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Regular Connections on Principal Fiber Bundles over the Infinitesimal Punctured Disc

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Abstract. This paper concerns regular connections on trivial algebraic G-principal fiber bundles over the infinitesimal punctured disc, where G is a connected reductive linear algebraic group over an algebraically closed field of characteristic zero. We show that the pull-back of every regular connection to an appropriate covering of the infinitesimal punctured disc is gauge equivalent to a connection of the form $Xz^{-1}\,\mathrm{d}z$ for some X in the Lie algebra of G. We may even arrange that the only rational eigenvalue of $\mathrm{ad}\,X$ is zero. Our results allow a classification of regular SL_n-connections up to gauge equivalence.
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1. Introduction

Let G be a linear algebraic group over an algebraically closed field k of characteristic zero, and let $\mathfrak{g}$ be its Lie algebra. The loop group $G((z)) = G(k((z)))$ acts as a gauge group on the set $\mathcal{G} = \mathfrak{g} \otimes_k k((z))\,\mathrm{d}z$ of connections on the trivial algebraic G-principal fiber bundle over the infinitesimal punctured disc. For $G = \mathrm{GL}_n$, this action is given by

$$g\,[A\,\mathrm{d}z] = \left(gAg^{-1} + \tfrac{\partial}{\partial z}(g)g^{-1}\right)\,\mathrm{d}z,$$

where $g \in \mathrm{GL}_n((z))$, $A \in \mathfrak{gl}_n \otimes_k k((z))$, and $\frac{\partial}{\partial z}$ acts on each entry of the matrix g. If $G \subset \mathrm{GL}_n$ is a closed subgroup, this action induces an action of $G((z))$ on $\mathcal{G}$. According to [1, §8.2, Definition], a connection is regular if it is gauge equivalent to an element of $\mathfrak{g} \otimes_k k[[z]]z^{-1}\,\mathrm{d}z$. For a positive integer m, we define the inclusion $m^* : k((z)) \hookrightarrow k((z))$ by $m^*(f(z)) = f(z^m)$. Geometrically, it corresponds to an m-fold covering of the infinitesimal punctured disc. We can pull back every connection A to a connection $m^*(A)$. Now the main results of this article can be formulated.

Theorem 4.2. *Let G be a connected reductive linear algebraic group. There exists a positive integer m, such that for every regular connection A the pull-back connection $m^*(A)$ is gauge equivalent to $Xz^{-1}\,\mathrm{d}z$ for a suitable $X \in \mathfrak{g}$.*

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In [1, §8.4 (c)], a similar statement is given for any affine algebraic group over the complex numbers. Its proof, however, uses analytic methods. Here, we give a purely algebraic proof for a connected reductive group.

Theorem 4.10. *Let G be a connected reductive linear algebraic group. For every regular connection A, there exists a positive integer m and an element $X \in \mathfrak{g}$, such that the only rational eigenvalue of $\operatorname{ad} X$ is zero and the pull-back connection $m^*(A)$ is gauge equivalent to $Xz^{-1}\,\mathrm{d}z$.*

The proofs of Theorems 4.2 and 4.10 are mainly based on the structure theory of the group G and its Lie algebra $\mathfrak{g}$. We use these theorems and Galois cohomology in order to get a classification of regular SL_n-connections up to $\mathrm{SL}_n((z))$-equivalence, see Theorem 7.10 and Remarks 7.11, 7.12. Our classification strategy is motivated by [1]. Some steps of this strategy can also be applied to other connected reductive linear algebraic groups, see [5].

This paper is organized as follows. Using results from [2], we define in Section 2 the action of the gauge group on the space of connections intrinsically, i. e., without choosing a closed embedding of G into some GL_n. We repeat the definitions of regular and aligned connections given in [1] and recall that every regular connection is gauge equivalent to an aligned connection. In Section 3, we explain how to pull back connections. We call two connections related, if they become gauge equivalent in some covering. Using Steinberg's theorem (cf. [10]), we show that for a connected group G the relatives of a connection A up to gauge equivalence correspond bijectively to a set $\mathrm{H}^1(K;A)$ defined via Galois cohomology. We prove our main Theorems 4.2 and 4.10 in Section 4. Section 5 contains the classification of regular GL_n-connections up to $\mathrm{GL}_n((z))$-equivalence. This is classical, see for example [1, §3]. We explain the relation between n-dimensional (Fuchsian) D-modules (see [4]) and (regular) GL_n-connections. If we translate the classification into the language of D-modules, we obtain a result of Manin ([4]). In Section 6 we give an explicit description of the semisimple conjugacy classes in the centralizer $\mathrm{Z}_{\mathrm{GL}_n}(X)$, where X is an element of the Lie algebra $\mathfrak{gl}_n$. We use the results of the previous sections in Section 7 in order to classify regular SL_n-connections up to relationship and up to gauge equivalence. The definition of a Fuchsian connection in Section 8 appears to be more natural than the definition of a regular connection. It is based upon the notion of a Fuchsian D-module given in [4]. We show that regular connections are Fuchsian, and that for $G = \mathrm{GL}_n$ and $G = \mathrm{SL}_n$, Fuchsian connections are regular.

This paper is a condensed version of my diploma thesis [5], written in Freiburg in 2002/2003. I would like to thank Wolfgang Soergel. He taught me a lot of the mathematics I know.

2. Connections and Gauge Group

2.1. Conventions. We fix an algebraically closed field k of characteristic zero and write $\otimes = \otimes_k$, $\mathrm{Hom} = \mathrm{Hom}_k$, and so on. If we discuss vector spaces, Lie algebras, linear algebraic groups, or the like, we always mean the corresponding structures defined over k. We denote the positive integers by $\mathbb{N}^+$ and define $\mathbb{N} = \mathbb{N}^+ \cup \{0\}$.

Let $\mathcal{O} = k[[z]]$ be the ring of formal power series over k, with maximal ideal $\mathfrak{m} = zk[[z]]$ and the induced $\mathfrak{m}$-topology. The quotient field of $\mathcal{O}$ is the field $K = k((z))$ of Laurent series over k. Let $D = k((z))\,\partial_z$ denote the continuous k-linear derivations from K to K, where we abbreviate $\partial_z = \frac{\partial}{\partial z}$. Define $\Omega = \mathrm{Hom}_K(D, K) = k((z))\,\mathrm{d}z$, where $\mathrm{d}z$ is dual to ∂_z, i. e., $\mathrm{d}z(\partial_z) = 1$.

2.2. Action of the Gauge Group on the Space of Connections. We collect some results from [2, II, §4], in particular a definition of the Lie algebra. These will enable us to define the action of the gauge group on the space of connections in an intrinsic way. The reader who is not interested in this intrinsic definition may use Equation (2) in Example 2.3 as the definition for closed subgroups of GL_n. He may check that this is well defined and does not depend on the closed embedding.

Let G be a linear algebraic group. We consider G as an affine algebraic group scheme. Suppose that R is a k-algebra. We denote the algebra of dual numbers by $R[\varepsilon] = R \oplus R\varepsilon$. By applying the group functor G to the unique R-algebra homomorphism from R to $R[\varepsilon]$, we consider $G(R)$ as a subgroup of $G(R[\varepsilon])$. Define the R-algebra homomorphism $p : R[\varepsilon] \to R$ by $p(\varepsilon) = 0$. As explained in [2], the kernel $\mathfrak{g}(R)$ of the group homomorphism $G(p) : G(R[\varepsilon]) \to G(R)$ is endowed with a Lie algebra structure over R. The Lie algebra $\mathfrak{g} = \mathfrak{g}(k)$ is canonically isomorphic to the standard Lie algebra of the linear algebraic group G. The obvious inclusion $k \hookrightarrow R$ induces a homomorphism of Lie algebras $\mathfrak{g} \to \mathfrak{g}(R)$. Tensoring with R gives a canonical homomorphism $R \otimes \mathfrak{g} \to \mathfrak{g}(R)$ of Lie algebras over R. This is an isomorphism, as our group scheme G is locally algebraic. Hence we identify $R \otimes \mathfrak{g} = \mathfrak{g}(R)$.

We define an action of the gauge group $G(K)$ on the set

$$\mathcal{G} = \mathrm{Hom}_K(D, \mathfrak{g}(K))$$

of connections on the trivial algebraic G-principal fiber bundle over the infinitesimal punctured disc. Every derivation $s \in D$ gives rise to a k-algebra homomorphism $\hat{s} : K \to K[\varepsilon]$ defined by $\hat{s}(f) = f + s(f)\varepsilon$. We get a group homomorphism $\hat{s} = G(\hat{s}) : G(K) \to G(K[\varepsilon])$. For $g \in G(K)$, define $\dot{g} : D \to G\,(K[\varepsilon])$ by $\dot{g}(s) = \hat{s}(g)$. For all $g \in G(K)$ and $s \in D$, we deduce from $p \circ \hat{s} = \mathrm{id}_K$ that $\dot{g}(s)g^{-1}$ is an element of the kernel of $G(p)$, i. e., an element of $\mathfrak{g}(K)$.

Proposition 2.1. *The map*

$$G(K) \times \mathcal{G} \to \mathcal{G}, \quad (g, A) \mapsto g[A] = (\mathrm{Ad}\, g) \circ A + (\cdot g^{-1}) \circ \dot{g},$$

defines an action of the gauge group $G(K)$ on the space of connections $\mathcal{G}$. Here, $(\cdot h)$ denotes right multiplication by a group element h.

If $f : G \to H$ is a homomorphism of linear algebraic groups, we have $f_*(g[A]) = f(g)[f_*(A)]$, where $f_* : \mathcal{G} \to \mathcal{H}$ is the obvious map.

Proof. We have to show that, for $g \in G(K)$ and $A \in \mathcal{G}$, the map $g[A]$ from D to $\mathfrak{g}(K)$ is K-linear. In order to do this, use the definition in [2] of the K-vector space structure on $\mathfrak{g}(K)$. The explicit arguments can be found in [5]. The remaining claims are obvious.

There is another way of proving the K-linearity of the map $g[A]$: By choosing a closed embedding $G \hookrightarrow \mathrm{GL}_n$, we reduce to the case $G = \mathrm{GL}_n$. Then an easy calculation based on Equation (2) in Example 2.3 shows the K-linearity. ■

Definition 2.2. Let $H \subset G(K)$ be a subgroup. Two connections A, $B \in \mathcal{G}$ are *H-equivalent*, if there is an element $h \in H$ such that $h[A] = B$. They are *gauge equivalent*, if they are $G(K)$-equivalent.

Note that canonically

$$\mathcal{G} = \operatorname{Hom}_K(D, \mathfrak{g}(K)) = \mathfrak{g}(K) \otimes_K \operatorname{Hom}_K(D, K) = \mathfrak{g}(K) \otimes_K \Omega.$$

Since $\Omega = K\,\mathrm{d}z$, each element of $\mathfrak{g}(K) \otimes_K \Omega$ can be written uniquely in the form $A \otimes \mathrm{d}z$ with $A \in \mathfrak{g}(K)$. We abbreviate $A\,\mathrm{d}z = A \otimes \mathrm{d}z$ and write $\mathcal{G} = \mathfrak{g}(K)\,\mathrm{d}z$ accordingly. We define $\operatorname{dlog} z = z^{-1}\,\mathrm{d}z \in \Omega$ and write similarly $A \operatorname{dlog} z$ and $\mathcal{G} = \mathfrak{g}(K) \operatorname{dlog} z$. If $B \in \mathcal{G} = \operatorname{Hom}_K(D, \mathfrak{g}(K))$ is a connection, we have $B = B(\partial_z)\,\mathrm{d}z = B(z\,\partial_z) \operatorname{dlog} z$.

We use the abbreviations $G((z))$ (resp. $\mathfrak{g}((z))$, $\mathfrak{g}[z]$) for $G(k((z)))$ (resp. $\mathfrak{g}(k((z)))$, $\mathfrak{g}(k[z])$), and so on.

For a connection $A \operatorname{dlog} z \in \mathfrak{g}((z)) \operatorname{dlog} z$ and a gauge transformation $g \in G((z))$, we get

$$g[A \operatorname{dlog} z] = \Big((\operatorname{Ad} g)(A) + \widehat{z\,\partial_z}(g) g^{-1} \Big) \operatorname{dlog} z. \tag{1}$$

Example 2.3. Consider the case $G = \mathrm{GL}_n$. Let $A \in \operatorname{Mat}_n(k((z))) = \mathfrak{gl}_n((z))$ and $g \in \mathrm{GL}_n((z))$ be given. As $\widehat{z\,\partial_z}(g) = g + z\,\partial_z(g)\varepsilon$ is satisfied in $\mathrm{GL}_n(k((z))[\varepsilon])$, we get

$$g\,[A \operatorname{dlog} z] = \big(gAg^{-1} + z\,\partial_z(g) g^{-1} \big) \operatorname{dlog} z. \tag{2}$$

2.3. Alignment of Regular Connections. Let $X \in \mathfrak{g}$ and $\lambda \in k$. We denote the eigenspace of $\operatorname{ad} X$ corresponding to λ by $\operatorname{Eig}(\operatorname{ad} X; \lambda) = \ker(\operatorname{ad} X - \lambda)$. Let $X = X_{\mathrm{s}} + X_{\mathrm{n}}$ be the Jordan decomposition in $\mathfrak{g} = \operatorname{Lie} G$.

Definition 2.4. (cf. [1, §8.2, Definition, and §8.5, Definition]) The elements of $\mathfrak{g}[[z]] \operatorname{dlog} z$ are called *connections of the first kind*. A connection is *regular*, if it is gauge equivalent to a connection of the first kind. A connection of the first kind $A = \sum_{r \in \mathbb{N}} A_r z^r \operatorname{dlog} z$ is *aligned*, if $A_r \in \operatorname{Eig}(\operatorname{ad} A_{0,\mathrm{s}}; r)$ for all $r \in \mathbb{N}$.

Theorem 2.5. ([1, §8.5, Proposition]) *Every regular connection is gauge equivalent to an aligned connection.*

For a worked out proof that uses the exponential map as defined in [2, II, §6, 3], see [5].

3. Relatives and Galois Cohomology

3.1. Pull-Back of Gauge Transformations and Connections.

Definition 3.1. Let E and F be fields, $\phi : E \to F$ a map, V an E-vector space, and W an F-vector space. By a *ϕ-linear* map $f : V \to W$ we mean a group homomorphism $f : V \to W$ such that $f(ev) = \phi(e) f(v)$ for all $e \in E$, $v \in V$.

Definition 3.2. For $m \in \mathbb{N}^+$ we define the field extension $m^* : K \hookrightarrow K = M$ by $m^*(f(z)) = f(z^m)$.

Let G be a linear algebraic group, let $l,\ m \in \mathbb{N}^+$, and let $\phi : l^* \to m^*$ be a morphism of field extensions, i.e. $\phi : K \to K$ is a ring homomorphism satisfying $\phi \circ l^* = m^*$. The map $\phi = G(\phi) : G(K) \hookrightarrow G(K)$ is called ϕ-pull-back of gauge transformations. There is a unique ϕ-linear map $\Phi : \Omega \to \Omega$ such that $\mathrm{d} \circ \phi = \Phi \circ \mathrm{d}$, where the derivation $\mathrm{d} : K \to \Omega$ is defined by $\mathrm{d}x(\delta) = \delta(x)$, for $x \in K$, $\delta \in D$. We denote this injective map by ϕ and call it ϕ-pull-back of 1-forms. This map ϕ and the ϕ-linear homomorphism of Lie algebras $\mathfrak{g}(\phi) : \mathfrak{g}(K) \hookrightarrow \mathfrak{g}(K)$ induce an injective ϕ-linear map

$$\mathfrak{g}(\phi) \otimes \phi : \mathcal{G} = \mathfrak{g}(K) \otimes_K \Omega \hookrightarrow \mathcal{G} = \mathfrak{g}(K) \otimes_K \Omega.$$

We denote this map simply by ϕ and call it ϕ-pull-back of connections.

Example 3.3. In particular, for $l = 1$ and $\phi = m^*$, we have $m^*(\operatorname{dlog} z) = m \operatorname{dlog} z$, and therefore, for $A(z) \in \mathfrak{g}((z))$,

$$m^*(A(z) \operatorname{dlog} z) = mA(z^m) \operatorname{dlog} z. \tag{3}$$

Proposition 3.4. *Under the above assumptions we have* $\phi(g)[\phi(A)] = \phi(g[A])$ *for all* $g \in G(K)$, $A \in \mathcal{G}$.

Proof. Use a closed embedding $G \subset \mathrm{GL}_n$ and Equation (2) in Example 2.3. For a more intrinsic proof we refer to [5]. ∎

Definition 3.5. Two connections A and B are *related* if there is $m \in \mathbb{N}^+$ such that $m^*(A)$ and $m^*(B)$ are gauge equivalent. This defines an equivalence relation on the set of connections, and we define $\mathrm{Rel}(A)$ to be the equivalence class containing A. The elements of $\mathrm{Rel}(A)$ are called the *relatives* of A.

3.2. Connections and Galois Cohomology. The results of this subsection are used only in Section 7.

For $m \in \mathbb{N}^+$ let $\Gamma_m = \mathrm{Gal}(m^*)$ be the Galois group of the field extension $m^* : K \hookrightarrow K = M$. The map $\Gamma_m \times \mathcal{G} \to \mathcal{G}$, $(\sigma, A) \mapsto \sigma(A)$, given by the pull-back of connections, is an action of Γ_m on $\mathcal{G}$.

Lemma 3.6. *Let* $m \in \mathbb{N}^+$. *A connection* C *is* Γ_m*-invariant if and only if there is a connection* D *such that* $C = m^*(D)$.

Proof. From $\sigma \circ m^* = m^*$, we deduce that $\sigma(m^*(D)) = m^*(D)$ for all $\sigma \in \Gamma_m$ and all connections D. Conversely, suppose that $C(z) \operatorname{dlog} z \in \mathcal{G}$ is Γ_m-invariant. For ω a primitive m-th root of unity, we define $(\omega\cdot)^* \in \Gamma_m$ by $(\omega\cdot)^*(f(z)) = f(\omega z)$. From $(\omega\cdot)^*(C(z) \operatorname{dlog} z) = C(\omega z) \operatorname{dlog} z$ we see that $C(z) = B(z^m)$ for some $B \in \mathfrak{g}((z))$. Equation (3) implies that the connection $D = m^{-1}B(z) \operatorname{dlog} z$ satisfies $m^*(D) = C(z) \operatorname{dlog} z$. ∎

Definition 3.7. Let C be a connection, $m \in \mathbb{N}^+$. An *m-form* of C is a connection D such that $m^*(D)$ and C are gauge equivalent.

Fix a Γ_m-invariant connection A. The action of the Galois group Γ_m on $K = M$ defines an action on $G(M)$ by group automorphisms. By Proposition 3.4, this action restricts to an action on the stabilizer $G(M)_A$ of the Γ_m-invariant connection A. We define a map

$$p = p(A) : \{m\text{-forms of } A\} \to \mathrm{H}^1(\Gamma_m; G(M)_A), \quad B \mapsto p^B,$$

as follows. Given an m-form B of A, choose $b \in G(M)$ such that $b[m^*(B)] = A$. Then the map $p^b : \Gamma_m \to G(M)_A$ defined by $p^b_\sigma = b\sigma(b^{-1})$ is a 1-cocycle, and its cohomology class $p^B = [p^b]$ does not depend on the choice of b.

Theorem 3.8. *For $m \in \mathbb{N}^+$, consider the field extension $m^* : K \hookrightarrow K = M$. If A is a Γ_m-invariant connection, the map $p = p(A)$ defined above descends to an injection*

$$\overline{p} = \overline{p(A)} : \{m\text{-forms of } A\} / G(K) \hookrightarrow \mathrm{H}^1(\Gamma_m; G(M)_A).$$

If $\mathrm{H}^1(\Gamma_m; G(M)) = \{1\}$ this map $\overline{p}$ is bijective.

Proof. Let $\Gamma = \Gamma_m$. Let B and C be gauge equivalent m-forms of A. Let $g \in G(K)$ with $B = g[C]$, and let $b \in G(M)$ with $b[m^*(B)] = A$. Consequently, we have $(bm^*(g))[m^*(C)] = A$. Now

$$p^{bm^*(g)}_\sigma = bm^*(g)\sigma(m^*(g)^{-1}b^{-1}) = b\sigma(b^{-1}) = p^b_\sigma$$

implies that $p^B = p^C$. Thus our map p descends.

Let B and C be m-forms of A. Choose b, $c \in G(M)$ with $b[m^*(B)] = c[m^*(C)] = A$. If p^b and p^c become equal in $\mathrm{H}^1(\Gamma, G(M)_A)$, there is an element $f \in G(M)_A$ such that $b\sigma(b^{-1}) = fc\sigma(c^{-1}f^{-1})$ for all $\sigma \in \Gamma$. We deduce that the element $c^{-1}f^{-1}b$ is Γ-invariant and hence equal to $m^*(h)$ for some $h \in G(K)$. But then

$$m^*(h[B]) = m^*(h)[m^*(B)] = c^{-1}f^{-1}b[m^*(B)] = m^*(C)$$

and $h[B] = C$ since m^* is injective. So $\overline{p}$ is injective.

Suppose that $\mathrm{H}^1(\Gamma; G(M)) = \{1\}$. We now prove that p is surjective. Let $a \in \mathrm{Z}^1(\Gamma; G(M)_A)$ be a 1-cocycle. By assumption, a considered as an element of $\mathrm{Z}^1(\Gamma; G(M))$ is cohomologous to the trivial 1-cocycle, i. e., there is $g \in G(M)$ such that $a_\sigma = g\sigma(g^{-1})$ for all $\sigma \in \Gamma$. Let $C = g^{-1}[A]$. For any $\sigma \in \Gamma$, we get

$$\sigma(C) = \sigma(g^{-1})[A] = (g^{-1}a_\sigma)[A] = C.$$

According to Lemma 3.6, there is a connection D such that $m^*(D) = C$. As $g[m^*(D)] = A$, we have $p^D = [p^g] = [a]$. ∎

Remark 3.9. In the cases $G = \mathrm{GL}_n$, $G = \mathrm{SL}_n$ or $G = \mathrm{Sp}_{2n}$ we know that $\mathrm{H}^1(\Gamma_m; G(M)) = \{1\}$, according to [6, X, §1 and §2].

We now explain that, more generally, for every connected linear algebraic group G we have $\mathrm{H}^1(\Gamma_m; G(M)) = \{1\}$. It is well known [6, IV, §2, Proposition 8 and Corollary] that the union $\overline{K} = \bigcup_{m \in \mathbb{N}^+} k((z^{1/m}))$ is an algebraic closure of $K = k((z))$. Thus, every algebraic extension of K ramifies, i. e., K itself is the maximal unramified extension of K. According to [3, Theorem 12] and [7, II, §3.2 Corollary], the maximal unramified extension of a field that is complete with respect to a discrete valuation and that has perfect residue class field has dimension ≤ 1. Consequently, we have $\dim K \leq 1$.

Let G be a connected group. Since $\overline{K} \otimes k[G]$ is an integral domain, $G(\overline{K})$ is a connected linear algebraic group over $\overline{K}$, and $K \otimes k[G]$ defines a K-structure on $G(\overline{K})$ in the sense of [9]. By [9, Theorem 17.10.2] (cf. [10] and [7, III, §2.3, Theorem 1']), we know that

$$\mathrm{H}^1(\mathrm{Gal}(\overline{K}/K); G(\overline{K})) = \{1\}.$$

For $m \in \mathbb{N}^+$, we view the field extension $m^* : K \hookrightarrow K = M$ as a subextension of $K \subset \overline{K}$ via the embedding $K = M \hookrightarrow \overline{K}$, $f(z) \mapsto f(z^{1/m})$. As the canonical map from $\mathrm{H}^1(\Gamma_m; G(M))$ to the direct limit

$$\mathrm{H}^1(\mathrm{Gal}(\overline{K}/K); G(\overline{K})) = \varinjlim_{m \in \mathbb{N}^+} \mathrm{H}^1(\Gamma_m; G(M))$$

is injective, we deduce that $\mathrm{H}^1(\Gamma_m; G(M)) = \{1\}$.

An easy calculation, left to the reader, Theorem 3.8, and Remark 3.9 yield

Proposition 3.10. *Let l, $m \in \mathbb{N}^+$ with $d = m/l \in \mathbb{N}^+$. Consider the commutative diagram of field extensions*

$$\begin{array}{ccc} K = L & \xrightarrow{d^*} & K = M \\ & {\scriptstyle l^*}\nwarrow \quad \nearrow{\scriptstyle m^*} & \\ & K & \end{array}$$

If A is a connection, the injection

$$d^* : \mathrm{Z}^1(\Gamma_l; G(L)_{l^*(A)}) \hookrightarrow \mathrm{Z}^1(\Gamma_m; G(M)_{m^*(A)}),$$

defined by $(d^*(p) : \sigma \mapsto d^*(p_{\sigma|_L})$, induces an injection $d^* = (m/l)^*$ on cohomology. Furthermore, the diagram

$$\begin{array}{ccc} \{l\text{-forms of } l^*(A)\}/G(K) & \xrightarrow{\overline{p(l^*(A))}} & \mathrm{H}^1(\Gamma_l; G(L)_{l^*(A)}) \\ \big\downarrow & & \big\downarrow {\scriptstyle (m/l)^*} \\ \{m\text{-forms of } m^*(A)\}/G(K) & \xrightarrow{\overline{p(m^*(A))}} & \mathrm{H}^1(\Gamma_m; G(M)_{m^*(A)}) \end{array}$$

commutes. If G is connected, the horizontal maps in this diagram are bijective.

Definition 3.11. Let A be a connection. The pointed sets $\mathrm{H}^1(\Gamma_m; G(M)_{m^*(A)})$, for $m \in \mathbb{N}^+$, together with the maps $(m/l)^*$, for l, $m \in \mathbb{N}^+$ with l divides m, form a directed system. We denote its direct limit by $\mathrm{H}^1(K; A)$.

Proposition 3.12. *Let A be a connection. The map* $\mathrm{Rel}(A)/G(K) \hookrightarrow \mathrm{H}^1(K;A)$ *that is induced by the maps* $\overline{p(m^*(A))}$, *for* $m \in \mathbb{N}^+$, *is injective. It is bijective if* G *is a connected group.*

Proof. The set of relatives of A is the union, i. e., the direct limit, of all m-forms of all connections $m^*(A)$ for $m \in \mathbb{N}^+$. Now use Proposition 3.10. ∎

4. Transforming Regular Connections

4.1. Transforming Regular Connections to Standard Form.

Definition 4.1. Let G be a linear algebraic group. Elements of $\mathfrak{g}\,\mathrm{dlog}\,z$ are called *connections in standard form.*

Theorem 4.2. *Let G be a connected reductive linear algebraic group. There exists a positive integer $m \in \mathbb{N}^+$, such that for every regular connection A the pull-back connection $m^*(A)$ is gauge equivalent to a connection in standard form.*

Corollary 4.3. *If G is connected reductive, each regular connection is related to a connection in standard form.*

Proof. If the pull-back $m^*(A)$ of a connection A is gauge equivalent to $X\,\mathrm{dlog}\,z$ for some $X \in \mathfrak{g}$, then Equation (3) shows that A is related to $m^{-1}X\,\mathrm{dlog}\,z$. ∎

Examples 4.4. 1. For $G = \mathrm{GL}_n$, we can choose $m = 1$, as one can see from the choice of m in the proof of Theorem 4.2.

2. Let $k = \mathbb{C} \subset K = \mathbb{C}((z))$ and $G = \mathrm{SL}_2$. By the proof of Theorem 4.2, we are able to choose $m = 2$. But one has to choose $m > 1$. Given $n \in \mathbb{N}^+$, we define the regular aligned SL_2-connection

$$A_n = \begin{bmatrix} \frac{n}{2} & z^n \\ 0 & -\frac{n}{2} \end{bmatrix} \mathrm{dlog}\,z.$$

Thanks to [1, §8.2 Example] we know that A_n is $\mathrm{SL}_2(K)$-equivalent to a connection in standard form if and only if n is even. If k is algebraically closed of characteristic $\neq 2$, the same statement can be verified by direct computation.

We now recall some results from [9] and [8]. Let G be a connected reductive linear algebraic group, $T \subset G$ a maximal torus and $(\mathcal{X}(T), \mathcal{R}, \mathcal{X}^\vee(T), \mathcal{R}^\vee)$ the associated root datum. The derived group $G' = (G, G)$ is semisimple. Let T' be the subgroup of T generated by the images of all $\alpha^\vee$, $\alpha \in \mathcal{R}$. To the maximal torus T' in G' we associate the root datum $(X(T'), R, X^\vee(T'), R^\vee)$. The restriction map $\mathcal{X}(T) \twoheadrightarrow X(T')$ induces a canonical identification $\mathcal{R} = R$.

On the Lie algebra level, the reductive Lie algebra $\mathfrak{g}$ is the direct sum of its center $\mathfrak{z}$ and the semisimple Lie algebra $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}'$. We have $\mathfrak{t} = \mathfrak{z} \oplus \mathfrak{t}'$. We associate to the Cartan subalgebra $\mathfrak{t}'$ in $\mathfrak{g}'$ the roots $\underline{R}$ in $\mathfrak{t}'^* = \mathrm{Hom}(\mathfrak{t}', k)$. Let $\underline{R}^\vee \subset \mathfrak{t}'$ denote the coroots. There are canonical identifications $R = \underline{R}$ and $R^\vee = \underline{R}^\vee$.

By taking the derivative at the unit element, every root $\alpha \in R = \mathcal{R}$ can be considered as an element of $\mathfrak{t}^*$. For $H \in \mathfrak{t}$ and $\alpha \in R$, we have $\langle \alpha, H \rangle = \langle \alpha, H' \rangle$ if we decompose H as $H = H'' + H' \in \mathfrak{z} \oplus \mathfrak{t}' = \mathfrak{t}$.

Suppose that $H \in \mathfrak{t}$ is arbitrary. Let

$$R_H^{\mathbb{Z}} = \{\alpha \in R \mid \langle \alpha, H \rangle \in \mathbb{Z}\} \subset R$$

denote the roots integral on H, and let

$$R_H^{\mathbb{Z}\vee} = \{\alpha^\vee \mid \alpha \in R_H^{\mathbb{Z}}\} \subset R^\vee$$

denote the corresponding coroots. Define

$$V_H = \mathbb{Q}R_H^{\mathbb{Z}} \subset \mathfrak{t}'^* \quad \text{and} \quad V_H^\vee = \mathbb{Q}R_H^{\mathbb{Z}\vee} \subset \mathfrak{t}'.$$

The roots integral on H are a root system $R_H^{\mathbb{Z}}$ in V_H. The canonical map

$$V_H^\vee \xrightarrow{\sim} \operatorname{Hom}_{\mathbb{Q}}(V_H, \mathbb{Q}), \quad \lambda \mapsto \lambda|_{V_H},$$

is an isomorphism of $\mathbb{Q}$-vector spaces. Therefore, we identify $V_H^\vee = \operatorname{Hom}_{\mathbb{Q}}(V_H, \mathbb{Q})$.

To the dual root system $R_H^{\mathbb{Z}\vee}$ in $V_H^\vee$, we associate the root lattice $Q(R_H^{\mathbb{Z}\vee}) = \mathbb{Z}R_H^{\mathbb{Z}\vee}$. It is a subgroup of finite index in the weight lattice

$$P(R_H^{\mathbb{Z}\vee}) = \{x \in V_H^\vee \mid \langle \alpha, x \rangle \in \mathbb{Z} \quad \text{for all } \alpha \in R_H^{\mathbb{Z}}\}.$$

From $Q(R_H^{\mathbb{Z}\vee}) \subset X^\vee(T') \subset \mathcal{X}^\vee(T)$, we obtain that

$$\left|P(R_H^{\mathbb{Z}\vee})/Q(R_H^{\mathbb{Z}\vee})\right| \in \{m \in \mathbb{N}^+ \mid m \cdot P(R_H^{\mathbb{Z}\vee}) \subset \mathcal{X}^\vee(T)\}.$$

In particular, the set on the right hand side is not empty. We denote its minimum by m_H. Note that the sets $\{R_H^{\mathbb{Z}\vee} \mid H \in \mathfrak{t}\}$ and $\{m_H \mid H \in \mathfrak{t}\}$ are finite.

Proposition 4.5. *Let $m = \operatorname{lcm}\{m_H \mid H \in \mathfrak{t}\}$ be the least common multiple of all m_H. For every $H \in \mathfrak{t}$, there is a cocharacter $\psi \in \mathcal{X}^\vee(T)$, such that $\langle \alpha, \psi \rangle = m\langle \alpha, H \rangle$ for all $\alpha \in R_H^{\mathbb{Z}}$.*

This proposition is a consequence of

Lemma 4.6. *Let $H \in \mathfrak{t}$. There is a cocharacter $\phi \in \mathcal{X}^\vee(T)$, such that $\langle \alpha, \phi \rangle = m_H\langle \alpha, H \rangle$ for all $\alpha \in R_H^{\mathbb{Z}}$.*

Proof. Let $B_H = \{\alpha_1, \ldots, \alpha_l\}$, where $\alpha_i \neq \alpha_j$ for $i \neq j$, be a basis of the root system $R_H^{\mathbb{Z}}$ in V_H. Let $\{\varpi_1^\vee, \ldots, \varpi_l^\vee\}$ be the basis dual to B_H in $V_H^\vee$, characterized by $\langle \alpha_i, \varpi_j^\vee \rangle = \delta_{ij}$ for all $i, j \in \{1, \ldots, l\}$. Define $\tau = \sum_{i=1}^l \langle \alpha_i, H \rangle \varpi_i^\vee$. As τ is an element of $P(R_H^{\mathbb{Z}\vee}) = \bigoplus_{i=1}^l \mathbb{Z}\varpi_i^\vee$, the definition of m_H shows that $\phi = m_H\tau$ is an element of $\mathcal{X}^\vee(T)$. We have $\langle \alpha_j, \phi \rangle = m_H\langle \alpha_j, H \rangle$ for all $j \in \{1, \ldots, l\}$. As B_H is a basis of $R_H^{\mathbb{Z}}$, the claim follows. ∎

Proof. (of Theorem 4.2) Let $m \in \mathbb{N}^+$ be the smallest positive integer that satisfies the following condition.

For all $H \in \mathfrak{t}$, there is $\psi \in \mathcal{X}^\vee(T)$, such that $\langle \alpha, \psi \rangle = m\langle \alpha, H \rangle$ for all $\alpha \in R_H^{\mathbb{Z}}$.

By Proposition 4.5, this is well defined. Let A be a regular connection. By Theorem 2.5, we may assume that A is aligned. Therefore, there are elements $N \in \mathbb{N}$ and $A_0, \ldots, A_N \in \mathfrak{g}$ such that

$$A = \left(A_0 + A_1 z + \cdots + A_N z^N\right) \operatorname{dlog} z \quad \text{and}$$
$$A_r \in \operatorname{Eig}(\operatorname{ad} A_{0,\mathrm{s}}; r) \quad \text{for all } r \in \{0, \ldots, N\}.$$

Decompose $A_r = A_r'' + A_r' \in \mathfrak{z} \oplus \mathfrak{g}' = \mathfrak{g}$ for $r \in \{0, \ldots, N\}$. Since all Cartan subalgebras of a semisimple Lie algebra are conjugate under the adjoint group, we find $x \in G'$ such that $(\operatorname{Ad} x)(A_{0,\mathrm{s}}') \in \mathfrak{t}'$. As $x \in G'(k) \subset G(K)$, we have $\widehat{z\,\partial_z}(x) = x$, i. e., $\widehat{z\,\partial_z}(x)x^{-1} = 0$ in $\mathfrak{g}(K)$. Using Equation (1), we see that

$$x[A] = \left((\operatorname{Ad} x)(A_0) + (\operatorname{Ad} x)(A_1)z + \cdots + (\operatorname{Ad} x)(A_N)z^N\right) \operatorname{dlog} z$$

is also aligned. Thus, by replacing A by $x[A]$, we may assume that $A_{0,\mathrm{s}}' \in \mathfrak{t}'$. We define $H = A_{0,\mathrm{s}}$ and have $H = A_0'' + A_{0,\mathrm{s}}' \in \mathfrak{z} \oplus \mathfrak{t}' = \mathfrak{t}$. As A is aligned, we know that $A_r \in \operatorname{Eig}(\operatorname{ad} H; r)$ for all $r \in \{0, \ldots, N\}$.

By the definition of m, we find $\psi \in \mathcal{X}^\vee(T)$ such that

$$\langle \alpha, \psi \rangle = m\langle \alpha, H \rangle \quad \text{for all } \alpha \in R_H^{\mathbb{Z}}.$$

Note that $\psi : G_{\mathrm{m}} \to T$ is a homomorphism of group schemes from the multiplicative group to our torus. Defining $t = \psi(z^{-1}) \in T(K)$, we get

$$\alpha(t) = z^{-\langle \alpha, \psi \rangle} = z^{-m\langle \alpha, H \rangle} \quad \text{for all } \alpha \in R_H^{\mathbb{Z}}.$$

We claim that $t[m^*(A)]$ is an element of $\mathfrak{g} \operatorname{dlog} z$. Equations (1) and (3) yield that

$$t[m^*(A)] = \Big(m(\operatorname{Ad} t)\,(A_0) + \cdots + m(\operatorname{Ad} t)\left(A_N z^{mN}\right) + \widehat{z\,\partial_z}(t)t^{-1}\Big) \operatorname{dlog} z.$$

Lemma 4.7 implies that we have $(\operatorname{Ad} t)(A_r z^{mr}) = A_r$ for all $r \in \{0, \ldots, N\}$. As $\widehat{z\,\partial_z}(t)t^{-1}$ is an element of $\mathfrak{t}$, by Lemma 4.8, our claim follows. ∎

Lemma 4.7. *Let $H \in \mathfrak{t}$, $m \in \mathbb{N}^+$, and $t \in T(K)$ be such that $\alpha(t) = z^{-m\langle \alpha, H \rangle}$ for all $\alpha \in R_H^{\mathbb{Z}}$. Then, for all $r \in \mathbb{N}$ and all $B \in \operatorname{Eig}(\operatorname{ad} H; r)$, we have $(\operatorname{Ad} t)(Bz^{mr}) = B$.*

Proof. We decompose $B = B_0 + \sum_{\alpha \in R} B_\alpha \in \mathfrak{t} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}'_\alpha$. Applying $\operatorname{ad} H$ gives $(\operatorname{ad} H)(B) = \sum_{\alpha \in R} \langle \alpha, H \rangle B_\alpha$. Combined with $B \in \operatorname{Eig}(\operatorname{ad} H; r)$, this implies that

$$B = B_0 \delta_{r0} + \sum_{\substack{\alpha \in R \\ \langle \alpha, H \rangle = r}} B_\alpha.$$

Now $(\operatorname{Ad} t)(Bz^{mr})$ is equal to

$$\Bigg(B_0 \delta_{r0} + \sum_{\substack{\alpha \in R \\ \langle \alpha, H \rangle = r}} \alpha(t) B_\alpha\Bigg) z^{mr} = B_0 \delta_{r0} z^{mr} + \sum_{\substack{\alpha \in R \\ \langle \alpha, H \rangle = r}} B_\alpha = B.$$

∎

Lemma 4.8. *Let $m \in \mathbb{N}^+$, $H \in \mathfrak{t}$, and $\psi \in \mathcal{X}^\vee(T)$ be such that $\langle \alpha, \psi \rangle = m\langle \alpha, H\rangle$ for all $\alpha \in R_H^{\mathbb{Z}}$. Let $t = \psi(z^{-1}) \in T(K)$ and $Y = \widehat{z\,\partial_z}(t)t^{-1} \in \mathfrak{t}(K)$. Then we have $Y \in \mathfrak{t}$, $\langle \alpha, Y\rangle \in \mathbb{Z}$ for all $\alpha \in R$, and $\langle \alpha, Y \rangle = -m\langle \alpha, H\rangle$ for all $\alpha \in R_H^{\mathbb{Z}}$.*

Proof. Let d be the dimension of the torus T. We choose an isomorphism from T to $(G_{\mathrm{m}})^d$ and identify $T = (G_{\mathrm{m}})^d$. For $i \in \{1, \dots, d\}$, we define the cocharacter $\delta_i : G_{\mathrm{m}} \to T$ by $\delta_i(x) = (1, \dots, 1, x, 1, \dots, 1)$. Let the character $\epsilon_i : T \to G_{\mathrm{m}}$ be defined by $\epsilon_i(x_1, \dots, x_d) = x_i$. We have $\mathcal{X}^\vee(T) = \bigoplus_{i=1}^d \mathbb{Z}\delta_i$ and $\mathcal{X}(T) = \bigoplus_{i=1}^d \mathbb{Z}\epsilon_i$.

Let $\psi = \sum f_i\delta_i$ for suitable $f_i \in \mathbb{Z}$. From $t = \psi(z^{-1}) = (z^{-f_1}, \dots, z^{-f_d})$, we deduce that $Y = z\,\partial_z(t)t^{-1} = (-f_1, \dots, -f_d) \in \mathfrak{t}$. An arbitrary $\alpha \in R$ can be written as $\alpha = \sum a_i\epsilon_i$ for suitable $a_i \in \mathbb{Z}$. Then we have $\langle \alpha, Y\rangle = -\sum a_i f_i \in \mathbb{Z}$. If $\alpha \in R_H^{\mathbb{Z}}$, we obtain $m\langle \alpha, H\rangle = \langle \alpha, \psi\rangle = \sum a_i f_i = -\langle \alpha, Y\rangle$. ∎

4.2. Transforming Regular Connections to Zero Standard Form.

Definition 4.9. Let G be a linear algebraic group. We denote by $\mathfrak{g}^{\mathrm{zero}}$ the set of all $X \in \mathfrak{g}$ such that zero is the only rational eigenvalue of $\operatorname{ad} X$. The elements of $\mathfrak{g}^{\mathrm{zero}} \operatorname{dlog} z$ are called *connections in zero standard form.*

Theorem 4.10. *Let G be a connected reductive linear algebraic group. For every regular connection A, there exists a positive integer $n \in \mathbb{N}^+$, such that the pull-back connection $n^*(A)$ is gauge equivalent to a connection in zero standard form.*

Corollary 4.11. *If G is connected reductive, each regular connection is related to a connection in zero standard form.*

Proof. Similar to the proof of Corollary 4.3. Note that $\mathfrak{g}^{\mathrm{zero}}$ is stable under multiplication by rational numbers. ∎

Example 4.12. Let $G = \mathrm{SL}_2$. For $n \in \mathbb{N}^+$, we define the regular SL_2-connection

$$B_n = \tfrac{1}{2n}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \operatorname{dlog} z.$$

We claim that $n^*(B_n)$ is not gauge equivalent to a connection in zero standard form. Otherwise suppose that

$$n^*(B_n) = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{bmatrix} \operatorname{dlog} z$$

is $\mathrm{SL}_2(K)$-equivalent to $X \operatorname{dlog} z$ for some $X \in \mathfrak{sl}_2^{\mathrm{zero}}$. We may assume that $X \in \mathfrak{sl}_2^{\mathrm{zero}}$ has Jordan normal form. According to Proposition 7.8, there is $l \in \mathbb{Z}$ such that

$$X = \begin{bmatrix} \frac{1}{2} + l & 0 \\ 0 & -\frac{1}{2} - l \end{bmatrix}.$$

Then $1 + 2l \in \mathbb{Q} - \{0\}$ is a rational eigenvalue of $\operatorname{ad} X$. This contradicts our assumption $X \in \mathfrak{sl}_2^{\text{zero}}$.

This example shows that there is no $n \in \mathbb{N}^+$, such that for every regular SL_2-connection B, the connection $n^*(B)$ is gauge equivalent to a connection in zero standard form.

We prepare for the proof of Theorem 4.10 by showing some results for semisimple linear algebraic groups and semisimple Lie algebras.

Proposition 4.13. *Let G be a semisimple linear algebraic group, $B \subset G$ a Borel subgroup, and $T \subset B$ a maximal torus in G. The set of semisimple elements in $\mathfrak{b}$ is equal to $\operatorname{Ad}(B)(\mathfrak{t})$.*

Proof. Suppose that $H \in \mathfrak{b}$ is a semisimple element. Let $\mathfrak{t}'$ be a maximal toral subalgebra of $\mathfrak{g}$, containing H. As $\mathfrak{g}$ is semisimple, $\mathfrak{t}'$ is a Cartan subalgebra of $\mathfrak{g}$. All Cartan subalgebras are conjugate under the adjoint group of $\mathfrak{g}$. This adjoint group is the identity component of $\operatorname{Ad}(G)$. Therefore, we find $g \in G$ such that $(\operatorname{Ad} g)(\mathfrak{t}) = \mathfrak{t}'$. Obviously, $\mathfrak{t}'$ is the Lie algebra of $T' = gTg^{-1}$. The identity component D of $T' \cap B$ is a torus. We find $H \in \mathfrak{t}' \cap \mathfrak{b} = \operatorname{Lie}(T' \cap B) = \mathfrak{d}$. Let S be a maximal torus in B that contains D. The maximal tori T and S in B are conjugate under B, so we find $b \in B$ such that $bTb^{-1} = S$. Now we see that $(\operatorname{Ad} b)(\mathfrak{t}) = \mathfrak{s} \supset \mathfrak{d} \ni H$. ■

Corollary 4.14. *For every $X \in \mathfrak{g}$, there is a group element $g \in G$ such that $(\operatorname{Ad} g)(X) \in \mathfrak{b}$ and $(\operatorname{Ad} g)(X_{\mathrm{s}}) \in \mathfrak{t}$.*

Proof. Given $X \in \mathfrak{g}$, let $\mathfrak{b}'$ be a Borel subalgebra containing X_{s} and X_{n}. Since all Borel subalgebras are conjugate under the adjoint group, we find an element $h \in G$ with $(\operatorname{Ad} h)(\mathfrak{b}') = \mathfrak{b}$. By Proposition 4.13, there is $b \in B$ such that $(\operatorname{Ad} bh)(X_{\mathrm{s}}) \in \mathfrak{t}$. We have $(\operatorname{Ad} bh)(X) \in \mathfrak{b}$. ■

Lemma 4.15. *Suppose that $\mathfrak{t}$ is a Cartan subalgebra of a semisimple Lie algebra $\mathfrak{g}$, let $R = R(\mathfrak{g}, \mathfrak{t})$ be the roots and choose a system of positive roots $R^+ \subset R$. Define $\mathfrak{u}^+ = \bigoplus_{\alpha \in R^+} \mathfrak{g}_\alpha$ and $\mathfrak{b} = \mathfrak{t} \oplus \mathfrak{u}^+$. Let $X \in \mathfrak{b}$ and $X = X_{\mathrm{s}} + X_{\mathrm{n}}$ be its Jordan decomposition in $\mathfrak{g}$. Then $X_{\mathrm{s}} \in \mathfrak{t}$ implies that $X_{\mathrm{n}} \in \mathfrak{u}^+$.*

Proof. This follows from the root space decomposition, the nilpotency of $\operatorname{ad} X_{\mathrm{n}}$, and $\mathfrak{t}^* = kR^+$. ■

Once again, we use the notation introduced in Subsection 4.1. In particular, G is connected reductive and $G' = (G, G)$ is semisimple. Let $B' \subset G'$ be a Borel subgroup containing T', and let $R^+ = R^+(B') \subset R$ be the corresponding positive roots. The set $U'^+ \subset B'$ of all unipotent elements of B' is a closed nilpotent connected subgroup. For the corresponding Lie algebras, we have $\mathfrak{u}'^+ = \bigoplus_{\alpha \in R^+} \mathfrak{g}'_\alpha$ and $\mathfrak{b}' = \mathfrak{t}' \oplus \mathfrak{u}'^+$.

Proposition 4.16. *For every $X \in \mathfrak{g}$, there is $g \in G$ such that $(\operatorname{Ad} g)(X_{\mathrm{s}}) \in \mathfrak{t}$ and $(\operatorname{Ad} g)(X_{\mathrm{n}}) \in \mathfrak{u}'^+$.*

Proof. This follows from Corollary 4.14, Lemma 4.15 and the adaption to the reductive situation. ■

Lemma 4.17. *Let $H \in \mathfrak{t}$ and $N \in \mathfrak{u}'^{+}$. The eigenvalues of* $\operatorname{ad}(H+N)$ *are given by* $\{\langle \alpha, H\rangle \mid \alpha \in R \cup \{0\}\}$.

Proof. The endomorphisms $\operatorname{ad} H$ and $\operatorname{ad}(H+N)$ have the same characteristic polynomial. ■

Proof. (of Theorem 4.10) Let A be a regular connection. According to Theorem 4.2, there are a positive integer $m \in \mathbb{N}^{+}$ and an element $X \in \mathfrak{g}$, such that $m^{*}(A)$ is gauge equivalent to the connection $C = X \operatorname{dlog} z$.

By Proposition 4.16, we may assume that $X_{\mathrm{s}} \in \mathfrak{t}$ and $X_{\mathrm{n}} \in \mathfrak{u}'^{+}$. Let

$$R^{\mathbb{Q}}_{X_{\mathrm{s}}} = \{\alpha \in R \mid \langle \alpha, X_{\mathrm{s}}\rangle \in \mathbb{Q}\}$$

denote the roots rational on X_{s}. Choose $l \in \mathbb{N}^{+}$ such that $\langle \alpha, lX_{\mathrm{s}}\rangle \in \mathbb{Z}$ for all $\alpha \in R^{\mathbb{Q}}_{X_{\mathrm{s}}}$. Define $H = lX_{\mathrm{s}}$ and $N = lX_{\mathrm{n}}$. If a root attains a rational value on H, this value is already integral, which means $R^{\mathbb{Q}}_{H} = R^{\mathbb{Z}}_{H}$.

By Lemma 4.6, there is $\phi \in \mathcal{X}^{\vee}(T)$ such that

$$\langle \alpha, \phi\rangle = m_H \langle \alpha, H\rangle \quad \text{for all } \alpha \in R^{\mathbb{Z}}_{H}.$$

Define $t = \phi(z^{-1}) \in T(K)$. Because

$$\alpha(t) = z^{-\langle \alpha, \phi\rangle} = z^{-m_H\langle \alpha, H\rangle} \quad \text{for all } \alpha \in R^{\mathbb{Z}}_{H},$$

and $m_H l X \in \operatorname{Eig}(\operatorname{ad} H; 0)$, Lemma 4.7 implies that $(\operatorname{Ad} t)(m_H l X) = m_H l X$.

Set $Y = \widehat{z\,\partial_z}(t)t^{-1}$. Lemma 4.8 gives $Y \in \mathfrak{t}$,

$$\langle \alpha, Y\rangle \in \mathbb{Z} \quad \text{for all } \alpha \in R, \text{ and} \tag{4}$$
$$\langle \alpha, Y\rangle = -\, m_H\langle \alpha, H\rangle \quad \text{for all } \alpha \in R^{\mathbb{Z}}_{H}. \tag{5}$$

We apply the gauge transformation t to $(m_H l)^{*}(C) = m_H l X \operatorname{dlog} z$ and get

$$t\,[(m_H l)^{*}(C)] = (\underbrace{m_H H + Y}_{\in \mathfrak{t}} + \underbrace{m_H N}_{\in \mathfrak{u}'^{+}}) \operatorname{dlog} z.$$

Let $\alpha \in R$ with $\langle \alpha, m_H H + Y\rangle \in \mathbb{Q}$. We claim that $\langle \alpha, m_H H + Y\rangle = 0$. Equation (4) shows that $\alpha \in R^{\mathbb{Q}}_{H} = R^{\mathbb{Z}}_{H}$. Then Equation (5) proves our claim. Thus, by Lemma 4.17, $(m_H H + Y) + m_H N$ is an element of $\mathfrak{g}^{\mathrm{zero}}$. ■

Corollary 4.18. *Let G be connected reductive with maximal torus $T \subset G$. Then every connection of the form* $X \operatorname{dlog} z$ *with $X \in \mathfrak{t}$ is related to a connection in* $\mathfrak{t} \cap \mathfrak{g}^{\mathrm{zero}} \operatorname{dlog} z$.

Proof. If $X \in \mathfrak{t}$, it is obvious from the proof of Theorem 4.10 that there exists $n \in \mathbb{N}^{+}$ such that $n^{*}(X \operatorname{dlog} z)$ is gauge equivalent to $X' \operatorname{dlog} z$ for some $X' \in \mathfrak{t} \cap \mathfrak{g}^{\mathrm{zero}}$. So $X \operatorname{dlog} z$ is related to $n^{-1}X' \operatorname{dlog} z$. ■

5. Regular GL_n-Connections

5.1. Classification of Regular GL_n-Connections. For $a \in \mathbb{N}^+$ and $x \in k$, we denote by $\mathrm{E}_a \in \mathrm{End}(k^a) = \mathrm{Mat}_a(k)$ the identity matrix and by $\mathrm{J}(x,a) \in \mathrm{Mat}_a(k)$ the $a \times a$-Jordan block with diagonal entries equal to x. For example, we have

$$\mathrm{J}(x,3) = \begin{bmatrix} x & 1 & 0 \\ 0 & x & 1 \\ 0 & 0 & x \end{bmatrix}.$$

Let $n \in \mathbb{N}$. We denote the set of all $n \times n$-matrices in Jordan normal form by $\mathcal{J}_n$.

Definition 5.1. Let X, $Y \in \mathcal{J}_n$ be given,

$$\begin{aligned} X &= \mathrm{blockdiag}\,(\mathrm{J}(x_1,a_1),\dots,\mathrm{J}(x_r,a_r)), \\ Y &= \mathrm{blockdiag}\,(\mathrm{J}(y_1,b_1),\dots,\mathrm{J}(y_s,b_s)). \end{aligned} \tag{6}$$

The matrices X and Y *differ integrally* (resp. *rationally*) *after block permutation*, if $s = r$ and there is a permutation $\tau \in \mathrm{Sym}_r$ such that

$$a_i = b_{\tau(i)} \quad \text{and} \quad x_i \equiv y_{\tau(i)} \mod \mathbb{Z} \quad (\text{resp.} \mod \mathbb{Q}) \quad \text{for all } i \in \{1,\dots,r\}.$$

Now we can classify regular GL_n-connections up to gauge equivalence. Our proof of the following theorem is more or less the same as that in [1, §3].

Theorem 5.2. (Regular GL_n-Connections up to Gauge Equivalence) *The map*

$$\mathcal{J}_n \twoheadrightarrow \{\textit{regular } \mathrm{GL}_n\textit{-connections}\}/\,\mathrm{GL}_n(K), \quad X \mapsto [X\,\mathrm{dlog}\,z],$$

is a surjection. For X, $Y \in \mathcal{J}_n$, the connections $X\,\mathrm{dlog}\,z$ and $Y\,\mathrm{dlog}\,z$ are $\mathrm{GL}_n(K)$-equivalent if and only if X and Y differ integrally after block permutation.

Proof. The surjectivity follows from Theorem 4.2 and the example $G = \mathrm{GL}_n$ in Examples 4.4. Let X, $Y \in \mathcal{J}_n$ be given in the form (6).

Assume that X and Y differ integrally after block permutation. Performing the block permutation by an element of $\mathrm{GL}_n(k)$ (or $\mathrm{SL}_n(k)$), we may assume that Y has the form

$$Y = \mathrm{blockdiag}\,(\mathrm{J}(x_1+n_1,a_1),\dots,\mathrm{J}(x_r+n_r,a_r))$$

for suitable $n_1,\dots,n_r \in \mathbb{Z}$. We define

$$g = \mathrm{blockdiag}\,\left(z^{n_1}\,\mathrm{E}_{a_1},\dots,z^{n_r}\,\mathrm{E}_{a_r}\right) \in \mathrm{GL}_n(K) \tag{7}$$

and deduce from Equation (2) that $g\,[X\,\mathrm{dlog}\,z] = Y\,\mathrm{dlog}\,z$.

Assume now that $X\,\mathrm{dlog}\,z$ and $Y\,\mathrm{dlog}\,z$ are gauge equivalent. We transform $X\,\mathrm{dlog}\,z$ by a gauge transformation g of the form (7) for suitable $n_1,\dots,n_r \in \mathbb{Z}$ and deal with $Y\,\mathrm{dlog}\,z$ similarly, and may so assume that

$$\lambda - \mu \in \mathbb{Z} \Rightarrow \lambda = \mu \quad \text{for all } \lambda,\ \mu \in \{x_1,\dots x_r,y_1,\dots,y_s\}. \tag{8}$$

Let $h \in \mathrm{GL}_n(K)$ with $h[Y\,\mathrm{dlog}\,z] = X\,\mathrm{dlog}\,z$. This implies $Xh - hY = z\,\partial_z(h)$. We write $h = \sum_{l \geq N} h_l z^l$ with $N \in \mathbb{Z}$ and $h_l \in \mathrm{Mat}_n(k)$ and get $Xh_l - h_lY = lh_l$ for all $l \geq N$. Since the eigenvalues of the linear map $\mathrm{Mat}_n(k) \to \mathrm{Mat}_n(k)$, $A \mapsto XA - AY$ are given by $x_i - y_j$, for $1 \leq i \leq r$, $1 \leq j \leq s$, (8) implies that $h = h_0 \in \mathrm{GL}_n(k)$. But then $h_0 Y h_0^{-1} = X$, so X and Y are conjugate and hence differ integrally (in fact, by 0) after block permutation. ∎

5.2. D-Modules and GL_n-Connections. We denote by $D_0 = k[[z]]z\,\partial_z$ the subspace of derivations $\delta \in D$ with $\delta(\mathfrak{m}) \subset \mathfrak{m}$.

Definition 5.3. ([4]) A *D-module* is a K-vector space M together with a map $\alpha : D \times M \to M$, $(\delta, m) \mapsto \delta m = \alpha(\delta, m)$, that is K-linear in the first argument and additive in the second one, and that satisfies $\delta(xm) = (\delta x)m + x(\delta m)$ for all $\delta \in D$, $x \in K$, and $m \in M$. A map α as above is a *D-module structure* on M. A *morphism of D-modules* $f : (M, \alpha) \to (N, \beta)$ is a K-linear map $f : M \to N$ satisfying $f(\alpha(\delta, m)) = \beta(\delta, f(m))$ for all $\delta \in D$, $m \in M$.

Let M be a D-module. For $m \in M$, let $E(m)$ be the smallest $\mathcal{O}$-submodule of M that contains m and is D_0-stable. A D-module M is *Fuchsian*, if $E(m)$ is finitely generated as an $\mathcal{O}$-module for all $m \in M$.

Let $a \in \mathbb{N}^+$ and $x \in k$. There is a unique D-module structure α on $M = K^a = \bigoplus_{i=1}^a Ke_i$ such that

$$\alpha(z\,\partial_z, e_i) = xe_i + e_{i-1} \quad \text{for all } i \in \{1, \dots a\},$$

where $e_0 = 0$. We denote this D-module by $M^{x,a}$. It is easy to see that $M^{x,a}$ is Fuchsian and indecomposable.

Fix $n \in \mathbb{N}$. The group $\mathrm{GL}_n(K)$ acts on the set of all D-module structures on K^n as follows: Given $g \in \mathrm{GL}_n(K)$ and a D-module structure α, we define $g.\alpha$ by $(\delta, w) \mapsto g(\alpha(\delta, g^{-1}(w)))$. Two D-module structures α and β are in the same orbit if and only if (K^n, α) and (K^n, β) are isomorphic.

Let A be a GL_n-connection. If we evaluate $A \in \mathcal{GL}_n = \mathrm{Hom}_K(D, \mathfrak{gl}_n(K))$ at $\delta \in D$, we get an element $A(\delta) \in \mathrm{End}_K(K^n) = \mathrm{Hom}(k^n, K^n)$. Then

$$\alpha_A(\delta, x \otimes v) = \delta(x) \otimes v - xA(\delta)(v),$$

where $x \in K$, $v \in k^n$, defines a D-module structure α_A on $K^n = K \otimes k^n$.

We omit the easy proof of

Proposition 5.4. *The map*

$$\{\mathrm{GL}_n\text{-}connections\} \xrightarrow{\sim} \{D\text{-}module\ structures\ on\ K^n\}, \quad A \mapsto \alpha_A,$$

is a $\mathrm{GL}_n(K)$-equivariant bijection and induces a bijection between regular connections and Fuchsian D-module structures. The regular GL_n-connection

$$\mathrm{blockdiag}\,(-\mathrm{J}(x_1, a_1), \dots, -\mathrm{J}(x_r, a_r))\,\mathrm{dlog}\,z$$

corresponds to the Fuchsian D-module $M^{x_1,a_1} \oplus \cdots \oplus M^{x_r,a_r}$.

We use Proposition 5.4 in order to translate Theorem 5.2 in the language of D-modules and obtain

Theorem 5.5. (cf. [4, Theorem 4]) *Every finite-dimensional Fuchsian D-module is a direct sum of indecomposable Fuchsian D-modules. The summands are unique up to permutation and isomorphism.*

The map $(x, a) \mapsto M^{x,a}$ induces a bijection from $k/\mathbb{Z} \times \mathbb{N}^+$ to the set of isomorphism classes of finite-dimensional indecomposable Fuchsian D-modules.

6. Semisimple Conjugacy Classes

Let $n \in \mathbb{N}$ and $X \in \mathfrak{gl}_n = \operatorname{End}(k^n)$. For $\lambda \in k$ and $i \in \mathbb{N}$, define

$$E_\lambda^i = \ker(X-\lambda) \cap \operatorname{im}(X-\lambda)^i.$$

Let $\mathrm{Z}_{\mathrm{GL}_n}(X)$ be the centralizer of X in $\mathrm{GL}_n = \mathrm{GL}_n(k)$ under the adjoint action. Every $g \in \mathrm{Z}_{\mathrm{GL}_n}(X)$ stabilizes all E_λ^i. Thus g induces maps $g|_{E_\lambda^i} \in \mathrm{GL}(E_\lambda^i)$ and $\overline{g|_{E_\lambda^i}} \in \mathrm{GL}(E_\lambda^i/E_\lambda^{i+1})$. The following theorem gives an explicit description of the semisimple conjugacy classes in $\mathrm{Z}_{\mathrm{GL}_n}(X)$.

Theorem 6.1. *Let $n \in \mathbb{N}$ and $X \in \mathfrak{gl}_n = \mathrm{Mat}_n(k)$ be in Jordan normal form. Let $T_n \subset \mathrm{GL}_n$ be the standard diagonal torus. Then we have the following:*

1. *$T = T_X = T_n \cap \mathrm{Z}_{\mathrm{GL}_n}(X)$ is a maximal torus in $\mathrm{Z}_{\mathrm{GL}_n}(X)$.*
2. *The homomorphism*

$$\pi : \mathrm{Z}_{\mathrm{GL}_n}(X) \twoheadrightarrow \prod_{\substack{\lambda\in k\\ i\in\mathbb{N}}} \mathrm{GL}(E_\lambda^i/E_\lambda^{i+1}), \quad g \mapsto \left(\overline{g|_{E_\lambda^i}}\right)_{\lambda,i},$$

is surjective. It induces, by restriction, an isomorphism $\pi|_T : T \xrightarrow{\sim} \pi(T)$, and $\pi(T)$ is a maximal torus in $\prod \mathrm{GL}(E_\lambda^i/E_\lambda^{i+1})$.

3. *The Weyl group W_X associated to the torus $\pi(T)$ in $\prod \mathrm{GL}(E_\lambda^i/E_\lambda^{i+1})$ acts via $\pi|_T$ on T, and the inclusion $T_X \hookrightarrow \mathrm{Z}_{\mathrm{GL}_n}(X)$ induces a bijection*

$$T_X/W_X \xrightarrow{\sim} \{\textit{semisimple conj. classes in } \mathrm{Z}_{\mathrm{GL}_n}(X)\}.$$

The proof is left to the reader. It can be found in [5].

7. Regular SL_n-Connections

Let $\mathcal{J}(\mathfrak{sl}_n) = \mathfrak{sl}_n \cap \mathcal{J}_n$ and $\mathcal{J}(\mathfrak{sl}_n^{\mathrm{zero}}) = \mathfrak{sl}_n^{\mathrm{zero}} \cap \mathcal{J}_n$. This section is organized as follows. First we describe the set $\mathrm{Rel}(X \operatorname{dlog} z)/\mathrm{SL}_n(K)$ of relatives up to gauge equivalence, for $X \in \mathcal{J}(\mathfrak{sl}_n^{\mathrm{zero}})$ (Bijection (12)). We deduce from this that the set of regular SL_n-connections is $\bigcup \mathrm{Rel}(X \operatorname{dlog} z)$, where X ranges over $\mathcal{J}(\mathfrak{sl}_n^{\mathrm{zero}})$ (Proposition 7.3, Corollary 7.5). Then we establish the classification up to relationship (Theorem 7.9) and up to gauge equivalence (Theorem 7.10). We conclude with a slightly different view on this classification (Remark 7.12) and explain the example SL_2 (Example 7.13).

Let $X \in \mathfrak{sl}_n^{\mathrm{zero}}$. Recall from Proposition 3.12 the bijection

$$\mathrm{Rel}(X \operatorname{dlog} z)/\mathrm{SL}_n(K) \xrightarrow{\sim} \mathrm{H}^1(K; X \operatorname{dlog} z). \tag{9}$$

Since $\mathrm{H}^1(K; X \operatorname{dlog} z)$ is the direct limit of the $\mathrm{H}^1(\Gamma_l; \mathrm{SL}_n(L)_{l^*(X \operatorname{dlog} z)})$, for $l \in \mathbb{N}^+$ and $l^* : K \hookrightarrow K = L$, we are interested in the stabilizer of $l^*(X \operatorname{dlog} z) = lX \operatorname{dlog} z$ in $\mathrm{SL}_n(L)$. As $\mathfrak{sl}_n^{\mathrm{zero}}$ is stable under multiplication by rational numbers, Proposition 7.1 shows that

$$\mathrm{SL}_n(L)_{lX \operatorname{dlog} z} = \mathrm{Z}_{\mathrm{SL}_n}(lX) = \mathrm{Z}_{\mathrm{SL}_n}(X) \subset \mathrm{SL}_n(k).$$

In particular, the action of the Galois group Γ_l on $\mathrm{SL}_n(L)_{lX \operatorname{dlog} z}$ is trivial.

Proposition 7.1. *Let* $X \in \mathfrak{sl}_n^{\text{zero}}$. *Then* $\mathrm{SL}_n(K)_{X\,\mathrm{dlog}\,z} = \mathrm{Z}_{\mathrm{SL}_n}(X)$, *where* $\mathrm{Z}_{\mathrm{SL}_n}(X)$ *is the centralizer of* X *in* $\mathrm{SL}_n = \mathrm{SL}_n(k)$ *under the adjoint action.*

Proof. The inclusion $\mathrm{Z}_{\mathrm{SL}_n}(X) \subset \mathrm{SL}_n(K)_{X\,\mathrm{dlog}\,z}$ is obvious. Let g be in $\mathrm{SL}_n(K)_{X\,\mathrm{dlog}\,z}$. As $\mathrm{SL}_n(K) \subset \mathrm{GL}_n(K) \subset \mathrm{Mat}_n(K)$, we find $N \in \mathbb{Z}$ and $g_i \in \mathrm{Mat}_n(k)$ such that $g = \sum_{i \geq N} g_i z^i$. From $g[X\,\mathrm{dlog}\,z] = X\,\mathrm{dlog}\,z$ and Equation (2) we get $Xg - gX = z\,\partial_z(g)$, or, equivalently, $(\mathrm{ad}_{\mathfrak{gl}_n} X)(g_i) = i g_i$ for all $i \geq N$. But then $X \in \mathfrak{sl}_n^{\text{zero}} \subset \mathfrak{gl}_n^{\text{zero}}$ implies that $g_i = 0$ for $i \neq 0$, in other words, $g = g_0 \in \mathrm{Z}_{\mathrm{SL}_n}(X)$. ■

Recall that $\overline{K} = \bigcup_{m \in \mathbb{N}^+} k((z^{1/m}))$ is an algebraic closure of $K = k((z))$. For $l \in \mathbb{N}^+$, we view the field extension $l^* : K \hookrightarrow K = L$ as a subextension of $K \subset \overline{K}$ via the embedding $K = L \hookrightarrow \overline{K}$, $f(z) \mapsto f(z^{1/l})$. The Galois group $\mathrm{Gal}(\overline{K}/K)$ is isomorphic to the procyclic group $\widehat{\mathbb{Z}}$. For the rest of this section, we fix a procyclic generator γ of $\mathrm{Gal}(\overline{K}/K)$. For $l \in \mathbb{N}^+$, the Galois group $\Gamma_l = \mathrm{Gal}(l^*)$ is generated by $\gamma|_L$.

Let $X \in \mathfrak{sl}_n^{\text{zero}}$ and $l \in \mathbb{N}^+$. Since Γ_l acts trivially on $\mathrm{SL}_n(L)_{lX\,\mathrm{dlog}\,z}$, the map

$$\mathrm{Z}^1(\Gamma_l; \mathrm{SL}_n(L)_{lX\,\mathrm{dlog}\,z}) \underset{\gamma}{\xrightarrow{\sim}} \{l\text{-torsion-elements in } \mathrm{Z}_{\mathrm{SL}_n}(X)\}, \quad p \mapsto p_{\gamma|_L}, \tag{10}$$

is bijective. It depends on γ. We indicate this here and in the following by writing a γ below the corresponding arrow. This Bijection (10) induces a bijection

$$\mathrm{H}^1(\Gamma_l; \mathrm{SL}_n(L)_{lX\,\mathrm{dlog}\,z}) \underset{\gamma}{\xrightarrow{\sim}} \{\text{conj. classes of } l\text{-torsion-elts. in } \mathrm{Z}_{\mathrm{SL}_n}(X)\}.$$

We use Proposition 3.10 and pass to the direct limit. We obtain for $X \in \mathfrak{sl}_n^{\text{zero}}$ a bijection

$$\mathrm{H}^1(K; X\,\mathrm{dlog}\,z) \underset{\gamma}{\xrightarrow{\sim}} \{\text{conj. classes of torsion elements in } \mathrm{Z}_{\mathrm{SL}_n}(X)\}. \tag{11}$$

Suppose that $X \in \mathcal{J}(\mathfrak{sl}_n)$. Let $T_X \subset \mathrm{Z}_{\mathrm{GL}_n}(X)$ be the diagonal maximal torus and W_X be the Weyl group as in Theorem 6.1. The Weyl group W_X stabilizes $D_X = T_X \cap \mathrm{SL}_n$.

Proposition 7.2. *For* $X \in \mathcal{J}(\mathfrak{sl}_n)$, *the inclusion* $D_X \hookrightarrow Z_{\mathrm{SL}_n}(X)$ *induces a bijection*

$$\{\textit{torsion elements in } D_X\}/W_X \xrightarrow{\sim} \{\textit{conj. classes of torsion elts. in } \mathrm{Z}_{\mathrm{SL}_n}(X)\}.$$

Proof. This follows from Theorem 6.1 (3) and the fact that, in characteristic zero, every element of finite order is semisimple. ■

Assume now that $X \in \mathcal{J}(\mathfrak{sl}_n^{\text{zero}})$. We combine Bijections (9), (11) and Proposition 7.2 in order to get the bijection

$$\{\text{torsion elements in } D_X\}/W_X \underset{\gamma}{\xrightarrow{\sim}} \mathrm{Rel}(X\,\mathrm{dlog}\,z)/\,\mathrm{SL}_n(K). \tag{12}$$

We denote this map by $\delta \mapsto \left[(X\,\mathrm{dlog}\,z)^{\delta}\right]$ and describe it explicitely in the proof of

Proposition 7.3. *All relatives of $X \operatorname{dlog} z$ are regular, for $X \in \mathcal{J}(\mathfrak{sl}_n^{\text{zero}})$.*

Proof. Write $X \in \mathcal{J}(\mathfrak{sl}_n^{\text{zero}})$ as

$$X = \text{blockdiag}\,(\mathrm{J}(x_1, a_1), \dots, \mathrm{J}(x_r, a_r))$$

for suitable $x_i \in k$ and $a_i \in \mathbb{N}^+$. Given a torsion element $d \in D_X$ we explain now how to construct an SL_n-connection in the orbit $\left[(X \operatorname{dlog} z)^{W_X d}\right]$. As this connection will be regular, this proves the proposition. For $l \in \mathbb{N}^+$, we view l^* as the field extension $K = k((z)) \hookrightarrow k((z^{1/l}))$. Let ω_l be the primitive l-th root of unity such that $\gamma(z^{1/l}) = \omega_l z^{1/l}$.

Let $d \in D_X$ be a torsion element. We find $l \in \mathbb{N}^+$ and $j_1, \dots, j_r \in \mathbb{N}$ such that

$$d = \text{blockdiag}\,\left(\omega_l^{j_1}\,\mathrm{E}_{a_1}, \dots, \omega_l^{j_r}\,\mathrm{E}_{a_r}\right).$$

Let $\omega = \omega_l$, $\zeta = z^{1/l}$, and $\Sigma = \sum_{s=1}^{r} j_s a_s \in \mathbb{N}$. As $1 = \det(d) = \omega^{\Sigma}$, we see that Σ is divisible by l. Define

$$g = \text{blockdiag}\,\left(\zeta^{j_1}\,\mathrm{E}_{a_1}, \dots, \zeta^{j_{r-1}}\,\mathrm{E}_{a_{r-1}}, \zeta^{j_r}, \dots, \zeta^{j_r}, \zeta^{j_r - \Sigma}\right) \in \mathrm{SL}_n((\zeta)).$$

Now $\omega^{\Sigma} = 1$ implies that $d = g^{-1}\gamma(g)$. Therefore, for any $m \in \mathbb{N}^+$, we get

$$d^m = d\gamma(d)\gamma^2(d)\cdots\gamma^{m-1}(d) = g^{-1}\gamma^m(g).$$

This means that d, regarded as an element of $\mathrm{Z}^1(\Gamma_l; \mathrm{SL}_n((\zeta))_{lX \operatorname{dlog} \zeta})$ via Bijection (10), is cohomologous to the trivial 1-cocycle in $\mathrm{Z}^1(\Gamma_l; \mathrm{SL}_n((\zeta)))$. It follows from the proof of Theorem 3.8 that the connection $g\,[lX \operatorname{dlog} \zeta]$ is invariant under the Galois group Γ_l. We define

$$(X \operatorname{dlog} z)^d = \text{blockdiag}\,\left(\mathrm{J}(x_1 + \tfrac{j_1}{l}, a_1), \dots, \mathrm{J}(x_{r-1} + \tfrac{j_{r-1}}{l}, a_{r-1}), C\right) \operatorname{dlog} z,$$

where $C \in \mathrm{Mat}_{a_r}(K)$ is given by

$$C = \begin{bmatrix} x_r + \frac{j_r}{l} & 1 & 0 & \dots & & & 0 \\ 0 & x_r + \frac{j_r}{l} & \ddots & \ddots & & & \\ \vdots & & \ddots & \ddots & 1 & 0 & \vdots \\ & & & 0 & x_r + \frac{j_r}{l} & 1 & 0 \\ & & & & 0 & x_r + \frac{j_r}{l} & z^{\frac{\Sigma}{l}} \\ 0 & & & & \dots & 0 & x_r + \frac{j_r - \Sigma}{l} \end{bmatrix}$$

If $a_r = 1$, this is to be interpreted as $C = x_r + \frac{j_r - \Sigma}{l}$. As $\frac{\Sigma}{l} \in \mathbb{N}$ is a nonnegative integer, $(X \operatorname{dlog} z)^d$ is regular. It is easy to verify that $l^*\left((X \operatorname{dlog} z)^d\right) = g\,[lX \operatorname{dlog} \zeta]$. We conclude that $(X \operatorname{dlog} z)^d$ is a connection in $\left[(X \operatorname{dlog} z)^{W_X d}\right]$. ■

Corollary 7.4. *If $X \in \mathfrak{sl}_n$ is a diagonal matrix, each connection related to $X \operatorname{dlog} z$ is gauge equivalent to a connection of the form $Y \operatorname{dlog} z$ with $Y \in \mathfrak{sl}_n$ a diagonal matrix.*

Proof. By Corollary 4.18 we may assume that X is diagonal and in $\mathfrak{sl}_n^{\text{zero}}$. Then our claim follows from the description of Bijection (12) in the above proof. ■

Corollary 7.5. *Every regular* SL_n*-connection is related to* $X \operatorname{dlog} z$*, for some* $X \in \mathcal{J}(\mathfrak{sl}_n^{\text{zero}})$*. All relatives of a regular* SL_n*-connection are regular. An* SL_n*-connection* A *is regular if and only if there is* $l \in \mathbb{N}^+$ *such that the connection* $l^*(A)$ *is regular.*

Proof. The first claim follows from Corollary 4.11, and then the second claim is a consequence of Proposition 7.3. If $l^*(A)$ is regular, we have just seen that it is related to $X \operatorname{dlog} z$ with $X \in \mathfrak{sl}_n$. So A is related to the regular connection $l^{-1}X \operatorname{dlog} z$ and therefore regular. ■

Remark 7.6. Very similar arguments prove that Corollary 7.5 with SL_n replaced by GL_n is true.

Problem 7.7. Are all relatives of a regular G-connection regular, if G is an arbitrary linear algebraic group?

Proposition 7.8. *For* X, $Y \in \mathcal{J}(\mathfrak{sl}_n)$*, the following are equivalent:*

1. $X \operatorname{dlog} z$ *and* $Y \operatorname{dlog} z$ *are* $\mathrm{SL}_n(K)$*-equivalent.*
2. $X \operatorname{dlog} z$ *and* $Y \operatorname{dlog} z$ *are* $\mathrm{GL}_n(K)$*-equivalent.*
3. X *and* Y *differ integrally after block permutation.*

Proof. The implication (1) $\Rightarrow$ (2) is obvious, and (2) $\Rightarrow$ (3) follows from Theorem 5.2. In the proof of Theorem 5.2 we proved the implication (3) $\Rightarrow$ (2). But we actually showed (3) $\Rightarrow$ (1): If the traces of X and Y vanish, the element g defined in Equation (7) is an element of $\mathrm{SL}_n(K)$. ■

Theorem 7.9. (Regular SL_n-Connections up to Relationship) *The map* $X \mapsto X \operatorname{dlog} z$ *induces a surjection*

$$\mathcal{J}(\mathfrak{sl}_n) \twoheadrightarrow \{\textit{regular } \mathrm{SL}_n\textit{-connections}\}/\textit{relationship}.$$

For X, $Y \in \mathcal{J}(\mathfrak{sl}_n)$*, the connections* $X \operatorname{dlog} z$ *and* $Y \operatorname{dlog} z$ *are related if and only if* X *and* Y *differ rationally after block permutation.*

Proof. Our map is surjective by Corollary 7.5. The second statement follows from Proposition 7.8 and the fact that the matrices $l\,\mathrm{J}(x,a)$ and $\mathrm{J}(lx,a)$ are $\mathrm{SL}_a(k)$-conjugate, for $l \in \mathbb{N}^+$. ■

Let $X \operatorname{dlog} z$ and $Y \operatorname{dlog} z$ be two related SL_n-connections with X, $Y \in \mathcal{J}(\mathfrak{sl}_n^{\text{zero}})$. From Bijection (12), we conclude that there is a unique map can_{YX} such that the diagram

$$\begin{array}{ccc} \{\text{torsion elements in } D_X\}/W_X & \xrightarrow[\gamma]{\sim} & \mathrm{Rel}(X \operatorname{dlog} z)/\mathrm{SL}_n(K) \\ {\scriptstyle \mathrm{can}_{YX}}\downarrow{\scriptstyle\sim} & & \| \\ \{\text{torsion elements in } D_Y\}/W_Y & \xrightarrow[\gamma]{\sim} & \mathrm{Rel}(Y \operatorname{dlog} z)/\mathrm{SL}_n(K) \end{array} \qquad (13)$$

commutes. This map can_{YX} can be described explicitly, see [5].

Theorem 7.10. (Regular SL_n-Connections up to Gauge Equivalence) *Let γ be a procyclic generator of* $\mathrm{Gal}(\overline{K}/K)$ *and*

$$\coprod_{X\in\mathcal{J}(\mathfrak{sl}_n^{\mathrm{zero}})} \{\textit{torsion-elts. in } D_X\}/W_X \underset{\gamma}{\twoheadrightarrow} \{\textit{regular } \mathrm{SL}_n\textit{-connections}\}\,/\,\mathrm{SL}_n(K)$$

be the map induced by the maps (12), $\delta \mapsto \left[(X \operatorname{dlog} z)^\delta\right]$. *Then this map is surjective, and we have* $\left[(X \operatorname{dlog} z)^\delta\right] = \left[(Y \operatorname{dlog} z)^\epsilon\right]$ *if and only if X and Y differ rationally after block permutation and* $\mathrm{can}_{YX}(\delta) = \epsilon$.

Remark 7.11. The sets $\{\text{torsion elements in } D_X\}/W_X$ are easy to describe. "Differing rationally after block permutation" is an equivalence relation on the set $\mathcal{J}(\mathfrak{sl}_n^{\mathrm{zero}})$. By choosing a complete system of representatives for this relation, and by using the explicit description of the map $\delta \mapsto \left[(X \operatorname{dlog} z)^\delta\right]$ given in the proof of Proposition 7.3, Theorem 7.10 enables us to give a list of all regular SL_n-connections up to $\mathrm{SL}_n(K)$-equivalence.

Proof. Proposition 7.3, Bijection (12) and Corollary 7.5 show that our map is well defined and surjective. The remaining claim follows from Theorem 7.9 and Diagram (13). ∎

Remark 7.12. We now explain a nice partial classification of regular connections up to gauge equivalence. We associate to the standard diagonal Cartan subalgebra $\mathfrak{t} \subset \mathfrak{sl}_n$ the coroots $R^\vee$ and the Weyl group W. This Weyl group acts naturally on $\mathfrak{t}$ and stabilizes the subgroups $\mathbb{Z}R^\vee$ and $\mathbb{Q}R^\vee$. The groups $W^{\mathbb{Z}} = \mathbb{Z}R^\vee \rtimes W$ and $W^{\mathbb{Q}} = \mathbb{Q}R^\vee \rtimes W$ act on $\mathfrak{t}$ by $(a, w).H = a + wH$. Two elements of $\mathfrak{t}$ are in the same $W^{\mathbb{Z}}$-orbit (resp. $W^{\mathbb{Q}}$-orbit) if and only if they differ integrally (resp. rationally) after block permutation. Let $\mathcal{N}_n = \mathcal{J}(\mathfrak{sl}_n) - \mathfrak{t}$. Consider the commutative diagram

$$\begin{array}{ccccc} \mathfrak{t}/W^{\mathbb{Z}} & \hookrightarrow & \{\text{regular } \mathrm{SL}_n\text{-connections}\}\,/\,\mathrm{SL}_n(K) & & \\ \downarrow & & \downarrow \pi & & \\ \mathfrak{t}/W^{\mathbb{Q}} & \hookrightarrow & \{\text{regular } \mathrm{SL}_n\text{-connections}\}\,/\text{relationship} & \xleftarrow{\nu} & \mathcal{N}_n \end{array} \tag{14}$$

with obvious vertical maps. The horizontal maps are induced by $X \mapsto X \operatorname{dlog} z$. The horizontal maps on the left are well-defined and injective by Proposition 7.8 and Theorem 7.9. They yield a partial classification. In the lower row of Diagram (14) the images of the horizontal maps are complementary by Theorem 7.9. From Corollary 7.4 follows that the image of the upper horizontal map is the complement of $\pi^{-1}(\nu(\mathcal{N}_n))$.

Example 7.13. We restrict now to the case $n = 2$. Then $\mathcal{N}_2 = \{\mathrm{J}(0, 2)\}$, and $\nu(\mathcal{N}_2)$ consists of one element, namely $\mathrm{Rel}(\mathrm{J}(0, 2) \operatorname{dlog} z)/\text{relationship}$. Its inverse image under π is $\mathrm{Rel}(\mathrm{J}(0, 2) \operatorname{dlog} z)/\,\mathrm{SL}_n(K)$. From the description of the map (12) in the proof of Proposition 7.3 we see that this set has precisely two elements, namely the orbits of the two connections (cf. example $G = \mathrm{SL}_2$ in Examples 4.4)

$$\begin{bmatrix} 0 & 1 \\ & 0 \end{bmatrix} \operatorname{dlog} z \quad \text{and} \quad \begin{bmatrix} \frac{1}{2} & z \\ & -\frac{1}{2} \end{bmatrix} \operatorname{dlog} z.$$

8. Fuchsian Connections

Let G be a linear algebraic group and $\rho : G \to \mathrm{GL}(V)$ be a (rational) representation of G in a finite-dimensional vector space V. If A is a G-connection, $\rho(A) = \rho_*(A) \in \mathcal{GL}(V)$ is a $\mathrm{GL}(V)$-connection and corresponds to a D-module structure $\alpha_{\rho(A)}$ on $K \otimes V$ (cf. Proposition 5.4).

Definition 8.1. A connection A is *Fuchsian* if for every finite-dimensional representation $\rho : G \to \mathrm{GL}(V)$ the D-module $(K \otimes V, \alpha_{\rho(A)})$ is Fuchsian.

Remark 8.2. According to Proposition 5.4, a connection A is Fuchsian if and only if for every finite-dimensional representation $\rho : G \to \mathrm{GL}(V)$ the connection $\rho(A)$ is regular.

Proposition 8.3. *Let G be a linear algebraic group. Every regular connection is Fuchsian. For $G = \mathrm{GL}_n$ or $G = \mathrm{SL}_n$, every Fuchsian connection is regular.*

Problem 8.4. Do the notions of Fuchsian and regular connection coincide for every linear algebraic group?

Remark 8.5. Using Remarks 7.6 and 8.2, it is easy to see that all relatives of a Fuchsian connection are Fuchsian. This shows that the answer "yes" to Problem 8.4 implies the same answer to Problem 7.7.

Proof. The first claim and the second one for $G = \mathrm{GL}_n$ are obvious. Let A be a Fuchsian SL_n-connection. Let $\rho : \mathrm{SL}_n \hookrightarrow \mathrm{GL}_n$ be the standard representation of SL_n. There are $g \in \mathrm{GL}_n(K)$ and $X(z) \in \mathfrak{gl}_n[[z]]$ such that $g[\rho(A)] = X(z)\,\mathrm{dlog}\, z$. Consider the field extension $n^* : K \hookrightarrow K = N$. Let $f \in N$ be an n-th root of $n^*(\det(g^{-1}))$. Then $h = fn^*(g)$ is an element of $\mathrm{SL}_n(N)$, and we have

$$\rho(h[n^*(A)]) = f\mathbf{1}\,[n^*(X(z)\,\mathrm{dlog}\, z)] = \big(nX(z^n) + z\,\partial_z(f)f^{-1}\mathbf{1}\big)\ \mathrm{dlog}\, z.$$

It is obvious that $z\,\partial_z(f)f^{-1} \in k[[z]]$. But then $h[n^*(A)]$ is regular, and Corollary 7.5 shows that A is regular. ■

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