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The Spherical Transform on Projective Limits of Symmetric Spaces

Andrew R. Sinton*

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Abstract. The theory of a spherical Fourier transform for measures on certain projective limits of symmetric spaces of non-compact type is developed. Such spaces are introduced for the first time and basic properties of the spherical transform, including a Levy-Cramer type continuity theorem, are obtained. The results are applied to obtain a heat kernel measure on the limit space which is shown to satisfy a certain cylindrical heat equation. The projective systems under consideration arise from direct systems of semi-simple Lie groups $\{G_j\}$ such that G_j is essentially the semi-simple component of a parabolic subgroup of G_{j+1}. This class includes most of the classical families of Lie groups as well as infinite direct products of semi-simple groups.
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Introduction

The goal of this paper is to lay the foundations for the study of the harmonic analysis and spherical transform of projective (inverse) limits of symmetric spaces of non-compact types. Consider the 'ladder' of real semi-simple Lie groups $G_1 \subset \cdots \subset G_j \subset G_{j+1} \subset \dots$ where each G_j is essentially the semi-simple part of a parabolic subgroup of G_{j+1}. This system gives rise to a parallel ladder of symmetric spaces $\mathbf{X}_j = G_j/K_j$ on which there are injective maps moving up the ladder and surjective projections moving down the ladder:

$$\eta_{j,j+1} : \mathbf{X}_j \hookrightarrow \mathbf{X}_{j+1}, \quad \pi_{j+1,j} : \mathbf{X}_{j+1} \twoheadrightarrow \mathbf{X}_j.$$

This in turn leads to the construction of two related infinite dimensional spaces, a direct limit and a projective limit of symmetric spaces

$$\mathbf{X}_\infty = \varinjlim\{\mathbf{X}_j, \eta_{j,j+1}\}, \quad \mathbf{X}^\infty = \varprojlim\{\mathbf{X}_j, \pi_{j+1,j}\}.$$

Both of these spaces are natural candidates for an extension of the theory of the spherical transform since they are built from objects for which the finite-dimensional theory is well-known.

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It turns out that the projective limit $\mathbf{X}^\infty$ is better suited to such a theory since, unlike $\mathbf{X}_\infty$, it possesses a clean theory of measures tying it to the finite-dimensional $\mathbf{X}_j$. Though there is not a nice analogue of the G_j-invariant measure on $\mathbf{X}_j$, there is a one-to-one correspondence between finite, semi-positive, bounded Borel measures on $\mathbf{X}^\infty$ and families $\{\mu_j\}$ where each μ_j is itself a finite, semi-positive, bounded Borel measure on $\mathbf{X}_j$ such that $\pi_{j,j-1,*}\mu_j = \mu_{j-1}$.

The main result presented here is the construction of a spherical transform $\mathbf{S}^\infty$ for such measures on $\mathbf{X}^\infty$ and an associated construction of a heat kernel measure. Success depends on the compatibility of the spherical transform (and heat kernel, respectively) with the push forwards $\pi^{j+1,j,*}$ at every rung of the ladder. Once this is established, the definitions of $\mathbf{S}^\infty$ (and ν_t^∞, respectively) follow from basic functorial considerations. The whole picture for the spherical transform can be summed up by the commutative diagram of Theorem 6.1. As is expected from the parallel theory on Euclidean spaces, the spherical transform of a measure on the projective limit $\mathbf{X}^\infty$ is a function on a *direct* limit of dual spaces.

The interplay of direct and inverse limits appearing here is hardly surprising but is a universal feature of harmonic analysis on limits of spaces, starting with $\mathbf{R}^\infty$. More recently, it has appeared in the beautiful works of Olshanski, Borodin, Vershik, Kerov and others on representations of the infinite symmetric and unitary groups. Cf. [19], [15], [4] and the references there. The present study can actually be viewed as a programmatic extension of these works, pursuing in the case of non-compact type symmetric spaces what has already been achieved for Euclidean and compact symmetric spaces. To be sure, Olshanski and company have pushed the theory much further and deeper than what is accomplished here. It is hoped that in the future similar results can be obtained for symmetric spaces of non-compact type as well.

In relating the current results to the rest of the literature, a number of questions present themselves immediately. What is the relationship between the spherical functions of direct limit groups in [18], the cylindrical spherical functions of section 5 (and the functions in the remark preceding Theorem 6.1), and the principal series representations of direct limit groups in [23]? Is there some kind of 'tangent space' relationship between the harmonic analysis of limits of Euclidean symmetric spaces in [20] and the limits of symmetric spaces of non-compact type discussed here? How does the heat kernel measure on $\mathbf{X}^\infty$ obtained here relate to the heat kernel on Hilbert-Schmidt groups obtained by M. Gordina in [9] and [10]? Also missing is a more systematic study of the geometry and structure of $\mathbf{X}^\infty$ beyond the Iwasawa-type coordinates that suffice for constructing the spherical transform.

Perhaps the most obvious gap is the lack of any kind of representation theory relating to $\mathbf{X}^\infty$. Both philosophically and practically, projective limits provide a natural geometric setting in which to realize the analogues of the regular representation for direct limit groups. However, unlike the situation for finite groups or compact and Euclidean symmetric spaces, $\mathbf{X}^\infty$ possesses no obvious measure that is invariant under the direct limit of the G_j so it is difficult to construct geometric representations.

The paper is organized as follows. Sections 1-3 set notation, review the requisite background of semi-simple Lie groups and their parabolic subgroups, and establish the relationship between the spherical transform on a group and a single parabolic subgroup. The details of the associated coordinates, projections, and inclusions are worked out in this setting, freed from the notational complexity of dealing with an infinite number of spaces simultaneously. Particularly important is the non-standard parametrization of the spherical functions and spherical transform in Section 3. The main results are the eigenfunction lifting property of Proposition 3.6 which is essential in proving the basic commutative diagram of Theorem 3.7. We follow Chapter 1 of [14] with the caveat that results here are phrased in terms of spherical transforms of measures instead of functions. This highlights the functorial nature of the results (see Theorem 3.7), which becomes critical in passing to the limits in the next part. Section 3 concludes with a brief discussion of the heat kernel as a primary application of Theorem 3.7.

Sections 4 begins the heart of the matter as the focus switches to the theory of an infinite ladder of groups with all the associated projections, inclusions, and limits. In it, we discuss an Iwasawa-type decomposition on both the direct and projective limits, before moving to the machinery related to limits of functions and measures in section 5. Section 6 defines the spherical transform $\mathbf{S}^\infty$ and proves some of its basic properties. The primary example of a measure and its spherical transform is provided by the heat kernel in section 7, for which it is also necessary to arrive at an appropriate notion of a Casimir (Laplacian) differential operator for $\mathbf{X}^\infty$. We conclude in section 8 with examples illustrating the two main types of ladders: direct products of semi-simple groups and limits of the classical groups.

Though the ideas are actually quite straightforward, the need to keep all the spaces, maps, and associated notation straight is one of the primary challenges in deciphering the results. To assist the reader we have tried to keep the notation as functorial as possible without abbreviations in an attempt to make the meaning more transparent.

I would like to thank Joe Wolf for advising the dissertation of which this paper is an adaptation, and for including me in his research into direct limits. Similar gratitude is due to Jay Jorgenson and Serge Lang for including me in their inquiries concerning the heat kernel on a group and a subgroup.

1. Background and Notation

We begin with some standard background to help fix notation. For a general reference, see [11], and Chapter 2 of [7].

Let G be a connected semi-simple finite dimensional real linear Lie group. We will always view G as a subgroup of $\mathrm{GL}(n, \mathbf{C})$, the group of all invertible linear transformations of $\mathbf{C}^n$. Let θ be a Cartan involution of G. Then after a conjugation if necessary, we may assume

$$\theta(g) = (g^*)^{-1}$$

where g^* denotes the transpose conjugate of g. We select an Iwasawa decomposition $G = NAK$ compatible with θ, so $K = G^\theta$ and θ fixes A pointwise. This determines all the usual data: the set of restricted roots $\mathcal{R}(\mathfrak{a}, \mathfrak{g})$, positive roots $\mathcal{R}^+(\mathfrak{a}, \mathfrak{g}) = \mathcal{R}(\mathfrak{a}, \mathfrak{n})$, simple roots $\mathcal{S}(\mathfrak{a}, \mathfrak{n}) = \mathcal{S}(\mathfrak{a}, \mathfrak{g})$, and minimal parabolic MAN, where M is the centralizer of A in K. We denote the half-trace of the regular (adjoint, bracket) representation of $\mathfrak{a}$ on $\mathfrak{n}$ by

$$\rho = \frac{1}{2} \sum_{\alpha \in \mathcal{R}(\mathfrak{a},\mathfrak{n})} m_\alpha \alpha$$

where m_α is the multiplicity of α.

Let B be a bilinear form on $\mathfrak{g}$ such that the restriction of B to any simple factor of $\mathfrak{g}$ is a positive multiple of the Killing form on that factor. Any positive multiple of the real trace form

$$B(Y_1, Y_2) = \mathrm{Re}\ \mathrm{Tr}(Y_1 Y_2)$$

will do. Since $\mathfrak{g}$ is semi-simple, B is non-degenerate. In particular, B also defines a positive definite bilinear form on $\mathfrak{a}^\vee$ which we denote $B^\vee$.

Define the Weyl group W of$\mathfrak{g}$ as the group of automorphisms of $\mathfrak{a}^\vee$ generated by reflections across hyperplanes perpendicular (under $B^\vee$) to elements of $\mathcal{R}(\mathfrak{a}, \mathfrak{g})$. We can also extend the action of W to the complexification $\mathfrak{a}^\vee_{\mathbf{C}}$ by making elements of W complex linear.

Let $\mathbf{X} = G/K$ and let $e \in \mathbf{X}$ be the identity coset eK. We can turn $\mathbf{X}$ into a symmetric space with G-invariant metric induced by B in the usual way. Cf. [22] and [11] for more detailed discussions. With this metric, $\mathbf{X}$ becomes a Cartan-Hadamard manifold (simply connected, complete, semi-negative curvature) and G acts isometrically on $\mathbf{X}$ by the natural left translation on cosets. The Iwasawa decomposition of G gives useful coordinates on $\mathbf{X}$, the natural map $(n, a) \mapsto naK$ yielding a differential isomorphism $N \times A \simeq \mathbf{X}$.

The projection to the Iwasawa A-component from both G and $\mathbf{X}$ will occur frequently, and for any $g = nak$ and $x = naK$ we abbreviate

$$(g)_A = (x)_A = a.$$

The Iwasawa projection can be used to turn characters on A into functions on G or $\mathbf{X}$. Given $\zeta \in \mathfrak{a}^\vee_{\mathbf{C}}$, we write

$$\zeta(g) = (g)_A^\zeta \text{ and } \zeta(x) = (x)_A^\zeta$$

when no confusion is possible. Here by the term 'character' we mean any homomorphism into $\mathbf{C}$.

Our interest lies primarily in studying the relationship between G and what are essentially semi-simple components of parabolic subgroups of G. Let Ω_Q be an arbitrary subset of $\mathcal{S}(\mathfrak{a}, \mathfrak{g})$ and let $\mathcal{R}(\mathfrak{a}_{\mathfrak{g}_\mathfrak{q}}, \mathfrak{g}_\mathfrak{q})$ (resp. $\mathcal{R}(\mathfrak{a}_{\mathfrak{g}_\mathfrak{q}}, \mathfrak{n}_{\mathfrak{g}_\mathfrak{q}})$) denote the set of restricted roots (resp. positive restricted roots) generated by Ω_Q. Let

$$\mathcal{R}(\mathfrak{a}_\mathfrak{q}, \mathfrak{n}_\mathfrak{q}) = \{\alpha \in \mathcal{R}(\mathfrak{a}, \mathfrak{n}) \mid \alpha \notin \mathcal{R}(\mathfrak{a}_{\mathfrak{g}_\mathfrak{q}}, \mathfrak{n}_{\mathfrak{g}_\mathfrak{q}})\}.$$

Then Ω_Q determines a unique standard parabolic subalgebra $\mathfrak{q} \subset \mathfrak{g}$ containing $\mathfrak{m}+\mathfrak{n}+\mathfrak{a}$. Cf. [7] §2.3. The Langlands decomposition of $\mathfrak{q}$ is given by $\mathfrak{q} = \mathfrak{n}_\mathfrak{q}+\mathfrak{a}_\mathfrak{q}+\mathfrak{m}_\mathfrak{q}$ where

- $\mathfrak{a}_\mathfrak{q} = \{H \in \mathfrak{a} \mid \alpha(H) = 0 \text{ for all } \alpha \in \mathcal{R}(\mathfrak{a}_\mathfrak{q}, \mathfrak{n}_\mathfrak{q})\}$;
- $\mathfrak{n}_\mathfrak{q} = \bigoplus_{\alpha \in \mathcal{R}(\mathfrak{a}_\mathfrak{q}, \mathfrak{n}_\mathfrak{q})} \mathfrak{g}_\alpha$ is the nilpotent radical of $\mathfrak{q}$;

- $\mathfrak{m}_\mathfrak{q} + \mathfrak{a}_\mathfrak{q}$ is the centralizer of $\mathfrak{a}_\mathfrak{q}$ in $\mathfrak{g}$.

The Lie algebra $\mathfrak{m}_\mathfrak{q}$ is reductive, θ-stable, centralizes $\mathfrak{a}_\mathfrak{q}$, and normalizes $\mathfrak{n}_\mathfrak{q}$. Let the algebra $\mathfrak{m}'_\mathfrak{q} = [\mathfrak{m}_\mathfrak{q}, \mathfrak{m}_\mathfrak{q}]$ denote the derived algebra, also called the *semi-simple component* of $\mathfrak{q}$. We will be exclusively interested in symmetric spaces, so to make our results as broad as possible we can ignore any compact factors occurring in $\mathfrak{m}'_\mathfrak{q}$. Since such factors will be contained in $\mathfrak{m} = \mathrm{Lie}(M)$, we let $\mathfrak{g}_\mathfrak{q}$ be any semi-simple Lie algebra contained in $\mathfrak{m}'_\mathfrak{q}$ that is equal to $\mathfrak{m}'_\mathfrak{q}$ modulo $\mathfrak{m}$. We will call any such $\mathfrak{g}_\mathfrak{q}$ a *reduced semi-simple component* of the parabolic subalgebra $\mathfrak{q}$ and say that such an algebra $\mathfrak{g}_\mathfrak{q}$ is *weakly parabolic* in $\mathfrak{g}$. Cf. [23] Section 8 for a further discussion and equivalent conditions.

On the group level we let Q be the normalizer of $\mathfrak{q}$ in G, and let A_Q, N_Q, and G_Q be be the unique connected subgroups of G with Lie algebras $\mathfrak{a}_\mathfrak{q}$, $\mathfrak{n}_\mathfrak{q}$, and $\mathfrak{g}_\mathfrak{q}$ respectively.

All notation for G now carries over to G_Q by inserting subscripts of G_Q everywhere. In particular G_Q has an Iwasawa decomposition given by

$$G_Q = N_{G_Q} A_{G_Q} K_{G_Q}.$$

A central feature of the structure theory is the way the groups N and A split nicely into products

$$N = N_Q N_{G_Q} \text{ and } A = A_Q A_{G_Q}. \tag{1}$$

In particular, note that N_Q is a normal subgroup of N and thus the first product is semidirect, while the second product is direct since A is abelian. On the Lie algebra level, the corresponding decomposition of $\mathfrak{a}$ is actually an orthogonal direct sum with respect to B, $\mathfrak{a} = \mathfrak{a}_{\mathfrak{g}_\mathfrak{q}} \oplus \mathfrak{a}_\mathfrak{q}$. This in turn induces a similar orthogonal decomposition on the dual space

$$\mathfrak{a}^\vee_\mathbf{C} = \mathfrak{a}^\vee_{\mathfrak{g}_\mathfrak{q},\mathbf{C}} \oplus \mathfrak{a}^\vee_{\mathfrak{q},\mathbf{C}}$$

where $\zeta \in \mathfrak{a}^\vee_{\mathfrak{q},\mathbf{C}}$ if and only if ζ is identically zero on $\mathfrak{a}_{\mathfrak{g}_\mathfrak{q}}$.

One effect of these splittings of A and N is to provide nice coordinates on the symmetric space $\mathbf{X} = G/K$.

Proposition 1.1. *Let $\mathbf{X}_Q = G_Q/K_{G_Q}$. There is a differential isomorphism*

$$N_Q \times A_Q \times \mathbf{X}_Q \simeq \mathbf{X}$$

Given by

$$(n_Q, a_Q, g_Q K_{G_Q}) \mapsto n_Q a_Q g_Q K.$$

Proof. By the Iwasawa coordinates of $\mathbf{X}$ and the splitting of N and A in (1) any element $x \in \mathbf{X}$ can be uniquely written as

$$x = n_Q n_{G_Q} a_Q a_{G_Q} K.$$

But a_Q commutes with n_{G_Q} and the proposition follows. ∎

We call the (n_Q, a_Q, x_Q) *Q-parabolic coordinates* on $\mathbf{X}$.

Given a group element $g = n\, a_Q\, a_{G_Q}\, k$ we denote the A_{G_Q} and A_Q Iwasawa projections by $(g)_{A_{G_Q}} = a_{G_Q}$ and $(g)_{A_Q} = a_Q$ respectively. We use the same notation for the projections from the symmetric space, with g replaced by x.

The focus of this paper will be on the natural maps associated to the Q-parabolic coordinates. In particular, these coordinates gives rise to maps in two directions: the natural inclusion $\eta_Q : \mathbf{X}_Q \hookrightarrow \mathbf{X}$ and the projection $\pi_Q : \mathbf{X} \to \mathbf{X}_Q$. For a specific example of such a projection see (3). The projection π_Q is surjective and identifying A and N with their orbits of eK, we can restrict π_Q to surjective projections from A and N

$$\pi_{Q,A} : A \twoheadrightarrow A_{G_Q}, \quad \pi_{Q,N} : N \twoheadrightarrow N_{G_Q}.$$

The former induces an injection on the dual space

$$\pi^\vee_{Q,A} : \mathfrak{a}^\vee_{\mathfrak{g}_\mathfrak{q},\mathbf{C}} \longrightarrow \mathfrak{a}^\vee_{\mathbf{C}}.$$

Recalling that $\mathcal{R}(\mathfrak{a}_{\mathfrak{g}_\mathfrak{q}}, \mathfrak{n}_{\mathfrak{g}_\mathfrak{q}}) \subset \mathcal{R}(\mathfrak{a}, \mathfrak{n})$, the map $\pi^\vee_{Q,A}$ is seen just be the linear extension of this inclusion.

A key fact concerns the way ρ splits into orthogonal pieces. Let

$$\rho_{G_Q} = \frac{1}{2} \sum_{\alpha \in \mathcal{R}(\mathfrak{a}_{\mathfrak{g}_\mathfrak{q}}, \mathfrak{n}_{\mathfrak{g}_\mathfrak{q}})} m_\alpha \alpha.$$

$$\rho_Q = \frac{1}{2} \sum_{\alpha \in \mathcal{R}(\mathfrak{a}_\mathfrak{q}, \mathfrak{n}_\mathfrak{q})} m_\alpha \alpha$$

be half of the trace of the regular representation of $\mathfrak{a}_{\mathfrak{g}_\mathfrak{q}}$ acting on $\mathfrak{n}_{\mathfrak{g}_\mathfrak{q}}$ and of $\mathfrak{a}_\mathfrak{q}$ acting on $\mathfrak{n}_\mathfrak{q}$ respectively.

Lemma 1.2. *There is an orthogonal decomposition*

$$\rho = \rho_{G_Q} + \rho_Q.$$

Furthermore, $\rho_Q \in \mathfrak{a}^\vee_\mathfrak{q}$. *I.e., for all* $H \in \mathfrak{a}_{\mathfrak{g}_\mathfrak{q}}$

$$\rho_Q(H) = 0.$$

Proof. The result is well-known. For a proof cf. [23] where it is proved that this condition is equivalent to $\mathfrak{g}_\mathfrak{q}$ being weakly parabolic in $\mathfrak{g}$. ∎

2. Integration, Measures, and Functions

Integration. We fix the following objects and normalizations:

- all non-discrete compact groups (M, K, K_{G_Q}) are given a Haar measure such that the total volume is 1;
- we give A, A_{G_Q}, and A_Q the Haar measures da, da_{G_Q}, and da_Q coming from the volume form associated to the restriction of B to $\mathfrak{a}$, $\mathfrak{a}_{\mathfrak{g}_\mathfrak{q}}$, and $\mathfrak{a}_\mathfrak{q}$ respectively;

- the bilinear form $(u,v) \mapsto B(u,v^*)$ is positive definite on $\mathfrak{g}$. Its restriction to $\mathfrak{n}$ defines a Euclidean measure on $\mathfrak{n}$, $\mathfrak{n}_{\mathfrak{g}_\mathfrak{q}}$, and $\mathfrak{n}_\mathfrak{q}$. The image of these measures under the exponential map are Haar measures denoted dn, dn_{G_Q}, and dn_Q on N, N_{G_Q}, and N_Q respectively;
- we define the B-*Iwasawa Haar measure* $d\mu_{\mathrm{Iw},B}(g) = dg$ on G such that for any $f \in C_c(G)$ we have
$$\int_G f(g)dg = \int_A \int_N \int_K f(ank)dkdnda = \int_A \int_N \int_K f(nak)a^{-2\rho}dkdnda.$$
where $a^{-2\rho} = e^{-2\rho(\log a)}$.

The Haar measure on G_Q is defined similarly.

The choice of Haar measures dg and dk determines a unique G-invariant measure dx on $\mathbf{X} = G/K$ such that
$$\int_{\mathbf{X}} \int_K f(xk)dkdx = \int_G f(g)dg$$
for every $f \in C_c(G)$. By standard Haar measure computations we obtain the *parabolic coordinates integration formula* on $\mathbf{X}$.

Proposition 2.1. *For any $f \in C_c(\mathbf{X})$ we have*
$$\int_{\mathbf{X}} f(x)dx = \int_{N_Q} \int_{A_Q} \int_{\mathbf{X}_Q} f(n_Q a_Q x_Q) a_Q^{-2\rho_Q} dn_Q\, da_Q\, dx_Q =$$
$$\int_{N_Q} \int_{A_Q} \int_{\mathbf{X}_Q} f(a_Q n_Q x_Q) dn_Q\, da_Q\, dx_Q \quad (2)$$

Proof. Cf. [7] Proposition 2.4.3. ∎

Borel Measures. We denote the set of all finite, semi-positive (i.e. non-negative) Borel measures on $\mathbf{X} = G/K$ by $M(\mathbf{X})$, and the subset of all K-invariant, finite, semi-positive Borel measures will be denoted by $M^\natural(\mathbf{X})$. In using the generic term *measure* we always mean a finite, semi-positive Borel measure unless otherwise specified (e.g. Haar measure).

When no confusion is possible we will identify elements of $M(\mathbf{X})$ and $M^\natural(\mathbf{X})$ with right K-invariant measures and K-bi-invariant measures on G respectively. Using this identification and the group law on G, we can define the convolution of two measures μ_1, $\mu_2 \in M^\natural(\mathbf{X})$ by the formula
$$\mu_1 \star \mu_2(U) = \int_G \mu_1(Ug^{-1})d\mu_2(g)$$
for any Borel set $U \subset G$. As is well known, (G,K) is a Gelfand pair, i.e., $M^\natural(\mathbf{X})$ is a commutative algebra under the convolution operation, cf. [6]. The same applies to G_Q, K_{G_Q}, and $\mathbf{X}_Q$ by inserting appropriate subscripts everywhere.

The spaces $M(\mathbf{X})$ and $M^\natural(\mathbf{X})$ can be viewed as subsets of the dual space of the bounded continuous functions $BC(\mathbf{X})$ with the weak topology, which they then inherit. In particular, if $\{\mu^{(i)}\}$ is a sequence of measures on $\mathbf{X}$, we say that $\{\mu^{(i)}\}$ *converges weakly* to a measure μ if for every $f \in BC(\mathbf{X})$

$$\lim_{i\to\infty}[f,\mu^{(i)}]_{\mathbf{X}} = [f,\mu]_{\mathbf{X}}.$$

We want to study the relationship between measures on $\mathbf{X}$ and measures on $\mathbf{X}_Q$. Fortunately, K-invariant measures push forward to K_{G_Q}-invariant measures under π_Q.

Lemma 2.2. *Let $\pi_{Q,*}$ denote the push forward of measures under π_Q. Then*

$$\pi_{Q,*} : M^\natural(G/K) \longrightarrow M^\natural(G_Q/K_{G_Q}).$$

Proof. Let $U_Q \subset \mathbf{X}_Q$ be an arbitrary Borel set, and $\mu \in M^\natural(\mathbf{X})$. Then by definition,

$$(\pi_{Q,*}\mu)(U_Q) = \mu\big(\pi_Q^{-1}(U_Q)\big).$$

But $\pi_Q^{-1}(U_Q) = N_Q A_Q U_Q$. Thus

$$(\pi_{Q,*}\mu)(k_{G_Q}U_Q) = \mu(N_Q A_Q k_{G_Q} U_Q).$$

But since k_{G_Q} commutes with A_Q, normalizes N_Q, and μ is K-invariant, the right hand side equals $(\pi_{Q,*}\mu)(U_Q)$ as desired. ∎

Functions. For any bounded continuous function $f \in BC(\mathbf{X})$ we denote it's (left)
K-average by

$$f^K(x) = \int_K f(kx)dk.$$

We then have the standard and useful fact

Lemma 2.3. *For any $f \in BC(\mathbf{X})$ and $\mu \in M^\natural(\mathbf{X})$*

$$\int_{\mathbf{X}} f^K(x)d\mu(x) = \int_{\mathbf{X}} f(x)d\mu(x).$$

Proof. Immediate from Fubini and the K-invariance of μ. ∎

Given a measure $\mu \in M(\mathbf{X})$ and a function $f \in \mathrm{Fu}(\mathbf{X})$ we sometimes use the notation

$$[f,\mu]_{\mathbf{X}} = \int_{\mathbf{X}} f(x)d\mu(x)$$

whenever the integral is defined. Then the preceding lemma simply reads

$$[f,\mu]_{\mathbf{X}} = [f^K,\mu]_{\mathbf{X}}$$

whenever $\mu \in M^\natural(\mathbf{X})$.

Let π_Q^* denote the pullback of functions from $\mathbf{X}_Q$ to $\mathbf{X}$ under the projection π_Q. Thus for $f \in BC(\mathbf{X}_Q)$ and $x \in \mathbf{X}$

$$(\pi_Q^* f)(x) = f(\pi_Q(x)).$$

It is important to remember the adjointness relation between $\pi_{Q,*}$ and $\pi_{Q,*}$

$$[f,\pi_{Q,*}\mu]_{\mathbf{X}_Q} = [\pi_Q^* f,\mu]_{\mathbf{X}}$$

for $f \in BC(\mathbf{X}_Q)$ and $\mu \in M^\natural(\mathbf{X})$.

Lemma 2.4. *The pull back π_Q^* commutes with K_{G_Q}- averaging. For any $f \in C(\mathbf{X}_Q)$*

$$\pi_Q^*(f^{K_{G_Q}}) = (\pi_Q^* f)^{K_{G_Q}}$$

as functions on $\mathbf{X}$.

Note that the average on the left occurs in $\mathbf{X}_Q$ while the average on the right occurs in $\mathbf{X}$.

Proof. The two sides clearly agree on $\mathbf{X}_Q \subset \mathbf{X}$. The left-hand side is left-invariant under A_Q and N_Q by definition. Since K_{G_Q} normalizes N_Q and commutes with A_Q, the right hand side is also easily seen to be left-invariant under A_Q and N_Q. By the Q-parabolic coordinates on $\mathbf{X}$, the two sides must be equal on all of $\mathbf{X}$. ∎

3. The Spherical Functions and Transform

For reasons that will become apparent when we consider an infinite ladder of groups, we break with 50 odd years of tradition and choose a non-standard parametrization of spherical functions and the spherical transform.

Given $\zeta \in \mathfrak{a}_{\mathbf{C}}^{\vee}$ we define the *spherical function* ϕ_ζ to be function on $\mathbf{X}$ given by the left K-average of the character ζ:

$$\phi_\zeta(x) = \int_K (kx)_A^\zeta dk.$$

The spherical functions ϕ_ζ and $\phi_{\zeta'}$ are equal if and only if there exists $w \in W$ such that

$$\zeta = w(\zeta' + \rho) - \rho.$$

This isn't the nice W-invariance of Harish-Chandra, but such expressions are common in the literature in connection with representations of compact groups, cf. [17] in the context of direct limits. Any object possessing this invariance for every $w \in W$ will be called (W, ρ)-invariant. For example, a function f on $\mathfrak{a}_{\mathbf{C}}^{\vee}$ satisfying

$$f(\zeta) = f(w(\zeta + \rho) - \rho)$$

for all $w \in W$ will be called (W, ρ)-invariant. The set of all such functions will be denoted $\mathrm{Fu}(\mathfrak{a}_{\mathbf{C}}^{\vee})^{(W,\rho)}$.

We define the *spherical transform* of a measure $\mu \in M^\natural(\mathbf{X})$ to be the function on $\mathfrak{a}_{\mathbf{C}}^{\vee}$ defined by

$$(\mathbf{S}\mu)(\zeta) = [\phi_\zeta, \mu]_{\mathbf{X}} = \int_X \phi_\zeta(x) d\mu(x)$$

whenever the integral is well defined. Using the definition of ϕ_ζ and Lemma 2.3 we can also write the spherical transform as

$$(\mathbf{S}\mu)(\zeta) = \int_{\mathbf{X}} (x)_A^\zeta d\mu(x).$$

In general, we are only guaranteed that the integral defining the spherical transform converges for values of ζ such that ϕ_ζ is bounded. Helgason and Johnson determined the bounded spherical functions, however we must shift their tube by ρ to make up for the lack of ρ in our definition of the spherical functions.

Let $\mathrm{Con}(W\rho)$ be the convex set in $\mathfrak{a}^\vee$ whose vertices are the W-orbit of ρ. Let $C_\rho = \mathrm{Con}(W\rho) + \rho$ be the convex set shifted by ρ. We define the tube

$$T_\rho = \mathbf{i}\mathfrak{a}^\vee + C_\rho.$$

Note that C_ρ and hence T_ρ are (W, ρ)-invariant sets.

Proposition 3.1. (Helgason-Johnson [12]) *The spherical function ϕ_ζ is bounded if and only if $\zeta \in T_\rho$.*

Corollary 3.2. *Given $\mu \in M^\natural(\mathbf{X})$, the spherical transform $\mathbf{S}\mu$ is a well defined function on T_ρ.*

Now that we have a well defined spherical transform, we take a slight pause to recall without proof some of its basic properties that will be generalized to the projective limit later on. Again, everything in this section applies to $\mathbf{X}_Q$ and its spherical transform $\mathbf{S}_Q$ with appropriate subscripts inserted everywhere. In particular, note that (W_{G_Q}, ρ_{G_Q}) takes the place of (W, ρ).

Proposition 3.3. (Gangolli [6]) *Let $\mu_1, \mu_2 \in M^\natural(\mathbf{X})$ and let $f_1 = \mathbf{S}(\mu_1)$, $g_2 = \mathbf{S}(\mu_2)$.*

1. *(Invariance) $\mathbf{S}(\mu_1)$ is (W, ρ)-invariant;*
2. *(Homomorphism) for $\zeta \in T_\rho$, $\mathbf{S}(\mu_1 \star \mu_2)(\zeta) = f_1(\zeta) f_2(\zeta)$;*
3. *(Injectivity) if $f_1 = f_2$, then $\mu_1 = \mu_2$.*

We cite Gangolli here as he was the first to prove statements about the spherical transform of measures as opposed to functions.

Gangolli also proved a Levy-Cramer type continuity theorem for the spherical transform on a symmetric space of non-compact type.

Theorem 3.4. (Gangolli [6]) *Let $\mu^{(i)} \in M^\natural(\mathbf{X})$ be a sequence of K-invariant, semi-positive, bounded Borel measures on $\mathbf{X}$ and let $\beta^{(i)} = \mathbf{S}(\mu^{(i)})$.*

1. *If $\mu^{(i)}$ converges weakly to μ then $\beta^{(i)}$ converges pointwise to $\beta = \mathbf{S}\mu$ on T_ρ.*
2. *Assume that $\beta^{(i)}$ converges pointwise to a function β on T_ρ and that $\lim_{i\to\infty} \mu^{(i)}(\mathbf{X})$ exists. Then there exists $\mu \in M^\natural(\mathbf{X})$ such that $\mathbf{S}\mu = \beta$. If in addition $\lim_{i\to\infty} \mu^{(i)}(\mathbf{X}) = \mu(\mathbf{X})$ then $\mu^{(i)}$ converges weakly to μ.*

Remark. Note that Gangolli uses the term *weak convergence* for convergence as functionals on the space of compactly supported functions. Our notion of weak convergence corresponds to his notion of *Bernoulli convergence*. Cf. [6] Theorem 4.2 for details.

We return to the relationship between the harmonic analysis on $\mathbf{X}$ and $\mathbf{X}_Q$. A critical fact concerns the relationships between the convex sets C_ρ and $C_{\rho_{G_Q}}$ and the associated tubes $T_{\rho_{G_Q}}$ and T_ρ. Because of our non-standard parametrization of the spherical functions we obtain the best possible relationship between convex sets: extreme points map to extreme points under $\pi^\vee_{Q,\mathfrak{a}}$.

Proposition 3.5. *Under the natural inclusion $\pi^\vee_{Q,A} : \mathfrak{a}^\vee_{\mathfrak{g}_\mathfrak{q},\mathbf{C}} \hookrightarrow \mathfrak{a}^\vee_{\mathbf{C}}$ we have*

1. *$T_{\rho_{G_Q}} \subset T_\rho$;*
2. *The vertices of $C_{\rho_{G_Q}}$ map to vertices of C_ρ.*

Proof. Clearly, 2 implies 1. Let λ_{G_Q} be a vertex of $C_{\rho_{G_Q}}$. Thus there exists $w'_{G_Q} \in W_{G_Q}$ such that

$$\lambda_{G_Q} = w'_{G_Q}(\rho_{G_Q}) + \rho_{G_Q}.$$

We must find an element $w' \in W$ such that $w'(\rho) + \rho = \lambda_{G_Q}$.

Let $w^*_{G_Q}$ (resp. w^*) denote the long element of W_{G_Q} (resp. W). Then $w^*_{G_Q}\rho_{G_Q} = -\rho_{G_Q}$ (similarly without subscripts). Also note that W_{G_Q} acts trivially on $\mathfrak{a}^\vee_\mathfrak{q}$, the orthogonal complement to $\mathfrak{a}^\vee_{\mathfrak{g}Q}$ and hence fixes ρ_Q. Thus we have

$$w^*_{G_Q} w^*(\rho) = w^*_{G_Q} w^*(\rho_{G_Q} + \rho_Q) = \rho_{G_Q} - \rho_Q.$$

Let $w' = w'_{G_Q} w^*_{G_Q} w^*$. Then

$$w'(\rho) + \rho = w'_{G_Q}(\rho_{G_Q}) + \rho_{G_Q} = \lambda_{G_Q}$$

as was to be proved. ∎

Before stating the main result of this chapter we need a statement about 'induced' spherical functions. Recall that we use $\pi^\vee_{Q,\mathfrak{a}}$ to embed $\mathfrak{a}^\vee_{\mathfrak{g}_\mathfrak{q}} \hookrightarrow \mathfrak{a}^\vee$.

Proposition 3.6. *Given $\zeta_{G_Q} \in \mathfrak{a}_{\mathfrak{g}_\mathfrak{q}}$, let $\psi_{\zeta_{G_Q}}$ denote the spherical function on $\mathbf{X}_Q$ corresponding to ζ_{G_Q} and $\phi_{\zeta_{G_Q}}$ denote the spherical function on $\mathbf{X}$ corresponding to ζ_{G_Q}. Then*

$$(\pi_Q^* \psi_{\zeta_{G_Q}})^K = \phi_{\zeta_{G_Q}}.$$

Proof. By definition

$$\psi_{\zeta_{G_Q}} = (\zeta_{G_Q})^{K_{G_Q}}.$$

By Lemma 2.4,

$$(\pi_Q^* \psi_{\zeta_{G_Q}})^K = \big((\pi_Q^* \zeta_{G_Q})^{K_{G_Q}}\big)^K.$$

But

$$(\pi_Q^* \zeta_{G_Q})(x) = \zeta_{G_Q}(x)$$

where on the left hand side we consider ζ_{G_Q} as a function on $\mathbf{X}_Q$ and on the right hand side as a function on $\mathbf{X}$. Finally,

$$\big((\zeta_{G_Q})^{K_{G_Q}}\big)^K = (\zeta_{G_Q})^K = \phi_{\zeta_{G_Q}}$$

as desired ∎

We now put together the way π_Q acts on measures and spherical functions to obtain the main result of this chapter: a commutative diagram relating $\mathbf{S}$ to $\mathbf{S}_Q$. Cf. [14] where the result appears for the case of $G = \mathrm{SL_n}(\mathbf{C})$.

Theorem 3.7. *The following diagram is commutative*

$$\begin{array}{ccc} M^\natural(\mathbf{X}) & \xrightarrow{\mathbf{S}} & \mathrm{Fu}(T_\rho)^{(W,\rho)} \\ \pi_{Q,*}\Big\downarrow & & \Big\downarrow \pi_Q^{\vee,*} \\ M^\natural(\mathbf{X}_Q) & \xrightarrow{\mathbf{S}_Q} & \mathrm{Fu}(T_{\rho_{G_Q}})^{(W_{G_Q},\rho_{G_Q})}. \end{array}$$

That is,

$$\pi_Q^{\vee,*} \circ \mathbf{S} = \mathbf{S}_Q \circ \pi_{Q,*}.$$

Remark. Observe that $\pi_Q^{\vee,*}$ is just restriction. We have used the present notation to highlight the functorial nature of the diagram.

Proof. The proof depends only on the functorial properties of the projection and Proposition 3.6. Let $\psi_{\zeta_{G_Q}}$ denote the spherical function on G_Q corresponding to ζ_{G_Q} and $\phi_{\zeta_{G_Q}}$ denote the spherical function on G corresponding to ζ_{G_Q}.

For $\zeta_{G_Q} \in T_{\rho_{G_Q}}$ we have

$$\big(\mathbf{S}_Q(\pi_{Q,*}\mu)\big)(\zeta_{G_Q}) = [\psi_{\zeta_{G_Q}}, \pi_{Q,*}\mu]_{\mathbf{X}_Q} = [\pi_Q^*\psi_{\zeta_{G_Q}}, \mu]_{\mathbf{X}} = [(\pi_Q^*\psi_{\zeta_{G_Q}})^K, \mu]_{\mathbf{X}}$$

where the last equality holds because μ is K-left-invariant (see Lemma 2.3). By Proposition 3.6

$$[(\pi_Q^*\psi_{\zeta_{G_Q}})^K, \mu]_{\mathbf{X}} = [\phi_{\zeta_{G_Q}}, \mu]_{\mathbf{X}} = (\mathbf{S}\mu)(\zeta_{G_Q}) = \big((\pi_Q^{\vee,*} \circ \mathbf{S})(\mu)\big)(\zeta_{G_Q})$$

as was to be shown. ∎

Remark. The reason for the change in normalization of the spherical functions and spherical transform is beginning to become clear. The above diagram is completely functorial with respect to the natural projection π_Q. Had we included the ρ shift in our definitions, factors of ρ_Q would show up in one of the vertical maps. When working only with two levels, i.e. a group and parabolic subgroup, the extra shift is no big deal, but when we construct an infinite ladder the extra ρ factors can grow out of control.

Example: The Heat Kernel. We assume the reader is familiar with the Casimir operator ω, a second-order differential operator on G. Since it is K-right-invariant, it descends to a differential operator on $\mathbf{X}$ which will be a positive multiple of the Laplacian. One of the fundamental objects associated to the symmetric space $\mathbf{X}$ is the heat semi-group of probability measures (or functions) $\nu_{t,\mathbf{X}}$ with $t > 0$. The family is determined uniquely by its spherical transform

$$(\mathbf{S}\nu_{t,\mathbf{X}})(\zeta) = \exp\Big(t\big(B^\vee(\zeta,\zeta) - 2B^\vee(\zeta,\rho)\big)\Big).$$

Note that the coefficient of t on the right is just the eigenvalue of the Casimir operator ω on ϕ_ζ as usual.

For each t, $\nu_{t,\mathbf{X}}$ is positive and absolutely continuous with respect to Haar measure. The family is a semi-group under convolution, i.e. $\nu_{t,\mathbf{X}} \star \nu_{s,\mathbf{X}} = \nu_{t+s,\mathbf{X}}$, and satisfies the heat equation

$$\partial_t [f, \nu_{t,\mathbf{X}}]_{\mathbf{X}} = [\omega f, \nu_t]_{\mathbf{X}}$$

for any bounded, smooth function f.

Applying Theorem 3.7 to $\nu_{t,\mathbf{X}}$ we see at once that

Corollary 3.8. *The parabolic projection $\pi_{Q,*}$ maps the heat kernel measure on $\mathbf{X}$ to the heat kernel measure on $\mathbf{X}_Q$. I.e.,*

$$\pi_{Q,*}(\nu_{t,\mathbf{X}}) = \nu_{t,\mathbf{X}_Q}.$$

Proof. By Theorem 3.7

$$\big(\mathbf{S}_Q(\pi_{Q,*}\nu_{t,\mathbf{X}})\big)(\zeta_{G_Q}) = \exp\Big(t\big(B^\vee(\zeta_{G_Q}, \zeta_{G_Q}) - 2B^\vee(\zeta_{G_Q}, \rho)\big)\Big).$$

But

$$B^\vee(\zeta_{G_Q}, \rho) = B^\vee(\zeta_{G_Q}, \rho_{G_Q})$$

so $\pi_{Q,*}(\nu_{t,\mathbf{X}})$ has the same spherical transform as $\nu_{t,\mathbf{X}_Q}$, hence they are equal by Proposition 3.3. ■

Compare this to [14] Chapter 1, Theorem 7.3.

Remark. Notice that with the present normalization, the spherical transform of $\nu_{t,\mathbf{X}}$ is only exponential linear in ρ, not exponential quadratic as usual. When we consider infinite ladders of symmetric spaces in the next section this will be important in ensuring that the spherical transform of the limit of heat kernels makes sense.

4. Weakly Parabolic Ladders

We now move on to consider infinite sequences or 'ladders' of such semi-simple groups and attempt to build a theory of harmonic analysis on the projective limit of the symmetric spaces associated to such a ladder. Though the ideas are fairly straightforward, the notation can be heavy at times, due largely to the need to keep track of both direct and inverse limits.

Direct Limits. Let $\{G_j, \eta_{j,l}\}_{j\leq l\in\mathbf{Z}^+}$ be a direct system satisfying the following conditions:

1. G_j is a real, linear, connected semi-simple Lie group with no compact factors;
2. $\eta_{j,l} : G_j \hookrightarrow G_l$ is an injective homomorphism such that $\eta_{j,l}(G_j)$ is a closed embedded Lie subgroup of G_l;
3. $d\eta_j(\mathfrak{g}_j)$ is a reduced semi-simple component of a parabolic subalgebra of $\mathfrak{g}_{j+1}$.

If the above conditions are satisfied, we call $\{G_j, \eta_{j,l}\}$ a *weakly parabolic system*, cf. [23]. In what follows we identify G_j with its image in G_l when no confusion is possible.

Remark. Other conditions on direct systems $\{G_j\}$ have occurred in the literature, depending on the types of theorems one wants to prove. The conditions here are chosen to enable the construction of a spherical transform on the inverse limit of the associated system of symmetric spaces. The first condition could be weakened to allow compact components, but this would not result in any more symmetric spaces of non-compact type.

All the results from Chapter 1 concerning reduced semi-simple components of parabolic subgroups $G_Q \subset G$ now apply for every $G_j \subset G_l, \quad j \leq l$. In particular, the commutative diagram Theorem 3.7 and the heat kernel results Corollary 3.8 will be of interest.

We can (and do) always choose compatible Iwasawa decompositions and Cartan involutions such that

$$A_j \hookrightarrow A_{j+1}, \;\; N_j \hookrightarrow N_{j+1}, \;\; K_j \hookrightarrow K_{j+1},$$

cf. [23]. For $j \leq l$ we let $Q_{j,l}$ be the unique parabolic subgroup of G_l containing $A_l N_l$ and having G_j as the connected piece of a reduced semi-simple component. The Langlands decomposition of $Q_{j,l}$ will be denoted by

$$Q_{j,l} = M_{j,l} A_{j,l} N_{j,l}$$

so that

$$N_l = N_{j,l} N_j, \;\; A_l = A_{j,l} A_j, \;\; \rho_l = \rho_j + \rho_{j,l}, \quad \mathfrak{g}_j \equiv [\mathfrak{m}_{j,l}, \mathfrak{m}_{j,l}] \bmod \mathfrak{m}_l.$$

Again, all statements about H, H_Q, and H_{G_Q} applies to H, $H_{j,l}$, and H_j where H stands for $N, A, K, \mathfrak{n}, \mathfrak{a}, \mathfrak{a}_{\mathbf{C}}^{\vee}$, or ρ. Note that $\eta_{l,m}(N_{j,l}) \subset N_{j,m}$ and $\eta_{l,m}(A_{j,l}) \subset A_{j,m}$
for $l \leq m$.

Let $G_\infty = \varinjlim \{G_j, \eta_{j,l}\}$ be the direct limit in the category of Lie groups and homomorphisms, i.e.

$$G_\infty = \bigcup G_j.$$

Let η_j denote the canonical inclusion $\eta_j : G_j \hookrightarrow G_\infty$. We give G_∞ the naive direct limit topology, i.e., open sets in G_∞ are those $U \subset G_\infty$ such that $U \cap G_j$ is open for all j. A function on G_∞ is continuous in this topology if and only if its restriction to G_j is continuous for all j. For a more complete discussion of direct limits and other topologies available on G_∞ cf. [16] and [8].

The group G_∞ also possesses an Iwasawa decomposition (cf. [23])

$$G_\infty = N_\infty A_\infty K_\infty$$

where

$$A_\infty = \varinjlim A_j = \bigcup_j A_j, \quad N_\infty = \varinjlim N_j = \bigcup_j N_j, \quad K_\infty = \varinjlim K_j = \bigcup_j K_j.$$

Of course, K is not compact and none of the limit groups are even locally compact.

Let

$$N_{j,\infty} = \bigcup_l N_{j,l}, \quad A_{j,\infty} = \bigcup_l A_{j,l}.$$

Applying (1) repeatedly, we see at once that

$$N = N_{j,\infty} N_j, \quad A = A_{j,\infty} A_j.$$

The $\eta_{j,l}$ also descend to the symmetric space level, yielding embeddings $\mathbf{X}_j \hookrightarrow \mathbf{X}_l$. Hence, we obtain a direct system $\{\mathbf{X}_j, \eta_{j,l}\}$ and the direct limit in the category of topological spaces and continuous maps

$$\mathbf{X}_\infty = \varinjlim\{\mathbf{X}_j, \eta_{j,l}\} = \bigcup_j \mathbf{X}_j \simeq G_\infty / K_\infty.$$

The group G_∞ acts continuously on $\mathbf{X}_\infty$ by left translation.

Everything above also has equivalent statements on the level of Lie algebras giving $\mathfrak{g}_\infty$, $\mathfrak{a}_\infty$, $\mathfrak{n}_\infty$, $\mathfrak{k}_\infty$, $\mathfrak{a}_{j,\infty}$, and $\mathfrak{n}_{j,\infty}$ the obvious meanings. On $\mathfrak{g}_\infty$ we can select a bilinear form B_∞ such that the restriction of B_∞ to any simple component of $\mathfrak{g}_j$ is a ppositivemultiple of the Killing form. Since we assumed our groups were linear, we can just select the real trace form

$$B_\infty(Y_1, Y_2) = \operatorname{Re}\operatorname{Tr}(Y_1 Y_2).$$

In general we cannot ensure that the restriction $B_\infty|_{\mathfrak{g}_j}$ is actually equal to the Killing form, only to a positive multiple thereof. In any event, the decomposition $\mathfrak{a}_\infty = \mathfrak{a}_{j,\infty} \oplus \mathfrak{a}_j$ becomes orthogonal with respect to B_∞.

The dual spaces $\mathfrak{a}_{j,\mathbf{C}}^\vee$ form an inverse system of vector spaces. The inverse limit

$$\mathfrak{a}_{\infty,\mathbf{C}}^\vee = \varprojlim\{\mathfrak{a}_{j,\mathbf{C}}^\vee, \eta_{j,l}^\vee\}$$

is the complex dual of $\mathfrak{a}_\infty$ and consists of families of characters $\{\zeta_j\}$ such that $\zeta_j|_{\mathfrak{a}_{j-1}} = \zeta_{j-1}$. We then define $\zeta^\infty = \varprojlim \zeta_j$ as the functional on $\mathfrak{a}_\infty$ such that $\zeta^\infty|_{\mathfrak{a}_j} = \zeta_j$.

The main example of such a functional ρ^∞. From Lemma 1.2 we see at once that $\rho_j|_{\mathfrak{a}_{j-1}} = \rho_{j-1}$. Thus, $\rho^\infty = \varprojlim \rho_j \in \mathfrak{a}_{\infty,\mathbf{C}}^\vee$ is a well defined functional on $\mathfrak{a}_\infty$ such that $\rho^\infty|_{\mathfrak{a}_j} = \rho_j$. Alternatively, we could write ρ^∞ as the infinite sum

$$\rho^\infty = \rho_1 + \sum_j \rho_{j,j+1}.$$

Since $\rho_{j,j+1}|_{\mathfrak{a}_j} = 0$, the sum is well defined as a functional on $\mathfrak{a}_\infty$ and we clearly have $\rho^\infty|_{\mathfrak{a}_j} = \rho_1 + \rho_{1,2} + \cdots + \rho_{j-1,j} = \rho_j$. We will sometimes write $\rho = \rho_j + \rho_j^\infty$ where

$$\rho_j^\infty = \sum_{l=j}^{\infty} \rho_{l,l+1}$$

so that $\rho_j^\infty|_{\mathfrak{a}_j} = 0$.

Projective Limits. As in Chapter 1, our primary interest lies not with inclusions on the group level, but rather the projections they yield on the associated symmetric spaces. Our general references for projective (inverse) limits are [5], [21] and [25].

For each $j < l$ the symmetric space $\mathbf{X}_l = G_l/K_l$ has parabolic coordinates

$$\mathbf{X}_l \simeq N_{j,l} \times A_{j,l} \times \mathbf{X}_j.$$

Projecting on to the $\mathbf{X}_j$ factor gives a surjective projection

$$\pi_{l,j} : \mathbf{X}_l \twoheadrightarrow \mathbf{X}_j.$$

The fibers of $\pi_{l,j}$ are all isomorphic to $N_{j,l}A_{j,l}$, which can be identified with the fiber at the identity coset. For an explicit example, see (3).

Remark. The projections $\pi_{l,j}$ are only on the symmetric space level and do not extend to the groups G_l in a natural way.

The projections give rise to an inverse system $\{\mathbf{X}_j, \pi_{l,j}\}$ in the category of topological spaces and continuous maps. Let $\mathbf{X}^\infty = \varprojlim\{\mathbf{X}_j, \pi_{l,j}\}$ be the projective (inverse) limit. We think of $\mathbf{X}^\infty$ as the subset of the infinite product $\prod \mathbf{X}_j$ compatible with the $\pi_{l,j}$, i.e.,

$$\mathbf{X}^\infty = \left\{ (x_1, x_2, \dots) \in \prod_j \mathbf{X}_j \;\middle|\; \pi_{l,j}(x_l) = x_j \text{ for all } l \geq j \right\}.$$

Let π_j denote the canonical map from $\mathbf{X}^\infty$ to $\mathbf{X}_j$. The topology on $\mathbf{X}^\infty$ is the inverse limit topology which is equivalent to the topology inherited from the infinite product. It can also be defined intrinsically as the coarsest topology for which the π_j are continuous. A base of open sets for this topology is given by the so-called 'cylinder sets'

$$\{U \in \mathbf{X}^\infty \mid U = \pi_j^{-1}(U_j) \text{ where } U_j \text{ open in } \mathbf{X}_j\ \}.$$

With this topology, $\mathbf{X}^\infty$ is second countable, Hausdorff, and normal. In particular the topology is metrizable, though the metric has nothing to do with the G_j-invariant metrics on $\mathbf{X}_j$ defined in Section 1.2.

The direct limit $\mathbf{X}_\infty$ can actually be embedded in the inverse limit $\mathbf{X}^\infty$ as the set of all sequences $(x_1, x_2, ...)$ that are eventually constant. Denote this embedding η_∞. It is not difficult to see that this η_∞ is continuous. Furthermore,

Proposition 4.1. *The image $\eta_\infty(\mathbf{X}_\infty)$ is a dense subspace of $\mathbf{X}^\infty$.*

Proof. Since $\mathbf{X}^\infty$ is second countable, we need only show sequential denseness. Let $(x_1, x_2, x_3, ...) = x \in \mathbf{X}^\infty$ and consider the sequence $\{\eta_\infty(x_1), \eta_\infty(x_2), \eta_\infty(x_3), ...\}$. It suffices to show that for every open cylinder set U containing x, there exists M such that $\eta_\infty(x_m) \in U$ whenever $m > M$. Assume $U = \pi_l^{-1}(U_l)$ where $U_l \in X_l$. Then clearly $x_l \in U_l$. In fact, for $m > l$ we have $\pi_l(x) = \pi_{m,l}(x_m) = \pi_l(\eta_\infty(x_m)) = x_l$ implying that $\eta_\infty(x_m)$ is in U for $m > l$ as desired. ∎

There is even an action of G_∞ on $\mathbf{X}^\infty$ extending the action on $\mathbf{X}_\infty$.

Proposition 4.2. *Given $x = (x_1, x_2, ...)$ and $g_j \in G_\infty$ let*

$$g_j x = (\pi_{j,1}(g_j x_j), \pi_{j,2}(g_j x_j), \dots, g_j x_j, g_j x_{j+1}, \dots).$$

This gives a well-defined homomorphism of G_∞ in to the automorphism group of $\mathbf{X}^\infty$.

Proof. If well defined, the action is obviously a homomorphism. Observe that since G_j normalizes $N_{l,m}A_{l,m}$, the projections $\pi_{m,l}$ are G_j equivariant for $m \geq l \geq j$, i.e., $\pi_{j,l}(g_j x_l) = g_j(\pi_{m,l}(x_l))$. This ensures that

$$g_j x = (\pi_{j,1}(g_j x_j),\, \pi_{j,2}(g_j x_j), \dots, g_j x_j,\, g_j x_{j+1}, \dots)$$

is consistent with the projections and is actually an element of $\mathbf{X}^\infty$.

Now for continuity. Let $U_l \subset \mathbf{X}_l$ be an arbitrary open set and let $\pi^{-1}(U_l)$ be the corresponding cylinder set. To show that the action of an individual g_j is continuous, it suffices to show that $g_j^{-1}\big(\pi_l^{-1}(U_l)\big)$ is open in $\mathbf{X}^\infty$. Assume first that $l \geq j$. Then

$$g_j^{-1}\big(\pi_l^{-1}(U_l)\big) = \pi_l^{-1}\big(g_j^{-1}(U_l)\big)$$

which is clearly open since the action of G_j on $\mathbf{X}_l$ is continuous. Now assume $l < j$. Then by definition

$$g_j^{-1}(\pi_l^{-1}(U_l)) = (\pi_j^{-1} \circ g_j^{-1} \circ \pi_{j,l}^{-1})(U_l)$$

which is also an open cylinder set since the $\pi_{j,l}$ are continuous. ∎

This action of G_∞ is of course not transitive on $\mathbf{X}^\infty$. However, we will see presently that $\mathbf{X}^\infty$ is actually a principal homogeneous space for the infinite dimensional Lie group obtained through the inverse systems of Iwasawa A and N components.

Iwasawa and Parabolic Coordinates on the Projective Limit. We can use various charts of $\mathbf{X}_j$ to obtain equivalent inverse systems which in turn yield new coordinates on $\mathbf{X}^\infty$. Of primary interest are the charts defined by Iwasawa coordinates. Recall the unique decompositions

$$A_l = A_{j,l}A_j, \quad N_l = N_{j,l}N_j$$

and the associated projections $\pi_{l,j,A}, \quad \pi_{l,j,N}$ which are just the restrictions of the projections on $\mathbf{X}_l$ to A_l and N_l (here we identify A_l and N_l with their orbits of the identity coset in $\mathbf{X}_l$). Remember that $N_{j,l}$ and $A_{j,l}$ are closed, normal subgroups of N_l and A_l respectively, so that the projections are in fact homomorphisms.

It will be more convenient to deal with both A_l and N_l simultaneously, so we form the semi-direct product groups denoted $V_l = N_l \ltimes A_l$. Let $V_{j,l} = N_{j,l}A_{j,l}$. Since $A_{j,l}$ commutes with N_j we can write

$$V_l = N_{j,l}N_jA_{j,l}A_j = N_{j,l}A_{j,l}N_jA_j = V_{j,l}V_j.$$

Observe that $V_{j,l}$ is a closed, normal subgroup of V_l and so again the product is actually semi-direct. We then have the natural projection

$$\pi_{l,j,V} : V_l \longrightarrow V_j$$

which is a homomorphism. We can thus take the inverse limits in the category of topological groups to obtain

$$A^\infty = \varprojlim A_j, \quad N^\infty = \varprojlim N_j, \quad V^\infty = \varprojlim V_j.$$

We can also make the topological product $N^\infty \times A^\infty$ into a group in a natural way. Given $a_l = a_{j,l}a_j$ and $n_l = n_{j,l}n_j$, since all of A_l normalizes $N_{j,l}$, and $A_{j,l}$ commutes with N_j we have

$$a_l n_l a_l^{-1} = n'_{j,l} a_j n_j a_j^{-1}$$

which implies

$$\pi_{l,j,N}(a_l n_l a_l^{-1}) = a_j n_j a_j^{-1}.$$

Thus, given $a = (a_1, a_2, \dots) \in A^\infty$ and $n = (n_1, n_2, \dots) \in N^\infty$ we see that

$$ana^{-1} = (a_1 n_1 a_1^{-1}, a_2 n_2 a_2^{-1}, \dots)$$

is a well defined element of N^∞. We can now form the semi-direct product $N^\infty \ltimes A^\infty$ which is naturally isomorphic as a group to V^∞ under the map $(n, a) \mapsto (n_1 a_1, n_2 a_2, \dots)$. That this isomorphism is actually a homeomorphism is immediate since each V_j is homeomorphic to $N_j \times A_j$ in a way that intertwines the connecting maps. We have thus shown

Proposition 4.3. *There is an isomorphism in the category of topological groups*

$$V^\infty \simeq N^\infty \ltimes A^\infty.$$

In other words

$$\varprojlim\{N_j \ltimes A_j\} \simeq \varprojlim\{N_j\} \ltimes \varprojlim\{A_j\}.$$

Next, we obtain an Iwasawa decomposition for $\mathbf{X}^\infty$.

Proposition 4.4. *There is a continuous transitive action*

$$V^\infty \times \mathbf{X}^\infty \longrightarrow \mathbf{X}^\infty.$$

The stabilizer of the base point is trivial and hence there is an isomorphism in the category of topological spaces and continuous maps

$$\mathbf{X}^\infty \simeq V^\infty \simeq N^\infty \times A^\infty.$$

Proof. The isomorphism between $\mathbf{X}^\infty$ and V^∞ is obvious since the Iwasawa coordinates give isomorphisms $\mathbf{X}_l \simeq V_l$ which intertwine the $\pi_{l,j,\mathbf{X}}$ and $\pi_{l,j,V}$. The continuous action of V^∞ on $\mathbf{X}^\infty$ is just the action of V^∞ on itself as a topological group, which is clearly continuous. ∎

Let the Iwasawa coordinate map on $\mathbf{X}^\infty$ be denoted $\mathrm{Iw}^\infty(x) = (n, a)$ and the Iwasawa projection on to the A^∞ component be denoted $(x)_{A^\infty}$.

If we consider the sequence of spaces $\{V_1, V_{1,2}, V_{2,3}, ...\}$ then the partial products are just

$$V_{l,l-1} \ltimes ... \ltimes V_{2,1} \ltimes V_1 = V_l.$$

A standard result (cf. [25] p. 17) gives

Proposition 4.5. *The inverse limit V^∞ is naturally isomorphic as a topological group to the infinite product*

$$... \ltimes V_{l-1,l} \ltimes ... \ltimes V_{1,2} \ltimes V_1.$$

Proof. The topological argument (with direct products) is given in [25] and the group structure is immediate. ∎

Identical statements are true substituting A or N for V. Letting

$$...A_{j+1,j+2}A_{j,j+1} = A_j^\infty, \quad ...N_{j+1,j+2}N_{j,j+1} = N_j^\infty, \quad ...V_{j+1,j+2}V_{j,j+1} = V_j^\infty$$

we obtain the decompositions

$$A^\infty = A_j^\infty A_j, \quad N^\infty = N_j^\infty N_j, \quad V^\infty = V_j^\infty V_j.$$

This can also be phrased in terms of *j-parabolic coordinates* on $\mathbf{X}^\infty$.

Proposition 4.6. *For each j there is a topological isomorphism*

$$\mathbf{X}^\infty \simeq N_j^\infty \times A_j^\infty \times \mathbf{X}_j.$$

In these coordinates, the projection π_j is just the projection on to the $\mathbf{X}_j$ factor and satisfies

$$\pi_j : \mathbf{X}^\infty \longrightarrow \mathbf{X}_j, \quad \pi_j(n_j^\infty a_j^\infty x_j) = x_j.$$

The j-parabolic coordinates on $\mathbf{X}^\infty$ often provide a convenient framework in which to work. For example, we can reformulate the action of G_∞ on $\mathbf{X}^\infty$ from Proposition 4.2 in the simple manner

$$g_j(n_j^\infty a_j^\infty x_j) = n_j^\infty a_j^\infty (g_j x_j).$$

5. Measures and Functions on $\mathbf{X}^\infty$

As before, the projections act on all the objects in sight. We will be mainly concerned with the behavior of functions and measures in the limit. Though here we choose smooth bounded cylindrical functions as our test functions, there are broader choices one could make,such as the locally cylindrical functions. The current choice is motivated by the desire to exert minimal effort to construct the spherical transform, for which cylindrical functions are sufficient.

Cylindrical Functions. The projections $\pi_{l,j}$ act contravariantly on functions, pulling back functions on $\mathbf{X}_j$ to left $N_{j,l}A_{j,l}$-invariant functions on $\mathbf{X}_l$. These pull backs are clearly injective, and we can form the direct limit of function spaces (in the category of vector spaces and linear maps), which is just the union, to obtain the space of all *cylindrical functions* on X^∞.

We will mostly be interested in the smooth and bounded cylindrical functions. More precisely, let $BC^\infty(\mathbf{X}_j)$ be the set of all infinitely differentiable functions f on $\mathbf{X}_j$, such that f and all its derivatives are bounded. With π_j^* denoting the pull back to $\mathbf{X}^\infty$, define

$$BCyl^\infty(\mathbf{X}^\infty) = \bigcup_j \pi_j^*(BC^\infty(\mathbf{X}_j)).$$

Thus a function f on $\mathbf{X}^\infty$ is in $BCyl^\infty(\mathbf{X}^\infty)$ if and only if there exists j and $f_j \in BC^\infty(\mathbf{X}_j)$ such that

$$f(x) = (\pi_j^* f_j)(x) = f_j(\pi_j(x))$$

for all $x \in \mathbf{X}^\infty$. Note that because of j-parabolic coordinates, the pull back $\pi_j^* f_j$ is a left $N_j^\infty A_j^\infty$-invariant function on $\mathbf{X}^\infty$.

Remark. Clearly, the union in the definition of $BCyl^\infty(\mathbf{X}^\infty)$ is not disjoint. In [1] the authors define the space of cylinder functions as the (disjoint) union of the $BC^\infty(\mathbf{X}_j)$ modulo an equivalence relation. They declare two functions, f_l and g_j, to be equal if $f_l = \pi_{l,j}^* g_j$. The current method obviates the need for the equivalence relation.

The most important examples of cylindrical functions come from the characters on $\mathfrak{a}^\infty$. Great care must be given to keeping straight the role of the direct and inverse systems of the dual spaces $\mathfrak{a}_{j,\mathbf{C}}^\vee$. We have already met the inverse limit

$$\mathfrak{a}_{\infty,\mathbf{C}}^\vee = \varprojlim \{\mathfrak{a}_{j,\mathbf{C}}^\vee, \eta_{j,l}^\vee\}$$

as the dual space to the direct limit $\mathfrak{a}_\infty$. We will be more concerned with are the direct limit associated to the dual of the inverse limit $\mathfrak{a}^\infty$. Since $\pi_{l,j,\mathfrak{a}}$ maps $\mathfrak{a}_l$ onto $\mathfrak{a}_j$ there is a dual map

$$\pi_{l,j}^\vee : \mathfrak{a}_{j,\mathbf{C}}^\vee \hookrightarrow \mathfrak{a}_{l,\mathbf{C}}^\vee$$

obtained in the usual way by forcing a functional to vanish on $\mathfrak{a}_{j,l} = \ker(\pi_{l,j,\mathfrak{a}})$. Let

$$\mathfrak{a}_{\mathbf{C}}^{\infty,\vee} = \varinjlim\{\mathfrak{a}_{j,\mathbf{C}}^\vee, \pi_{l,j}^\vee\} = \bigcup_j \mathfrak{a}_{j,\mathbf{C}}^\vee$$

be the direct limit, which is the complex dual of $\mathfrak{a}^\infty$. This should not be confused with the inverse limit $\mathfrak{a}_{\infty,\mathbf{C}}^\vee$ which is the complex dual of $\mathfrak{a}_\infty$, the direct limit of the $\mathfrak{a}_j$. We also have the associated direct limits of root systems and Weyl groups

$$W_\infty = \bigcup_j W_j, \quad \mathcal{R}(\mathfrak{a}^\infty, \mathfrak{n}^\infty) = \bigcup_j \mathcal{R}(\mathfrak{a}_j, \mathfrak{n}_j), \quad \mathcal{S}(\mathfrak{a}^\infty, \mathfrak{n}^\infty) = \bigcup_j \mathcal{S}(\mathfrak{a}_j, \mathfrak{n}_j).$$

There is even a well-defined positive definite bilinear form on $\mathfrak{a}_{\mathbf{C}}^{\infty,\vee}$ given by

$$B_\infty^\vee(\zeta_j, \chi_j) = B_j^\vee(\zeta_j, \chi_j)$$

for $\zeta_j,\ \chi_j \in \mathfrak{a}_{j,\mathbf{C}}^\vee$.

We also need to define some analogue of (W,ρ)-invariance for the direct limits. Unfortunately, given $w = w_j \in W_\infty$ and $\zeta \in \mathfrak{a}_{\mathbf{C}}^{\infty,\vee}$, the expression

$$w(\zeta + \rho^\infty) - \rho^\infty$$

does not make sense since ζ and ρ^∞ live in different spaces. However, since $\rho^\infty = \rho_j + \rho_j^\infty$ and $w_j(\rho_j^\infty) = \rho_j^\infty$, it makes sense to define

$$w(\zeta + \rho^\infty) - \rho^\infty = w(\zeta + \rho_j) - \rho_j.$$

Thus we arrive at the reasonable notion that an object will be said to possess (W_∞, ρ^∞)-invariance on $\mathfrak{a}_{\mathbf{C}}^{\infty,\vee}$ if it is (W_j, ρ_j)-invariant for all j.

Remark. Alternatively, we could make sense of the above discussion by embedding the direct limit $\mathfrak{a}_{\mathbf{C}}^{\infty,\vee}$ into the inverse limit $\mathfrak{a}_{\infty,\mathbf{C}}^\vee$ as we did with $\mathbf{X}_\infty$ inside $\mathbf{X}^\infty$. Then W_∞ acts on $\mathfrak{a}_{\infty,\mathbf{C}}^\vee$ in the same way G_∞ acts on $\mathbf{X}^\infty$ and the expression $w(\zeta + \rho^\infty) - \rho^\infty$ becomes well-defined.

If $\zeta \in \mathfrak{a}_{\mathbf{C}}^{\infty,\vee}$ belongs to $\mathfrak{a}_{j,\mathbf{C}}^\vee$ then we also denote it ζ_j. Then ζ_j defines a N_j-left-invariant function on $\mathbf{X}_j$ by $\zeta_j(x_j) = (x_j)_{A_j}^\zeta$. We can lift this function to a (possibly unbounded) cylindrical function on $\mathbf{X}^\infty$ by defining

$$\zeta(x) = \zeta_j\big(\pi_j(x)\big) = \big(\pi_j(x)\big)_{A_j}^{\zeta_j}.$$

In light of the A^∞ Iwasawa projection on $\mathbf{X}^\infty$ we can reformulate this more simply as

$$\zeta(x) = (x)_{A^\infty}^\zeta$$

for any $\zeta \in \mathfrak{a}_{\mathbf{C}}^{\infty,\vee} = \varinjlim \mathfrak{a}_{j,\mathbf{C}}^\vee$.

If we want to obtain an element of $BCyl^{\infty}(\mathbf{X}^{\infty})$ (i.e. a bounded function), we must be more careful. Let

$$T_{\rho^{\infty}} = \bigcup_{j} T_{\rho_j}.$$

For any $\zeta \in T_{\rho^{\infty}}$ there exists j such that $\zeta = \zeta_j \in T_{\rho_j}$. Then the spherical function $\phi_{\zeta} = \phi_{\zeta_j}$ is actually in $BC^{\infty}(\mathbf{X}_j)$ and hence the pull pack $\pi_j^*(\phi_{\zeta})$ is an element of $BCyl^{\infty}(\mathbf{X}^{\infty})$. Denote this pullback of a spherical function by ϕ_{ζ}^{∞} for any $\zeta \in \mathfrak{a}_{\mathbf{C}}^{\infty,\vee}$. These functions will play the role of spherical functions on $\mathbf{X}^{\infty}$.

Projective Limits of Measures. There is a covariant action of $\pi_{l,j}$ on the the spaces of measures $M(\mathbf{X}_j)$ and $M^{\natural}(\mathbf{X}_j)$ given by the push forward. A critical result ensures that the inverse limit of measures is a measure on the inverse limit. We refer to [25] and [24] for detailed proofs.

Let $M^{\natural}(\mathbf{X}^{\infty})$ denote the space of semi-positive, bounded, K_{∞}-invariant Borel measures on $\mathbf{X}^{\infty}$. Recall that the push-forward $\pi_{l,j,*}$ sends K_l-invariant measures on $\mathbf{X}_l$ to K_j-invariant measures on $\mathbf{X}_j$ by Lemma 2.2.

Definition. A family of semi-positive, bounded, K_j-invariant Borel measures $\{\mu_j\}$ such that $\mu_j \in M^{\natural}(\mathbf{X}_j)$ and $\pi_{l,j,*}(\mu_l) = \mu_j$ for all $l \geq j$ will be called a *consistent family of measures* or just a *consistent family.*

Theorem 5.1. *(Kolmogorov-Bochner) Let $\mu_j \in M^{\natural}(\mathbf{X}_j)$ be a consistent family of measures. Then there exists a unique semi-positive, bounded, K_{∞}-invariant Borel measure $\mu \in M^{\natural}(\mathbf{X}^{\infty})$ such that $\pi_{j,*}(\mu) = \mu_j$.*

In such a case, we write

$$\mu = \varprojlim\{\mu_j\}$$

and say that μ is the inverse (or projective) limit of $\{\mu_j\}$.

Proof. This is by now a standard result. Cf. [25] the Corollary on p. 39. That μ is actually K_{∞}-invariant is immediate from the K_j-invariance of μ_j and the fact that the the cylinder sets $\pi_j^{-1}(U_j)$ form a base for the topology on $\mathbf{X}^{\infty}$. ∎

Converseley, a measure $\mu \in M^{\natural}(\mathbf{X}^{\infty})$ gives rise to a consistent family of measures. Given $\mu \in M^{\natural}(\mathbf{X}^{\infty})$ let $\mu_j = \pi_{j,*}(\mu)$. Then $\{\mu_j\}$ is a consistent family of measures and $\varprojlim\{\mu_j\} = \mu$.

Remark. The construction of limits of consistent families of measures works equally well for non-K_j-invariant families. The only difference being, of course, that the limit is not K_{∞} invariant. If investigating the full Fourier transform and not just the spherical transform, they would be the natural families and limits to consider. For our purposes, the K_j-invariant families suffice.

The spherical transform will involve integrating elements of $BCyl^{\infty}(\mathbf{X}^{\infty})$ against measures in $M^{\natural}(\mathbf{X}^{\infty})$. Fortunately it is a straightforward process. Given $f \in BCyl^{\infty}(\mathbf{X}^{\infty})$ such that $f = \pi_j^*(f_j)$, i.e. f is the lift of the function f_j on $\mathbf{X}_j$, and $\mu \in M^{\natural}(\mathbf{X}^{\infty})$ we see from the definitions that

$$[\mu, f]_{\mathbf{X}^{\infty}} = \int_{\mathbf{X}^{\infty}} f(x)d\mu(x) = \int_{\mathbf{X}_j} f(x_j)d\mu_j(x_j) = [\mu_j, f_j]_{\mathbf{X}_j}.$$

Before stating the main theorem in the next section, we show how to define convolution in $M^\natural(\mathbf{X}^\infty)$. In the the finite dimensional setting, one depends on the the group law on G_j to define convolution of elements of $M^\natural(\mathbf{X}_j)$. There is no such group available for $\mathbf{X}^\infty$ and so we must build convolutions of measures by convolving the corresponding consistent families.

Proposition 5.2. *The map $\pi_{l,j,*} : M^\natural(\mathbf{X}_l) \longrightarrow M^\natural(\mathbf{X}_j)$ is a homomorphism for convolution. I.e.,*

$$\pi_{l,j,*}(\mu_l \star_l \kappa_l) = (\pi_{l,j,*}\mu_l) \star_j (\pi_{l,j,*}\kappa_l).$$

Proof. This can be shown directly from the definitions, but we prefer to use Theorem 3.7 instead. Let $f_l = \mathbf{S}_l(\mu_l)$ and $g_l = \mathbf{S}_l(\kappa_l)$. For $j \le l$ let $f_j = f_l|_{T_{\rho_j}}$ and similarly for g_j. By Theorem 3.7 we know that

$$f_j = \mathbf{S}_j(\pi_{l,j,*}\mu_l) \text{ and } g_j = \mathbf{S}_j(\pi_{l,j,*}\kappa_l).$$

Since the restriction of a product of two functions is just the product of the restrictions, after comparing the two sides in the statement of the proposition, we see that they both map to $f_j\, g_j$ under $\mathbf{S}_j$, and are equal by Proposition 3.3. ∎

We can now define convolution of measures $M^\natural(\mathbf{X}^\infty)$ as the limit of the convolution of the two families determined by the measures. Given $\mu, \kappa \in M^\natural(\mathbf{X}^\infty)$ define their convolution, $\mu \star \kappa$, to be the limit

$$\mu \star \kappa = \varprojlim \{\pi_{j,*}(\mu) \star_j \pi_{j,*}(\kappa)\}.$$

This convolution is easily seen to be commutative and with it $M^\natural(\mathbf{X}^\infty)$ becomes a commutative semi-group.

6. The Spherical Transform and its Properties

We are now in a position to define a spherical transform for the projective limit $\mathbf{X}^\infty$.

Definition. Given a measure $\mu \in M^\natural(\mathbf{X}^\infty)$ define its spherical transform $\mathbf{S}^\infty\mu$ to be the function on $\mathfrak{a}_{\mathbf{C}}^{\infty,\vee}$ given by

$$(\mathbf{S}^\infty\mu)(\zeta) = \int_{\mathbf{X}^\infty} \phi_\zeta^\infty(x) d\mu(x) = \int_{\mathbf{X}^\infty} (x)_{A^\infty}^\zeta d\mu(x)$$

whenever the integral is defined.

Remark. Since μ is assumed to be K_∞-left-invariant, we could use the function $\lim_{j\to\infty} \int_{K_j} \phi_\zeta^\infty(kx)dk$ instead of ϕ_ζ^∞ to define $\mathbf{S}^\infty$. Such functions warrant further study, see the introduction.

Clearly, we require that ϕ_ζ^∞ be bounded in order that the integral to be well defined for every μ. This occurs precisely when ζ belongs to the tube domain $T_{\rho^\infty} \subset \mathfrak{a}_{\mathbf{C}}^{\infty,\vee}$. As a generalization of Theorem 3.7 we obtain the following:

Theorem 6.1. *The following diagram commutes*

$$
\begin{tikzcd}
M^\natural(\mathbf{X}^\infty) \arrow[rrr, "\mathbf{S}^\infty"] \arrow[ddr, bend left, "\pi_{j,*}"'] \arrow[ddddr, bend right, "\pi_{2,*}"'] \arrow[dddddr, bend right, "\pi_{1,*}"'] & & & \mathrm{Fu}(T_{\rho^\infty})^{(W_\infty,\rho^\infty)} \arrow[ddl, bend right, "\pi_j^{\vee,*}"'] \arrow[ddddl, bend left, "\pi_2^{\vee,*}"'] \arrow[dddddl, bend left, "\pi_1^{\vee,*}"'] \\
& \vdots \arrow[d, "\pi_{j+1,j,*}"'] & \vdots \arrow[d, "\pi^{\vee,*}_{j+1,j}"] & \\
& M^\natural(\mathbf{X}_j) \arrow[r, "\mathbf{S}_j"] \arrow[d, "\pi_{j,j-1,*}"'] & \mathrm{Fu}(T_{\rho_j})^{(W_j,\rho_j)} \arrow[d, "\pi^{\vee,*}_{j,j-1}"] & \\
& \vdots \arrow[d, "\pi_{3,2,*}"'] & \vdots \arrow[d, "\pi^{\vee,*}_{3,2}"] & \\
& M^\natural(\mathbf{X}_2) \arrow[r, "\mathbf{S}_2"] \arrow[d, "\pi_{2,1,*}"'] & \mathrm{Fu}(T_{\rho_2})^{(W_2,\rho_2)} \arrow[d, "\pi^{\vee,*}_{2,1}"] & \\
& M^\natural(\mathbf{X}_1) \arrow[r, "\mathbf{S}_1"] & \mathrm{Fu}(T_{\rho_1})^{(W_1,\rho_1)} &
\end{tikzcd}
$$

Proof. That the spherical transform of a measure is (W_∞, ρ^∞)-invariant follows immediately if the diagram is commutative. The heavy lifting has already been done, and it is mainly a matter of unwinding the definitions. Theorem 3.7 tell us that the parts of the diagram on the finite dimensional rungs commute.

For the rest of the diagram, first recall that for $f_j \in BC^\infty(\mathbf{X}_j)$ and $\mu \in M^\natural(\mathbf{X}^\infty)$

$$[\pi_j^* f_j, \mu]_{\mathbf{X}^\infty} = [f_j, \mu_j]_{\mathbf{X}_j}$$

where $\mu_j = \pi_{j,*}\mu$ as usual. Now, If $\zeta \in \mathfrak{a}_{\mathbf{C}}^{\infty,\vee}$ comes from $\mathfrak{a}_{j,\mathbf{C}}^\vee$, i.e., $\zeta = \zeta_j$, then we see immediately from the definitions that

$$(\mathbf{S}^\infty\mu)(\zeta) = [\phi_\zeta^\infty, \mu]_{\mathbf{X}^\infty} = [\phi_\zeta, \mu_j]_{\mathbf{X}_j} = (\mathbf{S}_j\mu_j)(\zeta)$$

which is exactly the content of the diagram. ∎

Before discussing the main example of a consistent family, the heat kernel measures, we pause to give some of the main properties of $\mathbf{S}^\infty$ in parallel to those given in Section 3. The pattern of proof will quickly emerge: all the results follow easily from the above commutative diagram and the similar statements about $\mathbf{S}_j$.

Proposition 6.2. *Let $\mu_1, \mu_2 \in M^\natural(\mathbf{X}^\infty)$ and let $f_1 = \mathbf{S}^\infty(\mu_1), f_2 = \mathbf{S}^\infty(\mu_2)$.*

1. *(Invariance) $\mathbf{S}^\infty(\mu_1)$ is (W_∞, ρ^∞)-invariant;*
2. *(Homomorphism) For $\zeta \in T_{\rho^\infty}, \mathbf{S}^\infty(\mu_1 \star \mu_2)(\zeta) = f_1(\zeta)f_2(\zeta)$;*
3. *(Injectivity) if $f_1 = f_2$, then $\mu_1 = \mu_2$.*

Proof. The first two statements are immediate from the definitions, the commutative diagram, and Proposition 3.3. Injectivity follows as well, since any consistent family of measures has a unique limit in $M^\natural(\mathbf{X}^\infty)$. ∎

We define weak convergence just as for the finite-dimensional symmetric space. Let $\{\mu^{(i)}\}$ be a sequence of measures on $\mathbf{X}^\infty$. We say that $\{\mu^{(i)}\}$ *converges weakly* to a measure μ if for any $f \in BCyl^\infty(\mathbf{X}^\infty)$

$$\lim_{i\to\infty}[f,\mu^{(i)}]_{\mathbf{X}^\infty} = [f,\mu]_{\mathbf{X}^\infty}.$$

The set-up is rigid, and convergence on the finite rungs determines convergence on the limit.

Lemma 6.3. *A sequence of measures $\mu^{(i)} \in M^\natural(\mathbf{X}^\infty)$ converges weakly to a measure $\mu \in M^\natural(\mathbf{X}^\infty)$ if and only if for every j the sequence $\pi_{j,*}(\mu^{(i)})$ converges weakly to the measure $\mu_j = \pi_{j,*}(\mu)$.*

Proof. The proof, which is not difficult, will be omitted. Cf. [3] for details. ∎

We can now put together the above results to obtain a Levy-Cramer type continuity theorem for measures on $\mathbf{X}^\infty$.

Theorem 6.4. *Let $\mu^{(i)} \in M^\natural(\mathbf{X}^\infty)$ be a sequence of K-invariant measures on $\mathbf{X}^\infty$ and let $\beta^{(i)} = \mathbf{S}^\infty(\mu^{(i)})$.*

1. *If $\mu^{(i)}$ converges weakly to μ then $\beta^{(i)}$ converges pointwise to $\beta = \mathbf{S}^\infty\mu$ on T_{ρ^∞};*
2. *Assume that $\beta^{(i)}$ converges pointwise to a function β on T_{ρ^∞} and that $\lim_{i\to\infty}\mu^{(i)}(\mathbf{X}^\infty)$ exists. Then there exists $\mu \in M^\natural(\mathbf{X}^\infty)$ such that $\mathbf{S}^\infty\mu = \beta$. If in addition $\lim_{i\to\infty}\mu^{(i)}(\mathbf{X}^\infty) = \mu(\mathbf{X}^\infty)$ then $\mu^{(i)}$ converges weakly to μ.*

Proof. Let

$$\mu_j^{(i)} = \pi_{j,*}(\mu^{(i)}), \quad \mu_j = \pi_{j,*}\mu, \quad \beta_j^{(i)} = \beta^{(i)}|_{T_{\rho_j}}, \quad \beta_j = \beta|_{T_{\rho_j}}.$$

Under the conditions of 1, $\mu_j^{(i)}$ converges weakly to μ_j by the preceding Lemma and so $\beta_j^{(i)}$ converges pointwise to β_j for all j by Theorem 3.4. That proves 1.

As for 2, observe that if the limits exist, then

$$\lim_{i\to\infty}\mu^{(i)}(\mathbf{X}^\infty) = \lim_{i\to\infty}\mu_j^{(i)}(\mathbf{X}_j)$$

for all j. Hence, by Theorem 3.4 there exist μ_j such that $\mathbf{S}_j\mu_j = \beta_j$. Letting $\mu = \varprojlim \mu_j$ it is clear from the definitions that $\mathbf{S}^\infty\mu = \beta$. If in addition

$$\lim_{i\to\infty}\mu^{(i)}(\mathbf{X}^\infty) = \mu(\mathbf{X}^\infty)$$

then the identical statement with subscripts j also clearly holds. So Theorem 3.4 ensures that $\mu_j^{(i)}$ converges weakly to μ_j for all j and the preceding Lemma implies that $\mu^{(i)}$ converges weakly to μ. ∎

7. The Heat Kernel on the Projective Limit

To describe the heat kernel on $\mathbf{X}^\infty$, we need to arrive at a notion of differential operators on $BCyl^\infty(\mathbf{X}^\infty)$, or at least for some natural space of test functions. Our approach should by now be familiar: find differential operators compatible with the projections $\pi_{l,j}$ and pass to the limit. The prime example will be the second-order Casimir operator, which is essentially just the Laplacian.

Take a family of linear maps $\{T_j \mid T_j : BC^\infty(\mathbf{X}_j) \longrightarrow BC^\infty(\mathbf{X}_j)\}$ satisfying the compatibility condition

$$(\pi_{l,j,*}T_l)(f_j) \equiv T_l(f_j \circ \pi_{l,j})|_{\mathbf{X}_j} = T_j(f_j)$$

for every l and every $f_j \in BC^\infty(\mathbf{X}_j)$. The restriction on the left means that we view $T_l(f_j \circ \pi_{l,j})$ as a function on $\mathbf{X}_j$ so that the equality makes sense (this is possible since we embed $\mathbf{X}_j$ as a submanifold of $\mathbf{X}_l$ using the map $\eta_{j,l}$). Equivalently, we could require that

$$T_l(\pi_{l,j}^* f_j) = \pi_{l,j}^*(T_j f_j).$$

A family satisfying this condition will be called a *consistent family of maps.*

We can define the projective limit $T = \varprojlim\{T_j\}$ as a linear operator on the spaces of cylindrical functions $BCyl^\infty(\mathbf{X}^\infty)$ in a natural manner. Given $f \in BCyl^\infty(\mathbf{X}^\infty)$ such that $f = \pi_j^*(f_j)$ we let

$$T(f) = \pi_j^*(T_j f_j).$$

This is clearly well defined since the $\{T_j\}$ are a consistent family of maps.

Let ω_j denote the second-order Casimir operator on $\mathbf{X}_j$.

Proposition 7.1. *The family $\{\omega_j\}$ is a consistent family of maps. Hence the projective limit $\omega = \varprojlim\{\omega_j\}$ is a well defined map from $BCyl^\infty(\mathbf{X}^\infty)$ to $BCyl^\infty(\mathbf{X}^\infty)$.*

Proof. It is sufficient to show that $\pi_{l,j,*}(\omega_l) = \omega_j$. This is proved in [14] Theorem 4.2.5 for the case $G_j = \mathrm{SL}_j(\mathbf{C})$. The proof carries over to the general case essentially unchanged. ■

Remark. Most results in this paper would go through if we had chosen to work with the reductive parts of parabolic subgroups instead of reduced semi-simple components. However, Proposition 7.1 depends on the semi-simplicity condition. In projecting to the reductive part of a parabolic one picks up a first-order term related to the trace of $\mathfrak{a}_{j,l}$ acting on $\mathfrak{n}_{j,l}$.

Now that a suitable notion of a Casimir operator is available, we can formulate and solve the heat equation on $\mathbf{X}^\infty$. By applying Corollary 3.8 repeatedly, we obtain a consistent family of heat kernel measures $\{\nu_{t,j}\}$ associated to ω_j and $\mathbf{X}_j$. Let

$$\nu_t^\infty = \varprojlim\{\nu_{t,j}\}.$$

Fortunately, everything fits together nicely and we will see that ν_t^∞ is the heat kernel associated to ω and $\mathbf{X}^\infty$.

Just as at each finite dimensional rung on the ladder the heat kernel measure is determined by its spherical transform, so to ν_t^∞ can be characterized as the measure whose spherical transform is given by

$$(\mathbf{S}^\infty \nu_t^\infty)(\zeta) = \exp\Big(t\big(B_\infty^\vee(\zeta,\zeta) - 2B_\infty^\vee(\zeta,\rho^\infty)\big)\Big).$$

Now, the expression $B_\infty^\vee(\zeta,\rho^\infty)$ doesn't quite make sense since ρ^∞ is not in $\mathfrak{a}_{\mathbf{C}}^{\infty,\vee}$, but in $\mathfrak{a}_{\infty,\mathbf{C}}^\vee$. However, if $\zeta \in \mathfrak{a}_{j,\mathbf{C}}^\vee$, i.e. $\zeta = \zeta_j$, we observe that

$$B_\infty^\vee(\zeta_j, \rho_{j,l}) = 0$$

whenever $l > j$. Noting that $\rho^\infty = \rho_j + \rho_j^\infty$ makes clear that we can write

$$B_\infty^\vee(\zeta_j, \rho^\infty) = B_j^\vee(\zeta_j, \rho_j)$$

which is always well defined, if slightly incorrect.

Remark. There are natural isomorphisms $\mathfrak{a}_\infty^\vee \simeq \mathfrak{a}^\infty$ and $\mathfrak{a}^{\infty,\vee} \simeq \mathfrak{a}_\infty$ and hence we obtain $(\mathfrak{a}^{\infty,\vee})^\vee \simeq \mathfrak{a}_\infty^\vee$. We are using this last identification to turn ρ^∞ into a functional on $\mathfrak{a}_{\mathbf{C}}^{\infty,\vee}$ and simply writing $\rho^\infty(\zeta) = B_\infty^\vee(\zeta,\rho^\infty)$ to keep the similarity with the finite dimensional structures.

Now for some of the properties of ν_t^∞.

Proposition 7.2. *The family $\{\nu_t\}$ is a semi-group under convolution:*

$$\nu_t \star \nu_s = \nu_{t+s}.$$

Proof. Immediate from the fact that $\mathbf{S}^\infty$ is a homomorphism and injective. ■

To justify the name 'heat kernel' we should show that ν_t^∞ satisfies a heat equation.

Theorem 7.3. *The measure ν_t^∞ is the unique measure satisfying*

1. *For all $f \in BCyl^\infty(\mathbf{X}^\infty)$ $\lim_{t\to 0}\int_{\mathbf{X}^\infty} f(x)\,d\nu_t^\infty = f(e)$. I.e. ν_t^∞ converges weakly to the point mass at the identity as $t \to 0$;*
2. *For all $f \in BCyl^\infty(\mathbf{X}^\infty)$*

$$\partial_t[f,\nu_t^\infty]_{\mathbf{X}^\infty} = [\omega f, \nu_t^\infty]_{\mathbf{X}^\infty}.$$

Note that there is no analog of Haar measure on $\mathbf{X}^\infty$ and hence no way to express ν_t^∞ as a density function times another measure. Thus the differential equation condition must be expressed in this manner.

Proof. Assume $f = \pi_j^* f_j$. Then $\nu_t^\infty(f) = \nu_{t,j}(f_j)$ and $\nu_t^\infty(\omega f) = \nu_{t,j}(\omega_j f_j)$ and the two claims follow from the properties of $\nu_{t,j}$. For uniqueness, assume κ_t is another measure satisfying the two properties. Then $\kappa_{t,j} = \pi_{j,*}(\kappa_t)$ must be the heat kernel on $\mathbf{X}_j$ and is hence equal to $\nu_{t,j}$. Since a consistent family has a unique projective limit, $\kappa_t = \nu_t^\infty$ as desired. ■

8. Examples of Weakly Parabolic Ladders

We now give two quite different examples of weakly parabolic direct systems. Note that to be a non-trivial weakly parabolic direct system $\mathcal{S}(\mathfrak{a}_j, \mathfrak{n}_j)$ must be a proper subset of $\mathcal{S}(\mathfrak{a}_{j+1}, \mathfrak{n}_{j+1})$ for an infinite number of j, hence

$$\text{rank } \mathbf{X}_j < \text{rank } \mathbf{X}_{j+1}$$

for an infinite number of j and the rank of $\mathbf{X}^\infty$ (i.e. the dimension of $\mathfrak{a}^\infty$) must be infinite. This excludes families such as the classical hyperbolic spaces. Indeed the Lie algebra $\mathfrak{so}(1, j)$ is not weakly parabolic in $\mathfrak{so}(1, j+1)$ for $j > 2$ (even though the root systems are the same, the roots occur with different multiplicities) and hence the present theory cannot be applied directly. Though there are natural projections from H^{j+1} to H^j from which one could build a projective limit, it needs to be clarified how well each behaves with regards to the spherical transform and harmonic analysis in general.

Example. The first example is the simplest and is just the infinite direct product of a sequence of symmetric spaces. Let $G_{(i)}$ be a sequence of arbitrary (real, linear, connected) semi-simple Lie groups. Let

$$G_j = \prod_{i=1}^{j} G_{(i)}$$

be the j-th partial product. Let the embedding $\eta_{j,j+1} : G_j \longrightarrow G_{j+1}$ be given in the obvious manner by

$$\eta_{j,j+1}(g_1, ..., g_j) = (g_1, ..., g_j, 1).$$

With this embedding, $\mathfrak{g}_j \oplus \mathfrak{m}_{(j+1)}$ is the semi-simple part of the parabolic subalgebra $\mathfrak{g}_j \oplus \mathfrak{q}_{(j+1)} \subset \mathfrak{g}_{j+1}$ where $\mathfrak{q}_{(j+1)}$ is the minimal parabolic of $\mathfrak{g}_{(j+1)}$. Hence $\mathfrak{g}_j$ is weakly parabolic in $\mathfrak{g}_{j+1}$.

Clearly $\mathbf{X}_j = \prod_{i=1}^{j} \mathbf{X}_{(i)}$ (j-fold direct metric product), and similarly for A_j, N_j, and K_j. The projection $\pi_{l,j}$ projects onto the first j components. The projective limit $\mathbf{X}^\infty$ is just the infinite direct product

$$\mathbf{X}^\infty = \prod_{i=1}^{\infty} \mathbf{X}_{(i)}.$$

All the geometric objects discussed above (measures, differential operators, heat kernel) can also be thought of as direct products. In particular, the heat kernel is given by $\nu_t^\infty = \prod_{i=1}^{\infty} \nu_{t,(i)}$.

Though this example appears trivial in its simplicity, it is used as a starting point for the theory developed (with G compact) by Ashtekar and Lewandowski [1] where they consider the limit of products modulo a certain action.

Example. The main class of parabolic direct systems come from the infinite families of simple classical matrix groups. Let $G_j = \mathrm{SL}_j(\mathbf{C})$. We take G_j to be the standard complex special linear group, i.e., complex matrices of determinant 1. The groups A_j, N_j, and K_j consist of the diagonal, upper triangular, and unitary subgroups, respectively. The connecting map is $\eta_{j,j+1}(g) = diag(g, 1)$.

The symmetric space $\mathbf{X}_j = G_j/K_j$ is isometrically isomorphic to the space $\mathbf{SPos}_j$ of all positive definite Hermitian matrices of determinant 1.

Passing to the limits, we have the direct limit $\varinjlim \mathfrak{a}^{\vee}_{j,\mathbf{C}} = \mathfrak{a}^{\infty,\vee}_{\mathbf{C}}$ which can be identified with the infinite complex diagonal matrices with trace 0 and differing from 0 in a finite number of places. The projective limit $\varprojlim A_j = A^{\infty}$ can be identified with the group of all infinite diagonal matrices with positive entries modulo the scalar matrices. That is, we identify two diagonal matrices if they differ by a (positive) multiplicative constant. For the N components, the projective limit $\varprojlim N_j = N^{\infty}$ can be identified with the group of all infinite upper triangular matrices with complex entries and 1 on the diagonal. Observe that there is no problem forming the group $N^{\infty}A^{\infty}$ which is naturally isomorphic to the projective limit (in the category of topological groups and continuous homomorphisms) $\varprojlim N_j A_j$ (see Proposition 4.3). This group is just the set of all upper triangular matrices with positive real entries on the diagonal, subject to the equivalence that we identify two matrices that differ by a positive multiplicative constant.

As mentioned previously, $N^{\infty}A^{\infty}$ is topologically isomorphic to $\mathbf{X}^{\infty}$ and shares many properties of the Iwasawa coordinates on the finite dimensional symmetric spaces, e.g., N^{∞} is a normal subgroup, though it is not unipotent. One can even decompose $\mathfrak{n}^{\infty}$ under the action of $\mathfrak{a}^{\infty}$:

$$\mathfrak{n}^{\infty} = \sum_{\alpha \in \mathcal{R}(\mathfrak{a}^{\infty}, \mathfrak{n}^{\infty})} \mathfrak{n}_{\alpha}$$

where $\mathfrak{n}_{\alpha}$ is the eigenspace corresponding to α.

Furthermore, though we will not develop its properties, we mention that the matrix exponential can be manipulated to give a well defined map

$$\exp : \mathfrak{n}^{\infty} \times \mathfrak{a}^{\infty} \longrightarrow N^{\infty}A^{\infty}, \quad (U, H) \mapsto \exp(U)\exp(H).$$

This is slightly forced since it doesn't agree with the natural exponential $(U, H) \mapsto \exp(U + H)$ one would like. However, coupled with the Iwasawa coordinates one can obtain a global 'chart' of $\mathbf{X}^{\infty}$.

In terms of Hermitian matrices, the projections can be given explicitly by

$$\begin{aligned} \pi_{l,j} : \mathbf{SPos}_l &\longrightarrow \mathbf{SPos}_j \\ \begin{pmatrix} A & B \\ B^* & D \end{pmatrix} &\mapsto \frac{A - BD^{-1}B^*}{\det(A - BD^{-1}B^*)^{\frac{1}{j}}}. \end{aligned} \tag{3}$$

The projective limit cannot be described directly as any space of Hermitian matrices. However, taking the derivative at the identity gives a map

$$\begin{aligned} d\pi_{l,j} : \mathfrak{p}_l &\longrightarrow \mathfrak{p}_j \\ \begin{pmatrix} A & B \\ B^* & D \end{pmatrix} &\mapsto A - \tfrac{\mathrm{Tr}A}{j} I_j \end{aligned}$$

and the projective limit of the tangent spaces can be viewed as the space of all infinite Hermitian matrices modulo the equivalence relation identifying two matrices whose diagonals differ by an additive constant. The harmonic analysis on the space of all infinite Hermitian matrices was studied extensively in [4].

All the other classical weakly parabolic systems admit similar descriptions. For a list of the systems cf. [23].

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Andrew R. Sinton
Institute of Mathematics
The Hebrew University, Jerusalem
91904 Israel
sinton@gmail.com

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