d we have chosen $d = 10$. This is high enough to show how the machinery of the classification scheme works where other methods don't work any longer but low enough to avoid overly long calculations. We state the classification result in Theorem 4.7 at the end of this paper. Besides isolated nilpotent metric Lie algebras also 1-parameter families occur.

2. Admissible cohomology classes

In [12] we introduced the quadratic cohomology $\mathcal{H}^2_Q(\mathfrak{l}, \mathfrak{a})$ for a Lie algebra $\mathfrak{l}$ with values in an orthogonal $\mathfrak{l}$-module $\mathfrak{a}$. As mentioned in the introduction this cohomology is a special case of non-linear cohomology sets introduced by Grishkov [9]. Let us recall its definition. An orthogonal $\mathfrak{l}$-module is a triple $(\rho, \mathfrak{a}, \langle\cdot\,,\cdot\rangle_{\mathfrak{a}})$ (also $\mathfrak{a}$ in abbreviated notation) consisting of a finite-dimensional pseudo-Euclidean vector space $\mathfrak{a}$ and a representation ρ of $\mathfrak{l}$ on $\mathfrak{a}$ satisfying $\rho(L) \in \mathfrak{so}(\mathfrak{a}, \langle\cdot\,,\cdot\rangle_{\mathfrak{a}})$ for all $L \in \mathfrak{l}$. For $\mathfrak{l}$ and (any $\mathfrak{l}$-module) $\mathfrak{a}$ we have the standard cochain complex $(C^*(\mathfrak{l}, \mathfrak{a}), d)$ and corresponding cohomology groups $H^p(\mathfrak{l}, \mathfrak{a})$. If $\mathfrak{a}$ is the one-dimensional trivial representation, then we denote this cochain complex also by $C^*(\mathfrak{l})$. We define a product $\langle \cdot \wedge \cdot \rangle : C^p(\mathfrak{l}, \mathfrak{a}) \times C^q(\mathfrak{l}, \mathfrak{a}) \longrightarrow C^{p+q}(\mathfrak{l})$ by the composition

$$\langle \cdot \wedge \cdot \rangle : C^p(\mathfrak{l}, \mathfrak{a}) \times C^q(\mathfrak{l}, \mathfrak{a}) \xrightarrow{\wedge} C^{p+q}(\mathfrak{l}, \mathfrak{a} \otimes \mathfrak{a}) \xrightarrow{\langle\cdot\,,\cdot\rangle_{\mathfrak{a}}} C^{p+q}(\mathfrak{l}).$$

Let p be even. Then the group of quadratic $(p-1)$-cochains is the group

$$\mathcal{C}^{p-1}_Q(\mathfrak{l}, \mathfrak{a}) = C^{p-1}(\mathfrak{l}, \mathfrak{a}) \oplus C^{2p-2}(\mathfrak{l})$$

with group operation defined by

$$(\tau_1, \sigma_1) * (\tau_2, \sigma_2) = (\tau_1 + \tau_2, \sigma_1 + \sigma_2 + \tfrac{1}{2}\langle \tau_1 \wedge \tau_2 \rangle)\,.$$

Now we consider the set

$$\mathcal{Z}^p_Q(\mathfrak{l}, \mathfrak{a}) = \{(\alpha, \gamma) \in C^p(\mathfrak{l}, \mathfrak{a}) \oplus C^{2p-1}(\mathfrak{l}) \mid d\alpha = 0,\ d\gamma = \tfrac{1}{2}\langle \alpha \wedge \alpha \rangle\}$$

of so-called quadratic p-cocycles. The group $\mathcal{C}^{p-1}_Q(\mathfrak{l}, \mathfrak{a})$ acts on $\mathcal{Z}^p_Q(\mathfrak{l}, \mathfrak{a})$ by

$$(\alpha, \gamma)(\tau, \sigma) = (\,\alpha + d\tau, \gamma + d\sigma + \langle (\alpha + \tfrac{1}{2} d\tau) \wedge \tau \rangle\,).$$

and we define the quadratic cohomology set $\mathcal{H}^p_Q(\mathfrak{l}, \mathfrak{a}) := \mathcal{Z}^p_Q(\mathfrak{l}, \mathfrak{a})/\mathcal{C}^{p-1}_Q(\mathfrak{l}, \mathfrak{a})$. We denote the equivalence class of $(\alpha, \gamma) \in \mathcal{Z}^p_Q(\mathfrak{l}, \mathfrak{a})$ in $\mathcal{H}^p_Q(\mathfrak{l}, \mathfrak{a})$ by $[\alpha, \gamma]$.

Let us recall the definition of admissible cohomology classes from [12]. Admissible cohomology classes are certain elements of $\mathcal{H}^2_Q(\mathfrak{l}, \mathfrak{a})$ for a Lie algebra $\mathfrak{l}$ and a semi-simple orthogonal $\mathfrak{l}$-module $\mathfrak{a}$. Here we will give the definition of admissibility only for nilpotent Lie algebras. So we have the two following simplifications compared to [12]: Firstly, if $\mathfrak{l}$ is a nilpotent Lie algebra, then $H^*(\mathfrak{l}, \mathfrak{a}) = H^*(\mathfrak{l}, \mathfrak{a}^{\mathfrak{l}})$ holds for any semi-simple $\mathfrak{l}$-module $\mathfrak{a}$ (see [5]). This implies that $\mathcal{H}^2_Q(\mathfrak{l}, \mathfrak{a}) = \mathcal{H}^2_Q(\mathfrak{l}, \mathfrak{a}^{\mathfrak{l}})$ holds for any orthogonal semi-simple $\mathfrak{l}$-module $\mathfrak{a}$, too. Secondly, if $\mathfrak{l}$ is nilpotent, then its k-th nilpotent radical $R_k(\mathfrak{l})$ equals $\mathfrak{l}^{k+1}$, where $\mathfrak{l}^{k+1}$ is defined by $\mathfrak{l}^1 = \mathfrak{l}, \ldots, \mathfrak{l}^{k+1} = [\mathfrak{l}, \mathfrak{l}^k]$. As usual we often denote $\mathfrak{l}^2$ by $\mathfrak{l}'$.

Definition 2.1. Let $\mathfrak{l}$ be a nilpotent Lie algebra and let $(\rho, \mathfrak{a}, \langle\cdot\,,\cdot\rangle_{\mathfrak{a}})$ be a semi-simple orthogonal $\mathfrak{l}$-module. Let m be such that $\mathfrak{l}^{m+2} = 0$. Put $\mathfrak{l}_{(0)} = \mathfrak{z}(\mathfrak{l}) \cap \ker \rho$ and $\mathfrak{l}_{(k)} = \mathfrak{z}(\mathfrak{l}) \cap \mathfrak{l}^{k+1}$ for $k \geq 1$. Take a cohomology class in $\mathcal{H}^2_Q(\mathfrak{l}, \mathfrak{a})$ and represent it by a cocycle (α, γ) satisfying $\alpha(\mathfrak{l}, \mathfrak{l}) \subset \mathfrak{a}^{\mathfrak{l}}$. Then $[\alpha, \gamma] \in \mathcal{H}^2_Q(\mathfrak{l}, \mathfrak{a})$ is called admissible if and only if the following conditions (A_k) and (B_k) hold for all $0 \leq k \leq m$.

(A_k) Let $L_0 \in \mathfrak{l}_{(k)}$ be such that there exist elements $A_0 \in \mathfrak{a}$ and $Z_0 \in (\mathfrak{l}^{k+1})^*$ satisfying

(i) $\alpha(L, L_0) = 0$,

(ii) $\gamma(L, L_0, \cdot) = -\langle A_0, \alpha(L, \cdot)\rangle_{\mathfrak{a}} + \langle Z_0, [L, \cdot]_{\mathfrak{l}}\rangle$ as an element of $(\mathfrak{l}^{k+1})^*$,

for all $L \in \mathfrak{l}$, then $L_0 = 0$.

(B_k) The subspace $\alpha(\ker[\cdot\,,\cdot]_{\mathfrak{l}\otimes\mathfrak{l}^{k+1}}) \subset \mathfrak{a}$ is non-degenerate, where $\ker[\cdot\,,\cdot]_{\mathfrak{l}\otimes\mathfrak{l}^{k+1}}$ is the kernel of the map $[\cdot\,,\cdot] : \mathfrak{l} \otimes \mathfrak{l}^{k+1} \to \mathfrak{l}$.

This definition is independent of the choice of the cocycle representing $[\alpha, \gamma] \in \mathcal{H}^2_Q(\mathfrak{l}, \mathfrak{a})$. We denote the set of all admissible cohomology classes in $\mathcal{H}^2_Q(\mathfrak{l}, \mathfrak{a})$ by $\mathcal{H}^2_Q(\mathfrak{l}, \mathfrak{a})_\sharp$. A Lie algebra $\mathfrak{l}$ is called admissible if there is a semi-simple orthogonal $\mathfrak{l}$-module $\mathfrak{a}$ such that $\mathcal{H}^2_Q(\mathfrak{l}, \mathfrak{a})_\sharp \neq \emptyset$.

3. Nilpotent admissible Lie algebras with 2-dimensional radical

In this section we determine all nilpotent admissible Lie algebras $\mathfrak{l}$ with $\dim R_1(\mathfrak{l}) = \dim \mathfrak{l}' = 2$. We will often describe a Lie algebra by giving a basis and some of the Lie bracket relations. In this case we always assume that all other brackets of basis vectors vanish. If we do not mention the basis explicitly, then we assume that all basis vectors appear in one of the bracket relations (on the left or the right hand side). Using this convention we define

$$\begin{aligned} \mathfrak{h}(1) &= \{[X_1, X_2] = Y\}, \\ \mathfrak{g}_{4,1} &= \{[X_1, Z] = Y,\ [X_1, X_2] = Z\}, \end{aligned}$$

$$\begin{aligned} \mathfrak{g}_{5,2} &= \{[X_1, X_2] = Y,\ [X_1, X_3] = Z\}, \\ \mathfrak{g}_{6,4} &= \{[X_1, X_2] = Y,\ [X_1, X_3] = Z,\ [X_3, X_4] = Y\}, \\ \mathfrak{g}_{6,5} &= \{[X_1, X_2] = Y,\ [X_1, X_3] = Z,\ [X_2, X_4] = Z,\ [X_3, X_4] = -Y\}. \end{aligned}$$

Theorem 3.1. *If $\mathfrak{l}$ is an admissible nilpotent Lie algebra with $\dim \mathfrak{l}' = 2$, then $\mathfrak{l}$ is isomorphic to one of the (admissible) Lie algebras*

$$\mathfrak{h}(1) \oplus \mathfrak{h}(1) \oplus \mathbb{R}^k,\ \mathfrak{g}_{4,1} \oplus \mathbb{R}^k,\ \mathfrak{g}_{5,2} \oplus \mathbb{R}^k,\ \mathfrak{g}_{6,4} \oplus \mathbb{R}^k,\ \mathfrak{g}_{6,5} \oplus \mathbb{R}^k.$$

Proof. Let us first show that all these Lie algebras are admissible. In [12] we proved that $\mathfrak{h}(1)$ is admissible. In Props. 4.3 and 4.4 we will see that $\mathfrak{g}_{4,1}$ and $\mathfrak{g}_{5,2}$ are also admissible. Let us verify now that $\mathfrak{g}_{6,4}$ is admissible. Take $\mathfrak{a} = \mathbb{R}^{2,2}$ and $\rho = 0$. Let A_1, A_2, A_3, A_4 be a Witt basis of $\mathfrak{a}$, i.e. $\langle A_1, A_3\rangle = \langle A_2, A_4\rangle = 1$ and $\langle A_i, A_j\rangle = 0$ for the remaining pairs $1 \le i \le j \le 4$. We define $\alpha \in C^2(\mathfrak{l}, \mathfrak{a})$ by

$$\begin{gathered} \alpha(X_1, Y) = -\alpha(X_4, Z) = A_1,\ \alpha(X_3, Y) = \alpha(X_2, Z) = A_2,\ \alpha(X_3, Z) = A_3, \\ \alpha(X_1, Z) = A_4,\ \alpha(X_2, Y) = \alpha(X_4, Y) = \alpha(X_i, X_j) = \alpha(Y, Z) = 0. \end{gathered}$$

Then a direct calculation shows that $(\alpha, 0)$ is a cocycle and that $[\alpha, 0]$ is admissible. Similarly one can see that $\mathfrak{l} = \mathfrak{g}_{6,5}$ is admissible. Since $\mathbb{R}^k$ is also admissible and direct sums of admissible Lie algebras are admissible the assertion follows.

Now we prove that each admissible nilpotent Lie algebra $\mathfrak{l}$ with $\dim \mathfrak{l}' = 2$ is isomorphic to one of the mentioned Lie algebras. We distinguish between two cases: $\mathfrak{l}' \not\subset \mathfrak{z}(\mathfrak{l})$ (case I) and $\mathfrak{l}' \subset \mathfrak{z}(\mathfrak{l})$ (case II).

Case I: $\mathfrak{l}' \not\subset \mathfrak{z}(\mathfrak{l})$

In this case we can choose a basis Y, Z of $\mathfrak{l}'$ and vectors X_1, X_2 in $\mathfrak{l} \setminus \mathfrak{l}'$ such that

$$[X_1, Z] = Y,\ [X_1, X_2] = Z,\ [X_2, Z] = 0,\ [\mathfrak{l}, Y] = 0. \tag{2}$$

In particular it is possible to choose a vector space decomposition

$$\mathfrak{l} = \mathrm{span}\{X_1, X_2\} \oplus V \oplus \mathfrak{l}'$$

of $\mathfrak{l}$ such that $[X_1, V] = 0,\ [X_2, V] \subset \mathbb{R} \cdot Y,\ [V, Z] = 0$ and $[V, V] \subset \mathbb{R} \cdot Y$.

Now we distinguish between the cases $[X_2, V] \neq 0$ and $[X_2, V] = 0$.

Case I.1: $[X_2, V] \neq 0$

Claim. A Lie algebra $\mathfrak{l}$ which satisfies the conditions of case I.1 is not admissible.

Proof. By (2) and our choice of V we find a basis $X_3, \ldots, X_l$ of V such that

$$\mathfrak{l} = \{\, [X_1, X_2] = Z,\ [X_1, Z] = Y,\ [X_2, X_3] = Y,\ [X_i, X_j] = y_{ij} Y,\ i, j \geq 3 \,\}$$

for suitable $y_{ij} \in \mathbb{R}$. Assume that $\mathfrak{l}$ is admissible. Then we can choose a semi-simple orthogonal $\mathfrak{l}$-module $\mathfrak{a}$ and $[\alpha, \gamma] \in \mathcal{H}^2_Q(\mathfrak{l}, \mathfrak{a})$ such that $[\alpha, \gamma]$ is admissible. As explained above we may assume $\alpha(\mathfrak{l}, \mathfrak{l}) \subset \mathfrak{a}^{\mathfrak{l}}$. Hence $d\alpha = 0$ implies

$$\alpha(Y, Z) = 0,\ \alpha(Z, X_3) + \alpha(Y, X_1) = 0,\ \alpha(Y, X_j) = 0, \quad j \geq 2\,. \tag{3}$$

Because of $\langle \alpha \wedge \alpha \rangle = 2d\gamma$ we have $\langle \alpha \wedge \alpha \rangle(X_1, X_3, Y, Z) = 2d\gamma(X_1, X_3, Y, Z) = 0$ and by (3) this yields $\langle \alpha(Y, X_1), \alpha(Y, X_1) \rangle = 0$. Summarizing we obtain

$$\alpha(Y, Z) = 0, \quad \langle \alpha(Y, X_1), \alpha(Y, X_1) \rangle = 0, \quad \alpha(Y, X_j) = 0,\ j \geq 2\,. \tag{4}$$

Now let us consider Condition (B_2). Since $\mathfrak{l}^3 = \mathbb{R} \cdot Y \subset \mathfrak{z}(\mathfrak{l})$ it is satisfied if and only if the space $\alpha(\mathfrak{l}, Y)$ is non-degenerate. Now (4) implies that (B_2) holds if and only if $\alpha(Y, \mathfrak{l}) = 0$. But if $\alpha(Y, \mathfrak{l}) = 0$, then Condition (A_2) is not satisfied, a contradiction. ∎

Case I.2: $[X_2, V] = 0$

Claim. An admissible Lie algebra $\mathfrak{l}$ which satisfies the conditions of case I.2 is isomorphic to $\mathfrak{g}_{4,1} \oplus \mathbb{R}^k$.

Proof. By (2) and our choice of V we find a basis $X_3, \ldots, X_l$ of V such that

$$\mathfrak{l} = \{\, [X_1, X_2] = Z,\ [X_1, Z] = Y,\ [X_i, X_j] = y_{ij} Y,\ i, j \geq 3 \,\}\,.$$

for suitable $y_{ij} \in \mathbb{R}$, $i, j \geq 3$. Suppose $[\alpha, \gamma] \in \mathcal{H}^2_Q(\mathfrak{l}, \mathfrak{a})$ is admissible and $\alpha(\mathfrak{l}, \mathfrak{l}) \subset \mathfrak{a}^{\mathfrak{l}}$. The cocycle condition for α yields

$$\alpha(Y, Z) = 0,\ \alpha(Y, X_2) = 0,\ \alpha(Y, X_i) = 0,\ y_{ij}\alpha(Y, X_1) = 0,\ i, j \geq 3.$$

If $y_{ij} \neq 0$ for some $i, j \geq 3$, then $\alpha(Y, \mathfrak{l}) = 0$ holds. But then (A_2) is not satisfied, a contradiction. Consequently, $y_{ij} = 0$ for all $i, j \geq 3$, which proves the claim. ∎

Case II: $\mathfrak{l}' \subset \mathfrak{z}(\mathfrak{l})$

In this case we can choose a 3-dimensional subspace $\bar{\mathfrak{l}}$ of $\mathfrak{l}$ that satisfies $[\,\bar{\mathfrak{l}}, \bar{\mathfrak{l}}\,] = \mathfrak{l}'$. Moreover, we can fix a basis X_1, X_2, X_3 of $\bar{\mathfrak{l}}$ and a basis Y, Z of $\mathfrak{l}'$ satisfying

$$[X_1, X_2] = Y,\ [X_1, X_3] = Z,\ [X_2, X_3] = 0. \tag{5}$$

Lemma 3.2. *If $[\alpha, \gamma] \in \mathcal{H}^2_Q(\mathfrak{l}, \mathfrak{a})$ and $\alpha(\mathfrak{l}, \mathfrak{l}) \subset \mathfrak{a}^{\mathfrak{l}}$, then we have*

(i) $\alpha(Y, Z) = 0$*;*

(ii) $\alpha(Y, L) = 0$ *for all* $L \in \mathfrak{l}$ *satisfying* $[L, X_1] = [L, X_2] = 0$*;*

(iii) $\alpha(Z, L) = 0$ *for all* $L \in \mathfrak{l}$ *satisfying* $[L, X_1] = [L, X_3] = 0$*;*

(iv) $\langle \alpha(U_1, L_1), \alpha(U_2, L_2) \rangle = \langle \alpha(U_1, L_2), \alpha(U_2, L_1) \rangle$ *for all* $U_1, U_2 \in \mathfrak{l}'$, $L_1, L_2 \in \mathfrak{l}$.

Proof. Assertions (i), (ii), (iii) follow from the cocycle condition for α, from $\mathfrak{l}' \subset \mathfrak{z}(\mathfrak{l})$ and from the special conditions on $L \in \mathfrak{l}$ in (ii) and (iii), respectively.

As for assertion (iv) we first observe that

$$d\gamma(U_1, U_2, L_1, L_2) = -\gamma([L_1, L_2], U_1, U_2) = 0, \tag{6}$$

where the first equality follows from $U_1, U_2 \in \mathfrak{z}(\mathfrak{l})$ and the second equality follows from $[L_1, L_2] \in \mathfrak{l}'$ and $\dim \mathfrak{l}' = 2$. Combining (6) with the cocycle condition for (α, γ) we obtain

$$\langle \alpha(U_1, U_2), \alpha(L_1, L_2)\rangle + \langle \alpha(U_2, L_1), \alpha(U_1, L_2)\rangle + \langle \alpha(L_1, U_1), \alpha(U_2, L_2)\rangle = 0.$$

Since *(i)* implies $\alpha(U_1, U_2) = 0$ the first term vanishes and the assertion follows. ■

Lemma 3.3. *Let $[\alpha, \gamma] \in \mathcal{H}^2_Q(\mathfrak{l}, \mathfrak{a})$ be admissible and let α satisfy $\alpha(\mathfrak{l}, \mathfrak{l}) \subset \mathfrak{a}^{\mathfrak{l}}$.*

(i) If $K \in \mathfrak{l}'$ satisfies $\alpha(K, \bar{\mathfrak{l}}) = 0$, then $\alpha(K, \mathfrak{l}) = 0$.

(ii) If $L \in \mathfrak{l}$ satisfies $[L, \bar{\mathfrak{l}}] = 0$, then $\alpha([L, \mathfrak{l}], \mathfrak{l}) = 0$.

Proof. Let X_1, X_2, X_3, Y, Z be as in (5). For $L_1, L_2 \in \mathfrak{l}$ and $K \in \mathfrak{l}'$ we have

$$\begin{aligned}
\langle \alpha(K, L_1), \alpha(Y, L_2)\rangle &= \langle \alpha(K, L_1), \alpha([X_1, X_2], L_2)\rangle \\
&= \langle \alpha(K, L_1), \alpha([L_2, X_2], X_1)\rangle + \langle \alpha(K, L_1), \alpha([X_1, L_2], X_2)\rangle \\
&= \langle \alpha(K, X_1), \alpha([L_2, X_2], L_1)\rangle + \langle \alpha(K, X_2), \alpha([X_1, L_2], L_1)\rangle \\
&= 0,
\end{aligned}$$

where we first used the $d\alpha = 0$ and then Lemma 3.2. Similarly, we have

$$\langle \alpha(K, L_1), \alpha(Z, L_2)\rangle = \langle \alpha(K, L_1), \alpha([X_1, X_3], L_2)\rangle = 0.$$

This implies $\alpha(K, L_1) \perp \alpha(\mathfrak{l}', \mathfrak{l})$. Now (B_1) yields $\alpha(K, L_1) = 0$ for all $L_1 \in \mathfrak{l}$.

Let $L \in \mathfrak{l}$ satisfy $[L, \bar{\mathfrak{l}}] = 0$. By (i) it suffices to prove that $\alpha([L, L'], X) = 0$ holds for all $L' \in \mathfrak{l}$ and $X \in \bar{\mathfrak{l}}$. Since $[X, L] = 0$ the cocycle condition for α gives

$$d\alpha(L, L', X) = -\alpha([L, L'], X) - \alpha([L', X], L) = 0.$$

The assertion now follows since $\alpha([L', X], L) = 0$ by Lemma 3.2 (ii), (iii). ■

Lemma 3.4. *Let $\mathfrak{l}$ be admissible. Then $[L_1, L_2] = 0$ holds for all $L_1, L_2 \in \mathfrak{l}$ satisfying $[L_1, \bar{\mathfrak{l}}] = [L_2, \bar{\mathfrak{l}}] = 0$.*

Proof. We choose a semi-simple orthogonal $\mathfrak{l}$-module $\mathfrak{a}$ such that there is an admissible cohomology class $[\alpha, \gamma] \in \mathcal{H}^2_Q(\mathfrak{l}, \mathfrak{a})$. We may assume $\alpha(\mathfrak{l}, \mathfrak{l}) \subset \mathfrak{a}^{\mathfrak{l}}$. From $d\alpha = 0$ and $[\bar{\mathfrak{l}}, L_1] = [\bar{\mathfrak{l}}, L_2] = 0$ we obtain $\alpha([L_1, L_2], \bar{\mathfrak{l}}) = 0$. Lemma 3.3 (i) now implies

$$\alpha([L_1, L_2], \cdot) = 0. \tag{7}$$

Take X_1, X_2, X_3, Y, Z as in (5). Using $[L_1, L_2] \in \mathfrak{l}' \subset \mathfrak{z}(\mathfrak{l})$ and (7) we see that

$$\begin{aligned} \gamma(Y, [L_1, L_2], L_i) &= -d\gamma([L_1, L_2], X_1, X_2, L_i) = -\tfrac{1}{2}\langle \alpha \wedge \alpha \rangle([L_1, L_2], X_1, X_2, L_i) \\ &= 0 \end{aligned} \tag{8}$$

for $i = 1, 2$ and similarly we obtain

$$\gamma(Z, [L_1, L_2], L_i) = 0 \tag{9}$$

for $i = 1, 2$. Assume now that $[L_1, L_2] \neq 0$. By (8), (9) and $[L_1, L_2] \in \operatorname{span}\{Y, Z\}$ we obtain $\gamma(Y, Z, L_i) = 0$ for $i = 1, 2$. This yields

$$d\gamma(U, L_1, L_2, L) = -\gamma([L_1, L_2], U, L) \tag{10}$$

for all $L \in \mathfrak{l}$ and $U \in \mathfrak{l}'$. On the other hand,

$$d\gamma(U, L_1, L_2, L) = \langle \alpha(U, L), \alpha(L_1, L_2) \rangle. \tag{11}$$

by Lemma 3.2 $(ii), (iii)$. From (10) and (11) we get

$$\gamma(L, [L_1, L_2], \cdot) = \langle \alpha(L_1, L_2), \alpha(L, \cdot) \rangle$$

as an element of $(\mathfrak{l}')^*$. Hence Condition $(A_1)(ii)$ is satisfied for $L_0 = [L_1, L_2]$, $A_0 = -\alpha(L_1, L_2)$, $Z_0 = 0$. Since also $(A_1)(i)$ holds by (7) and $[\alpha, \gamma]$ is admissible we get $[L_1, L_2] = 0$, which is a contradiction. ■

Let X_1, X_2, X_3, Y, Z be as in (5). We extend these vectors to a basis $X_1, X_2, X_3, \ldots, X_l, Y, Z$ of $\mathfrak{l}$ satisfying

$$[X_1, X_2] = Y,\ [X_1, X_3] = Z,\ [X_2, X_3] = 0 \tag{12}$$

$$[X_1, X_4] = 0,\ [X_2, X_4] = \lambda Z,\ \lambda \in \{0, 1\}, \tag{13}$$

$$[X_1, X_j] = [X_2, X_j] = 0,\ j \geq 5. \tag{14}$$

Let j_0 be such that $[X_1, X_j] = [X_2, X_j] = [X_3, X_j] = 0$ for all $j \geq j_0$. Then we have $[X_r, X_s] = 0$ for all $r, s \geq j_0$ by Lemma 3.4.

We define $W := \operatorname{span}\{X_4, \ldots, X_l\}$ and $W' := \operatorname{span}\{X_5, \ldots, X_l\}$.

Case II.1: $\lambda = 0$

Case II.1.1: $[X_3, W] = 0$

Claim. An admissible Lie algebra $\mathfrak{l}$ that satisfies the conditions of case II.1.1 is isomorphic to $\mathfrak{g}_{5,2} \oplus \mathbb{R}^k$.

Proof. In this case we have $[X_1, W] = [X_2, W] = [X_3, W] = 0$ by assumption and $[W, W] = 0$ by Lemma 3.4. ■

Case II.1.2: $\dim[X_3, W] = 1$

Claim. An admissible Lie algebra $\mathfrak{l}$ that satisfies the conditions of case II.1.2 is isomorphic to $\mathfrak{g}_{6,4} \oplus \mathbb{R}^k$ or to $\mathfrak{h}(1) \oplus \mathfrak{h}(1)$.

Proof. We may assume $[X_3, X_4] = cY + dZ \neq 0$, $c, d \in \mathbb{R}$ and $[X_3, X_r] = 0$ for $r \geq 5$. Let us first consider the case $d \neq 0$. We may assume $c = 0$ and $d = 1$ by changing the basis suitably. We will prove that the admissibility of $\mathfrak{l}$ implies $[X_4, X_r] = 0$ for $r \geq 5$. Assume first that there is a vector $X \in \mathrm{span}\{X_5, \ldots, X_l\}$ such that $[X_4, X] = aY + bZ$, $a \neq 0$ and $b \neq 0$. Let $[\alpha, \gamma] \in \mathcal{H}^2_Q(\mathfrak{l}, \mathfrak{a})$ be admissible and choose α such that $\alpha(\mathfrak{l}, \mathfrak{l}) \subset \mathfrak{a}^{\mathfrak{l}}$. Then the cocycle condition for α on $\mathrm{span}\{X_1, \ldots, X_4\}$ yields

$$\alpha(Y, X_3) = \alpha(Z, X_2),\ \alpha(Z, X_4) + \alpha(Z, X_1) = 0,\ \alpha(Y, X_4) = \alpha(Z, X_2) = 0.$$

Moreover, Lemma 3.3 (ii) for $L = X$ yields $\alpha(aY + bZ, \cdot) = 0$. Since $a \neq 0$ and $b \neq 0$, this equation together with the cocycle conditions above implies $\alpha(\mathfrak{l}', X_i) = 0$ for $i \leq 4$. Lemma 3.2 (ii), (iii) now gives $\alpha(\mathfrak{l}', \mathfrak{l}) = 0$. In particular we obtain $d\gamma(\mathfrak{l}', \mathfrak{l}, \mathfrak{l}, \mathfrak{l}) = 0$. Using this condition one easily computes $\gamma(Y, Z, \cdot) = 0$. But then $[\alpha, \gamma]$ does not satisfy Condition (A_1). This contradicts the admissibility of $[\alpha, \gamma]$. Hence $[X_4, X_r] \in \mathbb{R}Z$ for all $r \geq 5$ or $[X_4, X_r] \in \mathbb{R}Y$ for all $r \geq 5$. If we are in the first case and if there is an $s \geq 5$ such that $[X_4, X_s] \neq 0$, then we may assume $[X_4, X_5] = Z$. If we replace our $\bar{\mathfrak{l}}$ by $\mathrm{span}\{X_1, X_2, X_3 + X_5\}$, then this new $\bar{\mathfrak{l}}$, $L_1 = X_4$ and $L_2 = X_5$ satisfy the assumptions of Lemma 3.4. But $[X_4, X_5] = Z \neq 0$ yields a contradiction. Similarly we can exclude the case $[X_4, X_r] \in \mathbb{R}Y$ for all $r \geq 5$ and $[X_4, X_s] \neq 0$ for some $s \geq 5$. We deduce $[X_4, X_r] = 0$ for $r \geq 5$. We conclude that in case $d \neq 0$ the Lie algebra $\mathfrak{l}$ is isomorphic to

$$\{[X_1, X_2] = Y,\ [X_1, X_3] = Z,\ [X_3, X_4] = Z\} \oplus \mathbb{R}^k \cong \mathfrak{h}(1) \oplus \mathfrak{h}(1) \oplus \mathbb{R}^k.$$

Now we consider the case $d = 0$. We may assume $c = 1$. Now we have a basis $X_1, X_2, X_3, \ldots, X_l, Y, Z$ of $\mathfrak{l}$ satisfying (12), (13), (14), $[X_3, X_4] = Y$ and $[X_3, X_r] = 0$ for $r \geq 5$. We will prove that $[X_4, X_r] = 0$ holds for $r \geq 5$. Assume that this is not true. Then we have without loss of generality $[X_4, X_5] = aY + bZ \neq 0$. If $b \neq 0$, then we replace X_3 by $X_3' := X_3 + X_5$. The basis $X_1, X_2, X_3', X_4, \ldots, X_l, Y, Z$ satisfies (12), (13), (14) with $\lambda = 0$, $[X_3', X_4] = (1 - a)Y - bZ$, and $[X_3', X_r] = 0$ for $r \geq 5$. Thus we are in the above case where $d \neq 0$. This implies $[X_4, X_5] = 0$, a contradiction. If $b = 0$, then we may assume $a = 1$. If we replace our $\bar{\mathfrak{l}}$ by $\mathrm{span}\{X_1, X_2, X_3 + X_5\}$, then this new $\bar{\mathfrak{l}}$, $L_1 = X_4$ and $L_2 = X_5$ satisfy the assumptions of Lemma 3.4, but $[X_4, X_5] \neq 0$, a contradiction. Therefore we have $[X_4, X_r] = 0$ for $r \geq 5$, thus $\mathfrak{l} \cong \mathfrak{g}_{6,4} \oplus \mathbb{R}^k$. ∎

Case II.1.3: $\dim[X_3, W] = 2$

Claim. A Lie algebra $\mathfrak{l}$ that satisfies the conditions of case II.1.3 is not admissible.

Proof. Obviously we may assume that $X_1, \ldots, X_l, Y, Z$ is a basis of $\mathfrak{l}$ which satisfies (12), (13), (14) with $\lambda = 0$, $[X_3, X_4] = Y$, $[X_3, X_5] = Z$, and $[X_3, X_r] = 0$ for $r > 5$. Moreover, we have $[X_4, X_5] = yY + zZ$ for suitable $y, z \in \mathbb{R}$. Let $[\alpha, \gamma] \in \mathcal{H}^2_Q(\mathfrak{l}, \mathfrak{a})$ be admissible and suppose that $\alpha(\mathfrak{l}, \mathfrak{l}) \subset \mathfrak{a}^{\mathfrak{l}}$. Then we get

$$\alpha(Y, X_2) = \alpha(Y, X_3) = \alpha(Z, X_2) = 0, \tag{15}$$

$$\alpha(Z, X_5) + \alpha(Z, X_1) = 0,\ \alpha(Z, X_4) + \alpha(Y, X_1) = 0, \tag{16}$$

$$\alpha(X_1, yY + zZ) = 0, \tag{17}$$

$$\alpha(Y, X_5) + \alpha(yY + zZ, X_3) - \alpha(Z, X_4) = 0 \tag{18}$$

using $d\alpha = 0$ on $\mathrm{span}\{X_1, \ldots, X_5\}$. Moreover, by Lemma 3.2 (ii) we have $\alpha(Y, X_j) = 0$ for $j \geq 4$. Together with (15) this yields $\alpha(Y, X_j) = 0$ for $j \geq 2$. Lemma 3.2 *(iv)* gives

$$\langle \alpha(Y, X_1), \alpha(Z, X_j) \rangle \;=\; \langle \alpha(Y, X_j), \alpha(Z, X_1) \rangle \;=\; 0$$

for $j \geq 2$. Together with (16) this implies $\alpha(Y, X_1) \perp \alpha(\mathfrak{l}', \mathfrak{l})$. Now we use that the admissibility condition (B_1) implies that $\alpha(\mathfrak{l}, \mathfrak{l}')$ is non-degenerate. Hence $\alpha(Y, X_1) = 0$. Consequently, $\alpha(Y, \mathfrak{l}) = 0$. Now we get $\alpha(Z, X_4) = 0$ from (16). Assume that $z \neq 0$. Then (17) would give $\alpha(Z, X_1) = 0$. Thus $\alpha(Z, X_5) = 0$ by (16). Finally, (18) would yield $\alpha(Z, X_3) = 0$. Since, moreover, $\alpha(Z, X_k) = 0$ for $k \geq 6$ by Lemma 3.2 *(ii)* we would obtain also $\alpha(Z, \mathfrak{l}) = 0$. From $2d\gamma(\mathfrak{l}', \mathfrak{l}, \mathfrak{l}, \mathfrak{l}) = \langle \alpha \wedge \alpha \rangle(\mathfrak{l}', \mathfrak{l}, \mathfrak{l}, \mathfrak{l}) = 0$ one now computes $\gamma(Y, Z, \mathfrak{l}) = 0$. This together with $\alpha(\mathfrak{l}', \mathfrak{l}) = 0$ gives a contradiction to admissibility. Hence $z = 0$ and $[X_4, X_5] = yY$. However, now we can apply Lemma 3.4 to $\bar{\mathfrak{l}} := \mathrm{span}\{X_3, X_4 + yX_2, X_5\}$, $L_1 = X_1 + X_5$, $L_2 = X_2$ and we obtain a contradiction to $[X_1 + X_5, X_2] = Y \neq 0$. ∎

Case II.2: $\lambda = 1$

Claim. An admissible Lie algebra $\mathfrak{l}$ which satisfies the conditions of case II.2 and which does not have a basis satisfying already the conditions of case II.1 is isomorphic to $\mathfrak{g}_{6,5} \oplus \mathbb{R}^k$.

Now $X_1, \ldots, X_l, Y, Z$ is a basis of $\mathfrak{l}$ that satisfies (12), (13), and (14) with $\lambda = 1$.

Lemma 3.5. *If* $[\alpha, \gamma] \in \mathcal{H}^2_Q(\mathfrak{l}, \mathfrak{a})_\sharp$ *and* $\alpha(\mathfrak{l}, \mathfrak{l}) \subset \mathfrak{a}^{\mathfrak{l}}$*, then* $\alpha([X_3, W'], \cdot) = 0$.

Proof. By Lemma 3.3 (i) it suffices to prove $\alpha([X_3, X_j], X_i) = 0$ for all $j \geq 5$ and $i = 1, 2, 3$. For $i = 2$ this follows obviously from the cocycle condition for α.

Let us consider $i = 3$. Using $[X_1, X_2] = Y$, the cocycle condition for α and Lemma 3.2 (iv) we see that

$$\begin{aligned} \langle \alpha([X_3, X_j], X_3), \alpha(Y, L) \rangle &= \alpha([X_3, X_j], L), \alpha(Y, X_3) \rangle \\ &= \langle \alpha([X_3, X_j], L), \alpha([X_1, X_2], X_3) \rangle \\ &= \langle \alpha([X_3, X_j], L), \alpha([X_1, X_3], X_2) \rangle \\ &= \langle \alpha([X_3, X_j], X_2), \alpha([X_1, X_3], L) \rangle \;=\; 0, \end{aligned}$$

where the last equality follows from the above considerations for $i = 2$. Similarly (using now $[X_2, X_4] = Z$) we obtain

$$\langle \alpha([X_3, X_j], X_3), \alpha(Z, L) \rangle = 0.$$

Since (B_1) implies that $\alpha(\mathfrak{l}, \mathfrak{l}')$ is non-degenerate we conclude $\alpha([X_3, X_j], X_3) = 0$.

Finally we consider the case $i = 1$. Note first that $d\alpha = 0$ implies

$$\alpha([X_3, X_j], X_1) = -\alpha(Z, X_j)\,. \tag{19}$$

Using now Lemma 3.2, (ii) and (iv) we obtain

$$\langle \alpha([X_3, X_j], X_1), \alpha(Y, L) \rangle = -\langle \alpha(Z, X_j), \alpha(Y, L) \rangle = -\langle \alpha(Z, L), \alpha(Y, X_j) \rangle = 0$$

for all $L \in \mathfrak{l}$. Consequently, $\alpha([X_3, X_j], X_1) \perp \alpha(Y, \mathfrak{l})$. Now we will prove that also $\alpha([X_3, X_j], X_1) \perp \alpha(Z, \mathfrak{l})$ and thus $\alpha([X_3, X_j], X_1) \perp \alpha(\mathfrak{l}', \mathfrak{l})$ holds. By (B_1) this will give $\alpha([X_3, X_j], X_1) = 0$. We observe that

$$\begin{aligned} \langle \alpha([X_3, X_j], X_1), \alpha(Z, L)\rangle = -\langle \alpha(Z, X_j), \alpha(Z, L)\rangle &= -\langle \alpha(Z, X_j), \alpha([X_2, X_4], L)\rangle \\ &= -\langle \alpha(Z, X_j), \alpha([X_2, L], X_4)\rangle - \langle \alpha(Z, X_j), \alpha([L, X_4], X_2)\rangle, \end{aligned} \tag{20}$$

where we used (19) and $d\alpha = 0$. By Equation (19) and Lemma 3.2 (iv) we have

$$\begin{aligned} -\langle \alpha(Z, X_j), \alpha([L, X_4], X_2)\rangle &= \langle \alpha([X_3, X_j], X_1), \alpha([L, X_4], X_2)\rangle \\ &= \langle \alpha([X_3, X_j], X_2), \alpha([L, X_4], X_1)\rangle = 0 \end{aligned}$$

since $\alpha([X_3, X_j], X_2) = 0$. Hence the last term in (20) vanishes and we get

$$\langle \alpha([X_3, X_j], X_1), \alpha(Z, L)\rangle = -\langle \alpha(Z, X_j), \alpha([X_2, L], X_4)\rangle = c\langle \alpha(Z, X_j), \alpha(Z, X_4)\rangle$$

for some real number $c \in \mathbb{R}$ since $[X_2, L] \in \operatorname{span}\{Y, Z\}$ and since

$$\langle \alpha(Z, X_j), \alpha(Y, X_4)\rangle = \langle \alpha(Z, X_4), \alpha(Y, X_j)\rangle = 0$$

by Lemma 3.2 (ii) and (iv). Furthermore, we have

$$\langle \alpha(Z, X_j), \alpha(Z, X_4)\rangle = \langle \alpha(Z, X_j), \alpha([X_1, X_3], X_4)\rangle = -\langle \alpha(Z, X_j), \alpha([X_3, X_4], X_1)\rangle\,.$$

Since we already know that $\alpha(Z, X_j) \perp \alpha(Y, X_1)$ the last equation implies that in order to prove $\alpha([X_3, X_j], X_1) \perp \alpha(Z, L)$ it suffices to show $\alpha(Z, X_j) \perp \alpha(Z, X_1)$. However, this follows from Lemma 3.2, (ii) and (iv):

$$\begin{aligned} \langle \alpha(Z, X_j), \alpha(Z, X_1)\rangle &= \langle \alpha(Z, X_j), \alpha([X_2, X_4], X_1)\rangle \\ &= -\langle \alpha(Z, X_j), \alpha([X_1, X_2], X_4)\rangle = -\langle \alpha(Z, X_j), \alpha(Y, X_4)\rangle \\ &= -\langle \alpha(Y, X_j), \alpha(Z, X_4)\rangle = 0\,. \end{aligned}$$

■

Lemma 3.6. *If $\mathfrak{l}$ is admissible, then $[X_3, W'] = 0$ and $[X_4, W'] = 0$.*

Proof. Recall that $X_1, \ldots, X_l, Y, Z$ is a basis of $\mathfrak{l}$ which satisfies (12), (13), (14) with $\lambda = 1$. Assume $[X_3, W'] \neq 0$. Then we may assume that

$$[X_3, X_5] = uY + vZ \neq 0\,. \tag{21}$$

holds. Take $[\alpha, \gamma] \in \mathcal{H}^2_Q(\mathfrak{l}, \mathfrak{a})_\sharp$ such that $\alpha(\mathfrak{l}, \mathfrak{l}) \subset \mathfrak{a}^{\mathfrak{l}}$. From Lemma 3.5 we get

$$\alpha(uY + vZ, \cdot) = 0\,, \tag{22}$$

which implies $d\gamma(uY + vZ, X_1, X_2, X_k) = 0$ for all k and therefore

$$\gamma(Y, uY + vZ, X_3) - \gamma(Z, uY + vZ, X_2) = 0 \tag{23}$$

$$\gamma(Y, uY + vZ, X_4) + \gamma(Z, uY + vZ, X_1) = 0 \tag{24}$$

$$\gamma(Y, uY + vZ, X_k) = 0,\ k \geq 5\,. \tag{25}$$

Furthermore, $d\alpha(X_1,X_2,X_3)=d\alpha(X_1,X_2,X_4)=0$ yields

$$\alpha(Y,X_3)=\alpha(Z,X_2),\ \alpha(Y,X_4)+\alpha(Z,X_1)=0\,. \tag{26}$$

Let us first consider the case $v\neq 0$ in (21). Then we may assume $u=0$ and $v=1$ in (21), i.e. $[X_3,X_5]=Z$. Now (22) says $\alpha(Z,\cdot)=0$. Hence $\alpha(Y,X_j)=0$ for $j\geq 3$ by Lemma 3.2 (*ii*) and (26). Equations (23), (24), and (25) imply $\gamma(Y,Z,X_j)=0$ for $j\geq 3$. Using all this we obtain

$$\gamma(Y,Z,X_i)=d\gamma(Y,X_i,X_3,X_5)=\langle\alpha(Y,X_i),\alpha(X_3,X_5)\rangle,\quad i=1,2.$$

In particular, $L_0=Z$, $A_0=-\alpha(X_3,X_5)$, and $Z_0=0$ satisfy the conditions (*i*) and (*ii*) of (A_1). Hence $Z=0$ by admissibility, which is a contradiction.

If $v=0$, then we may assume $u=1$, i.e. $[X_3,X_5]=Y$. Then (22) implies $\alpha(Y,\cdot)=0$ and (26) now gives $\alpha(Z,X_1)=\alpha(Z,X_2)=0$. Hence $\alpha(X_1,\mathfrak{l}')=\alpha(X_2,\mathfrak{l}')=0$ and therefore $0=d\alpha(X_2,X_4,X_3)=d\alpha(X_1,X_3,X_k)$ implies $\alpha(Z,X_k)=0$ for $k\geq 3$. This implies $\alpha(\mathfrak{l}',\mathfrak{l})=0$. From (23) and (24) we obtain $\gamma(Y,Z,X_j)=0$ for $j=1,2$. Using this we get

$$2\gamma(Y,Z,X_j)=2d\gamma(Z,X_1,X_2,X_j)=\langle\alpha\wedge\alpha\rangle(Z,X_1,X_2,X_j)=0$$

for all $j\geq 3$. Hence $\gamma(\mathfrak{l}',\mathfrak{l}',\mathfrak{l})=0$. Again this contradicts (A_1).

Let us now show $[X_4,W']=0$. Note that

$$X_1':=X_2,\ X_2':=X_1,\ X_3':=X_4,\ X_4':=X_3,\ X_j':=X_j,\ j\geq 5,\ Y':=-Y,\ Z':=Z$$

is also a basis satisfying (12), (13), (14) with $\lambda=1$. Hence $[X_4,X_j]=[X_3',X_j']=0$ for $j\geq 5$ by the first part of the lemma. ∎

Proof of the Claim. We know from Equations (12), (13), (14) and Lemma 3.6 that $\mathfrak{l}$ is isomorphic to $\mathfrak{l}_1\oplus\mathbb{R}^k$, where

$$\mathfrak{l}_1=\{[X_1,X_2]=Y,\ [X_1,X_3]=Z,\ [X_2,X_4]=Z,\ [X_3,X_4]=aY+bZ\}$$

for suitable $a,b\in\mathbb{R}$. It is not hard to verify that either $\mathfrak{l}_1$ is isomorphic to $\mathfrak{g}_{6,5}$ or it already satisfies the conditions of case II.1. ∎

This finishes the proof of Theorem 3.1. ∎

4. Nilpotent metric Lie algebras of dimension ≤ 10

In this section we will determine the isomorphism classes of indecomposable nilpotent metric Lie algebras of dimension ≤ 10. Recall that a metric Lie algebra is called indecomposable if it is not the direct sum of two non-trivial metric Lie algebras.

Let us first consider the following construction. We start with a nilpotent Lie algebra $\mathfrak{l}$, a pseudo-Euclidean vector space $(\mathfrak{a},\langle\cdot\,,\cdot\rangle_{\mathfrak{a}})$ which we consider as a trivial orthogonal $\mathfrak{l}$-module and a cocycle $(\alpha,\gamma)\in\mathcal{Z}^2_Q(\mathfrak{l},\mathfrak{a})$. Let $\mathfrak{d}$ be the vector space $\mathfrak{l}^*\oplus\mathfrak{a}\oplus\mathfrak{l}$. We define a Lie bracket $[\cdot\,,\cdot]$ on $\mathfrak{d}$ by $[\mathfrak{l}^*\oplus\mathfrak{a},\mathfrak{l}^*\oplus\mathfrak{a}]=0$ and

$$\begin{aligned}[L,Z]&=\mathrm{ad}^*(L)(Z)\\ [A,L]&=\langle A,\alpha(L,\cdot)\rangle\\ [L_1,L_2]&=\gamma(L_1,L_2,\cdot)+\alpha(L_1,L_2)+[L_1L_2]_{\mathfrak{l}}\end{aligned}$$

for all $L, L_1, L_2 \in \mathfrak{l}$, $A \in \mathfrak{a}$, and $Z \in \mathfrak{l}^*$ and an inner product $\langle \cdot\,,\cdot \rangle$ on $\mathfrak{d}$ by

$$\langle Z_1 + A_1 + L_1, Z_2 + A_2 + L_2 \rangle \;:=\; \langle A_1, A_2 \rangle_{\mathfrak{a}} + Z_1(L_2) + Z_2(L_1)$$

for Z_1, $Z_2 \in \mathfrak{l}^*$, A_1, $A_2 \in \mathfrak{a}$ and L_1, $L_2 \in \mathfrak{l}$. Then it is not hard to prove that $\mathfrak{d}_{\alpha,\gamma}(\mathfrak{l},\mathfrak{a}) := (\mathfrak{d}, [\cdot\,,\cdot], \langle \cdot\,,\cdot \rangle)$ is a nilpotent metric Lie algebra (see also [12] for the case of a general metric Lie algebra).

Let $\mathfrak{l}_i$, $i = 1, 2$ be Lie algebras and let $\mathfrak{a}_i$, $i = 1, 2$ be pseudo-Euclidean vector spaces which we consider as trivial orthogonal $\mathfrak{l}_i$-modules. Consider a pair (S, U) consisting of a homomorphism $S : \mathfrak{l}_1 \to \mathfrak{l}_2$ and an isometry $U : \mathfrak{a}_2 \to \mathfrak{a}_1$. Then $(S, U)^* : C^p(\mathfrak{l}_2, \mathfrak{a}_2) \to C^p(\mathfrak{l}_1, \mathfrak{a}_1)$ induces a map $(S, U)^* : \mathcal{H}^p_Q(\mathfrak{l}_2, \mathfrak{a}_2) \to \mathcal{H}^p_Q(\mathfrak{l}_1, \mathfrak{a}_1)$.

In particular, $G_{(\mathfrak{l},\mathfrak{a})} := \mathrm{Aut}(\mathfrak{l}) \times O(\mathfrak{a}, \langle \cdot\,,\cdot \rangle_{\mathfrak{a}})$ acts on $\mathcal{H}^2_Q(\mathfrak{l},\mathfrak{a})$.

Definition 4.1. Let $\mathfrak{l}$ be a nilpotent Lie algebra and let $(\mathfrak{a}, \langle \cdot\,,\cdot \rangle_{\mathfrak{a}})$ be a pseudo-Euclidean vector space considered as a trivial $\mathfrak{l}$-module. A cohomology class $\varphi \in \mathcal{H}^2_Q(\mathfrak{l},\mathfrak{a})$ is called decomposable if there are decompositions $\mathfrak{a} = \mathfrak{a}_1 \oplus \mathfrak{a}_2$ and $\mathfrak{l} = \mathfrak{l}_1 \oplus \mathfrak{l}_2$, at least one of them being non-trivial and cohomology classes $\varphi_i \in \mathcal{H}^2_Q(\mathfrak{l}_i, \mathfrak{a}_i)$, $i = 1, 2$ such that $\varphi = (q_1, j_1)^*\varphi_1 + (q_2, j_2)^*\varphi_2$, where $q_i : \mathfrak{l} \to \mathfrak{l}_i$ are the projections and $j_i : \mathfrak{a}_i \to \mathfrak{a}$ are the inclusions. Here we consider $\mathfrak{a}_i$ as trivial $\mathfrak{l}_i$-modules. We denote the subset of indecomposable admissible cohomology classes in $\mathcal{H}^2_Q(\mathfrak{l},\mathfrak{a})$ by $\mathcal{H}^2_Q(\mathfrak{l},\mathfrak{a})_0$.

One can check easily that $\mathcal{H}^2_Q(\mathfrak{l},\mathfrak{a})_0$ is invariant with respect to the action of $G_{(\mathfrak{l},\mathfrak{a})}$ on $\mathcal{H}^2_Q(\mathfrak{l},\mathfrak{a})$. The classification scheme (1) now gives

Proposition 4.2. *The set of isomorphism classes of indecomposable nilpotent metric Lie algebras of dimension at most 10 is in bijective correspondence with*

$$\bigcup_{\mathfrak{l}\in\mathfrak{L}} \bigcup_{\mathfrak{a}\in\mathfrak{A}_{\mathfrak{l}}} \mathcal{H}^2_Q(\mathfrak{l},\mathfrak{a})_0 / G_{(\mathfrak{l},\mathfrak{a})},$$

where $\mathfrak{L}$ is the set of isomorphism classes of nilpotent Lie algebras of dimension at most 5 and for a fixed $\mathfrak{l} \in \mathfrak{L}$ the set $\mathfrak{A}_{\mathfrak{l}}$ consists of all isometry classes of pseudo-Euclidean vector spaces of dimension at most $10 - 2\dim\mathfrak{l}$ which we consider as equivalence classes of trivial orthogonal $\mathfrak{l}$-modules.

As already explained it is not necessary to determine the whole set $\mathfrak{L}$. We are interested only in those $\mathfrak{l} \in \mathfrak{L}$ for which $\mathcal{H}^2_Q(\mathfrak{l},\mathfrak{a})_0 \neq \emptyset$ for some $\mathfrak{a} \in \mathfrak{A}_{\mathfrak{l}}$. If $\dim\mathfrak{l} \leq 4$ this is only possible for $\mathfrak{l} \in \{\mathfrak{g}_{4,1},\ \mathfrak{h}(1) \oplus \mathbb{R},\ \mathfrak{h}(1),\ \mathbb{R}^k, k = 1, \ldots, 4\}$ by Theorem 3.1 and [12], Prop. 5.2. Furthermore, if $\mathfrak{l}$ is nilpotent and if $\dim\mathfrak{l} = 5$ and $\mathfrak{a} = 0$, then $\mathcal{H}^2_Q(\mathfrak{l},\mathfrak{a})_0 \neq \emptyset$ implies $\mathfrak{l} = \mathbb{R}^5$ or $\mathfrak{l} = \mathfrak{g}_{5,2}$. Indeed, let $[0, \gamma] \in \mathcal{H}^2_Q(\mathfrak{l},\mathfrak{a})_0$ be such that $\gamma \neq 0$. Then we know from (A_k) that $\dim\mathfrak{l}^{k+1} \neq 1$ holds for all $k \geq 0$. Since $\mathfrak{l}$ is nilpotent the codimension of $\mathfrak{l}^2$ in $\mathfrak{l}$ cannot be one. Hence we have only the following possibilities:

(i) $\dim\mathfrak{l}^2 = 0$,

(ii) $\dim\mathfrak{l}^2 = 2, \dim\mathfrak{l}^3 = 0$,

(iii) $\dim\mathfrak{l}^2 = 3, \dim\mathfrak{l}^3 = 0$, or

(iv) $\dim\mathfrak{l}^2 = 3, \dim\mathfrak{l}^3 = 2, \dim\mathfrak{l}^4 = 0$.

If (i) holds, then $\mathfrak{l} \cong \mathbb{R}^5$. If (ii) holds, then $\mathfrak{l} \cong \mathfrak{g}_{5,2}$ by Proposition 3.1. The conditions in (iii) cannot be satisfied for a 5-dimensional Lie algebra $\mathfrak{l}$. In case (iv) the cocycle condition $d\gamma = 0$ implies $\gamma(\mathfrak{l}^3, \mathfrak{l}^3, \mathfrak{l})=0$, which contradicts Condition (A_2).

In the following we will often abbreviate $G_{(\mathfrak{l},\mathfrak{a})}$ to G. Furthermore, we will use the following conventions. An orthonormal basis of a pseudo-Euclidean vector space $(\mathfrak{a}, \langle\cdot\,,\cdot\rangle_{\mathfrak{a}})$ is a basis $A_1, \ldots, A_{p+q}$ consisting of pairwise orthogonal vectors satisfying $\langle A_i, A_i\rangle_{\mathfrak{a}} = -1$ for $1 \le i \le p$ and $\langle A_i, A_i\rangle_{\mathfrak{a}} = 1$ for $p+1 \le i \le p+q$. The pair (p, q) is called signature of $\mathfrak{a}$. We denote the standard pseudo-Euclidean vector space of signature (p, q) by $\mathbb{R}^{p,q}$. A Witt basis of $\mathbb{R}^{1,1}$ is a basis A_1, A_2, where A_1, A_2 are isotropic and $\langle A_1, A_2\rangle = 1$.

Proposition 4.3. *1. If $\mathfrak{l} = \mathbb{R}^5$ and $\mathfrak{a} = 0$, then $\mathcal{H}^2_Q(\mathfrak{l},\mathfrak{a})_0/G$ consists of one element. This element is represented by $[0,\gamma_0] \in \mathcal{H}^2_Q(\mathfrak{l},\mathfrak{a})_0$, where $\gamma_0 = (\sigma^1\wedge\sigma^2 + \sigma^3\wedge\sigma^4)\wedge\sigma^5$ for a fixed basis $\sigma^1, \ldots, \sigma^5$ of $\mathfrak{l}^*$.*

2. If $\mathfrak{l} = \mathfrak{g}_{5,2} = \{[X_1, X_2] = Y,\ [X_1, X_3] = Z\}$ and $\mathfrak{a} = 0$, then $\mathcal{H}^2_Q(\mathfrak{l},\mathfrak{a})_0/G$ consists of two elements. These elements are represented by $[0,\gamma_1], [0,\gamma_2] \in \mathcal{H}^2_Q(\mathfrak{l},\mathfrak{a})_0$, where $\gamma_1 = \sigma^1\wedge\sigma^Y\wedge\sigma^Z$ and $\gamma_2 = \sigma^1\wedge\sigma^Y\wedge\sigma^Z + \sigma^2\wedge\sigma^3\wedge\sigma^Z$ for the basis $\sigma^1,\sigma^2,\sigma^3,\sigma^Y,\sigma^Z$ of $\mathfrak{l}^*$ which is dual to X_1, X_2, X_3, Y, Z.

Proof. The statement for $\mathfrak{l} = \mathbb{R}^5$ is easy to prove. Take $\mathfrak{l} = \mathfrak{g}_{5,2}$. Let us first describe the automorphism group of $\mathfrak{g}_{5,2}$. For $c,\ A \in GL(2,\mathbb{R})$, $y = (y_1, y_2)$, $y_1, y_2 \in \mathbb{R}^2$, $x \in \mathfrak{gl}(2,\mathbb{R})$ we define a linear map $S(c, A, x, y) : \mathfrak{g}_{5,2} \to \mathfrak{g}_{5,2}$ by

$$S(c,A,x,y) = \begin{pmatrix} c & 0 & 0 \\ y_1 & A & 0 \\ y_2 & x & cA \end{pmatrix}$$

with respect to the basis X_1, X_2, X_3, Y, Z of $\mathfrak{g}_{5,2}$. Then one proves

$$\mathrm{Aut}(\mathfrak{g}_{5,2}) = \{S(c,A,x,y) \mid c \in \mathbb{R}\setminus 0, A \in GL(2,\mathbb{R}), y_1, y_2 \in \mathbb{R}^2, x \in \mathfrak{gl}(2,\mathbb{R})\}\,.$$

Obviously, we have $\mathcal{H}^2_Q(\mathfrak{l}, 0) = H^3(\mathfrak{g}_{5,2})$. Observe that $\mathfrak{q} := \mathrm{span}\{X_2, X_3, Y, Z\} \subset \mathfrak{g}_{5,2}$ is an abelian ideal. Using the Hochschild-Serre spectral sequence we see that $H^3(\mathfrak{g}_{5,2})$ is determined by the exact sequence

$$0 \longrightarrow H^1(\mathbb{R}\cdot X_1, C^2(\mathfrak{q})) \longrightarrow H^3(\mathfrak{g}_{5,2}) \longrightarrow H^0(\mathbb{R}\cdot X_1, C^3(\mathfrak{q})) \longrightarrow 0.$$

We have

$$H^1(\mathbb{R}\cdot X_1, C^2(\mathfrak{q})) = C^1(\mathbb{R}\cdot X_1, C^2(\mathfrak{q}))/B^1(\mathbb{R}\cdot X_1, C^2(\mathfrak{q})),$$

where

$$B^1(\mathbb{R}\cdot X_1, C^2(\mathfrak{q})) = \left\{ \sigma \in C^1(\mathbb{R}\cdot X_1, C^2(\mathfrak{q})) \;\middle|\; \begin{array}{l} \sigma(X_1)(X_2,Z) + \sigma(X_1)(X_3,Y) = 0 \\ \sigma(X_1)(X_2,Y) = \sigma(X_1)(X_3,Z) = 0 \\ \sigma(X_1)(Y,Z) = 0 \end{array} \right\},$$

and

$$H^0(\mathbb{R}\cdot X_1, C^3(\mathfrak{q})) = \{\sigma \in C^3(\mathfrak{q}) \mid \sigma(X_2,Y,Z) = \sigma(X_3,Y,Z) = 0\}\,. \qquad (27)$$

Observe that $\mathcal{H}^2_Q(\mathfrak{l},0)_0 = \mathcal{H}^2_Q(\mathfrak{l},0)_\sharp$ since $\mathfrak{l}$ is not the direct sum of two non-trivial Lie algebras. In particular, Condition (A_1) and Equation (27) imply

$$\mathcal{H}^2_Q(\mathfrak{l},0)_0 = \{[\gamma] \in H^3(\mathfrak{g}_{5,2}) \mid \gamma(X_1,Y,Z) \neq 0\}.$$

Using the description of $H^1(\mathbb{R}\cdot X_1, C^2(\mathfrak{q}))$ given above we see that

$$\{S(c,\mathrm{Id},x,0) \mid c \in \mathbb{R}\setminus 0, x \in \mathfrak{gl}(2,\mathbb{R})\} \subset \mathrm{Aut}(\mathfrak{g}_{5,2})$$

acts transitively on $\{[\sigma] \in H^1(\mathbb{R}\cdot X_1, C^2(\mathfrak{q})) \mid \sigma(X_1)(Y,Z) \neq 0\}$. Furthermore, using the description of $H^0(\mathbb{R}\cdot X_1, C^3(\mathfrak{q}))$ we see that the action of

$$\{S(1,A,0,0) \mid \det A = 1\} \subset \mathrm{Aut}(\mathfrak{g}_{5,2})$$

on $H^0(\mathbb{R}\cdot X_1, C^3(\mathfrak{q}))$ has two orbits represented by $\sigma_1 = 0$ and $\sigma_2 = \sigma^2\wedge\sigma^3\wedge\sigma^Z$. Moreover, this group leaves $\sigma^1\wedge\sigma^Y\wedge\sigma^Z$ invariant.

It is easy to check that the orbits of $[0,\gamma_1]$ and $[0,\gamma_2]$ are different. ∎

Proposition 4.4. *Take $\mathfrak{l} = \mathfrak{g}_{4,1} = \{[X_1,Z]=Y,\ [X_1,X_2]=Z\}$ and let $\mathfrak{a}$ be a trivial $\mathfrak{l}$-module. If $\mathfrak{a}=0$ or $\dim\mathfrak{a}\geq 3$, then $\mathcal{H}^2_Q(\mathfrak{l},\mathfrak{a})_0 = \emptyset$. If $\dim\mathfrak{a} = 1$, then $\mathcal{H}^2_Q(\mathfrak{l},\mathfrak{a})_0/G$ consists of four elements. They are represented by*

$$[\alpha,\gamma] = [\sigma^1\wedge\sigma^Y\otimes A, r\sigma^2\wedge\sigma^Y\wedge\sigma^Z + s\sigma^1\wedge\sigma^Y\wedge\sigma^Z],$$

where $(r,s)\in\{(0,0),\,(1,0),\,(0,1),\,(0,-1)\}$ and A is a fixed unit vector in $\mathfrak{a}$.

If $\mathfrak{a}\in\{\mathbb{R}^2,\mathbb{R}^{2,0}\}$, then $\mathcal{H}^2_Q(\mathfrak{l},\mathfrak{a})_0/G$ consists of two one-parameter families. They are represented by

$$[\alpha,\gamma] = [\sigma^1\wedge\sigma^Y\otimes A_1 + \sigma^2\wedge\sigma^Z\otimes A_2, s\sigma^1\wedge\sigma^Y\wedge\sigma^Z],\quad s\in\mathbb{R} \tag{28}$$

and

$$[\alpha,\gamma] = [\sigma^1\wedge\sigma^Y\otimes A_1 + \sigma^2\wedge\sigma^Z\otimes A_2, r\sigma^2\wedge\sigma^Y\wedge\sigma^Z],\quad r\in\mathbb{R}_+, \tag{29}$$

where A_1, A_2 is a fixed orthonormal basis in $\mathfrak{a}$.

If $\mathfrak{a}=\mathbb{R}^{1,1}$, then $\mathcal{H}^2_Q(\mathfrak{l},\mathfrak{a})_0/G$ consists of four one-parameter families. They are represented by (28) and (29), too, but now either A_1,A_2 or A_2,A_1 is an orthonormal basis.

Proof. Let us first describe the automorphism group of $\mathfrak{l}$. For $a,b,c\in\mathbb{R}$ and $x=(x_1,\dots,x_4)\in\mathbb{R}^4$ we define a linear map $S(a,b,c,x):\mathfrak{l}\to\mathfrak{l}$ by

$$S(a,b,c,x) = \begin{pmatrix} a & 0 & 0 & 0\\ x_1 & b & 0 & 0\\ x_2 & c & ab & 0\\ x_3 & x_4 & ac & a^2b \end{pmatrix}$$

with respect to the basis X_1, X_2, Z, Y of $\mathfrak{l}$. It is not hard to check that

$$\mathrm{Aut}(\mathfrak{g}_{4,1}) = \{S(a,b,c,x) \mid a,b\in\mathbb{R}\setminus 0, c\in\mathbb{R}, x\in\mathbb{R}^4\}.$$

Obviously $\mathfrak{q} = \mathrm{span}\{X_2, Y, Z\}$ is an abelian ideal of $\mathfrak{g}_{4,1}$. The cohomology group $H^2(\mathfrak{g}_{4,1}, \mathfrak{a})$ is determined by the exact sequence

$$0 \longrightarrow H^1(\mathbb{R}\cdot X_1, H^1(\mathfrak{q},\mathfrak{a})) \longrightarrow H^2(\mathfrak{g}_{4,1},\mathfrak{a}) \longrightarrow H^0(\mathbb{R}\cdot X_1, H^2(\mathfrak{q},\mathfrak{a})) \longrightarrow 0\,.$$

We have

$$H^1(\mathbb{R}\cdot X_1, H^1(\mathfrak{q},\mathfrak{a})) = C^1(\mathbb{R}\cdot X_1, C^1(\mathfrak{q},\mathfrak{a}))/B^1(\mathbb{R}\cdot X_1, C^1(\mathfrak{q},\mathfrak{a})),$$

where

$$B^1(\mathbb{R}\cdot X_1, C^1(\mathfrak{q},\mathfrak{a})) = \{\sigma \in C^1(\mathbb{R}\cdot X_1, C^1(\mathfrak{q},\mathfrak{a})) \mid \sigma(X_1)(Y) = 0\}\,,$$

and

$$H^0(\mathbb{R}\cdot X_1, H^2(\mathfrak{q},\mathfrak{a})) \;=\; C^2(\mathfrak{q},\mathfrak{a})^{X_1} = \{\sigma \in C^2(\mathfrak{q},\mathfrak{a}) \mid \sigma(X_2,Y) = \sigma(Y,Z) = 0\}\,.$$

In particular, (A_2) is equivalent to $\alpha(Y,X_1) \neq 0$. If $\alpha(Y,X_1)\neq 0$, then also (A_0) and (A_1) are satisfied. Since $d\gamma = 0$ for all $\gamma \in C^3(\mathfrak{l})$ the equation $2d\gamma = \langle \alpha\wedge\alpha\rangle$ holds if and only if $\alpha(Y,X_1) \perp \alpha(X_2,Z)$. Hence we obtain

$$\mathcal{H}^2_Q(\mathfrak{l},\mathfrak{a})_0 = \left\{ [\alpha,\gamma] \in \mathcal{H}^2_Q(\mathfrak{l},\mathfrak{a}) \;\middle|\; \begin{array}{l} \alpha = (\sigma^1\wedge\sigma^Y)\otimes A_1 + (\sigma^2\wedge\sigma^Z)\otimes A_2, \\ \mathfrak{a} = \mathrm{span}\{A_1,A_2\},\;\; A_1 \perp A_2,\;\; A_1 \neq 0 \end{array} \right\}.$$

Because of $B^3(\mathfrak{l}) = \{\gamma\in C^3(\mathfrak{l}) \mid \gamma(X_1,Y,Z) = \gamma(X_2,Y,Z) = 0\}$ we may assume

$$\gamma = r\sigma^2\wedge\sigma^Y\wedge\sigma^Z + s\sigma^1\wedge\sigma^Y\wedge\sigma^Z\,.$$

Suppose that $\dim\mathfrak{a} = 1$. Take $[\alpha,\gamma] \in \mathcal{H}^2_Q(\mathfrak{l},\mathfrak{a})_0$, $\alpha = (\sigma^1\wedge\sigma^Y)\otimes A_1$, $u := \langle A_1, A_1\rangle \neq 0$. Applying $\mathrm{Aut}(\mathfrak{g}_{4,1})$ to $[\alpha,\gamma]$ we see that we may assume $u = \pm 1$ and $(r,s) \in \{(0,0),(1,0),(0,1),(0,-1)\}$ without changing the G-orbit. The four orbits which correspond to different values of (r,s) are pairwise different. Similarly we determine the G-orbits in $\mathcal{H}^2_Q(\mathfrak{l},\mathfrak{a})_0$ for $\dim\mathfrak{a} = 2$. ∎

Now let $\mathfrak{l}$ be one of the Lie algebras $\mathfrak{h}(1)\oplus\mathbb{R} = \{[X_1,X_2] = X_3\}\oplus\mathbb{R}\cdot X_4$ or $\mathbb{R}^4 = \mathrm{span}\{X_1,\dots,X_4\}$. Let $\sigma^1,\dots,\sigma^4$ be a basis of $\mathfrak{l}^*$ that is dual to $X_1,\dots,X_4$. Let $A_1, A_2, \dots$ be a basis of a vector space $\mathfrak{a}$. We define the following 2-forms

$$\begin{array}{rcl}
\alpha_1 &=& (\sigma^1\wedge\sigma^3 + \sigma^2\wedge\sigma^4)\otimes A_1 + (\sigma^2\wedge\sigma^3 + \sigma^1\wedge\sigma^4)\otimes A_2 \\
\alpha_2 &=& (\sigma^1\wedge\sigma^3 - \sigma^2\wedge\sigma^4)\otimes A_1 + (\sigma^2\wedge\sigma^3 + \sigma^1\wedge\sigma^4)\otimes A_2 \\
\alpha_3 &=& (\sigma^1\wedge\sigma^3)\otimes A_1 + (\sigma^2\wedge\sigma^3 + \sigma^1\wedge\sigma^4)\otimes A_2 \\
\alpha_4 &=& (\sigma^1\wedge\sigma^3)\otimes A_1 + (\sigma^2\wedge\sigma^3)\otimes A_2 \\
\alpha_5 &=& (\sigma^1\wedge\sigma^3)\otimes A_1 + (\sigma^1\wedge\sigma^4)\otimes A_2, \quad \alpha_5' = (\sigma^1\wedge\sigma^4)\otimes A_1 + (\sigma^1\wedge\sigma^3)\otimes A_2 \\
\alpha_6 &=& (\sigma^1\wedge\sigma^3)\otimes A_1 + (\sigma^2\wedge\sigma^4)\otimes A_2, \quad \alpha_6' = (\sigma^2\wedge\sigma^4)\otimes A_1 + (\sigma^1\wedge\sigma^3)\otimes A_2 \\
\alpha_7 &=& (\sigma^1\wedge\sigma^3)\otimes A_1\,.
\end{array}$$

Moreover, we define the 3-form γ_0 on $\mathfrak{l}$ by $\gamma_0 = \sigma^2\wedge\sigma^3\wedge\sigma^4$.

Proposition 4.5. *Take $\mathfrak{l} = \mathfrak{h}(1) \oplus \mathbb{R} = \{[X_1, X_2] = X_3\} \oplus \mathbb{R} \cdot X_4$. Let $\mathfrak{a}$ be a trivial orthogonal $\mathfrak{l}$-module. If $\mathfrak{a} = \mathbb{R}^2$ or $\mathfrak{a} = \mathbb{R}^{2,0}$, then the elements in $\mathcal{H}^2_Q(\mathfrak{l}, \mathfrak{a})_0/G$ are represented by $[\alpha_1, 0]$, $[\alpha_5, 0]$, $[\alpha_5, \gamma_0]$, $[\alpha_6, 0]$, $[\alpha_6, \gamma_0]$, where A_1, A_2 is a fixed orthonormal basis of $\mathfrak{a}$.*

If $\mathfrak{a} = \mathbb{R}^{1,1}$, then $\mathcal{H}^2_Q(\mathfrak{l}, \mathfrak{a})_0/G$ has eleven elements, three of them are represented by $[\alpha_1, 0]$, $[\alpha_2, 0]$, $[\alpha_3, 0]$, where A_1, A_2 is a fixed Witt basis of $\mathfrak{a}$, eight further elements are represented by $[\alpha_5, 0]$, $[\alpha_5, \gamma_0]$, $[\alpha_6, 0]$, $[\alpha_6, \gamma_0]$, $[\alpha_5', 0]$, $[\alpha_5', \gamma_0]$, $[\alpha_6', 0]$, $[\alpha_6', \gamma_0]$, where A_1, A_2 is a fixed orthonormal basis of $\mathfrak{a}$.

If $\mathfrak{a} = \mathbb{R}^1$ or $\mathfrak{a} = \mathbb{R}^{1,0}$, then there is only one element in $\mathcal{H}^2_Q(\mathfrak{l}, \mathfrak{a})_0/G$. It is represented by $[\alpha_7, \gamma_0]$, where A_1 is a fixed unit vector in $\mathfrak{a}$.

If $\mathfrak{a} = 0$, then $\mathcal{H}^2_Q(\mathfrak{l}, \mathfrak{a})_0 = \emptyset$.

Proof. For $A, X \in \mathfrak{gl}(2, \mathbb{R})$ and $u = (u_1, u_2, u_3) \in \mathbb{R}^3$ we define

$$S(A, X, u) = \begin{pmatrix} A & 0 \\ X & U \end{pmatrix} \in \mathfrak{gl}(4, \mathbb{R}), \text{ where } U = \begin{pmatrix} u_1 & u_2 \\ 0 & u_3 \end{pmatrix} \in \mathfrak{gl}(2, \mathbb{R}).$$

Then the automorphism group of $\mathfrak{l} = \mathfrak{h}(1) \oplus \mathbb{R}$ equals

$$\mathrm{Aut}(\mathfrak{l}) = \left\{ S(A, X, u) \mid u \in \mathbb{R}^3,\ X \in \mathfrak{gl}(2, \mathbb{R}),\ A \in GL(2, \mathbb{R}),\ \det A = u_1,\ u_3 \neq 0 \right\},$$

where we consider all automorphisms with respect to the basis $X_1, \ldots, X_4$ of $\mathfrak{l}$. Using the Künneth formula and the description of $H^2(\mathfrak{h}(1), \mathfrak{a})$ in [12], we see that

$$\begin{array}{rcl} Z_{\mathfrak{l}} := \{\alpha \in C^2(\mathfrak{l}, \mathfrak{a}) \mid \alpha(X_1, X_2) = \alpha(X_3, X_4) = 0\} & \longrightarrow & H^2(\mathfrak{l}, \mathfrak{a}) \\ \alpha & \longmapsto & [\alpha] \end{array}$$

is a bijection. Now take $[\alpha, \gamma] \in \mathcal{H}^2_Q(\mathfrak{l}, \mathfrak{a})_0$, $\alpha \in Z_{\mathfrak{l}}$. Since, obviously, $d\gamma = 0$ we have $\langle \alpha \wedge \alpha \rangle = 0$. Condition (A_1) gives $\alpha(X_3, \mathfrak{l}) \neq 0$ and Condition (B_1) says that $\alpha(X_3, \mathfrak{l})$ is non-degenerate. By indecomposability we have $\alpha(\mathfrak{l}, \mathfrak{l}) = \mathfrak{a}$. Hence α is an element of the G-invariant subset $C \subset Z_{\mathfrak{l}}$ defined by

$$C := \{\alpha \in Z_{\mathfrak{l}} \mid \langle \alpha \wedge \alpha \rangle = 0,\ \alpha(\mathfrak{l}, \mathfrak{l}) = \mathfrak{a},\ 0 \neq \alpha(X_3, \mathfrak{l}) \subset \mathfrak{a} \text{ is non-degenerate}\}.$$

A cocycle $\alpha \in Z_{\mathfrak{l}}$ satisfies $\langle \alpha \wedge \alpha \rangle = 0$ if and only if

$$\langle \alpha(X_1, X_3), \alpha(X_2, X_4) \rangle = \langle \alpha(X_2, X_3), \alpha(X_1, X_4) \rangle. \tag{30}$$

Let us determine the G-orbits in C in the case that $\dim \mathfrak{a} \leq 2$. Take $\alpha \in C$. Then we have $\dim \alpha(X_3, \mathfrak{l}) = 1$ or $\dim \alpha(X_3, \mathfrak{l}) = 2$.

Let us first consider the case $\dim \alpha(X_3, \mathfrak{l}) = 1$. Replacing α by an element in the G-orbit of α we may assume $\alpha(X_1, X_3) = A_1$ and $\alpha(X_2, X_3) = 0$, where $\langle A_1, A_1 \rangle = \pm 1$. From (30) we obtain $\langle \alpha(X_1, X_3), \alpha(X_2, X_4) \rangle = 0$. Hence either $\alpha(X_2, X_4) = 0$ or $\alpha(X_2, X_4) =: A_2 \neq 0$ and $A_1 \perp A_2$. In the latter case A_2 cannot be isotropic since $\mathfrak{a}$ is at most two-dimensional. Hence, replacing α by an element in the same G-orbit we may assume that $\langle A_2, A_2 \rangle = \pm 1$ and $\alpha(X_1, X_4) = 0$, hence α is in the same G-orbit as α_6 or as α_6' for an orthonormal basis A_1, A_2. In the first case, where $\alpha(X_2, X_4) = 0$ we may assume that $\alpha(X_1, X_4) = 0$ or $\alpha(X_1, X_4) = A_2 \neq 0$ and $A_1 \perp A_2$, $\langle A_2, A_2 \rangle = \pm 1$, hence α is in the same orbit as α_5 or α_5' for an orthonormal basis A_1, A_2 or as α_7 for a unit vector A_1. Obviously, the orbit of α_7 contains neither α_5, α_5', α_6, nor α_6'. Also the orbits of α_5 and α_6 are different. Indeed, $\alpha_5(X_2,\mathfrak{l}) = 0$ and $\alpha_6(L,\mathfrak{l}) \neq 0$ for all $L \in \mathfrak{l}, L \neq 0$. Analogously, the orbits of α_5' and α_6' are different. Moreover, α_i and α_i', $i = 5,6$, are not on the same orbit, since X_3 plays a distinguished role in $\mathfrak{l}$.

Now we consider the case $\dim \alpha(X_3,\mathfrak{l}) = 2$. Then $\alpha(X_1,X_3) =: A_1$ and $\alpha(X_2,X_3) =: A_2$ are linearly independent. First we observe that we may assume that A_1, A_2 is an orthonormal basis of $\mathfrak{a}$ if $\mathfrak{a} = \mathbb{R}^2$ or $\mathfrak{a} = \mathbb{R}^{2,0}$ and that A_1, A_2 is a Witt basis if $\mathfrak{a} = \mathbb{R}^{1,1}$ (replacing α by an element in the same G-orbit). By (30) we have

$$\langle A_1, \alpha(X_2,X_4)\rangle = \langle A_2, \alpha(X_1,X_4)\rangle. \tag{31}$$

Replacing X_4 by a suitable linear combination of X_4 and X_3 we may assume that $\alpha(X_2,X_4)$ is a multiple of A_1.

Assume that $\mathfrak{a} = \mathbb{R}^2$ or $\mathfrak{a} = \mathbb{R}^{2,0}$. If $\alpha(X_2,X_4) = 0$, then (31) implies that $\alpha(X_1,X_4)$ is a multiple of A_1. Hence we may assume that either $\alpha(X_1,X_4) = 0$ or that $\alpha(X_1,X_4) = A_1$. Consequently, α is in the same orbit as α_4 or as the 2-form

$$(\sigma^1 \wedge \sigma^3 + \sigma^1 \wedge \sigma^4) \otimes A_1 + (\sigma^2 \wedge \sigma^3) \otimes A_2,$$

which is in the same orbit as α_1. If $\alpha(X_2,X_4) = rA_1$, $r \neq 0$, then rescaling X_4 we may assume that $\alpha(X_2,X_4) = A_1$. Now (31) yields $\alpha(X_1,X_4) = A_2 + sA_1$. We will show that we may assume $s = 0$. For $s \in \mathbb{R}$ we choose $t \in \mathbb{R}$ such that $s = 2\tan 2t$ and we define

$$a = \sin t,\ b = \cos t,\ u_2 = \sin 2t,\ u_3 = -\cos 2t$$

and

$$A = \begin{pmatrix} a & -b \\ b & a \end{pmatrix},\ u = (1, u_2, u_3) \in \mathbb{R}^3.$$

For this choice of A, and u we consider $S = S(A,0,u) \in \mathrm{Aut}(\mathfrak{l})$ and we define $X_i' = SX_i$, $i = 1,\dots,4$. Then we have

$$\begin{aligned}
\alpha(X_1',X_3') &= aA_1 + bA_2 =: A_1' \\
\alpha(X_2',X_3') &= -bA_1 + aA_2 =: A_2' \\
\alpha(X_1',X_4') &= (u_2a + au_3s + bu_3)A_1 + (u_2b + au_3)A_2 = A_2' \\
\alpha(X_2',X_4') &= (-u_2b - sbu_3 + au_3)A_1 + (u_2a - bu_3)A_2 = A_1',
\end{aligned}$$

where A_1', A_2' is again an orthonormal basis. Hence, α is in the same orbit as α_1. The 2-forms α_1 and α_4 are on different orbits, since $\alpha_4(X_4,\mathfrak{l}) = 0$ and $\alpha_1(L,\mathfrak{l}) \neq 0$ for all $L \in \mathfrak{l}, L \neq 0$.

Take now $\mathfrak{a} = \mathbb{R}^{1,1}$. Recall that we may assume $\alpha(X_1,X_3) = A_1$ and $\alpha(X_2,X_3) = A_2$ such that A_1, A_2 is a Witt basis and that $\alpha(X_2,X_4) = rA_1$, $r \in \{0,1\}$. From (31) we get $\alpha(X_1,X_4) = sA_2$ for a real number s. If $r = s = 0$, then α is in the same orbit as α_4. If $r = 0$, $s \neq 0$ or $r = 1$, $s = 0$, then α is in the same orbit as α_3. If $r = 1$, $s \neq 0$, then α is in the same orbit as α_1 or α_2. The 2-forms $\alpha_1,\dots,\alpha_4$ are on different orbits, since the elements of its orbits differ in the properties of their projections to the isotropic lines in $\mathfrak{a} = \mathbb{R}^{1,1}$.

We can summarize this as follows. If $\mathfrak{a} = \mathbb{R}^2$ or $\mathfrak{a} = \mathbb{R}^{2,0}$, then there are four G-orbits in C represented by $\alpha_1, \alpha_4, \alpha_5, \alpha_6$, where A_1, A_2 is a fixed orthonormal basis of $\mathfrak{a}$. If $\mathfrak{a} = \mathbb{R}^{1,1}$, then there are eight G-orbits in C, four of them are represented by $\alpha_1, \alpha_2, \alpha_3, \alpha_4$, where A_1, A_2 is a fixed Witt basis of $\mathfrak{a}$, four further orbits are represented by $\alpha_5, \alpha_5', \alpha_6, \alpha_6'$, where now A_1, A_2 is a fixed orthonormal basis of $\mathfrak{a}$. If $\mathfrak{a} = \mathbb{R}^1$ or $\mathfrak{a} = \mathbb{R}^{1,0}$, then $\alpha_7 \in C$ and G acts transitively on C.

Since $Z^1(\mathfrak{l}, \mathfrak{a}) = \{\tau \in C^1(\mathfrak{l}, \mathfrak{a}) \mid d\tau = 0\} = \{\tau \in C^1(\mathfrak{l}, \mathfrak{a}) \mid \tau(X_3) = 0\}$ and $B^3(\mathfrak{l}) = \{d\sigma \mid \sigma \in C^2(\mathfrak{l})\} = \mathbb{R} \cdot \sigma^1 \wedge \sigma^2 \wedge \sigma^4$ we have $B^3(\mathfrak{l}) + \langle \alpha_i \wedge Z^1(\mathfrak{l}, \mathfrak{a}) \rangle = C^3(\mathfrak{l})$ for $i = 1, \ldots, 4$. Hence $[\alpha_i, \gamma] = [\alpha_i, 0] \in \mathcal{H}^2_Q(\mathfrak{l}, \mathfrak{a})$ for $i = 1, \ldots, 4$ and arbitrary $\gamma \in C^3(\mathfrak{l})$. Note that $[\alpha_4, 0]$ is decomposable. If $\alpha \in \{\alpha_5, \alpha_5', \alpha_6, \alpha_6', \alpha_7\}$, then γ_0 spans a complement of $B^3(\mathfrak{l}) + \langle \alpha \wedge Z^1(\mathfrak{l}, \mathfrak{a}) \rangle$ in $C^3(\mathfrak{l})$. Hence, for all these α and for all $\gamma \in C^3(\mathfrak{l})$ there exists a real number c such that $[\alpha, \gamma] = [\alpha, c\gamma_0] \in \mathcal{H}^2_Q(\mathfrak{l}, \mathfrak{a})$. Moreover, $[\alpha, c\gamma_0]$ is in the same G-orbit as $[\alpha, 0]$ or $[\alpha, \gamma_0]$.

It remains to check admissibility and indecomposability. All cohomology classes $[\alpha, \gamma] \in \mathcal{H}^2_Q(\mathfrak{l}, \mathfrak{a})$ with $\alpha \in C$ satisfy (B_0), (A_1), and (B_1). Moreover, it is not hard to see that all cohomology classes listed in the proposition satisfy also (A_0) and are indecomposable. ■

A similar but easier computation gives the following result for the remaining Lie algebra $\mathfrak{l} = \mathbb{R}^4$.

Proposition 4.6. *Let $\mathfrak{l}$ be the abelian Lie algebra $\mathbb{R}^4 = \mathrm{span}\{X_1, \ldots, X_4\}$ and let $\mathfrak{a}$ be a trivial orthogonal $\mathfrak{l}$-module.*

If $\mathfrak{a} = \mathbb{R}^2$ or $\mathfrak{a} = \mathbb{R}^{2,0}$, then the elements in $\mathcal{H}^2_Q(\mathfrak{l}, \mathfrak{a})_0/G$ are represented by $[\alpha_1, 0]$ and $[\alpha_4, \sigma^1 \wedge \sigma^2 \wedge \sigma^4]$, where A_1, A_2 is a fixed orthonormal basis of $\mathfrak{a}$.

If $\mathfrak{a} = \mathbb{R}^{1,1}$, then the elements in $\mathcal{H}^2_Q(\mathfrak{l}, \mathfrak{a})_0/G$ are represented by $[\alpha_1, 0]$, $[\alpha_2, 0]$, $[\alpha_3, 0]$, $[\alpha_4, \sigma^1 \wedge \sigma^2 \wedge \sigma^4]$ where A_1, A_2 is a fixed Witt basis of $\mathfrak{a}$.

If $\mathfrak{a} = \mathbb{R}^1$ or $\mathfrak{a} = \mathbb{R}^{1,0}$, then $\mathcal{H}^2_Q(\mathfrak{l}, \mathfrak{a})_0/G$ contains exactly one element. This is represented by $[\alpha_7, \gamma_0]$, where A_1 is a fixed unit vector in $\mathfrak{a}$.

If $\mathfrak{a} = 0$, then $\mathcal{H}^2_Q(\mathfrak{l}, \mathfrak{a})_0 = \emptyset$.

Combining the description of the moduli space given in Prop. 4.2 with Props. 4.3 to 4.6 and the computations of $\mathcal{H}^2_Q(\mathfrak{l}, \mathfrak{a})_0$ for $\dim \mathfrak{l} \leq 3$ in [11] and [12] we obtain the following result. We use the 2-forms $\alpha_1, \ldots, \alpha_7, \alpha_5', \alpha_6'$ and the 3-form γ_0 introduced before Prop. 4.5.

Theorem 4.7. *If $(\mathfrak{g}, \langle \cdot, \cdot \rangle)$ is an indecomposable non-abelian nilpotent metric Lie algebra of dimension at most 10, then it is isomorphic to $\mathfrak{d}_{\alpha,\gamma}(\mathfrak{l}, \mathfrak{a})$ for exactly one of the data in the following list:*

1. $\mathfrak{l} = \mathbb{R}^5$

$$\mathfrak{a} = 0,\ (\alpha, \gamma) = (0, \sigma^1 \wedge \sigma^2 + \sigma^3 \wedge \sigma^4) \wedge \sigma^5);$$

2. $\mathfrak{l} = \mathfrak{g}_{5,2} = \{[X_1, X_2] = Y,\ [X_1, X_3] = Z\}$

$$\mathfrak{a} = 0,\ \alpha = 0,\ \gamma \in \{\sigma^1 \wedge \sigma^Y \wedge \sigma^Z,\ \sigma^1 \wedge \sigma^Y \wedge \sigma^Z + \sigma^2 \wedge \sigma^3 \wedge \sigma^Z\};$$

3. $\mathfrak{l} = \mathfrak{g}_{4,1} = \{[X_1, Z] = Y,\ [X_1, X_2] = Z\}$

   (a) $\mathfrak{a} \in \{\mathbb{R}^1,\ \mathbb{R}^{1,0}\}$, $A \in \mathfrak{a}$ *fixed unit vector,*
   $\alpha = \sigma^1 \wedge \sigma^Y \otimes A$,
   $\gamma \in \{0,\ \sigma^2 \wedge \sigma^Y \wedge \sigma^Z,\ \sigma^1 \wedge \sigma^Y \wedge \sigma^Z,\ -\sigma^1 \wedge \sigma^Y \wedge \sigma^Z\}$;

(b) $\mathfrak{a} \in \{\mathbb{R}^2,\ \mathbb{R}^{2,0}\}$ *with fixed orthonormal basis* A_1, A_2,
$\alpha = \sigma^1 \wedge \sigma^Y \otimes A_1 + \sigma^2 \wedge \sigma^Z \otimes A_2$,
$\gamma \in \{s\sigma^1 \wedge \sigma^Y \wedge \sigma^Z \mid s \in \mathbb{R}\} \cup \{r\sigma^2 \wedge \sigma^Y \wedge \sigma^Z \mid r \in \mathbb{R}_+\}$;

(c) $\mathfrak{a} = \mathbb{R}^{1,1}$ *with fixed orthonormal basis* A_1, A_2,
$\alpha \in \{\sigma^1 \wedge \sigma^Y \otimes A_1 + \sigma^2 \wedge \sigma^Z \otimes A_2,\ \sigma^2 \wedge \sigma^Z \otimes A_1 + \sigma^1 \wedge \sigma^Y \otimes A_2\}$,
$\gamma \in \{s\sigma^1 \wedge \sigma^Y \wedge \sigma^Z \mid s \in \mathbb{R}\} \cup \{r\sigma^2 \wedge \sigma^Y \wedge \sigma^Z \mid r \in \mathbb{R}_+\}$;

4. $\mathfrak{l} = \mathfrak{h}(1) \oplus \mathbb{R}^1$

(a) $\mathfrak{a} \in \{\mathbb{R}^2,\ \mathbb{R}^{2,0}\}$ *with fixed orthonormal basis* A_1, A_2
$(\alpha, \gamma) \in \{(\alpha_1, 0),\ (\alpha_5, 0),\ (\alpha_5, \gamma_0),\ (\alpha_6, 0),\ (\alpha_6, \gamma_0)\}$;

(b) $\mathfrak{a} = \mathbb{R}^{1,1}$,
$(\alpha, \gamma) \in \{(\alpha_1, 0),\ (\alpha_2, 0),\ (\alpha_3, 0)\}$, *where* A_1, A_2 *is a fixed Witt basis, or*
$(\alpha, \gamma) \in \{(\alpha_5, 0),\ (\alpha_5, \gamma_0),\ (\alpha_6, 0),\ (\alpha_6, \gamma_0),\ (\alpha_5', 0),\ (\alpha_5', \gamma_0),\ (\alpha_6', 0),\ (\alpha_6', \gamma_0)\}$,
where A_2, A_1 *is a fixed orthonormal basis of* $\mathfrak{a}$;

(c) $\mathfrak{a} \in \{\mathbb{R}^1,\ \mathbb{R}^{1,0}\}$,
$(\alpha, \gamma) = (\alpha_7, \gamma_0)$, *where* A_1 *is a fixed unit vector in* $\mathfrak{a}$;

5. $\mathfrak{l} = \mathbb{R}^4$

(a) $\mathfrak{a} \in \{\mathbb{R}^2, \mathbb{R}^{2,0}\}$ *with fixed orthonormal basis* A_1, A_2,
$(\alpha, \gamma) \in \{(\alpha_1, 0),\ (\alpha_4, \sigma^1 \wedge \sigma^2 \wedge \sigma^4)\}$;

(b) $\mathfrak{a} = \mathbb{R}^{1,1}$ *with fixed Witt basis* A_1, A_2,
$(\alpha, \gamma) \in \{(\alpha_1, 0),\ (\alpha_2, 0),\ (\alpha_3, 0),\ (\alpha_4, \sigma^1 \wedge \sigma^2 \wedge \sigma^4)\}$;

(c) $\mathfrak{a} \in \{\mathbb{R}^1,\ \mathbb{R}^{1,0}\}$,
$(\alpha, \gamma) = (\alpha_7, \gamma_0)$, *where* A_1 *is a fixed unit vector in* $\mathfrak{a}$;

6. $\mathfrak{l} = \mathfrak{h}(1) = \{[X_1, X_2] = Y\}$

(a) $\mathfrak{a} \in \{\mathbb{R}^1,\ \mathbb{R}^{1,0}\}$,
$(\alpha, \gamma) = (\sigma^1 \wedge \sigma^Y \otimes A,\ 0)$, *where* A *is a fixed unit vector in* $\mathfrak{a}$;

(b) $\mathfrak{a} \in \{\mathbb{R}^2,\ \mathbb{R}^{2,0},\ \mathbb{R}^{1,1}\}$ *with fixed orthonormal basis* A_1, A_2,
$(\alpha, \gamma) = (\sigma^1 \wedge \sigma^Y \otimes A_1 + \sigma^2 \wedge \sigma^Y \otimes A_2,\ 0)$;

7. $\mathfrak{l} = \mathbb{R}^3$

(a) $\mathfrak{a} = 0$, $(\alpha, \gamma) = (0, \sigma^1 \wedge \sigma^2 \wedge \sigma^3)$;

(b) $\mathfrak{a} \in \{\mathbb{R}^2,\ \mathbb{R}^{2,0},\ \mathbb{R}^{1,1}\}$ *with fixed orthonormal basis* A_1, A_2,
$(\alpha, \gamma) = (\sigma^1 \wedge \sigma^2 \otimes A_1 + \sigma^1 \wedge \sigma^3 \otimes A_2,\ 0)$;

(c) $\mathfrak{a} \in \{\mathbb{R}^3,\ \mathbb{R}^{2,1},\ \mathbb{R}^{1,2},\ \mathbb{R}^{3,0}\}$ *with fixed orthonormal basis* A_1, A_2, A_3,
$(\alpha, \gamma) = (\sigma^1 \wedge \sigma^2 \otimes A_1 + \sigma^1 \wedge \sigma^3 \otimes A_2 + \sigma^2 \wedge \sigma^3 \otimes A_3,\ 0)$;

8. $\mathfrak{l} = \mathbb{R}^2$

$\mathfrak{a} \in \{\mathbb{R}^1,\ \mathbb{R}^{1,0}\}$,
$(\alpha, \gamma) = (\sigma^1 \wedge \sigma^2 \otimes A,\ 0)$, *where* A *is a fixed unit vector in* $\mathfrak{a}$.

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