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Topological Properties of Ad-semisimple Conjugacy Classes in Lie Groups

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Abstract. We prove that every connected component of the zero locus in a connected Lie group G of any real polynomial without multiple roots is a conjugacy class. As applications, we prove that any Ad-semisimple conjugacy class C of G is a closed embedded submanifold, and that for any connected subgroup H of G, every connected component of $C \cap H$ is a conjugacy class of H. Corresponding results for adjoint orbits in real Lie algebras are also proved.
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1. Introduction

Conjugacy classes of algebraic groups have been extensively studied (see e.g. [6]). For instance, for a linear algebraic group $\mathbf{G}$ defined over an algebraically closed field, it is well-known that semisimple conjugacy classes of $\mathbf{G}$ are Zariski closed. If $\mathbf{H}$ a Zariski closed subgroup of $\mathbf{G}$, the well-known Richardson's Lemma asserts that the intersection of $\mathbf{H}$ and every semisimple conjugacy class of $\mathbf{G}$ is a finite union of conjugacy classes of $\mathbf{H}$. Based on these results, it is easy to prove similar assertions for a Lie group which is locally isomorphic to the group of $\mathbb{R}$-points of an algebraic group defined over $\mathbb{R}$, if we replace the Zariski topology with the Hausdorff topology (see e.g. [4, Proposition 10.1]).

On the contrary, few results are known for conjugacy classes of general Lie groups that are not algebraic. Here we recall the recent result of [1] (see also [2]) which asserts that if G is a connected Lie group, then every connected component of the set $E_n(G) = \{g \in G | g^n = e\}$ is a conjugacy class of G, where n is any positive integer. Note that if G is a linear Lie group, the set $E_n(G)$ can be viewed as the zero locus of the polynomial $\lambda^n - 1$ applied to G.

In this paper, we generalize this result to more general polynomials where addition and scalar multiplication are understood as being composed with an almost faithful representation. More precisely, we will prove the following assertion.

Theorem 1.1. *Let G be a connected Lie group, and let $\rho : G \to GL(\mathcal{V})$ be a representation of G in a finite dimensional real vector space $\mathcal{V}$. Suppose* $\ker(\rho)$

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is discrete. Let $f \in \mathbb{R}[\lambda]$ be a real polynomial without multiple roots in $\mathbb{C}$. Then every connected component of the set

$$Z_\rho(f) = \{g \in G | f(\rho(g)) = 0\}$$

is a conjugacy class of G.

As applications of Theorem 1.1, we prove the following two results about Ad-semisimple conjugacy classes in general Lie groups, which are the Lie-theoretic analogs of the Zariski closedness of semisimple conjugacy classes and Richardson's Lemma for algebraic groups.

Theorem 1.2. *Let G be a connected Lie group, and let C be an* Ad*-semisimple conjugacy class of G. Then we have*
(1) C is a closed embedded submanifold of G;
(2) For any connected closed subgroup H of G, every connected component of $C \cap H$ is a conjugacy class of H.

Here an element g of G is Ad*-semisimple* if $\mathrm{Ad}(g)$ is semisimple in $GL(\mathfrak{g})$, where $\mathfrak{g}$ is the Lie algebra of G, and a conjugacy class C of G is Ad*-semisimple* if a (hence every) element of C is Ad-semisimple. Similarly, an element X of $\mathfrak{g}$ is ad*-semisimple* if $\mathrm{ad}(X)$ is semisimple in $\mathfrak{gl}(\mathfrak{g})$, and an adjoint orbit O in $\mathfrak{g}$ is ad*-semisimple* if a (hence every) element of O is ad-semisimple.

To prove Theorem 1.2, we also need the following notion. For a conjugacy class C of a connected Lie group G, we define the set

$$\Gamma(C) = g^{-1}C \cap Z(G), \quad g \in C,$$

where $Z(G)$ is the center of G. It is easy to see that $\Gamma(C)$ is independent of the choice of $g \in C$ and is a subgroup of $Z(G)$. We will prove that if C is Ad-semisimple, then $\Gamma(C)$ is finite.

The arrangement of this paper is as follows. In Section 2 we will prove Theorem 1.1. In Section 3 the finiteness of $\Gamma(C)$ will be proved. The two parts of Theorem 1.2 will be proved in Sections 4 and 5 respectively. In each section, we will also prove the corresponding result for adjoint orbits in real Lie algebras in a parallel way.

2. Characterizations of conjugacy classes by polynomials

In this section we prove Theorem 1.1 and its Lie algebra counterpart.

Theorem 2.1. *Let G be a connected Lie group with Lie algebra $\mathfrak{g}$, $\rho : G \to GL(\mathcal{V})$ be a representation of G in a finite dimensional real vector space $\mathcal{V}$. Suppose* $\ker(\rho)$ *is discrete. Let $f \in \mathbb{R}[\lambda]$ be a real polynomial without multiple roots in $\mathbb{C}$. Then*
(1) Every connected component of

$$Z_\rho(f) = \{g \in G | f(\rho(g)) = 0\}$$

is a conjugacy class of G, and is a closed embedded submanifold of G;
(2) Every connected component of

$$\mathfrak{z}_\rho(f) = \{X \in \mathfrak{g} | f(d\rho(X)) = 0\}$$

is an adjoint orbit in $\mathfrak{g}$, and is a closed embedded submanifold of $\mathfrak{g}$.

Proof. (1) Denote $Z = Z_\rho(f)$. Firstly, we note that Z is invariant under the conjugation of G. So Z is the union of some conjugacy classes of G.

Let $g \in Z$. By the definition of the set Z, $f(\rho(g)) = 0$. Since f has no multiple roots, $\rho(g)$ is semisimple. We claim that g is Ad-semisimple. Indeed, since $\ker(\rho)$ is discrete, the differential $d\rho : \mathfrak{g} \to \mathfrak{gl}(\mathcal{V})$ of ρ is injective. So the action of $\mathrm{Ad}(g)$ on $\mathfrak{g}$ is equivalent to the action of $\mathrm{Ad}(\rho(g))|_{d\rho(\mathfrak{g})}$ on $d\rho(\mathfrak{g})$. Since $\rho(g)$ acts semisimply on $\mathcal{V}$, $\mathrm{Ad}(\rho(g))$ acts semisimply on $\mathfrak{gl}(\mathcal{V})$, and then $\mathrm{Ad}(\rho(g))|_{d\rho(\mathfrak{g})}$ acts semisimply on $d\rho(\mathfrak{g})$. This verifies the claim.

Denote $\mathfrak{a}_1 = \ker(1 - \mathrm{Ad}(g))$, $\mathfrak{a}_2 = \mathrm{Im}(1 - \mathrm{Ad}(g))$. Since $\mathrm{Ad}(g)$ is semisimple, $\mathfrak{g} = \mathfrak{a}_1 \oplus \mathfrak{a}_2$. Define a map $\varphi_g : \mathfrak{a}_1 \oplus \mathfrak{a}_2 \to G$ by

$$\varphi_g(Y_1, Y_2) = e^{Y_2} e^{Y_1} g e^{-Y_2}, \qquad Y_1 \in \mathfrak{a}_1, Y_2 \in \mathfrak{a}_2.$$

Then it is easy to compute the differential $(d\varphi_g)_{(0,0)} : \mathfrak{a}_1 \oplus \mathfrak{a}_2 \to T_g G$ of φ_g at $(0,0)$ as

$$(d\varphi_g)_{(0,0)}(Y_1, Y_2) = (dr_g)_e(Y_1 + (1 - \mathrm{Ad}(g))Y_2),$$

where r_g is the right translation on G induced by g. Since $\mathrm{Ad}(g)$ is semisimple, the restriction of $1 - \mathrm{Ad}(g)$ on $\mathfrak{a}_2 = \mathrm{Im}(1 - \mathrm{Ad}(g))$ is a linear automorphism. Hence $(d\varphi_g)_{(0,0)}$ is a linear isomorphism. By the Implicit Function Theorem, there exist an open neighborhood U_1 of $0 \in \mathfrak{a}_1$ and an open neighborhood U_2 of $0 \in \mathfrak{a}_2$ such that the restriction of φ_g to $U_1 \times U_2 \subset \mathfrak{a}_1 \oplus \mathfrak{a}_2$ is a diffeomorphism onto an open neighborhood $U = \varphi_g(U_1 \times U_2)$ of $g \in G$.

Define a map $\alpha_g : \mathfrak{a}_1 \to \mathfrak{gl}(\mathcal{V})$ by

$$\alpha_g(Y_1) = f(\rho(e^{Y_1} g)).$$

We claim that α_g is an immersion at $0 \in \mathfrak{a}_1$. Indeed, we have

$$(d\alpha_g)_0(Y_1) = \frac{d}{dt}\Big|_{t=0} \alpha_g(tY_1) = \frac{d}{dt}\Big|_{t=0} f(e^{td\rho(Y_1)}\rho(g)) = d\rho(Y_1)\rho(g) f'(\rho(g)),$$

where f' is the derivative of f. Here the last step holds because $e^{td\rho(Y_1)}$ commutes with $\rho(g)$. Since f has no multiple roots, $(f, f') = 1$. So there exist polynomials r, s such that $fr + f's = 1$. Substitute $\rho(g)$ for the indeterminate in this equality and notice that $f(\rho(g)) = 0$, we get $f'(\rho(g))s(\rho(g)) = 1$. So $f'(\rho(g))$ is invertible. Since $\rho(g)$ is also invertible and $d\rho$ is injective, $(d\alpha_g)_0(Y_1) = 0$ implies $Y_1 = 0$. Hence α_g is an immersion at $0 \in \mathfrak{a}_1$. Thus, shrinking U_1 if necessary, we may assume that $\alpha_g|_{U_1}$ is injective.

Now for $Y_1 \in U_1$, $Y_2 \in U_2$, we have

$$\begin{aligned} &f(\rho(\varphi_g(Y_1, Y_2))) = f(\rho(e^{Y_2})\rho(e^{Y_1} g)\rho(e^{Y_2})^{-1}) = \rho(e^{Y_2}) f(\rho(e^{Y_1} g))\rho(e^{Y_2})^{-1} \\ =&\rho(e^{Y_2})\alpha_g(Y_1)\rho(e^{Y_2})^{-1}. \end{aligned}$$

So $\varphi_g(Y_1, Y_2) \in Z \Leftrightarrow Y_1 = 0$, that is,

$$Z \cap U = \varphi_g(\{0\} \times U_2) = \{e^{Y_2} g e^{-Y_2} | Y_2 \in U_2\}.$$

This shows that every connected component of Z is an embedded submanifold of G, which is necessarily closed by the definition of Z, and that every conjugacy class contained in Z is open in Z. But the connectedness of G implies that conjugacy classes are connected. Hence every conjugacy class contained in Z is in fact a connected component of Z. This proves (1).

(2) Similar to the proof of (1), the set $\mathfrak{z} = \mathfrak{z}_\rho(f)$ is the union of some adjoint orbits in $\mathfrak{g}$. Let $X \in \mathfrak{z}$. Then $d\rho(X)$ and $\mathrm{ad}(X)$ are semisimple. Denote $\mathfrak{b}_1 = \ker(\mathrm{ad}(X))$, $\mathfrak{b}_2 = \mathrm{Im}(\mathrm{ad}(X))$. Then $\mathfrak{g} = \mathfrak{b}_1 \oplus \mathfrak{b}_2$. Define a map $\psi_X : \mathfrak{b}_1 \oplus \mathfrak{b}_2 \to \mathfrak{g}$ by

$$\psi_X(W_1, W_2) = \mathrm{Ad}(e^{W_2})(X + W_1), \qquad W_1 \in \mathfrak{b}_1, W_2 \in \mathfrak{b}_2.$$

Then

$$(d\psi_X)_{(0,0)}(W_1, W_2) = W_1 - \mathrm{ad}(X)(W_2).$$

Hence $(d\psi_X)_{(0,0)}$ is a linear isomorphism, and then there exist an open neighborhood V_1 of $0 \in \mathfrak{b}_1$ and an open neighborhood V_2 of $0 \in \mathfrak{b}_2$ such that the restriction of ψ_X to $V_1 \times V_2 \subset \mathfrak{b}_1 \oplus \mathfrak{b}_2$ is a diffeomorphism onto an open neighborhood $V = \psi_X(V_1 \times V_2)$ of $X \in \mathfrak{g}$.

Define $\beta_X : \mathfrak{b}_1 \to \mathfrak{gl}(\mathcal{V})$ by

$$\beta_X(W_1) = f(d\rho(X + W_1)).$$

Then

$$(d\beta_X)_0(W_1) = \frac{d}{dt}\Big|_{t=0} \beta_X(tW_1) = \frac{d}{dt}\Big|_{t=0} f(d\rho(X) + td\rho(W_1)) = d\rho(W_1)f'(d\rho(X)).$$

Similar to the proof of (1), we can prove $f'(d\rho(X))$ is invertible. So $(d\beta_X)_0(W_1) = 0$ implies $W_1 = 0$, that is, β_X is an immersion at $0 \in \mathfrak{b}_1$. Shrinking V_1 if necessary, we may assume that $\beta_X|_{V_1}$ is injective.

Now for $W_1 \in V_1$, $W_2 \in V_2$, we have

$$\begin{aligned} f(d\rho(\psi_X(W_1, W_2))) =& f(d\rho(\mathrm{Ad}(e^{W_2})(X + W_1))) = f(\rho(e^{W_2})d\rho(X + W_1)\rho(e^{W_2})^{-1}) \\ =& \rho(e^{W_2})f(d\rho(X + W_1))\rho(e^{W_2})^{-1} = \rho(e^{W_2})\beta_X(W_1)\rho(e^{W_2})^{-1}. \end{aligned}$$

So $\psi_X(W_1, W_2) \in \mathfrak{z} \Leftrightarrow W_1 = 0$, that is,

$$\mathfrak{z} \cap V = \psi_X(\{0\} \times V_2) = \{\mathrm{Ad}(e^{W_2})(X) | W_2 \in V_2\}.$$

Then an argument similar to the proof of (1) shows that every connected component of $\mathfrak{z}$ is a closed embedded submanifold of $\mathfrak{g}$, and is an adjoint orbit. This proves (2). ■

Theorem 2.1 has the following obvious corollary.

Corollary 2.2. *Let G be a connected Lie group with Lie algebra $\mathfrak{g}$, and let $\rho : G \to GL(\mathcal{V})$ be a representation of G in a finite dimensional real vector space $\mathcal{V}$. Suppose* $\ker(\rho)$ *is discrete. We have*
(1) If C is a conjugacy class of G such that $\rho(C)$ contains a semisimple element A of $GL(\mathcal{V})$, then C is a closed embedded submanifold of G, and is a connected component of the set

$$Z = \{g \in G | f_A(\rho(g)) = 0\},$$

where f_A is the minimal polynomial of A;
(2) If O is an adjoint orbit in $\mathfrak{g}$ such that $d\rho(O)$ contains a semisimple element B of $\mathfrak{gl}(\mathcal{V})$, then O is a closed embedded submanifold of $\mathfrak{g}$, and is a connected component of the set

$$\mathfrak{z} = \{X \in \mathfrak{g} | f_B(d\rho(X)) = 0\},$$

where f_B is the minimal polynomial of B.

3. Finiteness of $\Gamma(C)$

Let G be a connected Lie group with Lie algebra $\mathfrak{g}$. Let C be a conjugacy class of G, and let O be an adjoint orbit in $\mathfrak{g}$. The subset $\Gamma(C) = \Gamma_G(C)$ of the center $Z(G)$ of G is defined by

$$\Gamma_G(C) = g^{-1}C \cap Z(G), \quad g \in C.$$

The subset $\gamma(O) = \gamma_{\mathfrak{g}}(O)$ of the center $Z(\mathfrak{g})$ of $\mathfrak{g}$ is defined by

$$\gamma_{\mathfrak{g}}(O) = (-X + O) \cap Z(\mathfrak{g}), \quad X \in O.$$

For convenience, denote

$$\Gamma_0(C) = \Gamma(C) \cap Z(G)_0 = g^{-1}C \cap Z(G)_0, \quad g \in C,$$

where $Z(G)_0$ is the identity component of $Z(G)$. In this section we prove some properties of $\Gamma(C)$ and $\gamma(O)$, especially the finiteness of $\Gamma(C)$ and the triviality of $\gamma(O)$ under the Ad-semisimplicity or ad-semisimplicity condition.

Lemma 3.1. *(1) $\Gamma(C)$ is independent of the choice of the element $g \in C$;*
(2) $\gamma(O)$ is independent of the choice of the element $X \in O$.

Proof. (1) Let $g_1, g_2 \in C$. Then $g_1 = hg_2h^{-1}$ for some $h \in G$. Hence we have

$$\begin{aligned} &g_1^{-1}C \cap Z(G) = hg_2^{-1}h^{-1}C \cap Z(G) = hg_2^{-1}(h^{-1}Ch)h^{-1} \cap Z(G) \\ =&hg_2^{-1}Ch^{-1} \cap Z(G) = h(g_2^{-1}C \cap Z(G))h^{-1} = g_2^{-1}C \cap Z(G). \end{aligned}$$

(2) Let $X_1, X_2 \in O$. Then $X_1 = \mathrm{Ad}(g)X_2$ for some $g \in G$. Hence

$$\begin{aligned} & (-X_1 + O) \cap Z(\mathfrak{g}) = (-\mathrm{Ad}(g)X_2 + O) \cap Z(\mathfrak{g}) \\ = \ & \mathrm{Ad}(g)(-X_2 + O) \cap Z(\mathfrak{g}) = \mathrm{Ad}(g)((-X_2 + O) \cap Z(\mathfrak{g})) \\ = \ & (-X_2 + O) \cap Z(\mathfrak{g}). \end{aligned}$$

■

For an element g in G, we denote by $Z_G(g)$ the centralizer of g in G, and denote

$$N_G(g) = \{h \in G | g^{-1}hgh^{-1} \in Z(G)\}.$$

$N_G(g)$ is a closed subgroup of G containing $Z_G(g)$. In fact, if we let $\pi : G \to G/Z(G)$ be the quotient homomorphism, then $N_G(g) = \pi^{-1}(Z_{G/Z(G)}(\pi(g)))$. Similarly, for an element X in the Lie algebra $\mathfrak{g}$ of G, denote by $Z_G(X)$ the centralizer of X in G, and denote

$$N_G(X) = \{h \in G | -X + \mathrm{Ad}(h)X \in Z(\mathfrak{g})\}.$$

Then $N_G(X) = \pi^{-1}(Z_{G/Z(G)}(d\pi(X)))$ is a closed subgroup of G containing $Z_G(X)$.

Lemma 3.2. *(1) $\Gamma(C)$ is a Lie subgroup of $Z(G)$, and is isomorphic to $N_G(g)/Z_G(g)$ for every $g \in C$;*
(2) $\gamma(O)$ is a Lie subgroup of the vector group $Z(\mathfrak{g})$, and is isomorphic to $N_G(X)/Z_G(X)$ for every $X \in O$.

Proof. (1) Let $g \in C$. Define a smooth map $\alpha : N_G(g) \to Z(G)$ by

$$\alpha(h) = g^{-1}hgh^{-1}.$$

We claim that α is a homomorphism of Lie groups. Indeed, let $h_1, h_2 \in N_G(g)$, then

$$\begin{aligned} &\alpha(h_1)\alpha(h_2) = (g^{-1}h_1gh_1^{-1})(g^{-1}h_2gh_2^{-1}) \\ =&g^{-1}h_1g(g^{-1}h_2gh_2^{-1})h_1^{-1} = g^{-1}(h_1h_2)g(h_1h_2)^{-1} = \alpha(h_1h_2). \end{aligned}$$

It is obvious that the kernel of α is $Z_G(g)$, and the image of α is $\Gamma(C) = g^{-1}C \cap Z(G)$. So $\Gamma(C)$ is a Lie subgroup of $Z(G)$, and is isomorphic to $N_G(g)/Z_G(g)$.

(2) Let $X \in O$. Define $\beta : N_G(X) \to Z(\mathfrak{g})$ by

$$\beta(h) = -X + \mathrm{Ad}(h)X.$$

For $g_1, g_2 \in N_G(X)$, we have

$$\begin{aligned} &\beta(g_1g_2) = -X + \mathrm{Ad}(g_1g_2)X \\ =&(-X + \mathrm{Ad}(g_1)X) + (-\mathrm{Ad}(g_1)X + \mathrm{Ad}(g_1)\mathrm{Ad}(g_2)X) \\ =&\beta(g_1) + \mathrm{Ad}(g_1)(-X + \mathrm{Ad}(g_2)X) = \beta(g_1) + \mathrm{Ad}(g_1)\beta(g_2) = \beta(g_1) + \beta(g_2). \end{aligned}$$

So β is a homomorphism of Lie groups. The kernel of β is $Z_G(X)$, the image of β is $\gamma(O) = (-X+O) \cap Z(\mathfrak{g})$. So $\gamma(O)$ is a Lie subgroup of $Z(\mathfrak{g})$, and is isomorphic to $N_G(X)/Z_G(X)$. ■

For an adjoint orbit O in $\mathfrak{g}$, $\exp(O)$ is a conjugacy class of G. $\gamma(O)$ and $\Gamma(\exp(O))$ have the following relation.

Lemma 3.3. $\exp(\gamma(O)) \subset \Gamma_0(\exp(O))$.

Proof. Let $X \in O$. If $Y \in \gamma(O)$, then there exists $h \in G$ such that $Y = -X + \mathrm{Ad}(h)X$. Since $Y \in Z(\mathfrak{g})$, $he^Xh^{-1} = e^{\mathrm{Ad}(h)X} = e^{X+Y} = e^Xe^Y$. So $e^Y = e^{-X}he^Xh^{-1} \in e^{-X}\exp(O) \cap Z(G)_0 = \Gamma_0(\exp(O))$. This shows $\exp(\gamma(O)) \subset \Gamma_0(\exp(O))$. ■

Let $\pi : G \to G'$ be a covering homomorphism of Lie groups. Then for a conjugacy class C of G, $\pi(C)$ is a conjugacy class of G'. The next lemma relates $\Gamma_G(C)$ with $\Gamma_{G'}(\pi(C))$.

Lemma 3.4. $\pi(\Gamma_G(C)) = \Gamma_{G'}(\pi(C))$.

Proof. First we claim that $Z(G) = \pi^{-1}(Z(G'))$. Indeed, let $z \in \pi^{-1}(Z(G'))$, and let $\alpha : G \to G$ be the map defined by $\alpha(h) = hzh^{-1}z^{-1}$. Then $\alpha(G) \subset \ker(\pi)$. Since $\alpha(G)$ is connected containing the identity e of G, and $\ker(\pi)$ is discrete, we have $\alpha(G) = \{e\}$. So $z \in Z(G)$. This shows $\pi^{-1}(Z(G')) \subset Z(G)$. It is obvious that $Z(G) \subset \pi^{-1}(Z(G'))$. Hence $Z(G) = \pi^{-1}(Z(G'))$. Now we choose a $g \in C$, then

$$\begin{aligned}\pi(\Gamma_G(C)) &= \pi(g^{-1}C \cap Z(G)) = \pi(g^{-1}C \cap \pi^{-1}(Z(G')))\\ &= \pi(g^{-1}C) \cap Z(G') = \pi(g)^{-1}\pi(C) \cap Z(G') = \Gamma_{G'}(\pi(C)). \qquad \blacksquare\end{aligned}$$

The following lemma demonstrates a rough understanding of $\Gamma(C)$ and $\gamma(O)$ under the semisimplicity assumptions.

Lemma 3.5. *(1) If C is an* Ad*-semisimple conjugacy class of G, $g \in C$, then the Lie algebras of $N_G(g)$ and $Z_G(g)$ coincide, and $\Gamma(C)$ is a 0-dimensional Lie subgroup of $Z(G)$;*
(2) If O is an ad*-semisimple adjoint orbit in $\mathfrak{g}$, $X \in O$, then the Lie algebras of $N_G(X)$ and $Z_G(X)$ coincide, and $\gamma(O)$ is a 0-dimensional Lie subgroup of the vector group $Z(\mathfrak{g})$.*

Proof. (1) Since $Z_G(g) \subset N_G(g)$, to prove their Lie algebras coincide, it is sufficient to show that for every X in the Lie algebra of $N_G(g)$, X belongs to the Lie algebra of $Z_G(g)$. For such an X, we have $g^{-1}e^{tX}ge^{-tX} \in Z(G)$ for every $t \in \mathbb{R}$. So $e^{tX}e^{-t\mathrm{Ad}(g)X} = g(g^{-1}e^{tX}ge^{-tX})g^{-1} \in Z(G)$. This implies that $(1 - \mathrm{Ad}(g))X$ belongs to the Lie algebra of $Z(G)$, and then $(1 - \mathrm{Ad}(g))^2X = 0$. Since C is Ad-semisimple, $\mathrm{Ad}(g)$ is semisimple. So we in fact have $(1-\mathrm{Ad}(g))X = 0$. But the Lie algebra of $Z_G(g)$ is $\ker(1-\mathrm{Ad}(g))$. So X belongs to the Lie algebra of $Z_G(g)$. Hence the Lie algebras of $N_G(g)$ and $Z_G(g)$ coincide. As the image of the homomorphism α constructed in the proof of Lemma 3.2, $\Gamma(C)$ is a 0-dimensional Lie subgroup of $Z(G)$.

(2) Similar to the proof of (1), let Y be an element of the Lie algebra of $N_G(X)$. Then $-X + \mathrm{Ad}(e^{tY})X \in Z(\mathfrak{g})$ for every $t \in \mathbb{R}$. This implies that $\mathrm{ad}(Y)X \in Z(\mathfrak{g})$. So $\mathrm{ad}(X)^2Y = -\mathrm{ad}(X)(\mathrm{ad}(Y)X) = 0$. Since X is ad-semisimple, $\mathrm{ad}(X)Y = 0$. This shows that Y belongs to the Lie algebra of $Z_G(X)$. So the Lie algebras of $N_G(X)$ and $Z_G(X)$ coincide, and $\gamma(O)$ is a 0-dimensional Lie subgroup of $Z(\mathfrak{g})$. ■

Remark 3.6. We only need the discreteness of $\Gamma(C)$ in $Z(G)$ in the proof Theorem 1.2. By Lemma 3.5, $\Gamma(C)$ is 0-dimensional when C is Ad-semisimple. But this does not imply that $\Gamma(C)$ is discrete in $Z(G)$. To get the discreteness of $\Gamma(C)$, we have to show that it is finite. If fact, if $\Gamma(C)$ could be infinite for some connected Lie group G and some Ad-semisimple conjugacy class C of G, we would easily construct a discrete central subgroup D of $G\times\mathbb{R}$ such that $\Gamma_{(G\times\mathbb{R})/D}(\pi(C))$ is not discrete, where $\pi : G \to (G \times \mathbb{R})/D$ is the covering homomorphism.

The remaining of this section is devoted to the proof of the finiteness of $\Gamma(C)$ and the Lie algebra counterpart. Some results on real algebraic groups are needed.

We understand the Zariski topology on $GL_n(\mathbb{R})$ as the topology for which a closed set is the set of common zeros of a family of real polynomial functions on $GL_n(\mathbb{R})$ with indeterminates g_{ij} $(1 \le i, j \le n)$ and $\frac{1}{\det g}$, where $g = (g_{ij}) \in GL_n(\mathbb{R})$. It is obvious that if G is a Lie subgroup of $GL_n(\mathbb{R})$, then the Zariski closure $\overline{\overline{G}}$ of G is also a Lie subgroup of $GL_n(\mathbb{R})$.

Lemma 3.7. *Let G be a connected Lie subgroup of $GL_n(\mathbb{R})$ for some n, $\overline{\overline{G}}$ the Zariski closure of G in $GL_n(\mathbb{R})$. If $g \in G$ is* Ad*-semisimple in G, then it is* Ad*-semisimple in $\overline{\overline{G}}$.*

Proof. Let $g = g_s g_u$ be the multiplicative Jordan decomposition of g in $GL_n(\mathbb{R})$, where g_s is semisimple, g_u is unipotent. It is well-known that $g_s, g_u \in \overline{\overline{G}}$ (see, for example, [3, Chapter 1, Section 4]). Then $\mathrm{Ad}_{\overline{\overline{G}}}(g) = \mathrm{Ad}_{\overline{\overline{G}}}(g_s) \cdot \mathrm{Ad}_{\overline{\overline{G}}}(g_u)$ is the multiplicative Jordan decomposition of $\mathrm{Ad}_{\overline{\overline{G}}}(g)$ in $GL(\overline{\overline{\mathfrak{g}}})$, where $\overline{\overline{\mathfrak{g}}}$ is the Lie algebra of $\overline{\overline{G}}$. Since the Lie algebra $\mathfrak{g}$ of G in invariant under $\mathrm{Ad}_{\overline{\overline{G}}}(g)$, it is also invariant under $\mathrm{Ad}_{\overline{\overline{G}}}(g_s)$ and $\mathrm{Ad}_{\overline{\overline{G}}}(g_u)$. So $\mathrm{Ad}(g) = \mathrm{Ad}_{\overline{\overline{G}}}(g_s)|_{\mathfrak{g}} \cdot \mathrm{Ad}_{\overline{\overline{G}}}(g_u)|_{\mathfrak{g}}$ is the multiplicative Jordan decomposition of $\mathrm{Ad}(g)$ in $GL(\mathfrak{g})$. But by the assumption, $\mathrm{Ad}(g)$ is semisimple. So $\mathrm{Ad}_{\overline{\overline{G}}}(g_u)|_{\mathfrak{g}} = 0$. This implies that $G \subset Z_{\overline{\overline{G}}}(g_u)$. Since $Z_{\overline{\overline{G}}}(g_u)$ is Zariski closed, we have $\overline{\overline{G}} \subset Z_{\overline{\overline{G}}}(g_u)$, that is, $g_u \in Z(\overline{\overline{G}})$. So $\mathrm{Ad}_{\overline{\overline{G}}}(g_u) = 1$, and then $\mathrm{Ad}_{\overline{\overline{G}}}(g) = \mathrm{Ad}_{\overline{\overline{G}}}(g_s)$ is semisimple, that is, g is Ad-semisimple in $\overline{\overline{G}}$. ■

Lemma 3.8. *Let G be a connected Lie subgroup of $GL_n(\mathbb{R})$ for some n, C an* Ad*-semisimple conjugacy class of G. Then $\Gamma(C)$ is a finite subgroup of $Z(G)$.*

Proof. Let $\overline{\overline{G}}$ be the Zariski closure of G in $GL_n(\mathbb{R})$, and let C' be the conjugacy class of $\overline{\overline{G}}$ containing C. Choose a $g \in C$. Since g is Ad-semisimple in G, by Lemma 3.7, g is Ad-semisimple in $\overline{\overline{G}}$. by Lemma 3.5, the Lie algebras of $N_{\overline{\overline{G}}}(g)$ and $Z_{\overline{\overline{G}}}(g)$ coincide. Since $N_{\overline{\overline{G}}}(g)$ can be expressed as

$$N_{\overline{\overline{G}}}(g) = \{h \in \overline{\overline{G}} | (g^{-1}hgh^{-1})x = x(g^{-1}hgh^{-1}), \forall x \in \overline{\overline{G}}\},$$

which is algebraic, by Whitney's Theorem [9], $N_{\overline{\overline{G}}}(g)$ has finitely many connected components. So as a quotient group of the component group of $N_{\overline{\overline{G}}}(g)$, $N_{\overline{\overline{G}}}(g)/Z_{\overline{\overline{G}}}(g)$ is finite. Hence $\Gamma_{\overline{\overline{G}}}(C') \cong N_{\overline{\overline{G}}}(g)/Z_{\overline{\overline{G}}}(g)$ is finite.

We claim that $Z(G) \subset Z(\overline{\overline{G}})$. Indeed, if $z \in Z(G)$, then $Z_{GL_n(\mathbb{R})}(z)$ is an algebraic subgroup of $GL_n(\mathbb{R})$ containing G. So $Z_{GL_n(\mathbb{R})}(z)$ contains $\overline{\overline{G}}$, that is, $z \in Z(\overline{\overline{G}})$. This shows $Z(G) \subset Z(\overline{\overline{G}})$. Now we have

$$\Gamma_G(C) = g^{-1}C \cap Z(G) \subset g^{-1}C' \cap Z(\overline{\overline{G}}) = \Gamma_{\overline{\overline{G}}}(C').$$

Hence $\Gamma_G(C)$ is finite. This proves the lemma. ■

Lemma 3.9. *Let G be a connected semisimple Lie group, C an* Ad*-semisimple conjugacy class of G. Then $\Gamma(C)$ is a finite subgroup of $Z(G)$.*

Proof. Let $\mathrm{Aut}(\mathfrak{g})$ be the automorphism group of the Lie algebra $\mathfrak{g}$ of G. Since

$$\mathrm{Aut}(\mathfrak{g}) = \{A \in GL(\mathfrak{g}) | f_{X,Y}(A) = 0, \forall X, Y \in \mathfrak{g}\},$$

where

$$f_{X,Y}(A) = A[X,Y] - [AX, AY]$$

is algebraic, $\mathrm{Aut}(\mathfrak{g})$ is an algebraic subgroup of $GL(\mathfrak{g})$. Choose $g \in C$. Then $\mathrm{Ad}(g) \in \mathrm{Aut}(\mathfrak{g})$. By Whitney's Theorem, $Z_{\mathrm{Aut}(\mathfrak{g})}(\mathrm{Ad}(g))$ has finitely many connected components. Since G is semisimple, $\mathrm{Ad}(G)$ is the identity component of $\mathrm{Aut}(\mathfrak{g})$. So $Z_{\mathrm{Ad}(G)}(\mathrm{Ad}(g)) = Z_{\mathrm{Aut}(\mathfrak{g})}(\mathrm{Ad}(g)) \cap \mathrm{Ad}(G)$ has finitely many connected components. Since the kernel of the epimorphism $\mathrm{Ad} : G \to \mathrm{Ad}(G)$ is $Z(G)$, which is discrete, $N_G(g)/Z(G) \cong Z_{\mathrm{Ad}(G)}(\mathrm{Ad}(g))$ has finitely many connected components.

On the other hand, we have

$$\Gamma(C) \cong N_G(g)/Z_G(g) \cong (N_G(g)/Z(G))/(Z_G(g)/Z(G)).$$

By Lemma 3.5, the Lie algebras of $N_G(g)$ and $Z_G(g)$ coincide. So the Lie algebras of $N_G(g)/Z(G)$ and $Z_G(g)/Z(G)$ coincide. We have shown that $N_G(g)/Z(G)$ has finitely many connected components. So $\Gamma(C) \cong (N_G(g)/Z(G))/(Z_G(g)/Z(G))$ is finite. ■

Now we can prove the finiteness of $\Gamma(C)$ and the triviality of $\gamma(O)$ under the semisimplicity assumptions.

Theorem 3.10. *Let G be a connected Lie group with Lie algebra $\mathfrak{g}$. We have*
(1) If C is an Ad*-semisimple conjugacy class of G, then $\Gamma(C)$ is a finite subgroup of $Z(G)$;*
(2) If O is an ad*-semisimple adjoint orbit in $\mathfrak{g}$, then $\gamma(O)$ is trivial.*

Proof. (1) By Lemma 3.4, we may assume that G is simply connected. Let $R = \mathrm{Rad}(G)$. By Levi's Theorem, there is a connected semisimple subgroup L of G such that $G = R \rtimes L$. Note that R and L are simply connected.

We first prove that $R \cap \Gamma(C)$ is finite. Let $\Lambda(L)$ be the linearizer of L (by definition, $\Lambda(L)$ is the intersection of the kernels of all finite dimensional representations of L). By considering the adjoint representation of L in the Lie algebra of G, we know that $\Lambda(L) \subset Z(G)$. Since $L/\Lambda(L)$ admits a finite dimensional faithful representation (see [7, Chapter 5, Section 3, Theorem 8]), by a theorem of Harish-Chandra [5], $G/\Lambda(L) \cong R \rtimes (L/\Lambda(L))$ admits a finite dimensional faithful representation. Since C is Ad-semisimple in G, $\pi(C)$ is Ad-semisimple in $G/\Lambda(L)$, where $\pi : G \to G/\Lambda(L)$ is the quotient homomorphism. By Lemma 3.8, $\Gamma_{G/\Lambda(L)}(\pi(C))$ is finite. Since $\Lambda(L)$ is discrete, by Lemma 3.4, $\pi(\Gamma(C)) = \Gamma_{G/\Lambda(L)}(\pi(C))$ is finite. Since $R \cap \Lambda(L)$ is trivial, the restriction of π to $R \cap \Gamma(C)$ is injective. So $R \cap \Gamma(C)$ is finite.

Now consider the quotient homomorphism $\alpha : G \to G/R$. Since $\alpha(C)$ is Ad-semisimple in G/R, by Lemma 3.9, $\Gamma_{G/R}(\alpha(C))$ is finite. But the kernel of the homomorphism $\alpha|_{\Gamma(C)} : \Gamma(C) \to \Gamma_{G/R}(\alpha(C))$ is $R \cap \Gamma(C)$, which we have shown is finite. So $\alpha(C)$ is finite. This proves (1).

(2) We may assume that G is simply connected. Since O is ad-semisimple, $\exp(O)$ is an Ad-semisimple conjugacy class of G. By item (1) of the theorem, $\Gamma(\exp(O))$ is finite. So $\Gamma_0(\exp(O)) = \Gamma(\exp(O)) \cap Z(G)_0$ is a finite subgroup of $Z(G)_0$. But the simple connectedness of G implies that $Z(G)_0$ is simply connected (see [8, Corollary 3.18.6]), which is isomorphic to a vector group. So $\Gamma_0(\exp(O))$ is in fact trivial. By Lemma 3.3, $\exp(\gamma(O))$ is trivial. But the simple connectedness of $Z(G)_0$ implies that the restriction of the exponential map to $Z(\mathfrak{g})$ is injective. In particular, $\exp|_{\gamma(O)}$ is injective. So $\gamma(O)$ is trivial. ■

Remark 3.11. Our proof of item (2) of Theorem 3.10 is based on item (1) of that theorem. But one can also give a direct proof of item (2). To do this, one can embed $\mathfrak{g}$ into some $\mathfrak{gl}_n(\mathbb{R})$ using Ado's Theorem, consider the connected Lie subgroup G' of $GL_n(\mathbb{R})$ with Lie algebra $\mathfrak{g}$, and then consider the Zariski closure $\overline{\overline{G'}}$ of G'. In this course one need a result similar to Lemma 3.7, that is, if $X \in \mathfrak{g}$ is ad-semisimple in $\mathfrak{g}$, then it is ad-semisimple in the Lie algebra of $\overline{\overline{G'}}$. The details are similar to the proof of Lemma 3.8 and are omitted here.

4. Proof of Theorem 1.2 (1)

In this section we prove the closedness of Ad-semisimple conjugacy classes in connected Lie groups and ad-semisimple adjoint orbits in real Lie algebras.

Theorem 4.1. *Let G be a connected Lie group with Lie algebra $\mathfrak{g}$. Then*
(1) Ad*-semisimple conjugacy classes in G are closed embedded submanifolds of G;*
(2) ad*-semisimple adjoint orbits in $\mathfrak{g}$ are closed embedded submanifolds of $\mathfrak{g}$.*

Proof. Let $G' = G/Z(G)_0$, where $Z(G)_0$ is the identity component of the center $Z(G)$ of G. Let $\pi : G \to G'$ be the quotient homomorphism. Then the adjoint representation $\mathrm{Ad} : G \to GL(\mathfrak{g})$ induces naturally a representation $\rho : G' \to GL(\mathfrak{g})$, such that $\rho \circ \pi = \mathrm{Ad}$. Note that $\ker(\rho)$ is discrete in G'.

(1) Let C be an Ad-semisimple conjugacy class of G. Then $C' = \pi(C)$ is a conjugacy class of G'. Since all elements of $\rho(C') = \mathrm{Ad}(C)$ are semisimple in $GL(\mathfrak{g})$, by Corollary 2.2, C' is a closed embedded submanifold of G'. So

$$M = \pi^{-1}(C') = C \cdot Z(G)_0$$

is a closed embedded submanifold of G.

Now consider the transitive action of $G \times Z(G)_0$ on the manifold M, defined by

$$(h, z).x = hxh^{-1}z.$$

Choose a $g \in C \subset M$, and let $L \subset G \times Z(G)_0$ be the isotropic group of g. Then the map

$$\varphi : (G \times Z(G)_0)/L \to M$$

defined by

$$\varphi((h, z)L) = hgh^{-1}z$$

is a diffeomorphism.

By Theorem 3.10, $\Gamma(C) = g^{-1}C \cap Z(G)$ is a finite subgroup of $Z(G)$. So $\Gamma_0(C) = g^{-1}C \cap Z(G)_0 = \Gamma(C) \cap Z(G)_0$ is a finite subgroup of $Z(G)_0$. Then $Z(G)_0/\Gamma_0(C)$ is a Lie group. Let

$$\alpha : G \times Z(G)_0 \to Z(G)_0/\Gamma_0(C)$$

be the epimorphism defined by

$$\alpha(h, z) = [z],$$

where $[z]$ is the image of z under the quotient homomorphism

$$Z(G)_0 \to Z(G)_0/\Gamma_0(C).$$

For $(h, z) \in L$, $hgh^{-1}z = (h, z).g = g$, so $z^{-1} = g^{-1}hgh^{-1} \in \Gamma_0(C)$, and then $[z]$ is trivial in $Z(G)_0/\Gamma_0(C)$. This shows that $L \subset \ker(\alpha)$. Then α induces a smooth map

$$\widetilde{\alpha} : (G \times Z(G)_0)/L \to Z(G)_0/\Gamma_0(C)$$

defined by

$$\widetilde{\alpha}((h, z)L) = [z].$$

It is obvious that $(G \times Z(G)_0)/L$ is a fiber bundle with base space $Z(G)_0/\Gamma_0(C)$, fiber type $\widetilde{\alpha}^{-1}([e])$, and projection $\widetilde{\alpha}$.

We claim that

$$\widetilde{\alpha}^{-1}([e]) = \varphi^{-1}(C).$$

Firstly, let $(h, z)L \in \widetilde{\alpha}^{-1}([e])$. Then $[z] = [e]$, that is, $z \in \Gamma_0(C)$. So there exists $k \in G$ such that $z = g^{-1}kgk^{-1}$. Then

$$\varphi((h, z)L) = hgh^{-1}z = hgzh^{-1} = hg(g^{-1}kgk^{-1})h^{-1} = (hk)g(hk)^{-1} \in C,$$

that is, $(h, z)L \in \varphi^{-1}(C)$. Conversely, let $(h', z')L \in \varphi^{-1}(C)$. Then there exists $k' \in G$ such that $\varphi((h', z')L) = h'gz'h'^{-1} = k'gk'^{-1}$. This implies $z' = g^{-1}(h'^{-1}k')g(h'^{-1}k')^{-1}$. So $z' \in \Gamma_0(C)$. Hence $(h', z')L \in \widetilde{\alpha}^{-1}([e])$. This verifies the claim.

As the fiber above $[e]$, $\varphi^{-1}(C) = \widetilde{\alpha}^{-1}([e])$ is a closed embedded submanifold of $(G \times Z(G)_0)/L$. Since φ is a diffeomorphism, C is a closed embedded submanifold of M, hence a closed embedded submanifold of G. Item (1) is proved.

(2) Let O be an ad-semisimple adjoint orbit in $\mathfrak{g}$. Then $O' = d\pi(O)$ is an adjoint orbit in $\mathfrak{g}'$, the Lie algebra of G'. Since all elements of $d\rho(O') = \mathrm{ad}(O)$ is semisimple in $\mathfrak{gl}(\mathfrak{g})$, by Corollary 2.2, O' is a closed embedded submanifold of $\mathfrak{g}'$. So

$$N = (d\pi)^{-1}(O') = O + Z(\mathfrak{g})$$

is a closed embedded submanifold of $\mathfrak{g}$.

Consider the transitive action of $G \times Z(\mathfrak{g})$ on the manifold N, defined by

$$(h, Y).W = \mathrm{Ad}(h)W + Y.$$

Choose an $X \in O \subset N$, and let $K \subset G \times Z(\mathfrak{g})$ be the isotropic group of X. Then the map

$$\psi : (G \times Z(\mathfrak{g}))/K \to N$$

defined by

$$\psi((h, Y)K) = \mathrm{Ad}(h)X + Y$$

is a diffeomorphism.

Let

$$\beta : G \times Z(\mathfrak{g}) \to Z(\mathfrak{g})$$

be the projection to the second factor. For $(h, Y) \in K$, $\mathrm{Ad}(h)X + Y = (h, Y).X = X$, so $-Y = -X + \mathrm{Ad}(h)X \in \gamma(O)$. But by Theorem 3.10, $\gamma(O)$ is trivial. So $Y = 0$. This shows that $K \subset \ker(\beta)$. Then β induces a smooth map

$$\widetilde{\beta} : (G \times Z(\mathfrak{g}))/K \to Z(\mathfrak{g})$$

defined by

$$\widetilde{\beta}((h, Y)K) = Y.$$

It is obvious that $(G \times Z(\mathfrak{g}))/K$ is a fiber bundle with base space $Z(\mathfrak{g})$, fiber type $\widetilde{\beta}^{-1}(0)$, and projection $\widetilde{\beta}$. Similar to the proof of (1), we have $\widetilde{\beta}^{-1}(0) = \psi^{-1}(O)$.

As the fiber above $0 \in Z(\mathfrak{g})$, $\psi^{-1}(O) = \widetilde{\beta}^{-1}(0)$ is a closed embedded submanifold of $(G \times Z(\mathfrak{g}))/K$. Since ψ is a diffeomorphism, O is a closed embedded submanifold of N, hence a closed embedded submanifold of $\mathfrak{g}$. This proves (2). ∎

5. Proof of Theorem 1.2 (2)

In this section we prove Theorem 1.2 (2) in a more general setting. We first prove a lemma.

Lemma 5.1. *Let $\pi : G \to G'$ be a covering homomorphism of connected Lie groups. If C' is an* Ad*-semisimple conjugacy class of G', then every connected component of $\pi^{-1}(C')$ is a conjugacy class of G.*

Proof. By Theorem 4.1, C' is a closed embedded submanifold of G'. Let $\widetilde{C}$ be a connected component of $\pi^{-1}(C')$. Then $\widetilde{C}$ is a closed embedded submanifold of G. Since G is connected, $\widetilde{C}$ is invariant under the conjugation of G. Let C be a conjugacy class of G contained in $\widetilde{C}$. By Theorem 4.1, C is a closed embedded submanifold of G, hence a closed embedded submanifold of $\widetilde{C}$. But $\dim C = \dim C' = \dim \widetilde{C}$. By the connectedness of $\widetilde{C}$, we must have $C = \widetilde{C}$. ∎

Remark 5.2. Lemma 5.1 does not hold without the Ad-semisimplicity assumption.

Theorem 1.2 (2) and its Lie algebra counterpart are obvious corollaries of the following theorem.

Theorem 5.3. *Let $\alpha : H \to G$ be a homomorphism of connected Lie groups. Suppose* $\ker(\alpha)$ *is discrete. Let the Lie algebras of G and H be $\mathfrak{g}$ and $\mathfrak{h}$, respectively. We have*
(1) If C is an Ad*-semisimple conjugacy class of G, then every connected component of $\alpha^{-1}(C)$ is a conjugacy class of H;*
(2) If O is an ad*-semisimple adjoint orbit in $\mathfrak{g}$, then every connected component of $(d\alpha)^{-1}(O)$ is an adjoint orbit in $\mathfrak{h}$.*

Proof. (1) We first observed that $\alpha^{-1}(C)$ is invariant under the conjugation of H. So $\alpha^{-1}(C)$ is the union of a family of conjugacy classes of H. But the connectedness of H implies that conjugacy classes of H are connected. So every connected component of $\alpha^{-1}(C)$ is the union of a family of conjugacy classes of H. We prove that every connected component of $\alpha^{-1}(C)$ is a conjugacy class of H. The proof is divided into three steps.

Step (a). We prove (1) under the additional assumptions that α is injective and $\Gamma(C)$ is trivial. In this case, H can be identified with $\alpha(H)$, which is a Lie subgroup of G, and then $\alpha^{-1}(C)$ is identified with $C \cap H$. Note that under this identification, the prior topology on H may be different from the subspace topology on H induced from G. We call the prior topology on H the *H-topology*, and call a connected component of $C \cap H$ with respect to the H-topology an *H-connected component* of $C \cap H$.

Let C_i be an H-connected component of $C \cap H$. Consider the adjoint homomorphism $\mathrm{Ad}_G = \mathrm{Ad} : G \to \mathrm{Ad}(G)$. Then $\mathrm{Ad}_G(C_i) \subset \mathrm{Ad}_G(C) \cap \mathrm{Ad}_G(H)$. Let C_i' be the $\mathrm{Ad}_G(H)$-connected component of $\mathrm{Ad}_G(C) \cap \mathrm{Ad}_G(H)$ containing $\mathrm{Ad}_G(C_i)$. Since all elements of $\mathrm{Ad}_G(C)$ are semisimple, by Corollary 2.2, the conjugacy class $\mathrm{Ad}_G(C)$ in $\mathrm{Ad}(G)$ is an $\mathrm{Ad}(G)$-connected component of $Z = \{A \in \mathrm{Ad}(G) | f(A) = 0\}$, where f is the minimal polynomial of $\mathrm{Ad}_G(h_0)$ for some $h_0 \in H$. So C_i' is an $\mathrm{Ad}_G(H)$-connected component of $Z \cap \mathrm{Ad}_G(H)$. By Corollary 2.2 again, we conclude that C_i' is a conjugacy class of $\mathrm{Ad}_G(H)$. Let $g_1, g_2 \in C_i$. Then $\mathrm{Ad}_G(g_1), \mathrm{Ad}_G(g_2) \in C_i'$, and then there exists $h \in H$ such that $\mathrm{Ad}_G(g_2) = \mathrm{Ad}_G(h)\mathrm{Ad}_G(g_1)\mathrm{Ad}_G(h)^{-1}$. So $g_2 = hg_1h^{-1}z$ for some $z \in Z(G)$. But g_1 and g_2 are conjugate in G. So there is $g \in G$ such that $g_2 = gg_1g^{-1}$. This implies $gg_1g^{-1} = hg_1h^{-1}z = hg_1zh^{-1}$. Hence $z = g_1^{-1}(h^{-1}g)g_1(h^{-1}g)^{-1} \in g_1^{-1}C \cap Z(G) = \Gamma(C)$. But we have assumed that $\Gamma(C)$ is trivial. So $z = e$, and then $g_2 = hg_1h^{-1}$. This shows that C_i is a conjugacy class of H.

Step (b). We prove (1) under the additional assumption that α is injective. As we have done in step (a), we identify H with $\alpha(H)$. Let $G' = G/\Gamma(C)$. By Theorem 3.10, $\Gamma(C)$ is finite. So the quotient homomorphism $\pi : G \to G'$ is a covering homomorphism. In particular, $\pi|_H : H \to \alpha(H)$ is a covering homomorphism. Let C_i be an H-connected component of $C \cap H$, and let C' be a conjugacy class of H contained in C_i. Then $\pi(C')$ is a conjugacy class of $\pi(H)$, and we have $\pi(C') \subset \pi(C_i) \subset \pi(C) \cap \pi(H)$. Let C_i' be the $\pi(H)$-connected component of $\pi(C) \cap \pi(H)$ containing $\pi(C_i)$. By Lemma 3.4, $\Gamma_{G'}(\pi(C)) = \pi(\Gamma(C))$ is trivial. So by step (a), C_i' is a conjugacy class of $\pi(H)$ containing $\pi(C')$. This forces $\pi(C') = \pi(C_i) = C_i'$. Hence $C' \subset C_i \subset (\pi|_H)^{-1}(C_i')$. By Lemma 5.1, C' is an H-connected component of $(\pi|_H)^{-1}(C_i')$. As an H-connected subset of $(\pi|_H)^{-1}(C_i')$ containing C', C_i must coincide with C'. So C_i is a conjugacy class of H.

Step (c). We finish the proof of item (1). Let C_i be a connected component of $\alpha^{-1}(C)$, and let C_i' be the $\alpha(H)$-connected component of $C \cap \alpha(H)$ containing $\alpha(C_i)$. Then C_i is a connected component of $\alpha^{-1}(C_i')$. But by step (b), C_i' is a conjugacy class of $\alpha(H)$. So by Lemma 5.1, C_i is a conjugacy class of H.

(2) Since $d\alpha$ is injective, $\mathfrak{h}$ can be viewed as a subalgebra of $\mathfrak{g}$. We want to prove that if $O \cap \mathfrak{h}$ is nonempty, then every connected component of $O \cap \mathfrak{h}$ is an adjoint orbit in $\mathfrak{h}$. Let O_i be a connected component of $O \cap \mathfrak{h}$. Then $\mathrm{ad}_{\mathfrak{g}}(O_i) \subset \mathrm{ad}_{\mathfrak{g}}(O) \cap \mathrm{ad}_{\mathfrak{g}}(\mathfrak{h})$. Let O_i' be the connected component of $\mathrm{ad}_{\mathfrak{g}}(O) \cap \mathrm{ad}_{\mathfrak{g}}(\mathfrak{h})$ containing

$\mathrm{ad}_{\mathfrak{g}}(O_i)$. Since all elements of $\mathrm{ad}_{\mathfrak{g}}(O)$ are semisimple, by Corollary 2.2, the adjoint orbit $\mathrm{ad}_{\mathfrak{g}}(O)$ in $\mathrm{ad}(\mathfrak{g})$ is a connected component of $\mathfrak{z} = \{B \in \mathrm{ad}(\mathfrak{g}) | p(B) = 0\}$, where p is the minimal polynomial of $\mathrm{ad}_{\mathfrak{g}}(Y_0)$ for some $Y_0 \in \mathfrak{h}$. So O'_i is a connected component of $\mathfrak{z} \cap \mathrm{ad}_{\mathfrak{g}}(\mathfrak{h})$. By Corollary 2.2 again, we conclude that O'_i is an adjoint orbit in $\mathrm{ad}_{\mathfrak{g}}(\mathfrak{h})$. Let $X_1, X_2 \in O_i$. Then $\mathrm{ad}_{\mathfrak{g}}(X_1), \mathrm{ad}_{\mathfrak{g}}(X_2) \in O'_i$, and then there exists $h \in H$ such that $\mathrm{ad}_{\mathfrak{g}}(X_2) = \mathrm{Ad}(\mathrm{Ad}_G(h))\mathrm{ad}_{\mathfrak{g}}(X_1)$. So $X_2 = \mathrm{Ad}_G(h)X_1 + Y$ for some $Y \in Z(\mathfrak{g})$. But X_1 and X_2 lie in the same adjoint orbit in $\mathfrak{g}$. So there is $g \in G$ such that $X_2 = \mathrm{Ad}_G(g)X_1$. This implies $Y = -\mathrm{Ad}_G(h)X_1 + \mathrm{Ad}_G(g)X_1 \in \gamma(O)$. By Theorem 3.10, $\gamma(O)$ is trivial. So $Y = 0$, and then $X_2 = \mathrm{Ad}_G(h)X_1 + Y$. This shows that O_i is an adjoint orbit in $\mathfrak{h}$. (2) is proved. ■

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