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On Compact Just-Non-Lie Groups

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Abstract. A compact group is called a *compact Just-Non-Lie group* or a *compact* JNL *group* if it is not a Lie group but all of its proper Hausdorff quotient groups are Lie groups. We show that a compact JNL group is profinite and a compact nilpotent JNL group is the additive group of p-adic integers for some prime. Examples show that this fails for compact pronilpotent and solvable groups.
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1. Introduction

Let $\mathfrak{X}$ be a class of groups. A group G which belongs to $\mathfrak{X}$ is said to be an $\mathfrak{X}$*-group.* A group G is said to be a *Just-Non-*$\mathfrak{X}$ *group*, or a $JN\mathfrak{X}$ *group* , if it is not an $\mathfrak{X}$-group but all of its proper quotients are $\mathfrak{X}$-groups. The structure of Just-Non-$\mathfrak{X}$ groups has already been studied for several choices of the class $\mathfrak{X}$, so there is a well developed theory about this topic (see [3]). Moreover the study of Just-Non-$\mathfrak{X}$ groups has been investigated both in theory of finite groups and theory of infinite groups and many techniques have general applications.

There are some cautionary observations necessary when we consider as class $\mathfrak{X}$ of topological groups such as the class of Lie groups. The literature on varieties of topological groups is comparatively recent, as shown in [4], [5], [5], [7], [8]. In particular most of the classical results of [4, Chapter 2] do not apply to topological groups. In order to speak about quotients in a meaningful way in the any category of Hausdorff topological groups such as the category of Lie groups, we must quotients modulo closed normal subgroups (see [1, Definition 1.9]). Since every compact group has enough Lie group quotients to separate the points the concept of a compact Just-Non-Lie group is meaningful. One source for facts on compact groups is [1]; basic properties of compact Lie groups, for instance, are summarized in [1, Corollary 2.40].

If G is a topological group, let $\mathcal{N}(G)$ denote the set of all normal subgroups of G such that $N \in \mathcal{N}(G)$ if and only if G/N is a Lie group. Then $G \in \mathcal{N}(G)$;

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further $\{1\} \in \mathcal{N}(G)$ if and only if G is a Lie group. If $N \in \mathcal{N}(G)$ and M is a closed normal subgroup of G such that $N \leq M$, then $M \in \mathcal{N}(G)$. If G is a compact group, then $\mathcal{N}(G)$ converges to 1 and the natural morphism $G \to \lim_{N\in\mathcal{N}(G)} G/N$ is an isomorphism of compact groups (see [1, p.17-23]).

A Hausdorff topological group G is said to be a *Just-Non-Lie group* or, more shortly, a JNL *group* if G is a non-Lie group such that all closed normal subgroups $N \neq \{1\}$ are contained in $\mathcal{N}(G)$,

Lemma 1.1. *A* JNL *group does not contain any nonsingleton closed normal Lie subgroup.*

Proof. If N is a nonsingleton closed normal Lie subgroup of a JNL group, then $N \in \mathcal{N}(G)$ and so G/N is a Lie group. But then G, as an extension of a Lie group by a Lie group is a Lie group, contrary to the definition of JNL group. ■

As a special consequence, a JNL group does not have a finite normal nonsingleton subgroup. In particular, an abelian JNL group is torsionfree.

In Section 2 we shall show that every compact JNL group is profinite and that a compact abelian JNL-group is a group of p-adic integers. We shall illustrate these facts by some limiting examples and counterexamples. In Section 3 we peruse a variety and additional observations and results.

Most of our notation is standard and has been referred to [1]. For the general properties of compact groups we refer to [1]. For profinite groups we have the source books [9, 11].

2. Compact Just-Non-Lie groups are profinite

We denote by $\mathbb{Z}_p$ the additive group of p-adic integers for a prime p.

Theorem 2.1. *For a compact abelian* JNL *group G there is a prime p such that $G \cong \mathbb{Z}_p$.*

Proof. A compact abelian group is a Lie group if an only if its character group is a finitely generated discrete group. Hence, by duality, $A = \widehat{G}$ is a discrete abelian group which is not finitely generated, while every proper subgroup is finitely generated (since every proper quotient of G is a Lie group). We write A additively. As a consequence of Lemma 1.1, G is torsionfree and so A is divisible (see [1, Corollary 8.5]). Thus $A \cong \mathbb{Q}^{(I)} \oplus \bigoplus_{p\in\mathbb{P}} \mathbb{Z}(p^\infty)^{(I_p)}$, where I and I_p are suitable sets and $\mathbb{P}$ denotes the set of all prime numbers (see [1, Theorem A1.42]).Suppose that $I \neq \varnothing$. Let $a \neq 0$, $a \in \mathbb{Q}^{(I)}$. Then $\frac{1}{2^\infty}\mathbb{Z} \cdot a$ is a proper subgroup of A that is not finitely generated. This is a contradiction. Thus A is a divisible torsion group.Write $A_p = \mathbb{Z}(p^\infty)^{(I_p)}$ for its p-primary components. Since A is nonzero, at least one A_p is nonzero. Let $A^{(p)}$ denote the sum $\bigoplus_{q\neq p} A_q$ of all q-primary components A_q for q prime number which is distinct by p. Suppose $A^{(p)} \neq \{0\}$. Then $A_p \neq A$ and thus A_p is finitely generated contradicting the fact that a Prüfer group is not finitely generated. Thus $A = \mathbb{Z}(p^\infty)^{(I_p)}$. Let $j \in I_p$, then $A \simeq \mathbb{Z}(p^\infty) \oplus \mathbb{Z}(p^\infty)^{(I_p\setminus\{j\})}$. If the second summand were nonzero, then

$\mathbb{Z}(p^\infty)$ would have to be finitely generated, which is not. Thus $I_p = \{j\}$ and A is a Prüfer group. But this causes its dual G to be isomorphic to the group $\mathbb{Z}_p$ of p-adic integers. ■

We recall from [1, 11, 13] that a compact group is profinite if and only if it is totally disconnected. The connected component of the identity will be denoted G_0 (see [1, p.23]).

Theorem 2.2. *A compact* JNL *group is totally disconnected, that is, it is profinite.*

Proof. Assume that G is a compact JNL group and $G_0 \neq \{1\}$. We shall derive a contradiction.

(a) Since G is a compact JNL group, G/G_0 is a Lie group and thus, as a totally disconnected group, is finite.

(b) We will denote with S the commutator subgroup $[G_0, G_0]$ of G_0 and with A the identity component $Z(G_0)_0$ of the center $Z(G_0)$ of G_0. Both of these subgroups are characteristic subgroups of G_0. Set $\Delta = S \cap A$. We claim that $\Delta = \{1\}$.

Suppose that $\Delta \neq \{1\}$. Then G/Δ is a Lie group. In particular, S/Δ is a Lie group, whence $S/Z(S)$ is a Lie group, since $\Delta \leq Z(S)$. The factor group $S/Z(S)$ is of the form $\prod_{j \in J} S_j$ for a family of centerfree compact connected simple Lie groups (see [1, Theorem 9.24]), and thus J is finite. Then [1, Theorem 9.19] allows us to conclude that Δ is finite. Then Lemma 1.1 implies $\Delta = \{1\}$ and thus we have a direct product decomposition $G_0 = S \times A$ (see [1, Theorem 9.24]).

(c) Suppose that $S \neq 1$. Then G/S, and therefore G_0/S, is a Lie group. Hence $A \simeq G_0/S$ is a Lie group. Then Lemma 1.1 implies $A = \{1\}$. Therefore $G_0 = S = \prod_{j \in J} S_j$. Also, G_0 is centerfree. By Lee's Theorem [1, Theorem 9.41] there is a finite group F such that $G = S \rtimes F$. Since the factors S_j are simple, the action of F induces a permutation group on J. But F is finite, then there is a finite subset I of J which is invariant under this action. Then $\prod_{j \in I} S_j$ is a nonsingleton normal subgroup of G and is a Lie group as a finite product of Lie groups. Now Lemma 1.1 again implies that $S = \{1\}$, and thus we know that $G_0 = A$ is abelian.

(d) In that case, $G \neq A$ by Theorem 2.1. Hence A is not a Lie group by Lemma 1.1. The factor group $\Gamma = G/A$ acts as a finite group of automorphisms on A and then, By Pontryagin Duality (see [1, Theorem 1.37]), Γ acts as a finite automorphism group on the torsionfree character group $\widehat{A}$ as well. A closed nonsingleton subgroup B of A is normal in G if and only if it is Γ-invariant in which case A/B is a torus group. Hence the annihilator $B^\perp$ in the character group $\widehat{A}$ of A is finitely generated free. Thus we know that

$(*)$ every proper Γ-invariant subgroup of $\widehat{A}$ is finitely generated free.

If n is any natural number such that $n{\cdot}\widehat{A} \neq \widehat{A}$ then $n{\cdot}\widehat{A}$ is a proper Γ-invariant subgroup of $\widehat{A}$ and is, therefore, finitely generated free by $(*)$. Since $\widehat{A}$ is torsionfree, this implies that $\widehat{A}$ is finitely generated free, and that contradicts the fact that A is not a Lie group. It follows that $\widehat{A}$ is divisible and thus is the additive group of a $\mathbb{Q}$-vector space. By the Theorem of Maschke and Schur, $\widehat{A}$ is a semisimple Γ-module, that is, it is a direct sum of finite dimensional Γ-invariant $\mathbb{Q}$-vector subspaces. From $(*)$ we conclude that such a sum can have only one finite simple summand. Let $0 \neq v \in \widehat{A}$. The abelian subgroup generated by $\{\gamma{\cdot}v : \gamma \in \Gamma\}$ is a finitely generated free, hence proper, Γ-invariant subgroup F of $\widehat{A}$. Now let $R = \frac{1}{p^\infty}{\cdot}\mathbb{Z}$ the subring of $\mathbb{Q}$ of all rational numbers of the form m/p^n, $m \in \mathbb{Z}$, $n \in \mathbb{N}$ for a fixed prime p. Let $R \otimes F$ denote all linear combinations of elements $r{\cdot}f$ with $r \in R$ and $f \in F$. Then $R\otimes F$ is a proper Γ-invariant subgroup of $\widehat{A}$. Thus on one hand it is finitely generated free by $(*)$, but on the other fails to be free since it contains a copy of the additive group of R which is not free. This contradiction shows that G_0 cannot be abelian and nonsingleton. Hence the assumption $G_0 \neq \{1\}$ is false, and this shows that G is totally disconnected, and thus profinite. ■

We may reformulate this theorem as

Corollary 2.3. *In a compact JNL group, every nonsingleton normal closed subgroup is open and of finite index.*

Proposition 2.4. *A compact* JNL *group with a nonsingleton center is a central extension of a group $\mathbb{Z}_p$ of p-adic integers by a finite group.*

Proof. Let $Z(G) \neq \{1\}$ denote the center of the compact JNL group G. If $G = Z(G)$, the assertion follows from Theorem 2.1. Assume $G \neq Z(G)$. By Lemma 1.1, the center $Z(G)$ is not a Lie group and by Corollary 2.3, $Z(G)$ is open, that is, $G/Z(G)$ is finite. Now let $N \neq \{1\}$ be a closed subgroup of $Z(G)$. Then N is a closed normal subgroup of G and thus G/N is finite. In particular, $Z(G)/N \leq G/N$ is finite. Thus $Z(G)$ is a compact abelian JNL group. Therefore $Z(G) \cong \mathbb{Z}_p$ for some prime p by Theorem 2.1 and the proposition is proved. ■

Among other things it follows at once that a compact JNL group without nonsingleton torsionfree normal subgroups is centerfree. More significantly, we have

Corollary 2.5. *A compact nilpotent* JNL *group is abelian and therefore is isomorphic to a p-adic group.*

Proof. If we can prove that G is abelian whenever G is nilpotent of class at most 2, then we are done, because the second center $Z_2(G)$ has class of nilpotence at most 2 and would have class 2 if G is nonabelian.
Thus, without loss of generality, we may assume that G is nilpotent of class at most 2. Then $[G,G] \leq Z(G)$, and

$$[g_1Z(G), g_2Z(G)] = \{[g_1, g_2]\},$$

where $g_1, g_2 \in G$, so that the bihomomorphic function

$$b\colon G \times G \to Z(G)$$

factors through a bihomomorphic function

$$B\colon G/Z(G) \times G/Z(G) \to Z(G).$$

Now $G/Z(G)$ is finite and $Z(G) \cong \mathbb{Z}_p$ by Proposition 2.4. Now $[G,G] = B(G/Z(G) \times G/Z(G))$ is a union of finite subgroups $\bigcup_{g\in G} B(G/Z(G), gZ(G))$. On the other hand, $Z(G) \cong \mathbb{Z}_p$ is torsionfree and thus does not contain any nonsingleton finite subgroups. Hence $B(G/Z(G), gZ(G)) = \{1\}$ for each $g \in G$ and thus $[G,G] = 1$. ■

This corollary does not extend to solvable compact JNL-groups and not even to metabelian ones:

Example 2.6. We take a prime number $p \neq 2$. Let $R \subseteq \mathbb{Z}_p \setminus p\mathbb{Z}$ denote the multiplicative group of $(p-1)$-th roots of unity.

(a) We form $G = \mathbb{Z}_p \rtimes R$ with R acting on $\mathbb{Z}_p$ by multiplication. Then G is a profinite centerfree metabelian group. Every nonsingleton normal subgroup of G contains one of the form $p^k\mathbb{Z}_p \times \{1\}$ and thus is open and has finite index. Therefore G is a compact JNL group which illustrates that solvability of compact JNL groups does not imply commutativity.

(b) Let $A = \mathbb{Z}_p^2$ be the free $\mathbb{Z}_p$-module of rank 2. Every closed (additive) subgroup of A is obviously a free $\mathbb{Z}_p$-module of rank at most 2. Any $\mathbb{Z}_p$-submodule of rank 2 is an open subgroup of A, and, equivalently, has finite index in A. A $\mathbb{Z}_p$-submodule of rank 1 of A is of the form $\mathbb{Z}_p \cdot (a,b)$ for some $a,b \in \mathbb{Z}_p$. Let Γ be a group of automorphisms of A with the matrix representations

$$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, \qquad \begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix}$$

where $a,b \in R$. We note that Γ is a group of monomial matrices and it is isomorphic to a semidirect product of the group R^2 by the cyclic group of order 2. In particular, $|\Gamma| = 2(p-1)^2$.

Now let

$$G = \Gamma \ltimes A$$

denote the semidirect product with respect to the natural action of Γ on A. We will see that

$$(**) \qquad G \text{ is a compact JNL group.}$$

Let N be a nonsingleton closed normal subgroup of G. We must show that N has finite index in G. Since it suffices to show that the normal subgroup $N \cap (A \times \{1\})$ has finite index in G, we may assume that $N = B \times \{1\}$, where B is a Γ-invariant $\mathbb{Z}_p$-submodule of A. We must show that rank B=2, for then B is open in A; therefore A/B is finite. Since $N \neq \{1\}$ we have rank $B > 0$. Now assume that rank B=1. We will derive a contradiction. Indeed we have $B = \mathbb{Z}_p \cdot (a,b)$ for suitable elements $a,b, \in \mathbb{Z}_p$, not both of which are zero. Since B is Γ-invariant, for each $\gamma \in \Gamma$ there is a nonzero $\lambda = \lambda_\gamma \in \mathbb{Z}_p$ such that $\gamma(a,b) = \lambda \cdot (a,b)$. If $b = 0$, then $a \neq 0$ and we let

$$\tau = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

whence $(\lambda_\tau a, 0) = \lambda_\tau(a, 0) = \tau(a, 0) = (0, a)$, which is impossible. Likewise $a = 0$ is impossible, and so $a \neq 0 \neq b$. Then $(\lambda_\tau a, \lambda_\tau b) = \lambda_\tau(a, b) = \tau(a, b) = (b, a)$, and so $\lambda_\tau = b/a = a/b$. We conclude that $(a+b)(a-b) = a^2 - b^2 = 0$, then either $a = b$ or $a = -b$. We set

$$\alpha = \begin{pmatrix} r & 0 \\ 0 & 1 \end{pmatrix}$$

for some $1 \neq r \in R$. The existence of such r is due to the fact that $p \neq 2$. Then, in the first case, $(\lambda_\alpha a, \lambda_\alpha a) = \lambda_\alpha(a, b) = \alpha(a, b) = (ra, b)$. We first conclude $\lambda_\alpha = 1$, then $a = ra$ and so $r = u$, a contradiction. In the second case, we obtain $\lambda_\alpha a = ra$, then $-\lambda_\alpha a = \lambda_\alpha b = b = -a$. So again we get $\lambda_\alpha = 1$ and $r = 1$. This final contradiction proves $(**)$.

By Theorem 2.1, a compact abelian JNL group is isomorphic to $\mathbb{Z}_p$. But in our case $A \not\cong \mathbb{Z}_p$ and so A is not a compact JNL group. The group G has the following properties:

(i) G is a solvable compact JNL group with a nonsingleton abelian normal open subgroup $A \times \{1\}$ of finite index in G which is not a compact JNL group.

(ii) G is a compact JNL group which is solvable of derived length 3. Moreover, G'' is abelian and $G'' \not\cong \mathbb{Z}_p$.

(iii) G is centerfree.

We saw that Corollary 2.5 on profinite nilpotent JNL groups does not extend to profinite solvable JNL groups. It does not extend to profinite pronilpotent JNL groups either as the example of the *Nottingham Group* shows:

Example 2.7. Let p be a prime number. Following [11, p. 66–67], $\mathbb{F}_p[t]$ denotes the formal power series algebra over the field with p elements $\mathbb{F}_p$ in an indeterminate t. Write A for the group of (continuous) automorphisms of $\mathbb{F}_p[t]$ and for each integer $n \geq 1$ let J_n be the kernel of the homomorphism from A to the automorphism group of $\mathbb{F}_p[t]/(t^{n+1})$, where (t^{n+1}) is the ideal generated by t^{n+1}. From [11, p. 66–67], we know that $G = J_1$ coincides with the inverse limit of J_1/J_n for $n \geq 1$ and each J_1/J_n is a finite group of order p^n. Thus G is a profinite pro-p and thus pronilpotent group. Moreover G is a centerfree profinite JNL group.

3. Miscellaneous results on compact JNL groups

We present a variety of observations which are apt to contribute to the development of an intuition for compact JNL groups. First the simple observation that a compact JNL group never has minimal normal subgroups.

Remark 3.1. Let G be a compact JNL group. If M is a nonsingleton closed normal subgroup of G, then G contains an open normal subgroup which is properly smaller than M.

Proof. The set $\mathcal{N}(G)$ of all normal subgroups such that G/N is a Lie group is a nontrivial filterbasis intersecting in $\{1\}$ while not containing $\{1\}$. If M is a nonsingleton closed normal subgroup of G, then $M \in \mathcal{N}(G)$. By Theorem 2.2, M is an identity neighborhood. Now there is an $N \in \mathcal{N}(G)$ such that $M \not\subseteq N$. Then $M \cap N \in \mathcal{N}(G)$ is properly smaller than M. ∎

One might surmise that in a compact JNL group an open normal subgroup of finite index should be itself a compact JNL-group. Example 2.6 (ii) shows that even in the solvable case this is not the case. We inspect a sufficient condition for the conjecture to hold. Let G be a compact group and H a closed subgroup. We call $H_G = \bigcap_{g\in G} gHg^{-1}$, that is, the largest normal subgroup contained in H, the *core* of H. We say that H is *core-free* if $H_G = \{1\}$. This terminology is standard for abstract groups and can be found for instance in [10].

Proposition 3.2. *Let G be a compact JNL group and $N \neq \{1\}$ a closed subgroup of G. If no nonsingleton closed normal subgroup of N is core-free in G, then N is a compact JNL group.*

Proof. Let $M \neq \{1\}$ be a closed normal subgroup of N. Then $M_G \neq \{1\}$ and so G/M_G is finite by Theorem 2.2. Thus the closed subgroup N/M_G is a Lie group, too. But then $N/M \cong (N/M_G)/(M/M_G)$ is finite as well. ∎

As we observed, even if N is an open normal subgroup it need not be a compact JNL group as Example 2.6 (ii) shows. The case that N is central, on the other hand is one in which 3.2 applies. as we saw in 2.4.

If G is as pronilpotent compact JNL-group, in view of 2.2 we see that G is the product of its Sylow subgroups $G \cong \prod_{p\in\mathbb{P}} S_p$ (see [11, Propositions 2.2.2, 2.3.2, 2.4.3]). It follows at once that only finitely many Sylow subgroups are nonsingleton and precisely one of them is infinite. This reduces the problem of classifying all compact pronilpotent JNL groups to the corresponding problem for pro-p groups (p is a prime). However, we appear to know next to nothing about such a classification.

If G is a compact JNL group and N a pronilpotent nonsingleton normal subgroup, then since the Sylow subgroups S_p of N are characteristic, they are normal in G and it then follows that G is a finite extension of an infinite pro-p group. In particular, this applies to the case that N is abelian.

The question when a compact JNL group G splits semidirectly over one of its open finite index subgroups N does not lead very far, except the very special case that N is abelian and, according to the preceding remark is isomorphic to a finite product $S_{p_0} \times S_{p_1} \times \cdots \times S_{p_n}$ where S_{p_0} is an infinite abelian pro-p_0 group and the S_{p_k} are finite p_k groups for $1 \leq k \leq n$. If each of the primes dividing G/N differs from $p_0, \ldots, p_n$, then indeed G splits semidirectly over N by [2, Satz III].

Still, the following general observation shows, for instance, that the weight of a compact JNL group is $\aleph_0$.

Proposition 3.3. *Let G be a compact* JNL *group. There is a descending sequence*

$$G = G_1 \geq G_2 \geq G_3 \geq \ldots \geq \{1\}$$

of closed normal subgroups of G converging to 1 such that G_n/G_{n+1} is a finite product of finite simple groups or groups of prime order, for each positive integer $n \geq 1$. In particular, G is a second countable and thus metric profinite group.

Proof. The totally disconnected compact JNL group G cannot be finite, since it is not a Lie group. Then it has a descending family of compact normal subgroups

$$G = G_1 \geq G_2 \geq G_3 \geq \ldots$$

converging to 1, such that each factor group G_n/G_{n+1} is a product of simple groups or groups of prime order, for each positive integer $n \geq 1$ (see [1, Theorem 9.91]). Since G is a compact JNL group, by 2.2 it follows recursively that G_n is a nonsingleton finite product of finite simple groups or groups of prime order and the claim follows. ■

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