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A Paley-Wiener Theorem for the Bessel-Laplace Transform, I: the case $\mathrm{SU}(n,n)/\mathrm{SL}(n,\mathbb{C})\times\mathbb{R}_+^*$

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Abstract. Let $\mathfrak{q}$ be the tangent space to the noncompact causal symmetric space $SU(n,n)/SL(n,\mathbb{C})\times\mathbb{R}_+^*$ at the origin. In this paper we give an explicit formula for the Bessel functions on $\mathfrak{q}$. We use this result to prove a Paley-Wiener theorem for the Bessel Laplace transform on $\mathfrak{q}$. Further, a flat analogue of the Abel transform is defined and inverted.
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1. Introduction

One of the central theorems in harmonic analysis on $\mathbb{R}$ is the Paley-Wiener theorem which characterizes the space $L^2[-R,R]$ in terms of its image under the Euclidean Fourier transform by showing that: a function is in $L^2[-R,R]$ if and only if its Fourier transform can be continued analytically to the whole complex plane as an entire function of exponential type R [29].

In the last thirty years, analogues of Paley-Wiener theorems for various integral transformations have received a good deal of attention. Among these analogues one may mention the following settings: A Paley-Wiener theorem for the spherical Fourier transform on noncompact Riemannian symmetric spaces has been proved independently by Helgason [14] and Gangolli [13]. Recently, the case of Riemannian symmetric spaces of the compact type with even multiplicities was done by Branson, Ólafsson, and Pasquale [6]. Helgason-Gangolli's Paley-Wiener theorem was generalized later by Opdam for the so-called Cherednik transform [26].

A second direction has been attempted to extend the theory of Paley-Wiener type theorems to the setting of noncompact causal symmetric spaces. In this setting, a Paley-Wiener theorem for the Laplace transform has been proved by Andersen and Ólafsson [2] for the rank-one case. The extension to noncompact causal symmetric spaces of Cayley type was given by Andersen and Unterberger [4]. The proof for arbitrary noncompact causal symmetric space with even multiplicities is

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due independently to Andersen, Ólafsson, and Schlichtkrull [1] and Ólafsson and Pasquale [28].

Another important setting is that of integral transformations on flat symmetric spaces. A Paley-Wiener theorem for the Bessel Fourier transform on the tangent space to a noncompact Riemannian symmetric space at the origin has been proved by Helgason [15]. This result was generalized by de Jeu [21] for the so-called Dunkl transform.

In the present paper we consider the Bessel Laplace transform on the tangent space, say $\mathfrak{q}$, to the noncompact causal symmetric space

$$SU(n,n)/SL(n,\mathbb{C}) \times \mathbb{R}_+^*$$

at the origin. The precise statement of the Paley-Wiener theorem is given in Theorem A. The main tools in the proof are the explicit formula of the Bessel function on $\mathfrak{q}$, and a Bessel Laplace inversion formula. To establish the first tool, our approach uses the explicit formula of the spherical functions on $SU(n,n)/SL(n,\mathbb{C}) \times \mathbb{R}_+^*$ proved in [3], by taking an appropriate zero-curvature limit. We mention that the contraction procedure has been carried out by several authors in different settings. See e.g. [22, 9, 30, 11]. In Remark 5.1 we show how a certain shift operator can be used to recover the explicit formula of the Bessel function on $\mathfrak{q}$ via the rank one case. Thus one can use this shift operator to give an alternative proof for Theorem A. In a forthcoming paper we shall develop this approach further for a larger class of noncompact causal symmetric spaces.

In the last section of this paper we define a flat analogue of the Abel transform on $\mathfrak{q}$. In Theorem B we give an inversion formula for the Abel transform by means of a differential operator.

2. Notation and background

Let $\underline{G} = SU(n,n)$ be the group of complex matrices with determinant 1 which preserve the Hermitian form

$$z_1\overline{w}_1 + \cdots + z_n\overline{w}_n - z_{n+1}\overline{w}_{n+1} - \cdots - z_{2n}\overline{w}_{2n},$$

for $z, w \in \mathbb{C}^{2n}$. The group $\underline{G}$ is a connected noncompact semi-simple Lie group with finite center. Its Lie algebra $\underline{\mathfrak{g}} = \mathfrak{su}(n,n)$ is given by

$$\underline{\mathfrak{g}} = \left\{ \begin{bmatrix} a & b \\ b^* & c \end{bmatrix} \mid a = -a^*,\ c = -c^*,\ \operatorname{tr}(a+c) = 0 \right\},$$

where $a, b, c \in M(n,\mathbb{C})$. It is well known that $\underline{\mathfrak{g}}$ is isomorphic to the Lie algebra

$$\mathfrak{g} := \left\{ \begin{bmatrix} \alpha & \beta \\ \gamma & -\alpha^* \end{bmatrix} \mid \beta = \beta^*,\ \gamma = \gamma^*,\ \operatorname{Im}(\operatorname{tr}(\alpha)) = 0 \right\}.$$

Denote by G the analytic subgroup of $GL(2n,\mathbb{C})$ with Lie algebra $\mathfrak{g}$.

Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the decomposition of $\mathfrak{g}$ into the (± 1)-eigenspaces of the Cartan involution $\theta(X) := -X^*$, with $X \in \mathfrak{g}$. More precisely

$$\mathfrak{k} = \left\{ \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix} \mid \alpha + \alpha^* = 0,\ \beta = \beta^*,\ \operatorname{Im}(\operatorname{tr}(\alpha)) = 0 \right\},$$

and

$$\mathfrak{p} = \left\{ \begin{bmatrix} \alpha & \beta \\ \beta & -\alpha \end{bmatrix} \mid \alpha = \alpha^*,\ \beta = \beta^* \right\}.$$

The analytic subgroup K of G with Lie algebra $\mathfrak{k}$ is isomorphic to $S(U(n)\times U(n))$. The quotient $\mathcal{M}^{\mathrm{d}} := G/K$ is a Riemannian symmetric space of the non-compact type.

Set $\mathfrak{h} := \mathfrak{sl}(n,\mathbb{C}) \oplus \mathbb{R} \cong \{\alpha \in \mathfrak{gl}(n,\mathbb{C}) \mid \mathrm{Im}(\mathrm{tr}(\alpha)) = 0\}$. We may embed $\mathfrak{h}$ in $\mathfrak{g}$ as following

$$\mathfrak{h} \ni \alpha \hookrightarrow \begin{bmatrix} \alpha & 0 \\ 0 & -\alpha^* \end{bmatrix} \in \mathfrak{g}.$$

In particular, the subalgebra $\mathfrak{h}$ corresponds to the $(+1)$-eigenspace of the involution $\sigma : \mathfrak{g} \to \mathfrak{g}$ defined by

$$\sigma\Big(\begin{bmatrix} \alpha & \beta \\ \gamma & -\alpha^* \end{bmatrix}\Big) := \begin{bmatrix} \alpha & -\beta \\ -\gamma & -\alpha^* \end{bmatrix}.$$

The (-1)-eigenspace $\mathfrak{q}$ of σ is given by

$$\mathfrak{q} = \left\{ \begin{bmatrix} 0 & \beta \\ \gamma & 0 \end{bmatrix} \mid \beta = \beta^*,\ \gamma = \gamma^* \right\}.$$

Thus $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}$ is the σ-eigenspace decomposition of $\mathfrak{g}$. Denote by H the analytic subgroup of G with Lie algebra $\mathfrak{h}$. Then $\mathcal{M} := G/H \cong SU(n,n)/SL(n,\mathbb{C}) \times \mathbb{R}^*_+$ is a noncompact causal symmetric space of Cayley type. We refer to [18, Chap. 3] for more details on the theory of causal symmetric spaces of Cayley type. The symmetric space $\mathcal{M}^{\mathrm{d}}$ is (isomorphic to) the so-called Riemannian dual of $\mathcal{M}$.

Let $\mathfrak{a} \subset \mathfrak{p} \cap \mathfrak{q}$ be the Cartan subspace given by

$$\mathfrak{a} := \left\{ a_t = \begin{bmatrix} 0 & t \\ t & 0 \end{bmatrix} \mid t := \mathrm{diag}(t_1/2, \ldots, t_n/2),\ t_1, \ldots, t_n \in \mathbb{R} \right\}.$$

Note that $\mathfrak{a}$ is also a Cartan subspace of $\mathfrak{p}$. From now on we will identify $\mathfrak{a}$ with $\mathbb{R}^n$ via the map

$$\mathbb{R}^n \ni t \mapsto a_t \in \mathfrak{a}.$$

For $1 \le i \le n$, let $\alpha_i \in \mathfrak{a}^*$ be defined by $\alpha_i(t) = -t_i$. Thus, the roots of $(\mathfrak{g}, \mathfrak{a})$ are given by the long ones $\pm\alpha_i$ $(1 \le i \le n)$ and the short ones $\pm(\alpha_j \pm \alpha_i)/2$ $(1 \le i < j \le n)$, with multiplicities 1 and 2, respectively. The root system $\Sigma := \Sigma(\mathfrak{g}, \mathfrak{a})$ is of type C_n. Choose an ordering on $\mathfrak{a}^*$ such that the set Σ^+ of positive roots is given by

$$\Sigma^+ = \left\{ \alpha_i\ (1 \le i \le n), \frac{1}{2}(\alpha_j \pm \alpha_i)\ (1 \le i < j \le n) \right\}.$$

Then the negative open Weyl chamber in $\mathfrak{a}$ on which all elements of Σ^+ are strictly negative is

$$\mathfrak{a}_- = \{t \in \mathbb{R}^n \mid 0 < t_1 < \cdots < t_n\}.$$

Denote by

$$\Sigma_\circ := \left\{ \pm\frac{1}{2}(\alpha_j - \alpha_i)\ (1 \le i < j \le n) \right\},$$

and let

$$\Sigma_\circ^+ := \Sigma^+ \cap \Sigma_\circ = \left\{ \frac{1}{2}(\alpha_j - \alpha_i) \ (1 \le i < j \le n) \right\}.$$

The Weyl groups for Σ and $\Sigma_\circ$ are respectively $\mathcal{W} \cong \mathbb{S}_n \times \{\pm 1\}^n$ and $\mathcal{W}_\circ \cong \mathbb{S}_n$, where $\mathbb{S}_n$ is the permutation group of n elements. The group $\mathcal{W}$ acts on $\mathfrak{a}$ by $t \mapsto (\tau_1 t_{\sigma(1)}, \ldots, \tau_n t_{\sigma(n)})$ with $\tau_i = \pm 1$ and $\sigma \in \mathbb{S}_n$.

For all $\lambda \in \mathbb{C}^n$, denote by φ_λ the Harish-Chandra spherical functions on $\mathcal{M}^{\mathrm{d}}$ with spectral parameter λ (cf. [17, Chap. IV]). In particular, if we use the identification of functions on $\mathcal{M}^{\mathrm{d}}$ with right K-invariant functions on G, then $\varphi_\lambda(kgk') = \varphi_\lambda(g)$ for all $k, k' \in K$ and $g \in G$. Thus, the spherical functions are completely determined by their restriction to $A_- = \exp(\mathfrak{a}_-)$. Furthermore, they are $\mathcal{W}$-invariant on the spectral parameter λ. In [5] Berezin and Karpelevič gave an explicit formula for the Harish-Chandra spherical functions on

$$SU(n,n)/S(U(n) \times U(n)).$$

A complete proof can be found in [19].

Theorem 2.1. (cf. [5, 19]) *There exists a constant that depends only on n such that the spherical functions φ_λ on $SU(n,n)/S(U(n) \times U(n))$ are given by*

$$\varphi_\lambda(\exp(t)) = \text{const.} \frac{\det\limits_{1 \le i,j \le n} \left(P_{\lambda_i - \frac{1}{2}}(\operatorname{ch} t_j) \right)}{\prod\limits_{1 \le i < j \le n} (\lambda_j^2 - \lambda_i^2) \prod\limits_{1 \le i < j \le n} (\operatorname{ch} t_j - \operatorname{ch} t_i)},$$

for all $\lambda \in \mathbb{C}^n$ such that $\prod_{\alpha \in \Sigma^+} \langle \alpha, \lambda \rangle \neq 0$, and for all $t \in \mathfrak{a}_-$. Here P_μ denotes the Legendre function of the first kind.

Remark 2.2. *For fixed t, the function $\lambda \mapsto \varphi_\lambda(\exp(t))$ has a holomorphic extension to $\mathbb{C}^n$.*

From now on we will identify K-bi-invariant functions on $\mathcal{M}^{\mathrm{d}}$ with $\mathcal{W}$-invariant functions on $\mathfrak{a}$. For $\lambda \in \mathbb{C}^n$, the spherical Fourier transform $\mathcal{F}^{\mathrm{d}}(f)$ of a function $f \in \mathcal{C}_c^\infty(\mathfrak{a})^{\mathcal{W}}$ can be written as

$$\mathcal{F}^{\mathrm{d}}(f)(\lambda) = \int_{\mathfrak{a}_-} f(t) \varphi_{-\lambda}(\exp(t)) \Delta(t) dt,$$

where

$$\Delta(t) = 2^{n(n-1)} \prod_{1 \le j \le n} \operatorname{sh} t_j \prod_{1 \le i < j \le n} (\operatorname{ch} t_j - \operatorname{ch} t_i)^2. \tag{1}$$

The inversion formula for $\mathcal{F}^{\mathrm{d}}$ is given by

$$f(t) = \text{const.} \int_{i\mathbb{R}^n} \mathcal{F}^{\mathrm{d}}(f)(\lambda) \varphi_\lambda(\exp(t)) \frac{d\lambda}{|c^{\mathrm{d}}(\lambda)|^2}, \qquad t \in \mathbb{R}^n, \tag{2}$$

where

$$c^{\mathrm{d}}(\lambda) = c(\mathrm{d}) \prod_{1 \le i \le n} \frac{\Gamma(-\lambda_i)}{\Gamma(-\lambda_i + 1/2)} \prod_{1 \le i < j \le n} (\lambda_j^2 - \lambda_i^2)^{-1}. \tag{3}$$

The constant "const" is positive and depends only on the normalization of the measures, and $c(\mathrm{d})$ is a positive constant which can be determined from $c^{\mathrm{d}}(\rho) = 1$, where $\rho = (1/2, 3/2, \ldots, 1/2 + n - 1)$. For more details on the theory of spherical Fourier transforms, we refer to [17, Chap. IV].

Let $c_{\max}$ be the maximal $\mathcal{W}_0$-invariant regular cone in $\mathfrak{a}$ ($\cong \mathbb{R}^n$) defined by

$$c_{\max} := \{t \in \mathbb{R}^n \mid t_i \geq 0 \; (1 \leq i \leq n)\}.$$

The subset $C_{\max} := \mathrm{Ad}(H)c_{\max} \subset \mathfrak{q}$ is a maximal H-invariant regular cone in $\mathfrak{q}$. Denote by $\Gamma(C_{\max}) := \exp(C_{\max})H$ the semi-group in $SU(n,n)$ with interior $\Gamma(C^\circ_{\max}) = \exp(C^\circ_{\max})H = H\exp(c^\circ_{\max})H$.

For $\lambda \in \mathbb{C}^n$, set ψ_λ to be the spherical function on $\mathcal{M}$ with spectral parameter λ (cf. [12]). Note that ψ_λ are only defined on $\Gamma(C^\circ_{\max})$, and H-bi-invariant functions. We mention that for an arbitrary noncompact causal symmetric space, the spherical functions are defined in [12] by an integral formula over H. In [23], the authors determine the exact set $\mathcal{E}$ of $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$ for which the integral is finite. Further, a Harish-Chandra expansion type formula for ψ_λ can be found in [27]. We also note that $\psi_{w\lambda} = \psi_\lambda$ for all $w \in \mathcal{W}_\circ$.

In view of the Berezin-Karpelevič formula for φ_λ, and the Harish-Chandra expansion type formula for ψ_λ, we have:

Theorem 2.3. (cf. [3]) *There exists a constant that depends only on n such that the spherical functions ψ_λ on $SU(n,n)/SL(n,\mathbb{C}) \times \mathbb{R}^*_+$ are given by*

$$\psi_\lambda(\exp(t)) = \mathrm{const.} \frac{\det\limits_{1\leq i,j\leq n}\Big(Q_{\lambda_i - 1/2}(\mathrm{ch}\, t_j)\Big)}{\prod\limits_{1\leq i<j\leq n}(\lambda_j^2 - \lambda_i^2)\prod\limits_{1\leq i<j\leq n}(\mathrm{ch}\, t_j - \mathrm{ch}\, t_i)},$$

for all $\lambda \in \mathbb{C}^n$ such that $\mathrm{Re}(\lambda_i) > 0$ $(1 \leq i \leq n)$ and for all $t \in \mathfrak{a}_-$. Here Q_μ denotes the Legendre function of the second kind.

Remark 2.4. *Recall the set $\mathcal{E}$ from [23]. In the $SU(n,n)/SL(n,\mathbb{C})\times\mathbb{R}^*_+$-case, we have*

$$\mathcal{E} = \{\, \lambda \in \mathbb{C}^n \mid \mathrm{Re}(\lambda_i) > -1/2 \; (1 \leq i \leq n),\; \mathrm{Re}(\lambda_i + \lambda_j) > 0 \; (1 \leq i \neq j \leq n) \,\}.$$

Thus, the statement of Theorem 2.3 remains valid for every λ in $\mathcal{E}$.

Remark 2.5. *Using [20, Theorem 1.2.4] and the fact that $\nu \mapsto Q_\nu(z)$ is a meromorphic function on $\mathbb{C}$ with poles at the points $\nu \in -\mathbb{N}$, one can see that for fixed t, the function $\lambda \mapsto \psi_\lambda(\exp(t))$ has a meromorphic extension to $\mathbb{C}^n$ with simple poles at $\lambda \in \mathbb{C}^n$ such that $\lambda_i \in -\mathbb{N} + 1/2$ $(1 \leq i \leq n)$ and $\lambda_i + \lambda_j = 0$, $(1 \leq i \neq j \leq n)$.*

We may identify the space $\mathcal{C}^\infty_c(H \setminus \Gamma(C^\circ_{\max})/H)$ with $\mathcal{C}^\infty_c(c^\circ_{\max})^{\mathcal{W}_\circ}$. Thus, the spherical Laplace transform $\mathcal{L}(f)$ of a function $f \in \mathcal{C}^\infty_c(c^\circ_{\max})^{\mathcal{W}_\circ}$ can be written as

$$\mathcal{L}(f)(\lambda) = \int_{\mathfrak{a}_-} f(t)\psi_\lambda(\exp(t))\Delta(t)dt,$$

where $\Delta(t)$ is given by (1). The inverse spherical Laplace transform is given by

$$f(t) = \text{const.} \int_{i\mathbb{R}^n} \mathcal{L}(f)(\lambda)\varphi_\lambda(\exp(t)) \frac{d\lambda}{c(\lambda)c^{\mathrm{d}}(-\lambda)}, \qquad t \in c^\circ_{\max} \tag{4}$$

where c^{d} is given by (3), and

$$c(\lambda) = c(\Omega) \prod_{1\le i\le n} \frac{\Gamma(\lambda_i + 1/2)}{\Gamma(\lambda_i + 1)} \prod_{1\le i<j\le n} (\lambda_j^2 - \lambda_i^2)^{-1}. \tag{5}$$

Here $c(\Omega)$ is a positive constant, see [23, Theorem III.5]. We refer to [12] and [18, Chap. 8] for more details on the theory of spherical Laplace transforms.

3. The Bessel Laplace transform

Recall the symmetric spaces

$$\mathcal{M}^{\mathrm{d}} = SU(n,n)/S(U(n)\times U(n)) \quad \text{and} \quad \mathcal{M} = SU(n,n)/SL(n,\mathbb{C})\times \mathbb{R}^*_+.$$

For $\epsilon > 0$, write $g_\epsilon = k\exp(\epsilon X)$ with $k \in K$ and $X \in \mathfrak{p}$. Denote by $\Phi(\lambda, X) := \lim_{\epsilon\to 0} \varphi_{\lambda/\epsilon}(g_\epsilon)$. In [24] the author proved that the limit $\Phi(\lambda, X)$ exists and it is a smooth function. The limiting functions are the so-called Bessel functions on the flat symmetric space $\mathfrak{p}$. In [11, 7] this result was generalized to arbitrary noncompact Riemannian symmetric space. In [8] a similar result (for arbitrary noncompact causal symmetric space) was proved when φ_λ is replaced by ψ_λ. More precisely, if $\gamma_\epsilon = \exp(\epsilon X)h$, with $X \in C^0_{\max}$ and $h \in H$, then for a certain range of $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$, the limit $\Psi(\lambda, X) := \lim_{\epsilon\to 0}\psi_{\lambda/\epsilon}(\exp(\epsilon X))$ and its derivatives exist. See [8] for the proof.

Theorem 3.1. (i) (cf. [24]) *For all $\lambda \in \mathbb{C}^n$ such that $\prod_{\alpha\in\Sigma^+}\langle\lambda,\alpha\rangle \neq 0$, and for all $t \in \mathfrak{a}_-$, there exists a constant which depends only on n such that*

$$\Phi(\lambda, t) = \text{const.} \frac{\det_{1\le i,j\le n}\Big(I_0(\lambda_i t_j)\Big)}{\prod_{1\le i<j\le n}(\lambda_j^2 - \lambda_i^2)\prod_{1\le i<j\le n}(t_j^2 - t_i^2)},$$

where $I_\nu(z) := e^{-i\nu\pi/2}J_\nu(iz)$ and J_ν is the Bessel function of the first kind. The Bessel function Φ extends to a holomorphic function on $\mathbb{C}^n \times \mathbb{C}^n$.

(ii) *For all $\lambda \in \mathbb{C}^n$ such that $\mathrm{Re}(\lambda_i) > 0$ $(1 \le i \le n)$, and for all $t \in \mathfrak{a}_-$, there exists a constant which depends only on n such that*

$$\Psi(\lambda, t) = \text{const.} \frac{\det_{1\le i,j\le n}\Big(K_0(\lambda_i t_j)\Big)}{\prod_{1\le i<j\le n}(\lambda_j^2 - \lambda_i^2)\prod_{1\le i<j\le n}(t_j^2 - t_i^2)},$$

where

$$K_0(z) := \lim_{\nu\to 0}\frac{\pi}{2}\frac{I_{-\nu}(z) - I_\nu(z)}{\sin\nu\pi}$$

denotes the Bessel function of the third kind. For fixed t, the function $\lambda \mapsto \Psi(\lambda, t)$ has a meromorphic extension to

$$D = \{\, \lambda \in \mathbb{C}^n \mid \lambda_i \in \mathbb{C}\backslash]-\infty, 0]\,\},$$

with simple poles at $\lambda \in D$ such that $\lambda_i + \lambda_j = 0$ for some $1 \le i \neq j \le n$.

Proof. (ii) For $\epsilon > 0$, write $\psi_{\lambda/\epsilon}(\exp(\epsilon t))$ as

$$\psi_{\lambda/\epsilon}(\exp(\epsilon t)) = \text{const.} \frac{\epsilon^{n(n-1)}}{\prod_{1\le i<j\le n}(\lambda_j^2-\lambda_i^2)\prod_{1\le i<j\le n}(\mathrm{sh}^2(\epsilon t_j/2)-\mathrm{sh}^2(\epsilon t_i/2))} \times \sum_{\sigma\in\mathbb{S}_n}(-1)^\sigma \prod_{1\le i\le n} Q_{\lambda_{\sigma(i)}/\epsilon-1/2}(\mathrm{ch}\,\epsilon t_i).$$

By [31, p.259], we have

$$\begin{aligned}&(\mathrm{sh}\,t)^{-\mu}\frac{\Gamma(\lambda-\mu+1/2)}{\Gamma(\lambda+\mu+1/2)}Q^\mu_{\lambda-1/2}(\mathrm{ch}\,t)\\ &= \frac{e^{i\pi\mu}}{2}\left\{\frac{\Gamma(-\mu)}{2^\mu}{}_2F_1\Big(\frac{1}{2}(\lambda+\mu+\frac{1}{2}),\frac{1}{2}(-\lambda+\mu+\frac{1}{2});1+\mu;-\mathrm{sh}^2\,t\Big)+\frac{\Gamma(\mu)}{2^{-\mu}}\right.\\ &\left.(\mathrm{sh}\,t)^{-2\mu}\frac{\Gamma(\lambda-\mu+1/2)}{\Gamma(\lambda+\mu+1/2)}{}_2F_1\Big(\frac{1}{2}(\lambda-\mu+\frac{1}{2}),\frac{1}{2}(-\lambda-\mu+\frac{1}{2});1-\mu;-\mathrm{sh}^2\,t\Big)\right\}.\end{aligned}$$

Using the well known formula

$$\frac{\Gamma(z+a)}{\Gamma(z+b)} = z^{a-b}(1+O(z^{-1})) \qquad \text{as} \quad z\to\infty, \tag{6}$$

together with the hypergeometric series of ${}_2F_1$, we obtain:

$$\lim_{\epsilon\to 0}{}_2F_1\Big(\frac{1}{2}(\frac{\lambda}{\epsilon}\pm\mu+\frac{1}{2}),\frac{1}{2}(-\frac{\lambda}{\epsilon}\pm\mu+\frac{1}{2});1\pm\mu;-\mathrm{sh}^2\,\epsilon t\Big) = \Gamma(\pm\mu+1)\Big(\frac{\lambda t}{2}\Big)^{\mp\mu}I_{\pm\mu}(\lambda t),$$

and

$$\lim_{\epsilon\to 0}\frac{\Gamma(\frac{\lambda}{\epsilon}-\mu+\frac{1}{2})}{\Gamma(\frac{\lambda}{\epsilon}+\mu+\frac{1}{2})}(\mathrm{sh}\,\epsilon t)^{-2\mu} = (\lambda t)^{-2\mu}.$$

Here I_μ denotes the modified Bessel function given in the statement (i) above. Thus

$$\begin{aligned}&\lim_{\epsilon\to 0}(\mathrm{sh}\,\epsilon t)^{-\mu}\frac{\Gamma(\lambda/\epsilon-\mu+1/2)}{\Gamma(\lambda/\epsilon+\mu+1/2)}Q^\mu_{\lambda/\epsilon-1/2}(\mathrm{ch}\,\epsilon t)\\ &= \frac{e^{i\pi\mu}}{2}\Big\{\frac{\Gamma(-\mu)\Gamma(1+\mu)}{2^\mu}\Big(\frac{\lambda t}{2}\Big)^{-\mu}I_\mu(\lambda t)+\frac{\Gamma(\mu)\Gamma(1-\mu)}{2^{-\mu}}\frac{(\lambda t)^{-\mu}}{2^\mu}I_{-\mu}(\lambda t)\Big\}\\ &= e^{i\pi\mu}(\lambda t)^{-\mu}\Big\{\frac{\pi}{2}\frac{I_{-\mu}(\lambda t)-I_\mu(\lambda t)}{\sin(\pi\mu)}\Big\},\end{aligned}$$

and therefore

$$\lim_{\epsilon\to 0}Q_{\lambda/\epsilon-1/2}(\mathrm{ch}\,\epsilon t) = \lim_{\mu\to 0}\frac{\pi}{2}\frac{I_{-\mu}(\lambda t)-I_\mu(\lambda t)}{\sin(\pi\mu)} = K_0(\lambda t).$$

Now one may use [20, Theorem 1.2.4] to prove that the only singularities of $\Psi(\lambda,t)$ in $\lambda\in D$ are those for which $\lambda_i+\lambda_j=0$, with $i\neq j$. ■

Remark 3.2. *(i) The Bessel function Φ is symmetric in its arguments. Further, since $I_0(z)$ is an even function, clearly we have $\Phi(w\lambda, t) = \Phi(\lambda, wt) = \Phi(\lambda, t)$ for all $w \in \mathcal{W} = \mathbb{S}_n \times \{\pm 1\}^n$. For general results in the theory of Bessel functions associated with Cartan motion groups, we refer to [17, 25].*

(ii) The Bessel function Ψ is symmetric in λ and t, with $\Psi(w_0\lambda, t) = \Psi(\lambda, w_0 t) = \Psi(\lambda, t)$ for all $w_0 \in \mathcal{W}_\circ = \mathbb{S}_n$.

Following [16], the Bessel Fourier transform $\widetilde{\mathcal{F}}(f)$ of a function $f \in \mathcal{C}_c^\infty(\mathfrak{a})^{\mathcal{W}}$ is given by

$$\widetilde{\mathcal{F}}(f)(\lambda) = \int_{\mathfrak{a}_-} f(t)\Phi(\lambda, t)\omega(t)dt,$$

where

$$\omega(t) := \prod_{1\le i\le n} t_i \prod_{1\le i<j\le n} (t_j^2 - t_i^2)^2, \qquad t \in \mathfrak{a}_-. \tag{7}$$

Further, there exists a positive constant depending only on the normalization of the measures such that

$$f(t) = \text{const.} \int_{i\mathbb{R}^n} \widetilde{\mathcal{F}}(f)(\lambda)\Phi(\lambda, t)\omega(\lambda)d\lambda, \tag{8}$$

where

$$\omega(\lambda) := \prod_{1\le i\le n} |\lambda_i| \prod_{1\le i<j\le n} |\lambda_j^2 - \lambda_i^2|^2. \tag{9}$$

Observe that one may recover the definition of $\widetilde{\mathcal{F}}$ and its inversion formula via $\mathcal{F}$, by applying a limit transition approach. Indeed, for $\epsilon > 0$, set $f_\epsilon(t) := f(\epsilon^{-1}t)$. Then

$$\begin{aligned}\mathcal{F}^{\mathrm{d}}(f_\epsilon)(\lambda/\epsilon) &= \int_{\mathfrak{a}_-} f_\epsilon(t)\varphi_{-\lambda/\epsilon}(\exp t) \prod_{1\le i\le n} \operatorname{sh} t_i \prod_{1\le i<j\le n} (2\operatorname{ch} t_j - 2\operatorname{ch} t_i)^2 dt \\ &\sim \epsilon^{n(2n-1)} \int_{\mathfrak{a}_-} f(t)\varphi_{-\lambda/\epsilon}(\exp \epsilon t) \prod_{1\le i\le n} t_i \prod_{1\le i<j\le n} (t_j^2 - t_i^2)^2 dt\end{aligned}$$

as $\epsilon \to 0$. Hence

$$\lim_{\epsilon\to 0} \epsilon^{-n(2n-1)} \mathcal{F}^{\mathrm{d}}(f_\epsilon)(\lambda/\epsilon) = \widetilde{\mathcal{F}}(f)(\lambda). \tag{10}$$

By virtue of (6), one can also use the inversion formula (2) for $\mathcal{F}$ to recover(8). We should mention that the Bessel Fourier transform has been carried out by several authors in different settings (see e.g. [15, 16, 24, 33]).

Remark 3.3. *In [10] Dunkl introduced an integral transformation on the space $L^2(\mathfrak{a}, d\mu)$ (where μ is some suitable measure) in terms of the eigenfunctions of the so-called Dunkl operators. This class of Dunkl transforms encloses the Bessel Fourier transforms on flat symmetric spaces.*

Define the Bessel Laplace transform $\widetilde{\mathcal{L}}(f)$ of a function $f \in \mathcal{C}_c^\infty(c_{\max}^\circ)^{\mathcal{W}_\circ}$ by

$$\widetilde{\mathcal{L}}(f)(\lambda) = \int_{\mathfrak{a}_-} f(t)\Psi(\lambda, t)\omega(t)dt, \qquad \forall f \in \mathcal{C}_c^\infty(c_{\max}^\circ)^{\mathcal{W}_\circ},$$

whenever this integral converges. Once again one may obtain the above natural definition of $\widetilde{\mathcal{L}}$ via the spherical Laplace transform $\mathcal{L}$.

By [27, Lemma 4.16], we know that if $f \in \mathcal{C}_c(c^\circ_{\max})^{\mathcal{W}_\circ}$, then there exists a unique function $f^{\mathrm{d}} \in \mathcal{C}_c(\mathfrak{a})^{\mathcal{W}}$ such that $f^{\mathrm{d}}_{|\mathfrak{a}_-} \equiv f_{|\mathfrak{a}_-}$. Thus, we may obtain the following relation between $\widetilde{\mathcal{F}}$ and $\widetilde{\mathcal{L}}$: for $t \in \mathfrak{a}_-$, we know that

$$\varphi_\lambda(\exp(-t)) = \sum_{\tau \in \{\pm 1\}^n} \frac{c^{\mathrm{d}}(\tau\lambda)}{c(\tau\lambda)} \psi_{\tau\lambda}(\exp(t)), \tag{11}$$

for almost every $\lambda \in \mathbb{C}^n$ (cf. [18, Theorem 8.4.4]). Further, in the light of (6), we have

$$c^{\mathrm{d}}(\lambda/\epsilon) \sim \epsilon^{n(n-1/2)} c(\mathrm{d}) \prod_{1\le i\le n} (-\lambda_i)^{-1/2} \prod_{1\le i<j\le n} (\lambda_j^2 - \lambda_i^2)^{-1} \qquad \text{as } \epsilon \to 0,$$

and

$$c(\lambda/\epsilon) \sim \epsilon^{n(n-1/2)} c(\Omega) \prod_{1\le i\le n} \lambda_i^{-1/2} \prod_{1\le i<j\le n} (\lambda_j^2 - \lambda_i^2)^{-1} \qquad \text{as } \epsilon \to 0.$$

Thus

$$\Phi(\lambda, t) = \frac{c(\mathrm{d})}{c(\Omega)} \sum_{\tau=(\tau_i)_i \in \{\pm 1\}^n} \prod_{1\le i\le n} \{(-\tau_i\lambda_i)^{-1/2} (\tau_i\lambda_i)^{1/2}\} \Psi(\tau\lambda, t) \qquad \forall\, t \in \mathfrak{a}_-, \tag{12}$$

for almost every $\lambda \in \mathbb{C}^n$. When $n = 1$, we have $c(\mathrm{d})c(\Omega)^{-1} = \pi^{-1}$, and the equality (12) coincides with the well known formula $K_0(z) - K_0(-z) = i\pi I_0(z)$ (cf. [31, p. 428]). Now the following is clear.

Corollary 3.4. *For almost every $\lambda \in \mathbb{C}^n$ and for all $f \in \mathcal{C}_c^\infty(c^\circ_{\max})^{\mathcal{W}_\circ}$*

$$\widetilde{\mathcal{F}}(f^{\mathrm{d}})(\lambda) = \frac{c(\mathrm{d})}{c(\Omega)} \sum_{\tau=(\tau_i)_i \in \{\pm 1\}^n} \prod_{1\le i\le n} \{(-\tau_i\lambda_i)^{-1/2} (\tau_i\lambda_i)^{1/2}\} \widetilde{\mathcal{L}}(f)(\tau\lambda).$$

In particular, the right hand side extends to an analytic function on $\mathbb{C}^n$.

The Bessel Laplace inversion formula is now immediate.

Theorem 3.5. *If $f \in \mathcal{C}_c^\infty(c^\circ_{\max})^{\mathcal{W}_\circ}$, then there exists a positive constant such that*

$$f(t) = \mathrm{const.} \int_{i\mathbb{R}^n} \widetilde{\mathcal{L}}(f)(\lambda) \Phi(\lambda, t) \omega(\lambda) \prod_{1\le i\le n} \frac{\lambda_i}{|\lambda_i|} d\lambda$$

for all $t \in \mathfrak{a}_-$. Here $\omega(\lambda)$ is as in (9).

Proof. For $t \in \mathfrak{a}_-$ we have

$$\begin{aligned} f(t) &= \mathrm{const.} \int_{i\mathbb{R}^n} \widetilde{\mathcal{F}}^{\mathrm{d}}(f^{\mathrm{d}})(\lambda) \Phi(\lambda, t) \omega(\lambda) d\lambda \\ &= \mathrm{const.} \int_{i\mathbb{R}^n} \Big\{ \sum_{\tau \in \{\pm 1\}^n} \big(\prod_{1\le i\le n} \tau_i \big) \widetilde{\mathcal{L}} f(\tau\lambda) \Big\} \Phi(\lambda, t) \omega(\lambda) \prod_{1\le i\le n} \frac{\lambda_i}{|\lambda_i|} d\lambda. \end{aligned}$$

The statement is now due to the $\mathcal{W}$-invariance of Φ and $\omega(\lambda) d\lambda$. ■

4. The Paley-Wiener theorem

For $R>0$, let $B_R := \{t \in \mathbb{R}^n \mid \|t\| \leq R\}$. Denote by $\mathcal{C}_R^\infty(\mathfrak{a})$ the space of smooth functions on $\mathfrak{a}$ with support contained in the closed ball B_R. Define the Paley-Wiener space $\mathcal{H}_\mathcal{W}^R(\mathbb{C}^n)$ as the space of $\mathcal{W}$-invariant holomorphic functions on $\mathbb{C}^n$ with the property that for each $M \in \mathbb{N}$ there exists a constant $c_M > 0$ such that

$$|g(\lambda)| \leq c_M(1+\|\lambda\|)^{-M} e^{R\|\mathrm{Re}(\lambda)\|}, \qquad \forall\, \lambda \in \mathbb{C}^n.$$

We will denote by $\mathcal{H}_\mathcal{W}(\mathbb{C}^n)$ the union of the spaces $\mathcal{H}_\mathcal{W}^R(\mathbb{C}^n)$ over all $R>0$.

Theorem 4.1. (cf. [15]) *The Bessel Fourier transform $f \mapsto \widetilde{\mathcal{F}}(f)$ is a bijection of $\mathcal{C}_c^\infty(\mathfrak{a})^\mathcal{W}$ onto $\mathcal{H}_\mathcal{W}(\mathbb{C}^n)$. The function f has support in the ball B_R if and only if $\widetilde{\mathcal{F}}(f) \in \mathcal{H}_\mathcal{W}^R(\mathbb{C}^n)$.*

Next we will discuss a Paley-Wiener theorem for $\widetilde{\mathcal{L}}$. For $0<r<R<\infty$, let $\mathcal{PW}_\circ^{r,R}(\mathbb{C}^n)$ be the space of $\mathcal{W}_\circ$-invariant meromorphic functions g on D with at most simple poles at $\lambda_i+\lambda_j=0$ $(1 \leq i \neq j \leq n)$ such that:

($\mathbb{P}_1$) the map

$$\lambda \mapsto \mathtt{av}(g)(\lambda) := \sum_{\tau \in \{\pm 1\}^n} \prod_{1 \leq i \leq n} \{(-\tau_i\lambda_i)^{-1/2}(\tau_i\lambda_i)^{1/2}\} g(\tau\lambda)$$

extends to a function in $\mathcal{H}_\mathcal{W}^R(\mathbb{C}^n)$.

($\mathbb{P}_2$) for all $M \in \mathbb{N}$, there exists a constant c_M such that for $\lambda \in D$ with $\mathrm{Re}(\lambda_i) \geq 0$ $(1 \leq i \leq n)$ we have

$$\prod_{1\leq i \leq n} |\lambda_i|^{1/2} \prod_{1 \leq i<j \leq n} |\lambda_i^2-\lambda_j^2|\, |g(\lambda)| \leq c_M(1+\|\lambda\|)^{-M} e^{-r\langle \mathrm{Re}(\lambda), t_0\rangle},$$

where $t_0 := (1,\ldots,1)$.

Denote by $\mathcal{PW}_\circ(\mathbb{C}^n)$ the union of the spaces $\mathcal{PW}_\circ^{r,R}(\mathbb{C}^n)$ over all $0<r<R<\infty$.

Lemma 4.2. *For all $\lambda \in D$ such that* $\mathrm{Re}(\lambda_i) \geq 0$ *$(1 \leq i \leq n)$, and for all $t \in \mathbb{R}^n$ such that $t_i \geq r > 0$ $(1 \leq i \leq n)$, we have*

$$|\Psi(\lambda,t)| \prod_{1 \leq i<j \leq n} |t_j^2-t_i^2| \prod_{1 \leq i \leq n} |\lambda_i|^{1/2} \prod_{1 \leq i<j \leq n} |\lambda_i^2-\lambda_j^2| \leq c e^{-r\langle \mathrm{Re}(\lambda), t_0\rangle},$$

where $t_0=(1,\ldots,1)$ and c is a constant which depends only on r and n.

Proof. For all $t \in \mathbb{R}^n$ we have

$$\begin{aligned} |\Psi(\lambda,t)| \prod_{1\leq i\leq n} |\lambda_i|^{1/2} \prod_{1 \leq i<j \leq n} |t_i^2-t_j^2|\, |\lambda_i^2-\lambda_j^2| &= \prod_{1 \leq i \leq n} |\lambda_i|^{1/2} \Big| \det_{1 \leq i,j \leq n} \Big(K_0(\lambda_i t_j)\Big)\Big| \\ &\leq \sum_{\sigma \in \mathbb{S}_n} \prod_{1 \leq i \leq n} |\lambda_i|^{1/2} \left|K_0(\lambda_i t_{\sigma(i)})\right|. \end{aligned}$$

It is well known that for $z \in \mathbb{C}\backslash]\infty,0]$ we have $K_0(z)=\sqrt{\frac{\pi}{2z}} W_{0,0}(2z)$, where $W_{0,0}$ denotes the Whittaker function. Using the expression (4), p. 317 of [31], and the asymptotic expression (1), p. 202 of [32], we get

$$|z|^{1/2}|K_0(z)| \leq \mathrm{const.}\, e^{-\mathrm{Re}(z)}, \qquad z \in \mathbb{C}\backslash]\infty,0].$$

Thus, if $t_{\sigma(i)} \geq r > 0$ and $\lambda_i \in \mathbb{C}\backslash]\infty, 0]$, then

$$|\lambda_i|^{1/2}|K_0(\lambda_i t_{\sigma(i)})| \leq c_r e^{-\mathrm{Re}(\lambda_i)t_{\sigma(i)}}.$$

If in addition $\mathrm{Re}(\lambda_i) \geq 0$, we obtain

$$|\lambda_i|^{1/2}|K_0(\lambda_i t_{\sigma(i)})| \leq c_r e^{-r\mathrm{Re}(\lambda_i)}.$$

Now the desired lemma is clear. ■

For $0 < r < \infty$, we set $C_r := \{\, t \in \mathbb{R}^n \mid t_i \geq r \ (1 \leq i \leq n) \,\}$. Denote by $\mathcal{C}^\infty_{r,R}(c^\circ_{\max})^{\mathcal{W}_\circ}$ the space of functions $f \in \mathcal{C}^\infty_c(c^\circ_{\max})^{\mathcal{W}_\circ}$ with support contained in $C_r \cap B_R$. Note that $\mathcal{C}^\infty_{r,R}(c^\circ_{\max})^{\mathcal{W}_\circ} = \{0\}$ if $R \leq r$. The union of the spaces $\mathcal{C}^\infty_{r,R}(c^\circ_{\max})^{\mathcal{W}_\circ}$ over all $0 < r < R < \infty$ coincides with $\mathcal{C}^\infty_c(c^\circ_{\max})^{\mathcal{W}_\circ}$.

Lemma 4.3. *For all $0 < r < R < \infty$, the transformation $\widetilde{\mathcal{L}}$ maps $\mathcal{C}^\infty_{r,R}(c^\circ_{\max})^{\mathcal{W}_\circ}$ injectively into $\mathcal{PW}^{r,R}_\circ(\mathbb{C}^n)$.*

Proof. Since the function $\lambda \mapsto \Psi(\lambda, t)$ is meromorphic on D with simple poles at $\lambda_i + \lambda_j = 0$ for $1 \leq i \neq j \leq n$, it follows that $\lambda \mapsto \widetilde{\mathcal{L}}(f)(\lambda)$ extends to a meromorphic function on D with simple poles at $\lambda_i + \lambda_j = 0$ for $i \neq j$. Further, the $\mathcal{W}_\circ$-invariance of the Bessel functions Ψ implies that $\lambda \mapsto \widetilde{\mathcal{L}}(f)(\lambda)$ is a $\mathcal{W}_\circ$-invariant map for all $f \in \mathcal{C}^\infty_c(c^\circ_{\max})^{\mathcal{W}_\circ}$. Moreover, by means of Corollary 3.4, the Bessel Laplace transform $\widetilde{\mathcal{L}}$ satisfies the property $(\mathbb{P}_1)$. One can also check that $\widetilde{\mathcal{L}}$ obeys the property $(\mathbb{P}_2)$. Indeed, for $f \in \mathcal{C}^\infty_{r,R}(c^\circ_{\max})^{\mathcal{W}_\circ}$ we have

$$\begin{aligned}
&\prod_{1\leq i\leq n} |\lambda_i|^{1/2} \prod_{1\leq i<j\leq n} |\lambda_i^2 - \lambda_j^2| \left|\widetilde{\mathcal{L}}(f)(\lambda)\right| \\
&\leq \int_{\mathfrak{a}_-\cap \mathrm{supp}(f)} |f(t)| \; |\Psi(\lambda, t)| \prod_{1\leq i\leq n} |\lambda_i|^{1/2} \prod_{1\leq i<j\leq n} |\lambda_i^2 - \lambda_j^2| \omega(t)dt \\
&\leq c_{r,R}\, e^{-r\langle \mathrm{Re}(\lambda), t_0\rangle}.
\end{aligned}$$

Above we used Lemma 4.2. To reach the conclusion, it is enough to recall that $\Psi(\lambda, t)$ satisfies a Bessel system of differential equations (cf. [8, (4.8)]).

The injectivity of $\widetilde{\mathcal{L}}$ follows from the inversion formula in Theorem 3.5. ■

Lemma 4.4. *If $\mathtt{av}(\widetilde{\mathcal{L}}(f)) \equiv 0$ with $f \in \mathcal{C}^\infty_c(c^\circ_{\max})^{\mathcal{W}_\circ}$, then $f \equiv 0$.*

Proof. The statement of Corollary 3.4 can also be written as $\widetilde{\mathcal{F}}(f^{\mathrm{d}})(\lambda) = \frac{c(\mathrm{d})}{c(\Omega)}\mathtt{av}(\widetilde{\mathcal{L}}(f))(\lambda)$, where $f^{\mathrm{d}}_{|\mathfrak{a}_-} \equiv f_{|\mathfrak{a}_-}$. Now the claim is an easy consequence of the injectivity of $\widetilde{\mathcal{F}}$. ■

The following statement can be proved in a similar way as Lemma 9.1 in [1]. The function g bellow plays the same role as g_1 in the proof of [1, Lemma 9.1].

Lemma 4.5. *Let g be a meromorphic function on D which satisfies the condition $(\mathbb{P}_2)$ for some $r > 0$. If $\mathtt{av}(g) \equiv 0$, then $g \equiv 0$.*

We have now all ingredients to state and prove the first main result of the paper. Our approach is similar to the one used in [1] for the spherical Laplace transform.

Theorem A. *The Bessel Laplace transform $\widetilde{\mathcal{L}}$ is a bijection from $\mathcal{C}^\infty_{r,R}(c^\circ_{\max})^{\mathcal{W}_\circ}$ onto $\mathcal{PW}^{r,R}_\circ(\mathbb{C}^n)$ for every $0 < r < R < \infty$, and from $\mathcal{C}^\infty_c(c^\circ_{\max})^{\mathcal{W}_\circ}$ onto $\mathcal{PW}_\circ(\mathbb{C}^n)$.*

Proof. By virtue of Lemma 4.3 we only need to prove the surjectivity of $\widetilde{\mathcal{L}}$ from $\mathcal{C}^\infty_{r,R}(c^\circ_{\max})^{\mathcal{W}_\circ}$ to $\mathcal{PW}^{r,R}_\circ(\mathbb{C}^n)$. By Theorem 3.1 part (i), we have

$$\begin{aligned}\Phi(\lambda,t) &= \frac{\displaystyle\sum_{\sigma\in\mathbb{S}_n}(-1)^\sigma \prod_{1\le i\le n} I_0\big(\lambda_{\sigma(i)}t_i\big)}{\displaystyle\prod_{1\le i<j\le n}(\lambda_j^2-\lambda_i^2)\prod_{1\le i<j\le n}(t_j^2-t_i^2)}\\ &= \sum_{\sigma\in\mathbb{S}_n}\frac{\displaystyle\prod_{1\le i\le n} I_0\big(\lambda_{\sigma(i)}t_i\big)}{\displaystyle\prod_{1\le i<j\le n}(t_j^2-t_i^2)\prod_{1\le i<j\le n}(\lambda_{\sigma(j)}^2-\lambda_{\sigma(i)}^2)}\\ &= \sum_{\sigma\in\mathbb{S}_n}\Xi(\sigma(\lambda),t),\end{aligned}$$

where

$$\Xi(\lambda,t) := \frac{\displaystyle\prod_{1\le i\le n} I_0(\lambda_i t_i)}{\displaystyle\prod_{1\le i<j\le n}(t_j^2-t_i^2)\prod_{1\le i<j\le n}(\lambda_j^2-\lambda_i^2)},$$

with $t_i \neq \pm t_j$ and $\lambda_i \neq \pm\lambda_j$ for $i \neq j$.

For $\lambda \in \mathbb{C}^n$, let

$$\vartheta(\lambda) := \prod_{1\le i<j\le n}(\lambda_j^2-\lambda_i^2).$$

Fix r and R, and define the wave packet of $g \in \mathcal{PW}^{r,R}_\circ(\mathbb{C}^n)$ by

$$\mathcal{I}g(t) = \int_{i\mathbb{R}^n} g(\lambda)\Phi(\lambda,t)\vartheta(\lambda)^2\prod_{1\le i\le n}\lambda_i d\lambda$$

when $t \in \mathfrak{a}_-$. The function $\mathcal{I}g$ is well defined and it belongs to $\mathcal{C}^\infty(c^\circ_{\max})^{\mathcal{W}_\circ}$. This follows from the growth behavior of $g \in \mathcal{PW}^{r,R}_\circ(\mathbb{C}^n)$, and the fact that

$$|\partial^{\alpha_1}_{t_1}\ldots\partial^{\alpha_n}_{t_n}\Phi(\lambda,t)| \le \text{const.}\,\|\lambda\|^{\alpha_1+\cdots+\alpha_n},$$

with $\lambda \in i\mathbb{R}^n$ and $t \in \mathbb{R}^n$. Here the constant "const" depends only on $\alpha_1,\ldots,\alpha_n$. Notice that for $\lambda \in i\mathbb{R}^n$, $\vartheta(\lambda)^2\prod_{1\le i\le n}\lambda_i = \omega(\lambda)\prod_{1\le i\le n}\frac{\lambda_i}{|\lambda_i|}$. Bellow we will prove that the support of $\mathcal{I}g$ is contained in $C_r \cap B_R$, i.e. $\mathcal{I}g \in \mathcal{C}^\infty_{r,R}(c^\circ_{\max})^{\mathcal{W}_\circ}$.

By the $\mathcal{W}_\circ$-invariance of g and $\vartheta(\lambda)^2\prod_{1\le i\le n}\lambda_i$, we have

$$\mathcal{I}g(t)=n!\int_{i\mathbb{R}^n} g(\lambda)\Xi(\lambda,t)\vartheta(\lambda)^2\prod_{1\le i\le n}\lambda_i d\lambda,\qquad t\in\mathfrak{a}_-.$$

On the other hand, using the expression of I_0 in [32, p. 77] and the asymptotic expression (2), p. 203 of [32], it follows that there exist two positive constants such that

$$\begin{array}{ll}|I_0(z)|\le \text{const.}\,, & 0\le |z|\le 1,\\ |I_0(z)|\le \text{const.}\,|z|^{-1/2}e^{\mathrm{Re}(z)}, & 1\le |z|.\end{array}$$

Thus, for fixed $t\in\mathfrak{a}_-$,

$$|\vartheta(\lambda)|\prod_{1\le i\le n}|\lambda_i|^{1/2}|\Xi(\lambda,t)|\le \text{const.}\frac{1}{\prod_{1\le i\le n}t_i^{1/2}\prod_{1\le i<j\le n}(t_j^2-t_i^2)}\tag{13}$$

if $|\lambda_i|\le t_i^{-1}$ for all i, and

$$|\vartheta(\lambda)|\prod_{1\le i\le n}|\lambda_i|^{1/2}|\Xi(\lambda,t)|\le \text{const.}\frac{e^{\langle \mathrm{Re}(\lambda),t\rangle}}{\prod_{1\le i\le n}t_i^{1/2}\prod_{1\le i<j\le n}(t_j^2-t_i^2)}\tag{14}$$

if $|\lambda_i|\ge t_i^{-1}$ for all i.

Now let $t\in\mathfrak{a}_-\backslash C_r$. By [1, p. 721], there exists an element $\lambda^\circ\in\mathbb{R}^n_+$ such that $\zeta:=\langle\lambda^\circ,t-rt_\circ\rangle<0$, where $t_\circ=(1,\dots,1)$. Hence, for arbitrary $\alpha\gg 0$, we have

$$\begin{aligned}|\vartheta(\lambda+\alpha\lambda^\circ)|\prod_{i=1}^n|\lambda_i+\alpha\lambda_i^\circ|^{1/2}|\Xi(\lambda+\alpha\lambda^\circ,t)| &= \frac{\prod_{i=1}^n|\lambda_i+\alpha\lambda_i^\circ|^{1/2}\left|I_0\big((\lambda_i+\alpha\lambda_i^\circ)t_i\big)\right|}{\prod_{1\le i<j\le n}(t_j^2-t_i^2)}\\ &\sim \frac{e^{\langle \mathrm{Re}(\lambda+\alpha\lambda^\circ),t\rangle}}{\prod_{i=1}^n t_i^{1/2}\prod_{1\le i<j\le n}(t_j^2-t_i^2)}\end{aligned}\tag{15}$$

as $\alpha\to\infty$. Here we used the fact that $I_0(z)\sim z^{-1/2}e^z$ as $z\to\infty$. In particular, if $\lambda\in i\mathbb{R}^n$ and $t\in\mathfrak{a}_-\setminus C_r$, there exists a constant not depending on λ such that the left hand side of (15) is bounded by $ce^{\alpha\langle\lambda^\circ,t\rangle}$ as α goes to infinity. That is

$$|\vartheta(\lambda+\alpha\lambda^\circ)|\prod_{1\le i\le n}|\lambda_i+\alpha\lambda_i^\circ|^{1/2}|\Xi(\lambda+\alpha\lambda^\circ,t)|\le ce^{\alpha\zeta}e^{r\alpha\langle\lambda^\circ,t_\circ\rangle}\qquad \text{as }\ \alpha\to\infty.\tag{16}$$

By virtue of (13), (14), (16), and the growth behavior of $g\in\mathcal{PW}_\circ^{r,R}(\mathbb{C}^n)$, Cauchy's theorem and a contour shift imply that

$$\begin{aligned}\mathcal{I}g(t) &= n!\int_{i\mathbb{R}^n} g(\lambda)\Xi(\lambda,t)\vartheta(\lambda)^2\prod_{1\le i\le n}\lambda_i d\lambda\\ &= n!\int_{i\mathbb{R}^n} g(\lambda+\alpha\lambda^\circ)\Xi(\lambda+\alpha\lambda^\circ,t)\vartheta(\lambda+\alpha\lambda^\circ)^2\prod_{1\le i\le n}(\lambda_i+\alpha\lambda_i^\circ)d\lambda\\ &\longrightarrow 0\qquad \text{as }\ \alpha\to\infty.\end{aligned}$$

Thus $\mathcal{I}g$ vanishes on $\mathfrak{a}_-\setminus C_r$, and, by the continuity and the $\mathcal{W}_\circ$-invariance of $\mathcal{I}g$, this is equivalent to $\mathcal{I}g \equiv 0$ on $c^\circ_{\max}\setminus C_r$. Furthermore, the wave packet vanishes also on $c^\circ_{\max}\setminus B_R$. One can see this as following: If one recalls that for $\lambda \in i\mathbb{R}^n$, $\vartheta(\lambda)^2\prod_{1\le i\le n}\lambda_i = \omega(\lambda)\prod_{1\le i\le n}\frac{\lambda_i}{|\lambda_i|}$, then by the $\mathcal{W}$-invariance of Φ and $\omega(\lambda)$, one has (for $t \in \mathfrak{a}_-$)

$$\begin{aligned}\mathcal{I}g(t) &= \int_{i\mathbb{R}^n} g(\lambda)\Phi(\lambda,t)\omega(\lambda)\prod_{1\le i\le n}\frac{\lambda_i}{|\lambda_i|}d\lambda\\ &= \frac{1}{2^n}\int_{i\mathbb{R}^n}\sum_{\tau\in\{\pm1\}^n}\prod_{1\le i\le n}\{(-\tau_i\lambda_i)^{-1/2}(\tau_i\lambda_i)^{1/2}\}g(\tau\lambda)\Phi(\lambda,t)\omega(\lambda)d\lambda\\ &= \frac{1}{2^n}\frac{c(\Omega)}{c(\mathrm{d})}\int_{i\mathbb{R}^n}\mathtt{av}(g)(\lambda)\Phi(\lambda,t)\omega(\lambda)d\lambda.\end{aligned}$$

Comparing this formula with (8), we get (up to a positive constant which does not depend on λ)

$$\widetilde{\mathcal{F}}(\mathcal{I}g)(\lambda) = \text{const.}\,\mathtt{av}(g)(\lambda). \tag{17}$$

Since $g \in \mathcal{PW}^{r,R}_\circ(\mathbb{C}^n)$, the property $(\mathbb{P}_1)$ implies that $\widetilde{\mathcal{F}}(\mathcal{I}g)$ belongs to the Paley-Wiener space $\mathcal{H}^R_{\mathcal{W}}(\mathbb{C}^n)$. Hence, by Theorem 4.1, $\mathrm{supp}(\mathcal{I}g) \subset B_R$, i.e. $\mathcal{I}g(t) = 0$ for all $t \in c^\circ_{\max}\setminus B_R$. Thus we draw the conclusion that $\mathcal{I}g \in \mathcal{C}^\infty_{r,R}(c^\circ_{\max})^{\mathcal{W}_\circ}$. Moreover, in view of Corollary 3.4, equation (17) yields

$$\frac{c(\mathrm{d})}{c(\Omega)}\mathtt{av}(\widetilde{\mathcal{L}}(\mathcal{I}g))(\lambda) = \widetilde{\mathcal{F}}(\mathcal{I}g)(\lambda) = \text{const.}\,\mathtt{av}(g)(\lambda),$$

for all $g \in \mathcal{PW}^{r,R}_\circ(\mathbb{C}^n)$. Now, Lemma 4.5 implies that (up to a constant) $\widetilde{\mathcal{L}}(\mathcal{I}(g)) = g$ for all $g \in \mathcal{PW}^{r,R}_\circ(\mathbb{C}^n)$. This finishes the proof. ■

5. A flat analogue of the Abel transform

Replacing the Cartan involution by the involution σ in the proof of [17, Theorem I.5.17], one can prove that for $f \in \mathcal{C}_c(C^\circ_{\max})$

$$\int_{C^\circ_{\max}} f(Y)dY = \text{const.}\int_{\mathfrak{a}_-}\int_H f(\mathrm{Ad}(h)X)\prod_{\alpha\in\Sigma^+}|\langle\alpha,X\rangle|^{m_\alpha}dhdX,$$

where "const" is some positive constant depending only on the normalization of the measures. Thus, for $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$ such that $\mathrm{Re}(\lambda_i) > 0$ $(1 \le i \le n)$, the Bessel Laplace transform of $f \in \mathcal{C}^\infty_c(c^\circ_{\max})^{\mathcal{W}_\circ} \cong \mathcal{C}^\infty_c(C^\circ_{\max})^{\mathrm{Ad}(H)}$ can be written as

$$\begin{aligned}\widetilde{\mathcal{L}}(f)(\lambda) &= \int_{\mathfrak{a}_-} f(X)\Psi(\lambda,X)\omega(X)dX\\ &= \int_{\mathfrak{a}_-} f(X)\Big(\int_H e^{-\lambda(\mathrm{Ad}(h)X)}dh\Big)\omega(X)dX\\ &= \text{const.}\int_{C^\circ_{\max}} f(X)e^{-\lambda(X)}dX.\end{aligned}$$

Above we used the following integral representation of the Bessel functions

$$\Psi(\lambda,X) = \int_H e^{-\lambda(\mathrm{Ad}(h)X)}dh,$$

(cf. [8, Theorem 4.12]). Let $\mathfrak{a}^\perp$ be the orthogonal complement of $\mathfrak{a}$ in $\mathfrak{q}$. Then for $\lambda \in \mathfrak{a}^*$ such that $\lambda_i > 0$ $(1 \le i \le n)$, we have

$$\begin{aligned} \widetilde{\mathcal{L}}(f)(\lambda) &= \text{const.} \int_{C^\circ_{\max}\cap\mathfrak{a}} e^{-\lambda(X)} \Big(\int_{C^\circ_{\max}\cap\mathfrak{a}^\perp} f(X+Y)dY \Big) dX \\ &= \text{const.} \int_{c^\circ_{\max}} e^{-\lambda(X)} \mathcal{A}(f)(X)dX, \end{aligned} \tag{18}$$

where

$$\mathcal{A}(f)(X) := \int_{C^\circ_{\max}\cap\mathfrak{a}^\perp} f(X+Y)dY$$

denotes (the flat analogue of) the Abel transform of $f \in \mathcal{C}^\infty_c(C^\circ_{\max})^{\mathrm{Ad}(H)} \cong \mathcal{C}^\infty_c(c^\circ_{\max})^{\mathcal{W}_\circ}$ at $X \in c^\circ_{\max}$. The expression (18) is similar to the one proved by Helgason in [15] for the Bessel Fourier transform on $\mathfrak{p}$. It follows that

$$\widetilde{\mathcal{L}}(f)(\lambda) = \text{const.} \int_{c^\circ_{\max}} e^{-\lambda(X)} \mathcal{A}(f)(X)dX = \text{const.}\, \mathfrak{F}\big(\mathcal{A}(f)\big)(\lambda), \tag{19}$$

where $\mathfrak{F}$ denotes the Euclidean Laplace transform associated with $c^\circ_{\max}$. Let

$$\mathbb{V}(x_1,\ldots,x_n) := \prod_{1\le i<j\le n} (x_j^2 - x_i^2).$$

One may write $\mathbb{V}(\lambda_1,\ldots,\lambda_n)\widetilde{\mathcal{L}}(f)(\lambda)$ in two different ways. First, by (19), we have

$$\begin{aligned} \mathbb{V}(\lambda_1,\ldots,\lambda_n)\widetilde{\mathcal{L}}(f)(\lambda) &= \text{const.}\, \mathbb{V}(\lambda_1,\ldots,\lambda_n)\mathfrak{F}\big(\mathcal{A}(f)\big)(\lambda) \\ &= \text{const.}\, \mathfrak{F}\Big[\mathbb{V}(\partial_1,\ldots,\partial_n)\mathcal{A}(f)\Big](\lambda). \end{aligned} \tag{20}$$

Second, for $f \in \mathcal{C}^\infty_c(c^\circ_{\max})^{\mathcal{W}_\circ}$, we have

$$\begin{aligned} &\mathbb{V}(\lambda_1,\ldots,\lambda_n)\widetilde{\mathcal{L}}(f)(\lambda) \\ &= \text{const.} \int_{\mathfrak{a}_-} f(t) \det_{1\le i,j\le n}\big(K_0(\lambda_i t_j)\big) \frac{\omega(t)}{\prod_{1\le i<j\le n}(t_j^2 - t_i^2)} dt \\ &= \text{const.} \int_{\mathfrak{a}_-} f(t) \det_{1\le i,j\le n}\big(K_0(\lambda_i t_j)\big) \prod_{1\le i\le n} t_i \prod_{1\le i<j\le n}(t_j^2 - t_i^2) dt \\ &= \text{const.} \sum_{\sigma\in\mathbb{S}_n} \int_{\mathfrak{a}_-} f(t) \prod_{1\le i\le n} t_{\sigma(i)} K_0\big(\lambda_i t_{\sigma(i)}\big) \prod_{i<j} (t^2_{\sigma(j)} - t^2_{\sigma(i)}) dt \\ &= \text{const.} \int_{c^\circ_{\max}} f(t)\mathbb{V}(t_1,\ldots,t_n) \prod_{1\le i\le n} t_i \prod_{1\le i\le n} K_0\big(\lambda_i t_i\big) dt. \end{aligned}$$

Since

$$K_0(zt) = \int_t^\infty \frac{e^{-zs}}{\sqrt{s^2-t^2}} ds, \qquad (\mathrm{Re}(z) > 0, \quad t > 0),$$

it follows that

$$\begin{aligned} &\mathbb{V}(\lambda_1,\ldots,\lambda_n)\widetilde{\mathcal{L}}(f)(\lambda) \\ &= \text{const.} \int_{c^\circ_{\max}} \prod_{1\le i\le n} e^{-\lambda_i s_i} \Big[\int_0^{s_1} \cdots \int_0^{s_n} f(t_1,\ldots,t_n)\mathbb{V}(t_1,\ldots,t_n) \prod_{1\le i\le n} \frac{t_i dt_i}{\sqrt{s_i^2 - t_i^2}} \Big] ds \\ &= \text{const.}\, \mathfrak{F}\Big(\mathbb{A}_1^{\otimes n}(f\mathbb{V})\Big)(\lambda), \end{aligned}$$

where $\mathbb{A}_1^{\otimes n}$ denotes the n-fold tensor product of the one dimensional integral transformation

$$\mathbb{A}_1(F)(s) := \int_0^s F(t) \frac{t}{\sqrt{s^2 - t^2}} dt, \qquad F \in \mathcal{C}_c^\infty(\mathbb{R}^+),\ s > 0.$$

The later transform satisfies

$$F(t) = \text{const.} \frac{1}{t} \frac{\mathrm{d}}{\mathrm{d}t} \int_0^t \mathbb{A}_1(F)(s) \frac{s}{\sqrt{t^2 - s^2}} ds. \tag{21}$$

Comparing (20) with (21), and using the injectivity of the Euclidean Laplace transform $\mathfrak{F}$, we get

$$\mathbb{V}(\partial_1, \ldots, \partial_n)\mathcal{A}(f)(t) = \text{const.}\, \mathbb{A}_1^{\otimes n}(f\mathbb{V})(t).$$

In view of (22), we obtain the second main result of the paper.

Theorem B. *Assume that $f \in \mathcal{C}_c^\infty(c_{\max}^\circ)^{\mathcal{W}_\circ}$. For every $t \in \mathfrak{a}_-$, the inverse Abel transform is expressed as*

$$\mathbb{V}(t_1, \ldots, t_n) f(t) = \text{const.} \prod_{i=1}^n \frac{1}{t_i} \frac{\mathrm{d}}{\mathrm{d}t_i} \int_0^{t_1} \cdots \int_0^{t_n} \mathbb{V}(\partial_1, \ldots, \partial_n)\mathcal{A}(f)(s) \prod_{i=1}^n \frac{s_i ds_i}{\sqrt{t_i^2 - s_i^2}},$$

where $\mathbb{V}(\partial_1, \ldots, \partial_n) = \prod_{1 \le i < j \le n}(\partial_j^2 - \partial_i^2)$.

Remark 5.1. (Another way of computing the Bessel function $\Psi(\lambda, t)$ via the rank one case.) Let $\mathcal{M}_{(1,0)}^{(1)}$ be the rank one symmetric space $SO_0(1,2)/SO_0(1,1)$. The associated restricted root system is given by $\{\pm\alpha\}$, where $\alpha(t) = -t$ defines the positive root. Here $\mathfrak{a} \cong \mathbb{R}$, and $m_\alpha = 1$. By [8, Example 4.13], the Bessel functions associated with $\mathcal{M}_{(1,0)}^{(1)}$ are given by

$$\Psi_{(1,0)}^{(1)}(\lambda, t) = K_0(\lambda t), \qquad \mathrm{Re}(\lambda) > 0,\ t > 0.$$

Let $\mathcal{M}_{(1,0)}^{(n)}$ be the product of n-copies of $\mathcal{M}_{(1,0)}^{(1)}$, and define on $\mathcal{M}_{(1,0)}^{(n)}$ the pseudo-Bessel function

$$\Psi_{(1,0)}^{(n)}(\lambda, t) := \sum_{\sigma \in \mathbb{S}_n} \prod_{1 \le i \le n} K_0(\lambda_{\sigma(i)} t_i).$$

On the other hand, recall that the restricted root system $\Sigma(\mathfrak{g}, \mathfrak{a})$ associated with $\mathcal{M} = SU(n,n)/SL(n,\mathbb{C}) \times \mathbb{R}_*^+$ consists of long roots with multiplicities 1 and short roots with multiplicities 2. By [26] we can prove that we may obtain the Bessel function $\Psi(\lambda, t)$ associated with $\mathcal{M}$ via $\Psi_{(1,0)}^{(n)}(\lambda, t)$ as

$$\Psi(\lambda, t) = \frac{\text{const.}}{\prod_{1 \le i < j \le n} (\lambda_j^2 - \lambda_i^2)^2} \mathbb{G}(0,2) \Psi_{(1,0)}^{(n)}(\lambda, t), \tag{22}$$

where $\mathbb{G}(0,2)$ denotes the shift operator

$$\prod_{1 \le i < j \le n} (t_j^2 - t_i^2)^{-1} \prod_{1 \le i < j \le n} \big(\mathcal{D}(t_j, \partial_{t_j}) - \mathcal{D}(t_i, \partial_{t_i})\big),$$

with

$$\mathcal{D} := \frac{d^2}{dt^2} + \frac{1}{t}\frac{d}{dt}.$$

Since $K_0(z)$ is a solution to

$$u'' + \frac{1}{z}u' - u = 0,$$

it follows that

$$\begin{aligned} \Psi(\lambda, t) &= \frac{\text{const.}}{\prod\limits_{1\le i<j\le n} (t_j^2 - t_i^2)(\lambda_j^2 - \lambda_i^2)^2} \sum_{\sigma\in\mathbb{S}_n} \prod_{1\le i\le n} K_0(\lambda_{\sigma(i)} t_i) \prod_{1\le i<j\le n} (\lambda_{\sigma(j)}^2 - \lambda_{\sigma(i)}^2) \\ &= \frac{\text{const.}}{\prod\limits_{1\le i<j\le n} (t_j^2 - t_i^2)(\lambda_j^2 - \lambda_i^2)} \sum_{\sigma\in\mathbb{S}_n} (-1)^\sigma \prod_{1\le i\le n} K_0(\lambda_{\sigma(i)} t_i) \\ &= \text{const.} \frac{\det\limits_{1\le i,j\le n} \Big(K_0(\lambda_i t_j)\Big)}{\prod\limits_{1\le i<j\le n} (t_j^2 - t_i^2) \prod\limits_{1\le i<j\le n} (\lambda_j^2 - \lambda_i^2)}, \end{aligned}$$

which coincides with Theorem 3.1 part (ii). Notice that one may use (23) to give another proof for Theorem A. In a forthcoming paper we shall develop this approach further to prove a Paley-Wiener theorem for a larger class of noncompact causal symmetric spaces.

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