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A Quantum Analogue of the Bernstein Functor

S. Sinel'shchikov, A. Stolin, and L. Vaksman

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Abstract. We consider Knapp-Vogan Hecke algebras in the quantum group setting. This allows us to produce a quantum analogue of the Bernstein functor as a first step towards the cohomological induction for quantum groups.
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The cohomological induction is one of the most important tools for producing unitarizable Harish-Chandra modules [7]. The present paper demonstrates that this method can be applied to modules over quantum universal enveloping algebras [6] as well. Our way to this circle of problems was as follows.

The papers [11, 9] introduce quantum analogues of the bounded symmetric domains and the associated Harish-Chandra modules. One special case considered in [10] leads to two geometric realizations for the ladder representation of the quantum universal enveloping algebra $U_q\mathfrak{su}_{2,2}$ and to the quantum Penrose transform [1]. One of the main tools used in [10] was studying q-analogues of the Čech cohomology. Unfortunately this method can be hardly generalized because it relates to non-trivial problems in non-commutative algebraic geometry. We hope to overcome these obstacles via replacing the q-analogues of the Čech cohomology by q-analogues of the Dolbeault cohomology. It is well-known that in the classical situation the Dolbeault cohomology can be constructed algebraically using the so-called cohomological induction (see [7]).

It is convenient to work with the Bernstein functor (the projective Zuckerman functor) for producing unitarizable Harish-Chandra modules. The definition of the Bernstein functor uses essentially the algebra of distributions on a real reductive group G with support in a maximal compact subgroup K. The principal obstacle to cope with in this work is to construct a q-analogue for this algebra.

In the present paper we construct a q-analogue of the algebra of distributions as above in certain cases. This allows us to construct a q-analogue of the Bernstein functor.

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1. The category $C(\mathfrak{l},\mathfrak{l})_q$ and the algebra $R(\mathfrak{l})_q$

We will use $\mathbb{C}$ as the ground field.

Let $\mathbf{a}=(a_{ij})_{i,j=1,2,\ldots,l}$, be the Cartan matrix for a simple complex Lie algebra of rank l. Its universal enveloping algebra is a unital algebra U determined by the generators $\{H_i,E_i,F_i\}_{i,j=1,2,\ldots,l}$ and the relations [6, p. 51]:

$$H_iH_j-H_jH_i=0,\qquad E_iF_j-F_jE_i=\delta_{ij}H_i,$$

$$H_iE_j-E_jH_i=a_{ij}E_j,\qquad H_iF_j-F_jH_i=-a_{ij}F_j,\qquad i,j=1,2,\ldots,l,$$

and

$$\sum_{m=0}^{1-a_{ij}}(-1)^m\binom{1-a_{ij}}{m}E_i^{1-a_{ij}-m}\cdot E_j\cdot E_i^m=0,$$

$$\sum_{m=0}^{1-a_{ij}}(-1)^m\binom{1-a_{ij}}{m}F_i^{1-a_{ij}-m}\cdot F_j\cdot F_i^m=0,$$

for all $i\neq j$.

The Cartan subalgebra spanned by $H_1,H_2,\ldots,H_l$ is denoted by $\mathfrak{h}$, and $\alpha_1,\alpha_2,\ldots,\alpha_l$,

$$\alpha_j(H_i)=a_{ij},\qquad i,j=1,2,\ldots,l,$$

are simple roots.

There exists a unique collection of coprime positive integers $d_1,d_2,\ldots,d_l$ such that

$$d_ia_{ij}=d_ja_{ji},\qquad i,j=1,2,\ldots,l.$$

The bilinear form in $\mathfrak{h}^*$ given by

$$(\alpha_i,\alpha_j)=d_ia_{ij},\qquad i,j=1,2,\ldots,l,$$

is positive definite.

Recall the definition of the quantum universal enveloping algebra U_q introduced by V. Drinfeld and M. Jimbo. We assume that $q\in(0,1)$.[1]

The unital algebra U_q is determined by its generators K_i, K_i^{-1}, E_i, F_i, $i=1,2,\ldots,l$, and the relations

$$K_iK_j=K_jK_i,\qquad K_iK_i^{-1}=K_i^{-1}K_i=1,$$

$$K_iE_j=q_i^{a_{ij}}E_jK_i,\quad K_iF_j=q_i^{-a_{ij}}F_jK_i,\quad E_iF_j-F_jE_i=\delta_{ij}(K_i-K_i^{-1})/(q_i-q_i^{-1}),$$

$$\sum_{s=0}^{1-a_{ij}}(-1)^s\begin{bmatrix}1-a_{ij}\\ s\end{bmatrix}_{q_i}E_i^{1-a_{ij}-s}E_jE_i^s=0,$$

[1]The purpose of so strong assumptions on q will become evident while investigating the unitarizability of the Harish-Chandra modules in question over U_q.

$$\sum_{s=0}^{1-a_{ij}}(-1)^s \begin{bmatrix} 1-a_{ij} \\ s \end{bmatrix}_{q_i} F_i^{1-a_{ij}-s} F_j F_i^s = 0,$$

with

$$q_i = q^{d_i}, \qquad 1 \le i \le l,$$

$$\begin{bmatrix} m \\ n \end{bmatrix}_q = \frac{[m]_q!}{[n]_q![m-n]_q!}, \qquad [n]_q! = [n]_q \ldots [2]_q[1]_q, \qquad [n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}.$$

The Hopf algebra structure on U_q is given by the following formulas:

$$\triangle(E_i) = E_i \otimes 1 + K_i \otimes E_i, \qquad \triangle(F_i) = F_i \otimes K_i^{-1} + 1 \otimes F_i, \qquad \triangle(K_i) = K_i \otimes K_i,$$

$$S(E_i) = -K_i^{-1}E_i, \qquad S(F_i) = -F_iK_i, \qquad S(K_i) = K_i^{-1},$$

$$\varepsilon(E_i) = \varepsilon(F_i) = 0, \qquad \varepsilon(K_i) = 1.$$

Any subset $\mathbb{L} \subset \{1, 2, \ldots, l\}$ determines a Hopf subalgebra $U_q\mathfrak{l} \subset U_q$ generated by

$$K_i^{\pm 1}, \quad i = 1, 2, \ldots, l; \qquad E_j, F_j, \quad j \in \mathbb{L}.$$

We use the following notation:

$$P = \mathbb{Z}^l, \qquad P_+ = \mathbb{Z}_+^l, \qquad P_+^{\mathbb{L}} = \{\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_l) \in P |\, \lambda_i \ge 0 \text{ for } i \in \mathbb{L}\}.$$

Let V be a $U_q\mathfrak{l}$-module and

$$V_\mu = \{v \in V |\, K_i v = q_i^{\mu_i} v, \quad i = 1, 2, \ldots, l\}$$

the weight space, where $\mu = (\mu_1, \mu_2, \ldots, \mu_l) \in P$.

Definition 1.1. We say V is a **weight** $U_q\mathfrak{l}$-module if $V = \bigoplus_{\mu \in P} V_\mu$.

Example 1.2. Consider the Verma module $U_q\mathfrak{l}$-module $M(\mathfrak{l}, \lambda)$ with the highest weight $\lambda \in P_+^{\mathbb{L}}$. This is a module determined by its generator $v(\mathfrak{l}, \lambda)$ and the relations

$$E_j v(\mathfrak{l}, \lambda) = 0, \quad j \in \mathbb{L}; \qquad K_i^{\pm 1} v(\mathfrak{l}, \lambda) = q_i^{\pm \lambda_i} v(\mathfrak{l}, \lambda), \quad i = 1, 2, \ldots, l.$$

It is easy to prove that its submodule $K(\mathfrak{l}, \lambda)$ generated by $F_j^{\lambda_j + 1} v(\mathfrak{l}, \lambda)$, $j \in \mathbb{L}$, is the unique submodule of finite codimension. In particular, the factor-module $L(\mathfrak{l}, \lambda) = M(\mathfrak{l}, \lambda)/K(\mathfrak{l}, \lambda)$ is simple (see [6]).

The following claims are well-known (see for instance [6]). The dimensions of the weight subspaces of the $U_q\mathfrak{l}$-modules $L(\mathfrak{l}, \lambda)$ are the same as in the classical case $q = 1$. The weight finite dimensional $U_q\mathfrak{l}$-modules are completely reducible. Irreducible weight finite dimensional $U_q\mathfrak{l}$-modules are isomorphic to $L(\mathfrak{l}, \lambda)$ with $\lambda \in P_+^{\mathbb{L}}$. The center of $U_q\mathfrak{l}$ (denoted by $Z(U_q\mathfrak{l})$) admits an easy description via the Harish-Chandra isomorphism [6, p. 109] and separates the simple weight finite dimensional $U_q\mathfrak{l}$-modules [6, p. 125]. In other words, for any $\lambda_1, \lambda_2 \in P_+^{\mathbb{L}}$ there exists $z \in Z(U_q\mathfrak{l})$ such that $z|_{L(\mathfrak{l}, \lambda_1)} \ne z|_{L(\mathfrak{l}, \lambda_2)}$.

A module V over $U_q\mathfrak{l}$ is called locally finite dimensional if $\dim(U_q\mathfrak{l}\,v) < \infty$ for all $v \in V$. Introduce the notation $C(\mathfrak{l},\mathfrak{l})_q$ for the full subcategory of weight locally finite dimensional $U_q\mathfrak{l}$-modules.

It is easy to prove that any $U_q\mathfrak{l}$-module $V \in C(\mathfrak{l},\mathfrak{l})_q$ is uniquely decomposable as a direct sum $V = \bigoplus_{\lambda \in P_+^{\mathbb{L}}} V_\lambda$ of its submodules. Each V_λ here is a multiple of $L(\mathfrak{l},\lambda)$, $\lambda \in P_+^{\mathbb{L}}$. The sum is direct since $Z(U_q\mathfrak{l})$ separates simple finite dimensional weight $U_q\mathfrak{l}$-modules.

Let $\operatorname{End} V$ be the algebra of all endomorphisms of V considered as a vector space. Now consider the $U_q\mathfrak{l}$-module $L^{\mathrm{univ}} = \bigoplus_{\lambda \in P_+^{\mathbb{L}}} L(\mathfrak{l},\lambda)$ and the projections P_λ in L^{univ} onto the isotypic component $L(\mathfrak{l},\lambda)$ parallel to the sum of all other isotypic components. The algebra $\operatorname{End} L^{\mathrm{univ}}$ is a $U_q\mathfrak{l}$-module algebra.[2] The natural homomorphism $U_q\mathfrak{l} \to \operatorname{End} L^{\mathrm{univ}}$ is injective [6, p. 76 – 77]; this allows one to identify $U_q\mathfrak{l}$ with its image in $\operatorname{End} L^{\mathrm{univ}}$. We are interested in the following $U_q\mathfrak{l}$-module subalgebras of $\operatorname{End} L^{\mathrm{univ}}$:

$$R(\mathfrak{l})_q \stackrel{\mathrm{def}}{=} \bigoplus_{\lambda \in P_+^{\mathbb{L}}} \operatorname{End} L(\mathfrak{l},\lambda), \qquad F(\mathfrak{l})_q \stackrel{\mathrm{def}}{=} R(\mathfrak{l})_q \oplus U_q\mathfrak{l}.^{3}$$

Obviously, $R(\mathfrak{l})_q$ is a two-sided ideal of $F(\mathfrak{l})_q$ generated by the projections P_λ.

$R(\mathfrak{l})_q$ is not a unital algebra, but it admits a distinguished approximate identity. Specifically, any finite subset $\Lambda \subset P_+^{\mathbb{L}}$ determines an idempotent $\chi_\Lambda = \sum_{\lambda \in \Lambda} P_\lambda$ in $R(\mathfrak{l})_q$. Obviously, $\chi_{\Lambda_1}\chi_{\Lambda_2} = \chi_{\Lambda_2}\chi_{\Lambda_1} = \chi_{\Lambda_1 \cap \Lambda_2}$, and for any $r \in R(\mathfrak{l})_q$ there exists a finite subset $\Lambda \subset P_+^{\mathbb{L}}$ such that $\chi_\Lambda r = r\chi_\Lambda = r$. With some abuse of terminology, one can say that in $F(\mathfrak{l})_q$ we have

$$\lim_{\Lambda \uparrow P_+^{\mathbb{L}}} \chi_\Lambda = 1.$$

A module V over $R(\mathfrak{l})_q$ is called approximately unital if for any $v \in V$ there exists a finite subset $\Lambda \subset P_+^{\mathbb{L}}$ such that $\chi_\Lambda v = v$.

We are going to demonstrate that the full subcategory of approximately unital $R(\mathfrak{l})_q$-modules is canonically isomorphic to $C(\mathfrak{l},\mathfrak{l})_q$.

Proposition 1.3. *(*). For any approximately unital $R(\mathfrak{l})_q$-module V and any $\xi \in F(\mathfrak{l})_q$, $v \in V$ there exists a limit*

$$\xi v \stackrel{\mathrm{def}}{=} \lim_{\Lambda \uparrow P_+^{\mathbb{L}}} (\xi\chi_\Lambda)v \tag{1}$$

in the sense that $(\xi\chi_\Lambda)v$ does not depend on Λ for large Λ.

(**). The relation (1) equips V with a structure of $F(\mathfrak{l})_q$-module and defines a functor from the category of approximately unital $R(\mathfrak{l})_q$-modules into the category $C(\mathfrak{l},\mathfrak{l})_q$.

(***). This functor is an isomorphism of categories.

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[2]If $\xi \in U_q\mathfrak{l}$ and $\triangle\xi = \sum_i \xi_i' \otimes \xi_i''$, then $(\xi A)v = \sum_i \xi_i' A S(\xi_i'')v$, $v \in L^{\mathrm{univ}}$, $A \in \operatorname{End} L^{\mathrm{univ}}$, with $\triangle$ and S being the comultiplication and the antipode of $U_q\mathfrak{l}$.

[3]It is easy to prove that $R(\mathfrak{l})_q \cap U_q\mathfrak{l} = 0$. In fact, $U_q\mathfrak{l}$ has no zero divisors [3], while the Harish-Chandra isomorphism allows one to produce for any $a \in R(\mathfrak{l})_q$ a non-zero element $z \in Z(U_q\mathfrak{l})$ such that $za = 0$.

Proof. Any approximately unital $R(\mathfrak{l})_q$-module V admits an embedding into an approximately unital $R(\mathfrak{l})_q$-module that is a multiple of L^{univ}. This observation implies the first two statements, since they can be easily verified for L^{univ}.

Any $U_q\mathfrak{l}$-module that is a multiple of L^{univ}, is a $F(\mathfrak{l})_q$-module. This allows one to construct an inverse functor by embedding any $U_q\mathfrak{l}$-module $V \in C(\mathfrak{l},\mathfrak{l})_q$ into a $U_q\mathfrak{l}$-module that is a multiple of L^{univ}. This implies (***). ■

Note that the action of $P_\lambda \in R(\mathfrak{l})_q$ on vectors $v \in V \in C(\mathfrak{l},\mathfrak{l})_q$ can be described without referring to an embedding into a $U_q\mathfrak{l}$-module multiple to L^{univ}. More exactly, for any $\lambda \in P_+^{\mathbb{L}}$ there exists a sequence $z_1(\lambda), z_2(\lambda), \ldots$ of elements of the center of $U_q\mathfrak{l}$ such that $P_\lambda v = \lim\limits_{n\to\infty} z_n(\lambda)v$ for all $v \in V \in C(\mathfrak{l},\mathfrak{l})_q$. To see this, note that for any finite subset $\lambda \in \Lambda \subset P_+^{\mathbb{L}}$ there exists such $z \in Z(U_q\mathfrak{l})$ that $z|_{L(\mathfrak{l},\mu)} = 0$ for $\lambda \in \Lambda \setminus \{\lambda\}$ and $z|_{L(\mathfrak{l},\lambda)} = 1$. It remains to choose a sequence $\Lambda_n \uparrow P_+^{\mathbb{L}}$.

2. The category $C(\mathfrak{g},\mathfrak{l})_q$ and the algebra $R(\mathfrak{g},\mathfrak{l})_q$

Let $\mathbb{G} \supset \mathbb{L}$ be a pair of subsets of $\{1,2,\ldots,l\}$ and let $U_q\mathfrak{g} \supset U_q\mathfrak{l}$ be the associated pair of Hopf subalgebras of U_q.

Consider the category of all $U_q\mathfrak{g}$-modules and its full subcategory $C(\mathfrak{g},\mathfrak{l})_q$ that consists of weight $U_q\mathfrak{l}$-locally finite dimensional $U_q\mathfrak{g}$-modules.

Let us turn to a description of the algebra $F(\mathfrak{g},\mathfrak{l})_q$. This algebra is an important tool in studying $U_q\mathfrak{g}$-modules from $C(\mathfrak{g},\mathfrak{l})_q$. First, we consider a simple special case, namely $\mathbb{G} = \mathbb{L}$. In this situation we define $F(\mathfrak{g},\mathfrak{l})_q$ as $F(\mathfrak{l})_q$. We intend to construct $F(\mathfrak{g},\mathfrak{l})_q$ along with embeddings $U_q\mathfrak{g} \hookrightarrow F(\mathfrak{g},\mathfrak{l})_q$, $F(\mathfrak{l})_q \hookrightarrow F(\mathfrak{g},\mathfrak{l})_q$, which possess the following properties.

(F1) The diagram

$$\begin{array}{ccc} U_q\mathfrak{l} & \hookrightarrow & U_q\mathfrak{g} \\ \downarrow & & \downarrow \\ F(\mathfrak{l})_q & \hookrightarrow & F(\mathfrak{g},\mathfrak{l})_q \end{array}$$

commutes[4], and the subalgebras $U_q\mathfrak{g}$ and $F(\mathfrak{l})_q$ generate the algebra $F(\mathfrak{g},\mathfrak{l})_q$.

(F2) The approximate identity $\{\chi_\Lambda\}$ of $R(\mathfrak{l})_q$ is also an approximate identity of the two-sided ideal $R(\mathfrak{g},\mathfrak{l})_q$ of $F(\mathfrak{g},\mathfrak{l})_q$, generated by $R(\mathfrak{l})_q$.

(F3) For any approximately unital $R(\mathfrak{g},\mathfrak{l})_q$-module V and any $\xi \in F(\mathfrak{g},\mathfrak{l})_q$, $v \in V$ there exists a limit

$$\xi v \stackrel{\mathrm{def}}{=} \lim_{\Lambda\uparrow P_+^{\mathbb{L}}} (\xi\chi_\Lambda)v, \qquad (2)$$

and the equation (2) equips V with a structure of $F(\mathfrak{g},\mathfrak{l})_q$-module.

The intersection of the kernels for all representations of $F(\mathfrak{g},\mathfrak{l})_q$ obtained in this way is 0.

(F4) The functor from the category of approximately unital $R(\mathfrak{g},\mathfrak{l})_q$-modules into $C(\mathfrak{g},\mathfrak{l})_q$ given by (2) is an isomorphism of categories.

[4]In what follows $U_q\mathfrak{g}$ and $F(\mathfrak{l})_q$ are identified to their images under the embedding into $F(\mathfrak{g},\mathfrak{l})_q$.

We are going to show that the algebra $F(\mathfrak{g},\mathfrak{l})_q$ and the embeddings $U_q\mathfrak{g} \hookrightarrow F(\mathfrak{g},\mathfrak{l})_q$, $F(\mathfrak{l})_q \hookrightarrow F(\mathfrak{g},\mathfrak{l})_q$ are essentially uniquely determined by (F1) – (F4), i.e. there exists a unique isomorphism of algebras that respects the embeddings.

Assign to any $U_q\mathfrak{l}$-module V a $U_q\mathfrak{g}$-module $P(V) = U_q\mathfrak{g} \otimes_{U_q\mathfrak{l}} V$. The following result is well-known in the classical case $q = 1$ (see [7]). It is proved in the Appendix (see Corollary A3).

Lemma 2.1. *1. If $V \in C(\mathfrak{l},\mathfrak{l})_q$ then $P(V)$ is a projective object in $C(\mathfrak{g},\mathfrak{l})_q$.*

2. For any $V \in C(\mathfrak{g},\mathfrak{l})_q$, the map

$$P(V) \mapsto V, \qquad \xi \otimes v \mapsto \xi v, \qquad \xi \in U_q\mathfrak{g}, \quad v \in V,$$

is a surjective morphism of $U_q\mathfrak{g}$-modules.

The projective objects $P(V)$ of the category $C(\mathfrak{g},\mathfrak{l})_q$ are called standard projective objects.

Introduce the $U_q\mathfrak{g}$-module $V^{\text{univ}} = P(L^{\text{univ}})$.

Corollary 2.2. *In the category of $U_q\mathfrak{l}$-modules one has*

$$V^{\text{univ}} = \bigoplus_{\lambda \in P_+^{\mathbb{L}}} V_\lambda^{\text{univ}},$$

where the $U_q\mathfrak{l}$-modules V_λ^{univ} are multiples of $L(\mathfrak{l},\lambda)$.

Corollary 2.3. *For any module $M \in C(\mathfrak{g},\mathfrak{l})_q$ there exists a surjective morphism $\widetilde{V}^{\text{univ}} \to M$ in the category $C(\mathfrak{g},\mathfrak{l})_q$, with $\widetilde{V}^{\text{univ}}$ being a multiple of the module V^{univ}.*

Let us introduce the notation $\mathcal{P}_\lambda$ for the projection in V^{univ} onto the $U_q\mathfrak{l}$-isotypic component V_λ^{univ} parallel to the sum of all other $U_q\mathfrak{l}$-isotypic components.

The category $C(\mathfrak{g},\mathfrak{l})_q$ and the category of approximately unital $R(\mathfrak{g},\mathfrak{l})_q$-modules are closed under the operations of direct sums, passage to submodules and factor-modules. This allows one, using Lemma 2.1 and Corollary 2.3, to prove the following statement, which implies uniqueness of $F(\mathfrak{g},\mathfrak{l})_q$ and the embeddings $U_q\mathfrak{g} \hookrightarrow F(\mathfrak{g},\mathfrak{l})_q$, $F(\mathfrak{l})_q \hookrightarrow F(\mathfrak{g},\mathfrak{l})_q$.

Proposition 2.4. *Consider the algebra $F(\mathfrak{g},\mathfrak{l})_q$ and the embeddings $U_q\mathfrak{g} \hookrightarrow F(\mathfrak{g},\mathfrak{l})_q$, $F(\mathfrak{l})_q \hookrightarrow F(\mathfrak{g},\mathfrak{l})_q$ with the properties (F1) – (F4). Then the following statements hold.*

1. The representation of $U_q\mathfrak{g}$ in V^{univ} determines an approximately unital representation of $R(\mathfrak{g},\mathfrak{l})_q$ and a faithful representation of $F(\mathfrak{g},\mathfrak{l})_q$ in V^{univ}.

2. The image of $F(\mathfrak{g},\mathfrak{l})_q$ under the embedding into $\operatorname{End} V^{\text{univ}}$ is the subalgebra of $\operatorname{End} V^{\text{univ}}$ generated by elements of $U_q\mathfrak{g} \subset \operatorname{End} V^{\text{univ}}$ and projections $\mathcal{P}_\lambda$, $\lambda \in P_+^{\mathbb{L}}$.

Now we return to the construction of the algebra $F(\mathfrak{g},\mathfrak{l})_q$. In what follows we will identify the algebras $U_q\mathfrak{g}$, $F(\mathfrak{l})_q$ with their images under the embeddings into End V^{univ}.[5] In particular,

$$P_\lambda \mapsto \mathcal{P}_\lambda, \quad \lambda \in P_+^{\mathbb{L}}.$$

The following auxiliary statement is proved in the Appendix.

Lemma 2.5. *Given any $\xi \in U_q\mathfrak{g}$, $\lambda \in P_+^{\mathbb{L}}$, there exists a finite subset $\Lambda \subset P_+^{\mathbb{L}}$ such that*

$$\xi \mathcal{P}_\lambda = \chi_\Lambda \xi \mathcal{P}_\lambda, \qquad \mathcal{P}_\lambda \xi = \mathcal{P}_\lambda \xi \chi_\Lambda.$$

Consider the smallest subalgebra $F(\mathfrak{g},\mathfrak{l})_q \subset \operatorname{End} V^{\mathrm{univ}}$ that contains all the elements of $U_q\mathfrak{g}$ and all the projections $\mathcal{P}_\lambda$, $\lambda \in P_+^{\mathbb{L}}$.

Theorem 2.6. *The algebra $F(\mathfrak{g},\mathfrak{l})_q$ along with the embeddings $U_q\mathfrak{g} \hookrightarrow F(\mathfrak{g},\mathfrak{l})_q$, $F(\mathfrak{l})_q \hookrightarrow F(\mathfrak{g},\mathfrak{l})_q$, satisfy (F1) – (F4).*

Proof. (F1) is obviously satisfied.

Let us prove that $\{\chi_\Lambda\}$ is a left approximate identity in $R(\mathfrak{g},\mathfrak{l})_q$. Every $a \in R(\mathfrak{g},\mathfrak{l})_q$ has the form $a = \sum_{i=1}^{N(a)} \xi_i \mathcal{P}_{\lambda_i} \eta_i$, with $\xi_i \in U_q\mathfrak{g}$, $\eta_i \in F(\mathfrak{g},\mathfrak{l})_q$, $\lambda_i \in P_+^{\mathbb{L}}$. It follows from Lemma 2.5 that $\chi_\Lambda \xi_i \mathcal{P}_{\lambda_i} = \xi_i \mathcal{P}_{\lambda_i}$ for some finite subset $\Lambda \subset P_+^{\mathbb{L}}$ and all $i = 1, 2, \ldots, N(a)$. Hence $\chi_\Lambda a = a$. One can prove in a similar way that $\{\chi_\Lambda\}$ is a right approximate identity in $R(\mathfrak{g},\mathfrak{l})_q$. Hence (F2) is satisfied.

Consider an approximately unital $R(\mathfrak{g},\mathfrak{l})_q$-module V. Given any vector $v \in V$, there exists such finite subset $\Lambda \subset P_+^{\mathbb{L}}$ that $\chi_\Lambda v = v$. Thus for all $\Lambda' \supset \Lambda$ we have

$$(\xi \chi_{\Lambda'})v = (\xi \chi_{\Lambda'})\chi_\Lambda v = (\xi \chi_{\Lambda'} \chi_\Lambda)v = (\xi \chi_\Lambda)v,$$

and hence the element

$$\xi v \stackrel{\mathrm{def}}{=} \lim_{\Lambda \uparrow P_+^{\mathbb{L}}} (\xi \chi_\Lambda)v$$

is well defined. It is easy to prove that $(\xi\eta)v = \xi(\eta v)$ for any $\xi, \eta \in F(\mathfrak{g},\mathfrak{l})_q$, $v \in V$. In fact, there exist such finite subsets $\Lambda', \Lambda'' \subset P_+^{\mathbb{L}}$ that

$$\chi_{\Lambda'} v = v, \qquad \chi_{\Lambda''} \eta \chi_{\Lambda'} = \eta \chi_{\Lambda'},$$

because $\eta \chi_{\Lambda'} \in R(\mathfrak{g},\mathfrak{l})_q$ and $\{\chi_\Lambda\}$ is an approximate identity. Hence,

$$(\xi\eta)v = \lim_{\Lambda \uparrow P_+^{\mathbb{L}}} (\xi \eta \chi_\Lambda)v = \xi(\eta \chi_{\Lambda'})v = (\xi \chi_{\Lambda''})(\eta \chi_{\Lambda'})v = \lim_{\Lambda \uparrow P_+^{\mathbb{L}}} (\xi \chi_\Lambda)(\eta \chi_{\Lambda'})v =$$
$$= \xi(\eta \chi_{\Lambda'} v) = \xi(\eta v).$$

It follows that V is an $F(\mathfrak{g},\mathfrak{l})_q$-module, and the the first of the conditions (F3) is satisfied. The second condition, which requires the existence of a faithful representation as above, obviously holds.

[5] The representation of $U_q\mathfrak{g}$ in V^{univ} is faithful since for any weight finite dimensional $U_q\mathfrak{g}$-module W there exists a surjective morphism of $U_q\mathfrak{g}$-modules $V^{\mathrm{univ}} \to W$ (Corollary 2.3). The representation of $F(\mathfrak{l})_q$ in V^{univ} is faithful since $L^{\mathrm{univ}} \hookrightarrow V^{\mathrm{univ}}$.

The arguments below show that the $F(\mathfrak{g},\mathfrak{l})_q$-modules in question are locally $U_q\mathfrak{l}$-finite dimensional and weight:

$$\dim(U_q\mathfrak{l}\, v)=\dim U_q\mathfrak{l}(\chi_{\Lambda'}v)=\dim(U_q\mathfrak{l}\,\chi_{\Lambda'})v\le\dim R(\mathfrak{l})_q v<\infty.$$

Thus we have constructed a functor from the category of approximately unital $R(\mathfrak{g},\mathfrak{l})_q$-modules into $C(\mathfrak{g},\mathfrak{l})_q$. What remains is to construct an inverse functor.

Suppose $V\in C(\mathfrak{g},\mathfrak{l})_q$ and let $V=\bigoplus\limits_{\mu\in P_+^{\mathbb{L}}}V_\mu$ be a decomposition of V into a sum of $U_q\mathfrak{l}$-isotypic components. Let π be the representation of $U_q\mathfrak{g}$ in V, and denote by Π_λ the projection in V onto the isotypic component of V_λ parallel to $\bigoplus\limits_{\substack{\mu\in P_+^{\mathbb{L}}\\ \mu\ne\lambda}}V_\mu$.

It suffices to prove existence and uniqueness of an extension $\widetilde{\pi}$ of π to $F(\mathfrak{g},\mathfrak{l})_q$ that possesses the following properties:

i) $\widetilde{\pi}(\mathcal{P}_\lambda)=\Pi_\lambda,\ \lambda\in P_+^{\mathbb{L}}$,

ii) $\widetilde{\pi}|_{R(\mathfrak{g},\mathfrak{l})_q}$ is an approximately unital representation of $R(\mathfrak{g},\mathfrak{l})_q$,

iii) $\pi(\xi)v=\lim\limits_{\Lambda\uparrow P_+^{\mathbb{L}}}\widetilde{\pi}(\xi\chi_\Lambda)v,\ v\in V,\ \xi\in U_q\mathfrak{g}$.

Uniqueness of such extension is obvious. To prove its existence, we consider subsequently the following cases:

1. $V=V^{\mathrm{univ}}$;
2. $V=\widetilde{V}^{\mathrm{univ}}$, with $\widetilde{V}^{\mathrm{univ}}$ being a multiple of the $U_q\mathfrak{g}$-module V^{univ};
3. V is a submodule of $\widetilde{V}^{\mathrm{univ}}$;
4. V is a standard projective object in $C(\mathfrak{g},\mathfrak{l})_q$;
5. (the general case) $V\in C(\mathfrak{g},\mathfrak{l})_q$.

In the case 1) the desired statement follows from the definition of $R(\mathfrak{g},\mathfrak{l})_q$. A passage from 1) to 2) and to 3) is obvious. To pass from 3) to 4), it is sufficient to use the fact that every $U_q\mathfrak{g}$-module $P(V)$, $V\in C(\mathfrak{l},\mathfrak{l})_q$, admits an embedding into a multiple $\widetilde{V}^{\mathrm{univ}}$ of the $U_q\mathfrak{g}$-module V^{univ}. This is because every locally finite dimensional $U_q\mathfrak{l}$-module admits an embedding into a $U_q\mathfrak{l}$-module $\widetilde{L}^{\mathrm{univ}}$, which is a multiple of L^{univ} and has the property $P(\widetilde{L}^{\mathrm{univ}})=\widetilde{V}^{\mathrm{univ}}$. Finally, to pass from 4) to 5), one can apply the existence of a surjective morphism $P(V)\to V$ in the category $C(\mathfrak{g},\mathfrak{l})_q$. ■

Recall that the Weyl group W is generated by simple reflections $s_1,s_2,\dots,s_l$ and acts on the weight lattice P.

Proposition 2.7. *1. There exists a unique one-dimensional representation $\widetilde{\varepsilon}$ of $F(\mathfrak{g},\mathfrak{l})_q$ with*

$$\widetilde{\varepsilon}(\xi)=\varepsilon(\xi),\quad \xi\in U_q\mathfrak{g};\qquad \widetilde{\varepsilon}(P_\lambda)=\begin{cases}1, & \lambda=0,\\ 0, & \lambda\in P_+^{\mathbb{L}}\setminus\{0\}.\end{cases}$$

2. There exists a unique anti-automorphism $\widetilde{S}$ of the algebra $F(\mathfrak{g},\mathfrak{l})_q$ such that

$$\widetilde{S}(\xi)=S(\xi),\quad \xi\in U_q\mathfrak{g};\qquad \widetilde{S}(P_\lambda)=P_{-w_0^{\mathbb{L}}\lambda},\quad \lambda\in P_+^{\mathbb{L}},$$

where $w_0^{\mathbb{L}}$ is the longest element of the Weyl group $W_{\mathbb{L}}\subset W$ generated by the simple reflections s_i, $i\in\mathbb{L}$.

Proof. Uniqueness of $\widetilde{\varepsilon}$ and $\widetilde{S}$ is obvious. Existence of $\widetilde{\varepsilon}$ has been demonstrated in the proof of Theorem 2.6. Turn to proving existence of $\widetilde{S}$. In the special case $\mathbb{G}=\mathbb{L}$ it follows from the definition of the algebra $F(\mathfrak{l})_q\hookrightarrow \operatorname{End}L^{\mathrm{univ}}$ and the well-known isomorphism [2, p. 168]

$$L(\mathfrak{l},\lambda)^*\xrightarrow{\sim}L(\mathfrak{l},-w_0^{\mathbb{L}}\lambda),\qquad v(\mathfrak{l},\lambda)\mapsto v(\mathfrak{l},w_0^{\mathbb{L}}\lambda).$$

To pass from the special case $\mathbb{G}=\mathbb{L}$ to the general case $\mathbb{G}\supset\mathbb{L}$, consider the algebra $F(\mathfrak{g},\mathfrak{l})_q^{\mathrm{op}}$ which is derived from $F(\mathfrak{g},\mathfrak{l})_q$ by replacement of the multiplication by the opposite one. $U_q\mathfrak{g}$ can be embedded into $F(\mathfrak{g},\mathfrak{l})_q^{\mathrm{op}}$:

$$U_q\mathfrak{g}\hookrightarrow F(\mathfrak{g},\mathfrak{l})_q^{\mathrm{op}},\qquad \xi\mapsto S(\xi).$$

The algebra $F(\mathfrak{l})_q$ also admits an embedding into $F(\mathfrak{g},\mathfrak{l})_q^{\mathrm{op}}$:

$$U_q\mathfrak{l}\hookrightarrow F(\mathfrak{g},\mathfrak{l})_q^{\mathrm{op}},\quad \xi\mapsto S(\xi);\qquad P_\lambda\mapsto P_{-w_0^{\mathbb{L}}\lambda}.$$

It is easy to verify (F1) – (F4) for this pair of embeddings. What remains is to use the uniqueness of such pair, Proposition 2.4. ■

The category $C(\mathfrak{g},\mathfrak{l})_q$ is canonically isomorphic to the category of approximately unital $R(\mathfrak{g},\mathfrak{l})_q$-modules. In what follows we identify these categories.

Consider the Hopf subalgebra $U_q\mathfrak{q}_{\mathbb{L}}^+\subset U_q\mathfrak{g}$ generated by

$$E_i,\quad i\in\mathbb{G};\qquad F_j,\quad j\in\mathbb{L};\qquad K_m^{\pm1},\quad m=1,2,\ldots,l.$$

and the Hopf subalgebra $U_q\mathfrak{q}_{\mathbb{L}}^-\subset U_q\mathfrak{g}$ generated by

$$E_i,\quad i\in\mathbb{L};\qquad F_j,\quad j\in\mathbb{G};\qquad K_m^{\pm1},\quad m=1,2,\ldots,l.$$

The following statement is well known in the classical case $q=1$ (see [7, p. 90]).

Proposition 2.8. *The linear maps*

$$U_q\mathfrak{g}\otimes_{U_q\mathfrak{l}}R(\mathfrak{l})_q\to R(\mathfrak{g},\mathfrak{l})_q,\qquad \xi\otimes r\mapsto\xi r,\tag{3}$$

$$R(\mathfrak{l})_q\otimes_{U_q\mathfrak{l}}U_q\mathfrak{g}\to R(\mathfrak{g},\mathfrak{l})_q,\qquad r\otimes\xi\mapsto r\xi,$$

are bijective.

Proof. By Proposition 2.7, it suffices to consider the linear map (3). Let us prove that it is onto. Note that for any $\lambda \in P_+^{\mathbb{L}}$ and any finite subset $\Lambda \subset P_+^{\mathbb{L}}$ containing λ there exists an element z in the center $Z(U_q\mathfrak{l})$ of the algebra $U_q\mathfrak{l}$ such that $P_\lambda = z\chi_\Lambda$. On the other hand, for any $\lambda \in P_+^{\mathbb{L}}$, $\xi \in U_q\mathfrak{g}$ there exist finite subsets $\Lambda', \Lambda'' \ni \lambda$ such that

$$P_\lambda \xi = P_\lambda \xi \chi_{\Lambda'}, \qquad \xi \chi_{\Lambda'} = \chi_{\Lambda''} \xi \chi_{\Lambda'}.$$

Hence $P_\lambda \xi = (P_\lambda \chi_{\Lambda''})\xi\chi_{\Lambda'} = z\chi_{\Lambda''}\xi\chi_{\Lambda'} = z\xi\chi_{\Lambda'}$. Thus for any $\xi \in U_q\mathfrak{g}$ and any finite subset $\Lambda \subset P_+^{\mathbb{L}}$ there exist such $\widetilde{\xi} \in U_q\mathfrak{g}$ and a finite subset $\widetilde{\Lambda} \subset P_+^{\mathbb{L}}$ that $\chi_\Lambda \xi = \widetilde{\xi}\chi_{\widetilde{\Lambda}}$. This can be readily used to prove that the linear map (3) is onto.

Prove that it is injective. One can use well-known results on bases in quantum universal enveloping algebras [6, Chapter 8] to derive a decomposition $U_q\mathfrak{g} = U_q\mathfrak{q}_{\mathbb{L}}^- \otimes_{U_q\mathfrak{l}} U_q\mathfrak{q}_{\mathbb{L}}^+$. It follows from this decomposition and Lemma A1 that $U_q\mathfrak{g}$ is a free right $U_q\mathfrak{l}$-module. Choose a free basis $\{e_i\}$. Let a be a non-zero element of $U_q\mathfrak{g} \otimes_{U_q\mathfrak{l}} R(\mathfrak{l})_q$. It has the form

$$a = \sum_i e_i \otimes r_i, \qquad r_i \in R(\mathfrak{l})_q.$$

Let $\widetilde{a}$ be the image of a under the map (3). We treat it as an element of $\operatorname{End} V^{\mathrm{univ}}$. It suffices to prove that the restriction of the linear map $\widetilde{a}$ to the subspace $\{1 \otimes v|\ v \in L^{\mathrm{univ}}\}$ is non-zero. It is easy to verify that $\widetilde{a}(1 \otimes v) = \sum_i e_i \otimes r_i v$ for any $v \in L^{\mathrm{univ}}$. It remains to use our choice of $\{e_i\}$ and the fact that the representation of $U_q\mathfrak{l}$ in L^{univ} is faithful. ■

3. The functors ind and Π

Consider two pairs of subsets

$$\mathbb{L} \subset \mathbb{G} \subset \{1, 2, \ldots, l\}, \qquad \mathbb{L}_1 \subset \mathbb{G}_1 \subset \{1, 2, \ldots, l\},$$

with $\mathbb{L}_1 \subset \mathbb{L}$, $\mathbb{G}_1 \subset \mathbb{G}$. Obviously, one has embeddings of the associated Hopf subalgebras

$$U_q\mathfrak{l}_1 \subset U_q\mathfrak{l}, \qquad U_q\mathfrak{g}_1 \subset U_q\mathfrak{g}.$$

$R(\mathfrak{g}, \mathfrak{l})_q$ is a left ideal in $F(\mathfrak{g}, \mathfrak{l})_q$ and hence is a left $U_q\mathfrak{g}_1$-module. We claim that $R(\mathfrak{g}, \mathfrak{l})_q$ is an approximately unital left $R(\mathfrak{g}_1, \mathfrak{l}_1)_q$-module. In fact, $R(\mathfrak{g}, \mathfrak{l})_q$ is an approximately unital left $R(\mathfrak{g}, \mathfrak{l})_q$-module, hence a $U_q\mathfrak{g}$-module of the category $C(\mathfrak{g}, \mathfrak{l})_q$ given by (1). Since $U_q\mathfrak{l}_1 \subset U_q\mathfrak{l}$, the module $R(\mathfrak{g}, \mathfrak{l})_q$ is also in the category $C(\mathfrak{g}, \mathfrak{l}_1)_q$. Finally, in view of $U_q\mathfrak{g}_1 \subset U_q\mathfrak{g}$, we are inside the category $C(\mathfrak{g}_1, \mathfrak{l}_1)_q$, which is equivalent to our claim. In a similar way, one can prove that $R(\mathfrak{g}, \mathfrak{l})_q$ is a right $R(\mathfrak{g}_1, \mathfrak{l}_1)_q$-module.[6]

Throughout the rest of this section we follow [7] and replace the groups involved therein with the corresponding quantum universal enveloping algebras. Introduce the functor $P_{\mathfrak{g}_1,\mathfrak{l}_1}^{\mathfrak{g},\mathfrak{l}}$ from the category $C(\mathfrak{g}_1, \mathfrak{l}_1)_q$ to the category $C(\mathfrak{g}, \mathfrak{l})_q$ by defining it on objects from $C(\mathfrak{g}_1, \mathfrak{l}_1)_q$ as follows:

$$P_{\mathfrak{g}_1,\mathfrak{l}_1}^{\mathfrak{g},\mathfrak{l}}(Z) = R(\mathfrak{g}, \mathfrak{l})_q \otimes_{R(\mathfrak{g}_1,\mathfrak{l}_1)_q} Z.$$

[6] One can see from the proof that $R(\mathfrak{g}, \mathfrak{l})_q$ is a $R(\mathfrak{g}_1, \mathfrak{l}_1)_q$-bimodule.

The action on morphisms is defined in an obvious way.

$C(\mathfrak{g}_1,\mathfrak{l}_1)_q$ has enough projectives and the functor $P^{\mathfrak{g},\mathfrak{l}}_{\mathfrak{g}_1,\mathfrak{l}_1}$ is covariant and right exact.[7] Hence one has well defined derived functors: $\left(P^{\mathfrak{g},\mathfrak{l}}_{\mathfrak{g}_1,\mathfrak{l}_1}\right)_j$, $j\in\mathbb{Z}_+$, from $C(\mathfrak{g}_1,\mathfrak{l}_1)_q$ to $C(\mathfrak{g},\mathfrak{l})_q$.

Consider the two special cases: $\mathfrak{l}_1=\mathfrak{l}$ and $\mathfrak{g}_1=\mathfrak{g}$. Start from the first one. Let $\mathrm{ind}^{\mathfrak{g},\mathfrak{l}}_{\mathfrak{g}_1,\mathfrak{l}}$ be the functor from the category $C(\mathfrak{g}_1,\mathfrak{l})_q$ to the category $C(\mathfrak{g},\mathfrak{l})_q$ defined on objects as follows:

$$\mathrm{ind}^{\mathfrak{g},\mathfrak{l}}_{\mathfrak{g}_1,\mathfrak{l}}(Z)=U_q\mathfrak{g}\otimes_{U_q\mathfrak{g}_1}Z.$$

The action on morphisms is defined in an obvious way. Just as in the classical case $q=1$, one gets an isomorphism of functors $P^{\mathfrak{g},\mathfrak{l}}_{\mathfrak{g}_1,\mathfrak{l}}$ and $\mathrm{ind}^{\mathfrak{g},\mathfrak{l}}_{\mathfrak{g}_1,\mathfrak{l}}$. Describe briefly this construction. The functor $\mathrm{ind}^{\mathfrak{g},\mathfrak{l}}_{\mathfrak{g}_1,\mathfrak{l}}$ is left adjoint to the forgetful functor $\mathcal{F}^{\mathfrak{g}_1,\mathfrak{l}}_{\mathfrak{g},\mathfrak{l}}:C(\mathfrak{g},\mathfrak{l})_q\to C(\mathfrak{g}_1,\mathfrak{l})_q$ determined by the embedding $U_q\mathfrak{g}_1\to U_q\mathfrak{g}$. On the other hand, $P^{\mathfrak{g},\mathfrak{l}}_{\mathfrak{g}_1,\mathfrak{l}}$ is left adjoint to the functor $(\mathcal{F}^\vee)^{\mathfrak{g}_1,\mathfrak{l}}_{\mathfrak{g},\mathfrak{l}}$, to be defined below (cf. [7, Proposition 2.34]). The functor $(\mathcal{F}^\vee)^{\mathfrak{g}_1,\mathfrak{l}}_{\mathfrak{g},\mathfrak{l}}:C(\mathfrak{g},\mathfrak{l})_q\to C(\mathfrak{g}_1,\mathfrak{l})_q$ is defined on objects of the category $C(\mathfrak{g},\mathfrak{l})_q$ as follows:

$$(\mathcal{F}^\vee)^{\mathfrak{g}_1,\mathfrak{l}}_{\mathfrak{g},\mathfrak{l}}(X)=\mathrm{Hom}_{R(\mathfrak{g},\mathfrak{l})_q}(R(\mathfrak{g},\mathfrak{l})_q,X)_{\mathfrak{l}}.$$

A structure of $U_q\mathfrak{g}_1$-module in $\mathrm{Hom}_{R(\mathfrak{g},\mathfrak{l})_q}(R(\mathfrak{g},\mathfrak{l})_q,X)$ is imposed via the structure of right $U_q\mathfrak{g}_1$-module in $R(\mathfrak{g},\mathfrak{l})_q$, and the subscript $\mathfrak{l}$ stands for distinguishing the maximal submodule in the category $C(\mathfrak{g}_1,\mathfrak{l})_q$. The action of $(\mathcal{F}^\vee)^{\mathfrak{g}_1,\mathfrak{l}}_{\mathfrak{g},\mathfrak{l}}$ on morphisms is defined in an obvious way. What remains is to construct an isomorphism of functors $\mathcal{F}^{\mathfrak{g}_1,\mathfrak{l}}_{\mathfrak{g},\mathfrak{l}}\xrightarrow{\sim}(\mathcal{F}^\vee)^{\mathfrak{g}_1,\mathfrak{l}}_{\mathfrak{g},\mathfrak{l}}$ (cf. [7, Proposition 2.33]). Let $X\in C(\mathfrak{g},\mathfrak{l})_q$ and $x\in X$. Associate to every x a morphism of $R(\mathfrak{g},\mathfrak{l})_q$-modules given by

$$R(\mathfrak{g},\mathfrak{l})_q\to X,\qquad r\mapsto rx.$$

One can verify that the map $X\to\mathrm{Hom}_{R(\mathfrak{g},\mathfrak{l})_q}(R(\mathfrak{g},\mathfrak{l})_q,X)$, which arises this way, is a morphism of $U_q\mathfrak{g}_1$-modules and provides the desired isomorphism of functors.

An important consequence is the observation that for any $V\in C(\mathfrak{l},\mathfrak{l})_q$ the standard projective object $P(V)$ in the category $C(\mathfrak{g},\mathfrak{l})_q$ is canonically isomorphic to $P^{\mathfrak{g},\mathfrak{l}}_{\mathfrak{l},\mathfrak{l}}(V)$.

In the second special case $\mathfrak{g}_1=\mathfrak{g}$ the functor in question is called the Bernstein functor and it is denoted by $\Pi^{\mathfrak{g},\mathfrak{l}}_{\mathfrak{g},\mathfrak{l}_1}$:

$$\Pi\equiv\Pi^{\mathfrak{g},\mathfrak{l}}_{\mathfrak{g},\mathfrak{l}_1}(Z)=R(\mathfrak{g},\mathfrak{l})_q\otimes_{R(\mathfrak{g},\mathfrak{l}_1)_q}Z,\qquad Z\in C(\mathfrak{g}_1,\mathfrak{l}_1)_q.$$

In the classical case $q=1$ the derived functors Π_j are crucial in constructing unitarizable Harish-Chandra modules via the Vogan-Zuckerman cohomological induction [7]. Turn to describing a quantum analogue for this method.

4. A quantum analogue for cohomological induction

We assume in the sequel that $\mathbb{G}=\{1,2,\ldots,l\}$, and hence $U_q\mathfrak{g}=U_q$.

[7]See [7, p. 840]

Recall (see [4, 8]) a definition of the pairs $(\mathfrak{g}, \mathfrak{k})$ used in a construction of bounded symmetric domains via the Harish-Chandra embedding. Here $\mathfrak{g}$ is a simple complex Lie algebra of type A, B, C, D, E_6, E_7. A Hopf subalgebra $U_q\mathfrak{k} \subset U_q\mathfrak{g}$ is generated by E_i, F_i, $i \in \mathbb{K}$; $K_j^{\pm 1}$, $j = 1, 2, \dots, l$, with $\mathbb{K} = \{1, 2, \dots, l\} \setminus \{l_0\}$. We assume also that the simple root α_{l_0} has coefficient 1 in the decomposition of the maximal root

$$\delta = \sum_{i=1}^{l} n_i \alpha_i, \qquad n_{l_0} = 1. \tag{4}$$

In what follows the notation $(\mathfrak{g}, \mathfrak{k})$ will stand only for such pairs.

To clarify the precise nature of condition (4), equip the Lie algebra $\mathfrak{g}$ with a grading as follows: $\deg E_{l_0} = 1$, $\deg F_{l_0} = -1$, $\deg H_{l_0} = 0$, $\deg E_j = \deg F_j = \deg H_j = 0$ for $j \neq l_0$. The automorphism θ of $\mathfrak{g}$ given by the formula

$$\theta(\xi) = (-1)^{\deg \xi} \xi$$

is an involution. Now (4) implies that

$$\mathfrak{g} = \mathfrak{p}^- \oplus \mathfrak{k} \oplus \mathfrak{p}^+, \qquad \mathfrak{p}^\pm = \{\xi \in \mathfrak{g} |\ \deg \xi = \pm 1\}.$$

It is worthwhile to note that the pairs $(\mathfrak{g}, \mathfrak{k})$ in question are complexifications of the pairs $(\mathfrak{g}_0, \mathfrak{k}_0)$, with $\mathfrak{g}_0$ being the Lie subalgebra of the automorphism group of an irreducible bounded symmetric domain, and $\mathfrak{k}_0$ being the Lie algebra of the stabilizer of a point in this domain [4].

Equip $U_q\mathfrak{g}$ with a structure of Hopf $*$-algebra via the involution $*$ defined as follows:

$$E_j^* = \begin{cases} -K_j F_j, & j = l_0 \\ K_j F_j, & j \neq l_0 \end{cases}; \qquad F_j^* = \begin{cases} -E_j K_j^{-1}, & j = l_0 \\ E_j K_j^{-1}, & j \neq l_0 \end{cases};$$

$$(K_j^{\pm 1})^* = K_j^{\pm 1}, \qquad j = 1, 2, \dots, l.$$

A module V over a Hopf $*$-algebra A is said to be unitarizable if it admits a positive definite invariant form $(\cdot, \cdot)$:

$$(av_1, v_2) = (v_1, a^* v_2), \qquad a \in A,\ v_1, v_2 \in V.$$

The cohomological induction is among the tools for constructing unitarizable modules of the category $C(\mathfrak{g}, \mathfrak{k})$ in the case $q = 1$. Describe a q-analogue for this method.

Let $\mathbb{L} \subset \{1, 2, \dots, l\}$, $\mathbb{L} \not\subset \mathbb{K}$, and $U_q\mathfrak{l}$ is the Hopf subalgebra corresponding to the subset $\mathbb{L}$. Obviously, $U_q\mathfrak{l}$ inherits the structure of Hopf $*$-algebra. We intend to derive a unitarizable $U_q\mathfrak{g}$-module of the category $C(\mathfrak{g}, \mathfrak{k})_q$ starting from a unitarizable $U_q\mathfrak{l}$-module of the category $C(\mathfrak{l}, \mathfrak{l} \cap \mathfrak{k})_q$. [8]

If $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_l) \in P$ and $\lambda_j = 0$ for all $j \in \mathbb{L}$, one has the following well defined one-dimensional $U_q\mathfrak{l}$-module $\mathbb{C}_\lambda$:

$$E_j \mathbf{1} = F_j \mathbf{1} = 0, \quad j \in \mathbb{L}; \qquad K_j^{\pm 1} \mathbf{1} = q_j^{\pm \lambda_j} \mathbf{1}, \quad j = 1, 2, \dots, l.$$

[8] $U_q(\mathfrak{l} \cap \mathfrak{k})$ is a Hopf subalgebra corresponding to the subset $\mathbb{L} \cap \mathbb{K}$.

As an important example of such linear functional one should mention the difference $\rho_{\mathfrak{u}} = \rho - \rho_{\mathfrak{l}}$ between the half-sum ρ of positive roots of the Lie algebra $\mathfrak{g}$ and the half-sum $\rho_{\mathfrak{l}}$ of positive roots of the Lie algebra $\mathfrak{l}$:

$$\rho_{\mathfrak{u}}(H_j) = \begin{cases} 1, & j \notin \mathbb{L}, \\ 0, & j \in \mathbb{L}. \end{cases}$$

Every $U_q\mathfrak{l}$-module $Z \in C(\mathfrak{l}, \mathfrak{l} \cap \mathfrak{k})_q$ determines a $U_q\mathfrak{l}$-module $Z^{\#} = Z \otimes \mathbb{C}_{2\rho_{\mathfrak{u}}} \in C(\mathfrak{l}, \mathfrak{l} \cap \mathfrak{k})_q$. Equip it with a structure of $U_q\mathfrak{q}_{\mathbb{L}}^-$-module via the surjective morphism of algebras $U_q\mathfrak{q}_{\mathbb{L}}^- \to U_q\mathfrak{l}$:

$$F_i \mapsto \begin{cases} F_i, & i \in \mathbb{L} \\ 0, & i \in \{1, 2, \ldots, l\} \setminus \mathbb{L} \end{cases}, \qquad E_j \mapsto E_j, \quad K_j^{\pm 1} \mapsto K_j^{\pm 1},\ j \in \{1, 2, \ldots, l\}.$$

The generalized Verma module $\mathrm{ind}_{\mathfrak{q}_{\mathbb{L}}^-}^{\mathfrak{g}} Z^{\#} \stackrel{\mathrm{def}}{=} U_q\mathfrak{g} \otimes_{U_q\mathfrak{q}_{\mathbb{L}}^-} Z^{\#}$ belongs to the category $C(\mathfrak{g}, \mathfrak{l})_q$ (see Lemma A1). Hence the equality

$$\mathfrak{L}_j(Z) = \left(\Pi_{\mathfrak{g}, \mathfrak{l} \cap \mathfrak{k}}^{\mathfrak{g}, \mathfrak{k}}\right)_j \left(\mathrm{ind}_{\mathfrak{q}_{\mathbb{L}}^-}^{\mathfrak{g}}(Z^{\#})\right)$$

determines functors $\mathfrak{L}_j$, $j \in \mathbb{Z}_+$, from $C(\mathfrak{l}, \mathfrak{l} \cap \mathfrak{k})_q$ to $C(\mathfrak{g}, \mathfrak{k})_q$.

In the classical case $q = 1$ under suitable dominance assumptions on Z the only non-zero modules $\mathfrak{L}_s(Z)$ are those with $s = \frac{1}{2}(\dim \mathfrak{k} - \dim(\mathfrak{k} \cap \mathfrak{l}))$ [7, p. 369]. So, a particular interest is in considering the $U_q\mathfrak{g}$-modules

$$A_{\mathfrak{q}}(\lambda) \stackrel{\mathrm{def}}{=} \mathfrak{L}_s(\mathbb{C}_\lambda), \qquad s = \frac{1}{2}(\dim \mathfrak{k} - \dim(\mathfrak{k} \cap \mathfrak{l})),$$

with $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l) \in P$ and $\lambda_j = 0$ for $j \in \mathbb{L}$.

Appendix. Proofs of the Lemmas

Let $\mathbb{G} \supset \mathbb{L}$ be subsets of $\{1, 2, \ldots, l\}$. Equip $U_q\mathfrak{g}$ with a grading by setting

$$\deg E_j = \begin{cases} 1, & j \in \mathbb{G} \setminus \mathbb{L} \\ 0, & j \in \mathbb{L} \end{cases}, \qquad \deg F_j = \begin{cases} -1, & j \in \mathbb{G} \setminus \mathbb{L} \\ 0, & j \in \mathbb{L} \end{cases},$$

$$\deg K_j^{\pm 1} = 0, \qquad j = 1, 2, \ldots, l.$$

Let $\left(U_q\mathfrak{q}_{\mathbb{L}}^{\pm}\right)_j = \left\{\xi \in U_q\mathfrak{q}_{\mathbb{L}}^{\pm} \,\middle|\, \deg \xi = j\right\}$. Obviously, $\left(U_q\mathfrak{q}_{\mathbb{L}}^{\pm}\right)_0 = U_q\mathfrak{l}$.

Lemma A1. *The homogeneous components $\left(U_q\mathfrak{q}_{\mathbb{L}}^{\pm}\right)_j$ are free left and free right $U_q\mathfrak{l}$-modules of finite rank.*

Proof. Recall that the subsets $\mathbb{L} \subset \mathbb{G} \subset \{1, 2, \ldots, l\}$ determine the subgroups $W_{\mathbb{L}} \subset W_{\mathbb{G}}$ of the Weyl group W. Those subgroups are exactly the Weyl groups of the Lie algebras $\mathfrak{l}$ and $\mathfrak{g}$, respectively. Let $w_{0,\mathbb{L}}$ (respectively, $w_{0,\mathbb{G}}$) be the longest element of the Weyl group $W_{\mathbb{L}}$ (respectively, $W_{\mathbb{G}}$).

We start from the larger subset $\mathbb{G}$. Choose a reduced expression

$$w_{0,\mathbb{G}} = s_{i_1} s_{i_2} s_{i_3} \dots s_{i_M}, \tag{5}$$

with s_{i_k} being the reflection corresponding to the simple root α_{i_k}. Consider the Lusztig automorphisms T_i, $i \in \mathbb{G}$, of the algebra $U_q\mathfrak{g}'$ generated by E_i, F_i, $K_i^{\pm 1}$, $i \in \mathbb{G}$, [6, Chapter 8]. They have the form

$$\begin{gathered} T_i(K_j) = K_j K_i^{-a_{ij}}, \\ T_i(E_i) = -F_i K_i, \qquad T_i(F_i) = -K_i^{-1} E_i, \\ T_i(E_j) = \sum_{r+s=-a_{ij}} \mathrm{const}\, E_i^s E_j E_i^r, \qquad i \neq j, \\ T_i(F_j) = \sum_{r+s=-a_{ij}} \mathrm{const}\, F_i^r F_j F_i^s, \qquad i \neq j. \end{gathered} \tag{6}$$

In a similar way, consider the algebra $U_q\mathfrak{l}'$ associated to $\mathbb{L}$. Let $W^{\mathbb{L}} = \{w \in W_{\mathbb{G}} |\ l(ws) > l(w)$ for all simple reflections $s \in W_{\mathbb{L}}\}$, where $l(w)$ stands for the length of w. One has $w_{0,\mathbb{G}} = w^{\mathbb{L}} w_{0,\mathbb{L}}$ with $w^{\mathbb{L}} \in W^{\mathbb{L}}$ and $l(w_{0,\mathbb{G}}) = l(w^{\mathbb{L}}) + l(w_{0,\mathbb{L}})$ [5, Proposition 1.10(c)]. This allows one to rewrite the reduced expression (5) in the form

$$w_{0,\mathbb{G}} = s_{i_1} s_{i_2} s_{i_3} \dots s_{i_{M'}} s_{i_{M'+1}} s_{i_{M'+2}} \dots s_{i_M}, \tag{7}$$

with $w_{0,\mathbb{L}} = s_{i_{M'+1}}, s_{i_{M'+2}}, \dots, s_{i_M}$ and $w^{\mathbb{L}} = s_{i_1} s_{i_2} s_{i_3} \dots s_{i_{M'}}$.

We are about to apply the Lusztig theorem [6, Theorem 8.24]. Observe that the special reduced expression (7), in view of the explicit form of the Lusztig automorphisms (6) implies that all the monomials

$$T_{i_1} T_{i_2} \cdots T_{i_{M'-1}}(E_{i_{M'}}^{a_{M'}}) \cdot T_{i_1} T_{i_2} \cdots T_{i_{M'-2}}(E_{i_{M'-1}}^{a_{M'-1}}) \cdot \ldots \cdot T_{i_1} T_{i_2}(E_{i_3}^{a_3}) \cdot T_{i_1}(E_{i_2}^{a_2}) \cdot E_{i_1}^{a_1}$$

are in $U_q\mathfrak{l}'$. It follows from the Lusztig theorem [6, Theorem 8.24] that the monomials

$$\begin{aligned} T_{i_1} T_{i_2} \cdots T_{i_{M-1}}(E_{i_M}^{a_M}) \cdot T_{i_1} T_{i_2} \cdots T_{i_{M-2}}(E_{i_{M-1}}^{a_{M-1}}) \cdot \ldots \\ \ldots \cdot T_{i_1} T_{i_2} \cdots T_{i_{M'+1}}(E_{i_{M'+2}}^{a_{M'+2}}) \cdot T_{i_1} T_{i_2} \cdots T_{i_{M'}}(E_{i_{M'+1}}^{a_{M'+1}}) \end{aligned} \tag{8}$$

with all $a_{i_k} \in \mathbb{Z}_+$, form a free basis in the right $U_q\mathfrak{l}$-module $U_q\mathfrak{q}_{\mathbb{L}}^+$. As one can readily separate out for each $j \in \mathbb{Z}_+$ finitely many such monomials that span the j-th homogeneous component, we thus get our claim for $(U_q\mathfrak{q}_{\mathbb{L}}^+)_j$ as a right $U_q\mathfrak{l}$-module. All other claims can be proved in a similar way. ■

Corollary A2. *Every $U_q\mathfrak{g}$-module V contains the largest submodule $V_\mathfrak{l}$ of the category $C(\mathfrak{g}, \mathfrak{l})_q$.*

Proof. Obviously, V possesses the largest weight submodule $V_\mathfrak{h}$. What remains is to prove that the subspace $V_\mathfrak{l} = \{v \in V_\mathfrak{h} |\ \dim(U_q\mathfrak{l}v) < \infty\}$ is a submodule of the $U_q\mathfrak{g}$-module V. Let $\xi \in \left(U_q\mathfrak{q}_{\mathbb{L}}^{\pm}\right)_j$, $v \in V_\mathfrak{l}$. It follows from Lemma A1 that for some $\{\eta_1, \eta_2, \dots, \eta_{N(j)}\}$

$$\left(U_q\mathfrak{q}_{\mathbb{L}}^{\pm}\right)_j = \sum_{k=1}^{N(j)} \eta_k U_q\mathfrak{l}.$$

Hence,

$$\dim(U_q\mathfrak{l}\,\xi v) \le \dim\left(\left(U_q\mathfrak{q}^{\pm}_{\mathbb{L}}\right)_j v\right) = \dim\left(\sum_{k=1}^{N(j)} \eta_k U_q\mathfrak{l}\, v\right) < \infty.$$

Thus $\xi v \in V_{\mathfrak{l}}$. ■

Corollary A3. *For any $V \in C(\mathfrak{l},\mathfrak{l})_q$, the module $P(V)$ belongs to the category $C(\mathfrak{g},\mathfrak{l})_q$.*

Proof. Since $U_q\mathfrak{g}$ is a free right $U_q\mathfrak{l}$-module, one has an embedding of $U_q\mathfrak{l}$-modules

$$i : V \hookrightarrow P(V), \qquad i : v \mapsto 1 \otimes v.$$

The relation $P(V) = P(V)_{\mathfrak{l}}$ is due to

$$P(V) = U_q\mathfrak{g}\, i(V) \subset U_q\mathfrak{g}\, P(V)_{\mathfrak{l}} = P(V)_{\mathfrak{l}}.$$ ■

The rest of the statements of Lemma 2.1 can be proved in the same way as in the classical case $q = 1$.

To prove Lemma 2.5 we need the following auxiliary statement.

Lemma A4. *Let $\lambda \in P^{\mathbb{L}}_{+}$ and $\left(U_q\mathfrak{q}^{\pm}_{\mathbb{L}}\right)_j$ be the homogeneous components of the graded algebras $U_q\mathfrak{q}^{\pm}_{\mathbb{L}}$.*

1. *The vector spaces $\left(U_q\mathfrak{q}^{\pm}_{\mathbb{L}}\right)_j \cdot \mathcal{P}_\lambda \subset \operatorname{End} V^{\mathrm{univ}}$ are finite dimensional.*
2. *The vector spaces $\mathcal{P}_\lambda \cdot \left(U_q\mathfrak{q}^{\pm}_{\mathbb{L}}\right)_j \subset \operatorname{End} V^{\mathrm{univ}}$ are finite dimensional.*[9]

Proof. Prove the first statement. We consider $\operatorname{End} V^{\mathrm{univ}}$ as a $U_q\mathfrak{l}$-module with respect to the action as follows:

$$(\xi a) : v \mapsto \xi(av), \qquad v \in V^{\mathrm{univ}}, \quad a \in \operatorname{End} V^{\mathrm{univ}}, \quad \xi \in U_q\mathfrak{l}.$$

Obviously, $\left(U_q\mathfrak{q}^{\pm}_{\mathbb{L}}\right)_j \cdot \mathcal{P}_\lambda$ is a submodule of the $U_q\mathfrak{l}$-module $\operatorname{End} V^{\mathrm{univ}}$.

Let $\pi_\lambda : U_q\mathfrak{l} \to \operatorname{End} L(\mathfrak{l},\lambda)$ be the representation of the Hopf subalgebra $U_q\mathfrak{l}$ corresponding to the $U_q\mathfrak{l}$-module $L(\mathfrak{l},\lambda)$. If $\xi \in \operatorname{Ker}\pi_\lambda$, then $\xi \cdot \mathcal{P}_\lambda = 0$. Hence the diagram

$$\begin{array}{ccc} U_q\mathfrak{l} & \xrightarrow{\;\xi\mapsto\xi\cdot\mathcal{P}_\lambda\;} & \operatorname{End} V^{\mathrm{univ}} \\ {\scriptstyle\pi_\lambda}\downarrow & \nearrow & \\ \operatorname{End} L(\mathfrak{l},\lambda) & & \end{array}$$

can be completed up to a commutative one, and

$$\dim(U_q\mathfrak{l} \cdot \mathcal{P}_\lambda) \le (\dim(L(\mathfrak{l},\lambda)))^2.$$

Thus the first statement of the Lemma is proved in the case $j = 0$. It remains to elaborate the fact that $(U_q\mathfrak{q}^{\pm}_{\mathbb{L}})_j$ is a right $U_q\mathfrak{l}$-module of finite rank (see Lemma A1).

[9] A dot is used here to denote the product of elements in $\operatorname{End} V^{\mathrm{univ}}$.

Now the first statement is proved. The second one can be proved in a similar way using the commutative diagram

$$\begin{array}{ccc} U_q\mathfrak{l} & \xrightarrow{\xi\mapsto\mathcal{P}_\lambda\cdot\xi} & \operatorname{End}V^{\mathrm{univ}} \\ {\scriptstyle \pi_\lambda}\downarrow & \nearrow & \\ \operatorname{End}L(\mathfrak{l},\lambda) & & \end{array}$$

in the category of $U_q\mathfrak{l}^{\mathrm{op}}$-modules.[10] ■

Turn to the proof of Lemma 2.5. Consider the subspace $A \subset U_q\mathfrak{g}$ of such elements $\xi \in U_q\mathfrak{g}$ that for any $\lambda \in P_+^{\mathbb{L}}$ there exists a finite subset $\Lambda \subset P_+^{\mathbb{L}}$ with

$$\xi \cdot \mathcal{P}_\lambda = \chi_\Lambda \cdot \xi \cdot \mathcal{P}_\lambda, \tag{9}$$

$$\mathcal{P}_\lambda \cdot \xi = \mathcal{P}_\lambda \cdot \xi \cdot \chi_\Lambda. \tag{10}$$

One has to prove that $A = U_q\mathfrak{g}$. For that, it suffices to demonstrate that A is a subalgebra of $U_q\mathfrak{g}$ and $A \supset \left(U_q\mathfrak{q}_{\mathbb{L}}^{\pm}\right)_j$ for all j.

Prove the first statement. It is obvious that A is a vector subspace. Now let $\xi, \eta \in A$, and consider the product $\xi\eta$. A double application of (9) first w.r.t. η and then ξ allows one to deduce that the image of the linear operator $\xi \cdot \eta \cdot \mathcal{P}_\lambda \in \operatorname{End}V^{\mathrm{univ}}$ is accommodated by the sum of finitely many isotypic components $V_\lambda^{\mathrm{univ}}$. Let this sum be $\bigoplus\limits_{\lambda' \in \Lambda'} V_{\lambda'}^{\mathrm{univ}}$, then $\xi \cdot \eta \cdot \mathcal{P}_\lambda = \chi_{\Lambda'} \cdot \xi \cdot \eta \cdot \mathcal{P}_\lambda$. Thus we get (9) for $\xi\eta$.

In a similar way, apply (10) twice to deduce that for some finite subset $\Lambda'' \subset P_+^{\mathbb{L}}$ one has $\operatorname{Ker}(\mathcal{P}_\lambda \cdot \xi \cdot \eta) \supset (\mathrm{id} - \chi_{\Lambda''})\,V^{\mathrm{univ}}$, hence $\mathcal{P}_\lambda \cdot \xi \cdot \eta = \mathcal{P}_\lambda \cdot \xi \cdot \eta \cdot \chi_{\Lambda''}$. Thus (10) holds for $\xi\eta$.

Turn to the second statement. We restrict ourselves to proving that (9) is valid for all $\xi \in \left(U_q\mathfrak{q}_{\mathbb{L}}^{\pm}\right)_j$. It follows from Lemma A4 that for every j, λ, the vector space $\left(U_q\mathfrak{q}_{\mathbb{L}}^{\pm}\right)_j \cdot \mathcal{P}_\lambda$ is finite dimensional. It is also a $U_q\mathfrak{l}$-module since $U_q\mathfrak{l}\left(U_q\mathfrak{q}_{\mathbb{L}}^{\pm}\right)_j \subset \left(U_q\mathfrak{q}_{\mathbb{L}}^{\pm}\right)_j$. Hence $\left(U_q\mathfrak{q}_{\mathbb{L}}^{\pm}\right)_j \cdot \mathcal{P}_\lambda$ is a sum of finitely many $U_q\mathfrak{l}$-isotypic components. On the other hand, if for some $a \in \operatorname{End}V^{\mathrm{univ}}$, the $U_q\mathfrak{l}$-module $U_q\mathfrak{l} \cdot a \subset \operatorname{End}V^{\mathrm{univ}}$ is a multiple of $L(\mathfrak{l}, \mu)$, $\mu \in P_+^{\mathbb{L}}$, one has $aV^{\mathrm{univ}} \subset V_\mu^{\mathrm{univ}}$ and $\mathcal{P}_\mu \cdot a = a$. It follows that $\mathcal{P}_{\lambda'} \cdot \left(U_q\mathfrak{q}_{\mathbb{L}}^{\pm}\right)_j \cdot \mathcal{P}_\lambda = 0$ for all but finitely many $\lambda' \in P_+^{\mathbb{L}}$. Hence $\xi \cdot \mathcal{P}_\lambda = \chi_\Lambda \cdot \xi \cdot \mathcal{P}_\lambda$ for some finite subset $\Lambda \subset P_+^{\mathbb{L}}$. ■

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S. Sinel'shchikov, L. Vaksman
Institute for Low Temperature Physics & Engineering
47 Lenin Ave
61103 Kharkov, Ukraine
sinelshchikov@ilt.kharkov.ua
vaksman@ilt.kharkov.ua

A. Stolin
Chalmers Tekniska Högskola, Mathematik,
412 96, Göteborg, Sweden
astolin@math.chalmers.se

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