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A Local-to-Global Principle for Convexity in Metric Spaces

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Abstract. We introduce an extension of the standard Local-to-Global Principle used in the proof of the convexity theorems for the momentum map to handle closed maps that take values in a length metric space. As an application, this extension is used to study the convexity properties of the cylinder valued momentum map introduced by Condevaux, Dazord, and Molino.
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1. Introduction

The Local-to-Global Principle [9, 16] is an important topological technique that has been used in the formulation of convexity theorems for momentum maps associated to compact group actions on non-compact symplectic manifolds with proper momentum maps. This approach has been further developed and applied to many interesting situations in [16] and has been generalized to closed maps in [4].

All the convexity results in the literature in which the Local-to-Global Principle has been used concern maps with values in vector spaces. The main goal of this paper is *the extension of this principle to closed maps with values in a length metric space*. Our approach is based on the extension of the classical Hopf-Rinow theorem to length metric spaces due to Cohn-Vossen. This circle of ideas also appears for compact spaces in [7]. In this context, we also show how the connectedness of the fibers of the map in question, a property that is usually addressed in symplectic convexity theorems, is ultimately a consequence of the uniquely geodesic character of the target metric space (a property always available for Euclidean spaces). A short appendix at the end of the paper contains all the definitions and results on convexity in length metric spaces necessary for the comprehension of the paper.

As an application of our extension of the Local-to-Global Principle we study the convexity properties of the cylinder valued momentum map. This map is a generalization of the usual momentum map that is always available for any Lie

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algebra action on a symplectic manifold and that generically takes values in an Abelian group isomorphic to a cylinder. The cylinder valued momentum map was introduced in [9] and carefully studied in [25, 26] in the context of reduction. Additionally, its local properties are as well understood as those for the standard momentum map [23, 26]. The generalized Local-to-Global Principle allows us to extend to the cylinder valued momentum map the knowledge that we have about the convexity properties of the classical momentum map. The metric approach seems to be the best adapted generalization of the classical setup to our problem since, under certain hypotheses related to the topological nature of the Hamiltonian holonomy of the problem (a concept defined carefully later on), the target space of the cylinder valued momentum map has an associated canonical length space structure.

2. Image convexity for maps with values in length spaces

One of the main goals in this paper is the study of the convexity properties of the image of a natural generalization of the momentum map. The notion of convexity is usually associated with vector spaces. However, the map considered in this paper has values in a manifold that is, in general, diffeomorphic to a cylinder. Thus, one is forced to work in a more general setting. As reviewed in the Appendix (see §4), most of the concepts pertaining to convexity can be extended to the context of the so called *length spaces*. It turns out that the target space of the map that we are going to study can be naturally endowed with a length space structure and hence convexity will be used in this context. We give in §4 a self-contained brief summary of all the definitions and results on length spaces necessary in this paper.

The convexity program has been successfully carried out for the standard momentum map by several means. One possible approach consists in determining certain local properties of the map that guarantee that it has a globally convex image. This strategy relies on a fundamental result called the *Local-to-Global Principle* which has been introduced in [9, 16] for maps whose target space is a Euclidean vector space. Since the extension of the standard momentum map with which we will be working does not map into a vector space but into a length space, a generalization of the Local-to-Global Principle is needed to handle this situation. This is the main goal of the present section.

Let $f : X \to Y$ be a continuous map between two connected Hausdorff topological spaces. Declare two points $x_1, x_2 \in X$ to be equivalent if and only if $f(x_1) = f(x_2) = y$ and they belong to the same connected component of $f^{-1}(y)$. The elements of the topological quotient space X_f are the connected components of the fibers of f. Let $\pi_f : X \to X_f$ be the projection and $\widetilde{f} : X_f \to Y$ the induced map which is uniquely characterized by $\widetilde{f} \circ \pi_f = f$. The map $\widetilde{f}$ is continuous and if the fibers of f are connected then it is also injective.

Definition 2.1. Let X and Y be two topological spaces and $f : X \to Y$ a continuous map. The subset $A \subset X$ satisfies the *locally fiber connected condition (LFC)* if A does not intersect two different connected components of the fiber $f^{-1}(f(x))$, for any $x \in A$.

Let X be an arcwise connected Hausdorff topological space. The continuous map $f : X \to Y$ is said to be *locally fiber connected* if for each $x \in X$, any open neighborhood of x contains a connected neighborhood U_x of x such that U_x satisfies the *(LFC)* condition.

The following consequences of the definition are useful later on. A subset of a set that satisfies (LFC) also satisfies (LFC). If $A \subset X$ satisfies (LFC), then its saturation $\pi_f^{-1}(\pi_f(A))$ also satisfies (LFC). If f is locally fiber connected, then any open neighborhood of $x \in X$ contains an open neighborhood U_x of x such that the restriction of $\widetilde{f}$ to $\pi_f(U_x)$ is injective.

Definition 2.2. A continuous map $f : X \to Y$ is said to be *locally open onto its image* if for any $x \in X$ there exists an open neighborhood U_x of x such that the restriction $f|_{U_x} : U_x \to f(U_x)$ is an open map, where $f(U_x)$ has the topology induced by Y. We say that such a neighborhood satisfies the *(LOI)* condition.

Benoist proved in Lemma 3.7 of [3] the following result that will be used later on.

Proposition 2.3. *Suppose $f : X \to Y$ is a continuous map between two topological spaces. If f is locally fiber connected and locally open onto its image then π_f is an open map.*

Corollary 2.4. *Under the conditions of the previous proposition, the induced map $\widetilde{f} : X_f \to Y$ is locally open onto its image.*

Proof. Let $x \in X$ be arbitrary and $[x] := \pi_f(x)$. Since f is locally open onto its image there exists a neighborhood U_x of x in X such that $f|_{U_x}$ is open onto its image. Let $U_{[x]} := \pi_f(U_x)$; since by the previous lemma π_f is an open map, we conclude that $U_{[x]}$ is an open neighborhood of $[x] \in X_f$. We shall prove that $\widetilde{f}|_{U_{[x]}}$ is open onto its image.

Let $V \subset U_{[x]}$ be an open subset of $U_{[x]}$. We will show that $\widetilde{f}(V)$ is open in $\widetilde{f}(U_{[x]}) = (\widetilde{f} \circ \pi_f)(U_x) = f(U_x)$ by proving that $\widetilde{f}(V) = f(\pi_f^{-1}(V) \cap U_x)$. Indeed, if $z \in \widetilde{f}(V)$ there is some $[y] = \pi_f(y) \in V$ such that $\widetilde{f}([y]) = z$; hence $y \in \pi_f^{-1}(V)$. Moreover y can be chosen in U_x because if this were not true, then $\pi_f^{-1}([y]) \cap U_x = \varnothing$. However, by the definition of the equivalence relation we know that $\pi_f^{-1}([y])$ is a connected component of the f-fiber of z that cannot possibly intersect the π_f-saturation of U_x (which is the union of all connected components of the fibers of f that intersect U_x). But this means that $[y] \notin U_{[x]} = \pi_f(U_x)$ which is a contradiction with the hypothesis $V \subset U_{[x]}$. The converse inclusion is straightforward. ∎

We need the following characterization of closed maps (see [10], Theorems 1.4.12 and 1.4.13).

Theorem 2.5. *Let $f : X \to Y$ be a continuous mapping.*

(i) *f is closed if and only if for every $B \subset Y$ and every open set $A \subset X$ which contains $f^{-1}(B)$, there exists an open set $C \subset Y$ containing B and such that $f^{-1}(C) \subset A$.*

(ii) *f is closed if and only if for every point $y \in Y$ and every open set $U \subset X$ which contains $f^{-1}(y)$, there exists a neighborhood V_y of the point y in Y such that $f^{-1}(V_y) \subset U$.*

Lemma 2.6. *Let X be a normal, first countable, arcwise connected, Hausdorff topological space and Y a Hausdorff topological space. Let $f: X \to Y$ be a continuous map that is locally open onto its image and is locally fiber connected. If f is a closed map, then*

(i) *the projection $\pi_f : X \to X_f$ is also a closed map,*

(ii) *the quotient X_f is a Hausdorff topological space.*

Proof. (i) Let $[x] \in X_f$ and $U \subset X$ an open set that includes $E_x := {\pi_f}^{-1}([x])$, the connected component of $f^{-1}(f(x))$ that contains x. Denote by $F := f^{-1}(f(x)) \setminus E_x$ the union of all (closed) connected components of $f^{-1}(f(x))$ different from E_x. We claim that F is a closed subset of X. Indeed, if $z \in \overline{F}$, by first countability of X, there exists a sequence $\{z_n\}_{n\in\mathbb{N}}$ in F which is convergent to z. Since $f(z_n) = f(x)$, by continuity of f we conclude that $f(x) = f(z_n) \to f(z)$ and hence $z \in f^{-1}(f(x))$. If $z \in E_x$ then any neighborhood of z intersects at least one other connected component of the fiber $f^{-1}(f(x))$ since $z \in \overline{F}$. This, however, contradicts the (LFC) condition. Therefore, $z \in F$ and hence F is closed. The same argument as above shows that the (LFC) condition implies that E_x is also closed in X.

Using the normality of X there exist two open sets U_{E_x} and W such that $E_x \subset U_{E_x}$, $F \subset W$, and $U_{E_x} \cap W = \varnothing$. After shrinking, if necessary, we can assume that $U_{E_x} \subset U$. Applying Theorem 2.5(ii), the closedness of f ensures the existence of an open neighborhood $V_{f(x)}$ of $f(x)$ in Y such that $E_x \subset f^{-1}(f(x)) \subset f^{-1}(V_{f(x)}) \subset U_{E_x} \cup W$.

The set $A := U_{E_x} \cap f^{-1}(V_{f(x)})$ is a nonempty open subset of X and is also saturated with respect to the equivalence relation that defines π_f or, equivalently, ${\pi_f}^{-1}(\pi_f(A)) = A$. Indeed, if a connected component of a fiber of f from $f^{-1}(V_{f(x)})$ intersects U_{E_x}, respectively W, then it is entirely contained either in U_{E_x} or in W since $U_{E_x} \cap W = \varnothing$.

Since A is open in X, by the definition of the quotient topology of X_f, it follows that $\pi_f(A)$ is an open neighborhood of $[x]$. Note that ${\pi_f}^{-1}(\pi_f(A)) \subset U$, which shows via Theorem 2.5(ii) that π_f is a closed map.

(ii) We prove that X_f is Hausdorff by showing that the projection π_f is an open map (which holds by Proposition 2.3) and that the graph of the equivalence relation that defines X_f is closed.

To show that the graph is closed, we need some preliminary considerations. For every $x \in X$ there exists a neighborhood U_x that satisfies (LOI) and (LFC). By normality of X there exists also a neighborhood U'_x of x with $\overline{U'_x} \subset U_x$. We shall prove that $\overline{{\pi_f}^{-1}(\pi_f(U'_x))} \subset {\pi_f}^{-1}(\pi_f(U_x))$ which shows that for every connected component E_x of a fiber there exists a saturated neighborhood of it which contains a smaller saturated neighborhood whose closure still satisfies (LFC). In order to prove the above inclusion observe that since π_f is continuous and closed we have that $\overline{\pi_f(U'_x)} = \pi_f(\overline{U'_x}) \subset \pi_f(U_x)$. By the continuity of π_f we obtain the inclusion $\overline{{\pi_f}^{-1}(\pi_f(U'_x))} \subset {\pi_f}^{-1}(\overline{\pi_f(U'_x)}) \subset {\pi_f}^{-1}(\pi_f(U_x))$.

We now prove the closedness of the graph of the equivalence relation that defines X_f. Take $\{x_n\}_{n\in\mathbb{N}}$ and $\{y_n\}_{n\in\mathbb{N}}$ two convergent sequences in X such that x_n and y_n are in the same equivalence class for all $n \in \mathbb{N}$. Suppose that $x_n \to x$ and $y_n \to y$. The continuity of f guarantees that $f(x) = f(y)$. Additionally, there exists $n_0 \in \mathbb{N}$ such that for $n > n_0$ all $x_n \in {\pi_f}^{-1}(\pi_f(U'_x))$, where U'_x has been chosen as above. Consequently $y_n \in {\pi_f}^{-1}(\pi_f(U'_x))$ since x_n and y_n are in the same equivalence class and ${\pi_f}^{-1}(\pi_f(U'_x))$ is saturated. Therefore, $x, y \in \overline{{\pi_f}^{-1}(\pi_f(U'_x))}$. But $\overline{{\pi_f}^{-1}(\pi_f(U'_x))}$ satisfies (LFC) and thus x and y belong to the same connected component of the fiber $f^{-1}(f(x))$. This shows that the graph of the equivalence relation is closed, as required. ■

Proposition 2.7. *Let X be a normal, first countable, arcwise connected, Hausdorff topological space and (Y, d) a metric space. Let $f : X \to Y$ be a continuous closed map that is also locally open onto its image and locally fiber connected. Define $\widetilde{d} : X_f \times X_f \to [0, \infty]$ as follows: given $[x], [y] \in X_f$, let $\widetilde{d}([x], [y])$ be the infimum of all the lengths $l_d(\widetilde{f} \circ \gamma)$ where γ is a continuous curve in X_f that connects $[x]$ and $[y]$. The length l_d is computed with respect to the distance d on Y. Then $\widetilde{d} : X_f \times X_f \to [0, \infty]$ is a metric on X_f. From the definition it follows that $d(\widetilde{f}([x]), \widetilde{f}([y])) \leq \widetilde{d}([x], [y])$. The open ball centered at $y \in Y$ of radius $r > 0$ is denoted $B_d(y, r)$ or $B_Y(y, r)$.*

Proof. The positivity, symmetry, and the triangle inequality of $\widetilde{d}$ are obvious from the definition of $\widetilde{d}$. We need to show that $\widetilde{d}([x], [y]) = 0$ implies $[x] = [y]$. Suppose that there exist $[x] \neq [y]$ with $\widetilde{d}([x], [y]) = 0$. Then $d(\widetilde{f}([x]), \widetilde{f}([y])) = 0$ and hence $f(x) = \widetilde{f}([x]) = \widetilde{f}([y]) = f(y)$. This implies that $[x]$ and $[y]$ are images under the the projection π_f of two different connected components of the same fiber.

Since $\widetilde{f}$ is locally open onto its image (Corollary 2.4) and the quotient topology of X_f is Hausdorff (Lemma 2.6), there exist two open neighborhoods $U_{[x]}$ and $U_{[y]}$ of $[x]$ and $[y]$, respectively, such that $U_{[x]} \cap U_{[y]} = \varnothing$ and $\widetilde{f}|_{U_{[x]}}$ and $\widetilde{f}|_{U_{[y]}}$ are open onto their images. Consequently, there exist two open neighborhoods $U'_{[x]} \subset U_{[x]}$ and $U'_{[y]} \subset U_{[y]}$ such that $\widetilde{f}(U'_{[x]}) \supset B_d(\widetilde{f}[x], r) \cap \widetilde{f}(U_{[x]})$ and $\widetilde{f}(U'_{[y]}) \supset B_d(\widetilde{f}([y]), r') \cap \widetilde{f}(U_{[y]})$ for two small enough constants r, $r' > 0$

Any curve γ in X_f connecting $[x]$ and $[y]$ is mapped by $\widetilde{f}$ to a loop in Y based at $\widetilde{f}([x]) = \widetilde{f}([y])$; γ exits $U'_{[x]}$ and enters $U'_{[y]}$ since $U'_{[x]} \cap U'_{[y]} = \varnothing$. Thus, the curve $\widetilde{f} \circ \gamma$ in Y exits the open ball $B_d(f([x]), r)$ and enters the open ball $B_d(f([y]), r')$ which implies that $l_d(\widetilde{f} \circ \gamma) \geq r + r'$. This is in contradicts the hypothesis that $\widetilde{d}([x], [y]) = 0$ for $[x] \neq [y]$. ■

In order to put the following definition in context, the reader is encouraged to look at the Appendix (see §4) where the concepts of length metric and geodesic metric space are discussed.

Definition 2.8. A subset C in a length metric space (X, d) is said to be *convex* if for any two points $x, y \in C$ there exists a rectifiable shortest path in (X, d) connecting x and y which is entirely contained in C.

Definition 2.9. Let X be a connected Hausdorff space and (Y, d) a length space. A continuous mapping $f : X \to Y$ is said to have *local convexity data* if for each $x \in X$ and every sufficiently small neighborhood U_x of x the set $f(U_x)$ is a convex subset of Y. Any open set U in X such that $f(U)$ is a convex subset of Y will be said to satisfy the *(LCD)* condition.

Proposition 2.10. *Let X be a normal, first countable, arcwise connected, Hausdorff topological space and (Y, d) a geodesic metric space (see Definition 4.2). Assume that $f : X \to Y$ is a continuous closed map that is also locally open onto its image, locally fiber connected, and has local convexity data. Then $(X_f, \widetilde{d})$ is a length space and the topology induced by $\widetilde{d}$ coincides with the quotient topology of X_f.*

The proof of this proposition is based on the following lemma.

Lemma 2.11. *Let $[x], [y] \in X_f$ be such that they are contained in an open set $\widetilde{U} \subset X_f$ (open in the quotient topology of X_f) such that $\widetilde{U} = \pi_f(U)$, where U open in X and satisfies the (LCD), (LOI), and (LFC) conditions. Then $d(\widetilde{f}([x]), \widetilde{f}([y])) = \widetilde{d}([x], [y])$.*

Proof. Let $\Omega := f(U) = \widetilde{f}(\widetilde{U})$. Since Ω is convex (because (LCD) holds for U), there exists a rectifiable shortest path γ_0 entirely contained in Ω that connects $\widetilde{f}([x])$ and $\widetilde{f}([y])$, that is, $l_d(\gamma_0) = d(\widetilde{f}([x]), \widetilde{f}([y]))$. Note that $\widetilde{f}|_{\widetilde{U}} : \widetilde{U} \to \Omega$ is a homeomorphism ($\widetilde{U}$ endowed with the quotient topology of X_f) because $\widetilde{f}|_{\widetilde{U}}$ is open, since U satisfies (LOI), and is injective, because U satisfies (LFC). The curve $c_0 := \widetilde{f}^{-1} \circ \gamma_0$ is continuous and connects $[x]$ with $[y]$. From the definition of $\widetilde{d}$ we have that $\widetilde{d}([x], [y]) \leq l_d(\widetilde{f} \circ c_0) = l_d(\gamma_0) = d(\widetilde{f}([x]), \widetilde{f}([y]))$. As the inequality $d(\widetilde{f}([x]), \widetilde{f}([y])) \leq \widetilde{d}([x], [y])$ always holds, we obtain the desired equality $d(\widetilde{f}([x]), \widetilde{f}([y])) = \widetilde{d}([x], [y])$. ■

Proof of Proposition 2.10. First, the previous lemma and Proposition 2.7 imply that the homeomorphism $\widetilde{f}|_{\widetilde{U}} : \widetilde{U} \to \Omega$ is also an isometry from $(\widetilde{U}, \widetilde{d}|_{\widetilde{U}})$ to $(\Omega, d|_\Omega)$ and thus the quotient topology of X_f coincides with the metric topology induced by $\widetilde{d}$.

Next we will prove that $(X_f, \widetilde{d})$ is a length space. Let $c : [a, b] \to X_f$ be a continuous curve connecting two arbitrary points $[x]$ and $[y]$ in X_f. For two partitions Δ_n and Δ_{n+1} of the interval $[a, b]$ with Δ_{n+1} finer then Δ_n we have that $\sum_{i=1}^{n} \widetilde{d}(c(t_i), c(t_{i+1})) \leq \sum_{i=1}^{n+1} \widetilde{d}(c(s_i), c(s_{i+1}))$ due to the triangle inequality. Therefore, in order to compute $l_{\widetilde{d}}(c)$ it suffices to work with partitions fine enough such that two consecutive points $c(t_i), c(t_{i+1})$, corresponding to a partition Δ_n, are close enough as above. Hence, by what was just proved, we have $d(\widetilde{f}(c(t_i)), \widetilde{f}(c(t_{i+1}))) = \widetilde{d}(c(t_i), c(t_{i+1}))$ and we conclude

$$l_{\widetilde{d}}(c) = \sup_{\Delta_n} \sum_{i=1}^{n} \widetilde{d}(c(t_i), c(t_{i+1})) = \sup_{\Delta_n} \sum_{i=1}^{n} d(\widetilde{f}(c(t_i)), \widetilde{f}(c(t_{i+1}))) = l_d(\widetilde{f} \circ c). \quad (1)$$

Thus, $\widetilde{f} \circ c$ is a rectifiable curve in (Y, d) if and only if c is rectifiable in $(X_f, \widetilde{d})$. The equality

$$\widetilde{d}([x], [y]) = \inf l_d(\widetilde{f} \circ c) = \inf l_{\widetilde{d}}(c) = \overline{\widetilde{d}}([x], [y]),$$

shows that $(X_f, \widetilde{d})$ is a length space. See the Appendix in §4 for the definition of $\overline{\widetilde{d}}$. ■

The following two results are Propositions 4.4.16 and 3.7.2 in [10].

Lemma 2.12. *(Vaĭnšteĭn) If $f : X \to Y$ is a closed mapping from a metrizable space X onto a metrizable space Y, then for every $y \in Y$ the boundary $\mathrm{bd}(f^{-1}(y)) := \overline{f^{-1}(y)} \cap \overline{(X \setminus f^{-1}(y))}$ is compact.*

Definition 2.13. Let X be a Hausdorff topological space and $f : X \to Y$ a continuous map. We call f a *proper map* if it is closed and all fibers $f^{-1}(y)$ are compact subsets of X.

Theorem 2.14. *If $f : X \to Y$ is a proper map, then for every compact subset $Z \subset Y$ the inverse image $f^{-1}(Z)$ is compact.*

A converse of this theorem is available when Y is a k-space (i.e., Y is a Hausdorff topological space that is the image of a locally compact space under a quotient mapping). For example every first countable Hausdorff space is a k-space (see [10], Theorem 3.3.20).

Proposition 2.15. *Let X be a normal, first countable, arcwise connected, Hausdorff topological space and (Y, d) a complete locally compact length space (and thus, by Hopf-Rinow-Cohn-Vossen a geodesic metric space; see Theorem 4.3). Assume that $f : X \to Y$ is a continuous closed map that is also locally open onto its image, locally fiber connected, and has local convexity data. Then $(X_f, \widetilde{d})$ is a complete locally compact length space and hence a geodesic metric space.*

Proof. First we will prove that $\widetilde{f}$ is a proper map. It is a closed map since f is a closed map. The local injectivity of $\widetilde{f}$ implies that its fibers are made of isolated points and hence $\mathrm{bd}(\widetilde{f}^{-1}(y)) = \widetilde{f}^{-1}(y)$. By Vaĭnšteĭn's Lemma 2.12 we conclude that the fibers of $\widetilde{f}$ are all compact and consequently $\widetilde{f}$ is a proper map.

Because local compactness is an inverse invariant for proper maps we obtain also that $(X_f, \widetilde{d})$ is locally compact since $\widetilde{f} : X_f \to Y$ is a proper map.

By the Hopf-Rinow-Cohn-Vossen Theorem 4.3 it suffices to show that every closed metric ball in X_f is compact in order to conclude that $(X_f, \widetilde{d})$ is a complete metric space. Let $\overline{B}([x], r)$ be the closure of the ball with center $[x]$ and radius $r > 0$ in X_f. By definition of the metric $\widetilde{d}$, the inclusion $\widetilde{f}(\overline{B}([x], r)) \subset \overline{B}(\widetilde{f}([x]), r)$ holds. By the Hopf-Rinow-Cohn-Vossen Theorem it follows that $\overline{B}(\widetilde{f}([x]), r)$ is a compact set in Y and, consequently, $\widetilde{f}^{-1}(\overline{B}(\widetilde{f}([x]), r))$ is compact in X_f due to properness of $\widetilde{f}$. Since $\overline{B}([x], r)$ is a closed subset of $\widetilde{f}^{-1}(\overline{B}(\widetilde{f}([x]), r))$ it is necessarily compact in X_f. ■

As a consequence of the above proposition, $(X_f, \widetilde{d})$ satisfies all the conditions of the Hopf-Rinow-Cohn-Vossen Theorem which implies that for any two points $[x], [y] \in X_f$ there exists a shortest geodesic connecting them.

Definition 2.16. Let (X, d) be a geodesic metric space. We say that C is *weakly convex* if for any two points $x, y \in C$ there exists a geodesic between x and y entirely contained in C.

Note that weak convexity does not require that the geodesic be a shortest one. Now we can present the main result of this section.

Theorem 2.17. **(Local-to-Global Principle)** *Let X be a normal, first countable, arcwise connected, Hausdorff topological space and (Y, d) a complete, locally compact length space. Assume that $f : X \to Y$ is a continuous closed map that is also locally open onto its image, locally fiber connected, and has local convexity data. Then the following hold:*

(i) *$f(X) \subset (Y, d)$ is a weakly convex subset of Y.*

(ii) *If, in addition, (Y, d) is uniquely geodesic (that is, any two points can be joined by a unique shortest path), then $f(X)$ is a convex subset of (Y, d), f has connected fibers, and f is open onto its image.*

Proof. (i) We have to prove that for any $y_1, y_2 \in f(X)$ there exists a geodesic (not necessary shortest) in (Y, d) completely included in $f(X)$. Indeed, take $[x_1], [x_2] \in X_f$ such that $\widetilde{f}([x_1]) = y_1$ and $\widetilde{f}([x_2]) = y_2$. As was explained above, there exists a shortest geodesic $c : [a, b] \to X_f$ with the properties $c(a) = [x_1]$, $c(b) = [x_2]$, and $\widetilde{d}([x_1], [x_2]) = l_{\widetilde{d}}(c)$. We will show that $\widetilde{f} \circ c \subset f(X)$ is a geodesic that connects y_1 with y_2. Since f is locally open onto its image, locally fiber connected, and has local convexity data, each $[x] \in X_f$ admits by Corollary 2.4 an open neighborhood $U_{[x]}$ such that $\widetilde{f}|_{U_{[x]}} : U_{[x]} \to \widetilde{f}(U_{[x]})$ is injective, open onto its image, and $\widetilde{f}(U_{[x]})$ is convex in Y. Choose now $[x]$ in the image of c and let t_0 be such that $c(t_0) = [x]$. Then the intersection of the image of c with $U_{[x]}$ is the image of a curve $c' : I \to U_{[x]}$ with $I \subset [a, b]$ a subinterval. If we take $U_{[x]}$ small enough then for any subinterval $[t_1, t_2] \subset I$ with $t_0 \in [t_1, t_2]$ we have by Lemma 2.11 that $\widetilde{d}(c(t_1), c(t_2)) = d(\widetilde{f}(c(t_1)), \widetilde{f}(c(t_2)))$. Since c is a shortest geodesic we have that $\widetilde{d}(c(t_1), c(t_2)) = l_{\widetilde{d}}(c|_{[t_1,t_2]})$. Additionally, by (1) we have that $l_{\widetilde{d}}(c|_{[t_1,t_2]}) = l_d(\widetilde{f} \circ c|_{[t_1,t_2]})$ and we hence obtain the desired equality $d(\widetilde{f}(c(t_1)), \widetilde{f}(c(t_2))) = l_d(\widetilde{f} \circ c|_{[t_1,t_2]})$. This proves that $\widetilde{f} \circ c$ is a geodesic in (Y, d) because it is a local distance minimizer.

(ii) Let $y, z \in f(X)$. As Y is a geodesic metric space, there exists a shortest path γ_0 in Y connecting y to z. Since Y is a length space, a shortest path is also a geodesic. Consequently γ_0 is a geodesic. As we proved in (i), $f(X)$ is weakly convex and hence there exists a geodesic γ_1 included in $f(X)$ connecting the two points. By uniqueness of geodesics we obtain that $\gamma_0 = \gamma_1$. Thus, we have a shortest path connecting y and z which is completely included in $f(X)$.

We prove that f has connected fibers. Suppose the contrary, that is, there exist $[x] \neq [y]$ with $\widetilde{f}([x]) = \widetilde{f}([y])$. Since by Proposition 2.15 the length metric space $(X_f, \widetilde{d})$ is geodesic, there exists a shortest geodesic $c : [a, b] \to X_f$ that links $[x]$ and $[y]$ which is mapped by $\widetilde{f}$ to a loop based at $\widetilde{f}([x]) = \widetilde{f}([y])$. As was proved in (i), $\widetilde{f} \circ c$ is a geodesic in Y. Since (Y, d) is uniquely geodesic we obtain that $\widetilde{f} \circ c$ is the constant loop. Consequently, $\widetilde{f}(c(t)) = \widetilde{f}([x])$ for all $t \in [a, b]$. This implies that $c(t)$ and $[x]$ belong to the same fiber of $\widetilde{f}$ for all $t \in [a, b]$ which contradicts the local injectivity of $\widetilde{f}$ implied by the (LFC) property of f.

Since $\widetilde{f}$ is a closed injective map, it is also open onto its image. As $f = \widetilde{f} \circ \pi_f$ and by Proposition 2.3 π_f is open, it follows that f is open onto its image. ■

Remark 2.18. Unlike the situation encountered in the classical Local-to-Global Principle [9, 16] in which the target space of the map is a Euclidean vector space and hence uniquely geodesic, f could have, in general, a weakly convex image but disconnected fibers. See Remark 3.10 for an example.

Remark 2.19. If Y is a Euclidean vector space and C is a convex subset of Y then Theorem 2.17 applied to the map $f : X \to C$ yields the generalization of the classical Local-to-Global Principle introduced in Theorem 2.28 of [4].

3. Metric convexity for cylinder valued momentum maps

The goal of this section is to apply the general results obtained in §2 to study the convexity properties of the image of the *cylinder valued momentum map*. This object, introduced in [9], naturally generalizes the standard momentum map definition due to Kostant and Souriau. The standard momentum map is associated to certain symplectic Lie algebra actions on a symplectic manifold and its convexity properties have been extensively studied [2, 13, 14, 16, 28]. Unlike the standard momentum map, the cylinder valued momentum map always exists for any symplectic Lie algebra action. However, the convexity properties of the standard momentum map cannot be trivially extended to this object because it does not map into a vector space but into a manifold that is, in general, diffeomorphic to a cylinder. Thus, in order to study the convexity properties of the cylinder valued momentum map the notion of convexity introduced and studied in §2 is necessary.

3.1. The cylinder valued momentum map.

We quickly review below the elementary properties of the cylinder valued momentum map. For more information and detailed proofs see [9] or Chapter 5 of [24].

Let (M, ω) be a connected paracompact symplectic manifold and let $\mathfrak{g}$ be a Lie algebra that acts symplectically on M. Let $\pi : M \times \mathfrak{g}^* \to M$ be the projection onto M. Consider π as the bundle map of the trivial principal fiber bundle $(M \times \mathfrak{g}^*, M, \pi, \mathfrak{g}^*)$ that has $(\mathfrak{g}^*, +)$ as Abelian structure group. The group $(\mathfrak{g}^*, +)$ acts on $M \times \mathfrak{g}^*$ by $\nu \cdot (m, \mu) := (m, \mu - \nu)$, with $m \in M$ and $\mu, \nu \in \mathfrak{g}^*$. Let $\alpha \in \Omega^1(M \times \mathfrak{g}^*; \mathfrak{g}^*)$ be the connection one-form defined by

$$\langle \alpha(m, \mu)(v_m, \nu), \xi \rangle := (\mathbf{i}_{\xi_M}\omega)(m)(v_m) - \langle \nu, \xi \rangle, \tag{2}$$

where $(m, \mu) \in M \times \mathfrak{g}^*$, $(v_m, \nu) \in T_mM \times \mathfrak{g}^*$, $\langle \cdot, \cdot \rangle$ denotes the natural pairing between $\mathfrak{g}^*$ and $\mathfrak{g}$, and ξ_M is the infinitesimal generator vector field associated to $\xi \in \mathfrak{g}$. The connection α is flat. For $(z, \mu) \in M \times \mathfrak{g}^*$, let $(M \times \mathfrak{g}^*)(z, \mu)$ be the holonomy bundle through (z, μ) and let $\mathcal{H}(z, \mu)$ be the holonomy group of α with reference point (z, μ) (which is an Abelian zero dimensional Lie subgroup of $\mathfrak{g}^*$ by the flatness of α). The principal bundle $((M \times \mathfrak{g}^*)(z, \mu), M, \pi|_{(M\times\mathfrak{g}^*)(z,\mu)}, \mathcal{H}(z, \mu))$ is a reduction of the principal bundle $(M \times \mathfrak{g}^*, M, \pi, \mathfrak{g}^*)$. To simplify notation, we will write $(\widetilde{M}, M, \widetilde{p}, \mathcal{H})$ instead of $((M \times \mathfrak{g}^*)(z, \mu), M, \pi|_{(M\times\mathfrak{g}^*)(z,\mu)}, \mathcal{H}(z, \mu))$. Let $\widetilde{\mathbf{K}} : \widetilde{M} \subset M \times \mathfrak{g}^* \to \mathfrak{g}^*$ be the projection into the $\mathfrak{g}^*$-factor.

Let $\overline{\mathcal{H}}$ be the closure of $\mathcal{H}$ in $\mathfrak{g}^*$. Since $\overline{\mathcal{H}}$ is a closed subgroup of $(\mathfrak{g}^*, +)$, the quotient $C := \mathfrak{g}^*/\overline{\mathcal{H}}$ is a cylinder (that is, it is isomorphic to the Abelian Lie group $\mathbb{R}^a \times \mathbb{T}^b$ for some $a, b \in \mathbb{N}$). Let $\pi_C : \mathfrak{g}^* \to \mathfrak{g}^*/\overline{\mathcal{H}}$ be the projection. Define

$\mathbf{K} : M \to C$ to be the map that makes the following diagram commutative:

$$\begin{array}{ccc} \widetilde{M} & \xrightarrow{\widetilde{\mathbf{K}}} & \mathfrak{g}^* \\ {\scriptstyle \widetilde{p}}\big\downarrow & & \big\downarrow{\scriptstyle \pi_C} \\ M & \xrightarrow{\mathbf{K}} & \mathfrak{g}^*/\overline{\mathcal{H}}. \end{array} \tag{3}$$

Thus, $\mathbf{K}$ is defined by $\mathbf{K}(m) = \pi_C(\nu)$, where $\nu \in \mathfrak{g}^*$ is any element such that $(m, \nu) \in \widetilde{M}$.

We call $\mathbf{K} : M \to \mathfrak{g}^*/\overline{\mathcal{H}}$ a *cylinder valued momentum map* associated to the symplectic $\mathfrak{g}$-action on (M, ω) and $\mathcal{H}$ the *Hamiltonian holonomy* of the $\mathfrak{g}$-action on (M, ω).

Elementary properties. The cylinder valued momentum map is a strict generalization of the standard (Kostant-Souriau) momentum map since the $\mathfrak{g}$-action has a standard momentum map if and only if the holonomy group $\mathcal{H}$ is trivial. In such a case the cylinder valued momentum map is a standard momentum map. The cylinder valued momentum map satisfies Noether's Theorem, that is, for any $\mathfrak{g}$-invariant function $h \in C^\infty(M)^{\mathfrak{g}} := \{f \in C^\infty(M) \mid \mathbf{d}h(\xi_M) = 0 \text{ for all } \xi \in \mathfrak{g}\}$, the flow F_t of its associated Hamiltonian vector field X_h satisfies the identity $\mathbf{K} \circ F_t = \mathbf{K}|_{\mathrm{Dom}(F_t)}$. Additionally, for any $v_m \in T_m M$, $m \in M$, $T_m\mathbf{K}(v_m) = T_\mu \pi_C(\nu)$, where $\mu \in \mathfrak{g}^*$ is any element such that $\mathbf{K}(m) = \pi_C(\mu)$ and $\nu \in \mathfrak{g}^*$ is uniquely determined by $\langle \nu, \xi \rangle = (\mathbf{i}_{\xi_M}\omega)(m)(v_m)$, for any $\xi \in \mathfrak{g}$. Also, $\ker(T_m\mathbf{K}) = \left(\left(\mathrm{Lie}(\overline{\mathcal{H}})\right)^\circ \cdot m\right)^\omega$, where $\mathrm{Lie}(\overline{\mathcal{H}}) \subset \mathfrak{g}^*$ is the Lie algebra of $\overline{\mathcal{H}}$ and $\operatorname{range}(T_m\mathbf{K}) = T_\mu \pi_C\left((\mathfrak{g}_m)^\circ\right)$ (Bifurcation Lemma); the superscripts $\circ$ and ω in the previous expressions denote the annihilator and the symplectic orthogonal, respectively. In the first statement we use the fact that the annihilator $\left(\mathrm{Lie}(\overline{\mathcal{H}})\right)^\circ$ is a Lie subalgebra of $\mathfrak{g}$. The notation $\mathfrak{k} \cdot m$ for any Lie subalgebra $\mathfrak{k} \subset \mathfrak{g}$ means the vector subspace of $T_m M$ formed by evaluating all infinitesimal generators η_M at the point $m \in M$ for all $\eta \in \mathfrak{k}$.

Equivariance properties of the cylinder valued momentum map. Suppose now that the $\mathfrak{g}$-Lie algebra action on (M, ω) is obtained from a symplectic action of the Lie group G on (M, ω) by taking the infinitesimal generators of all elements in $\mathfrak{g}$. There is a G-action on the target space of the cylinder valued momentum map $\mathbf{K} : M \to \mathfrak{g}^*/\overline{\mathcal{H}}$ with respect to which it is G-equivariant. This action is constructed by noticing first that the Hamiltonian holonomy $\mathcal{H}$ is invariant under the coadjoint action, that is, $\mathrm{Ad}^*_{g^{-1}}\mathcal{H} \subset \mathcal{H}$, for any $g \in G$. Actually, if G is connected, then $\mathcal{H}$ is pointwise fixed by the coadjoint action [25]. Hence, there is a unique group action $\mathcal{A}d^* : G \times \mathfrak{g}^*/\overline{\mathcal{H}} \to \mathfrak{g}^*/\overline{\mathcal{H}}$ such that for any $g \in G$, $\mathcal{A}d^*_{g^{-1}} \circ \pi_C = \pi_C \circ \mathrm{Ad}^*_{g^{-1}}$. With this in mind, we define $\overline{\sigma} : G \times M \to \mathfrak{g}^*/\overline{\mathcal{H}}$ by $\overline{\sigma}(g, m) := \mathbf{K}(\Phi_g(m)) - \mathcal{A}d^*_{g^{-1}}\mathbf{K}(m)$. Since M is connected by hypothesis, it can be shown that $\overline{\sigma}$ does not depend on the points $m \in M$ and hence it defines a map $\sigma : G \to \mathfrak{g}^*/\overline{\mathcal{H}}$ which is a group valued one-cocycle: for any $g, h \in G$, it satisfies the equality $\sigma(gh) = \sigma(g) + \mathcal{A}d^*_{g^{-1}}\sigma(h)$. This guarantees that the map

$$\Theta : (g, \pi_C(\mu)) \in G \times \mathfrak{g}^*/\overline{\mathcal{H}} \longmapsto \mathcal{A}d^*_{g^{-1}}(\pi_C(\mu)) + \sigma(g) \in \mathfrak{g}^*/\overline{\mathcal{H}}$$

defines a G-action on $\mathfrak{g}^*/\overline{\mathcal{H}}$ with respect to which the cylinder valued momentum map $\mathbf{K}$ is G-equivariant, that is, for any $g \in G$, $m \in M$, we have $\mathbf{K}(\Phi_g(m)) =$

$\Theta_g(\mathbf{K}(m))$. We will refer to $\sigma : G \to \mathfrak{g}^*/\overline{\mathcal{H}}$ as the *non-equivariance one-cocycle* of the cylinder valued momentum map $\mathbf{K} : M \to \mathfrak{g}^*/\overline{\mathcal{H}}$ and to Θ as the *affine G-action* on $\mathfrak{g}^*/\overline{\mathcal{H}}$ induced by σ. The infinitesimal generators of the affine G-action on $\mathfrak{g}^*/\overline{\mathcal{H}}$ are given by the expression

$$\xi_{\mathfrak{g}^*/\overline{\mathcal{H}}}(\pi_C(\mu)) = -T_\mu \pi_C\left(\Psi(m)(\xi, \cdot)\right), \tag{4}$$

for any $\xi \in \mathfrak{g}$, $(m, \mu) \in \widetilde{M}$, where $\Psi : M \to Z^2(\mathfrak{g})$ is the *Chu map* defined by $\Psi(\xi, \eta) := \omega\left(\xi_M, \eta_M\right)$, for any $\xi, \eta \in \mathfrak{g}$.

3.2. A normal form for the cylinder valued momentum map.

A major technical tool in some proofs of the classical convexity theorems for the standard momentum map is a normal form ([19], [15]) which is the analogue of the classical Slice Theorem for proper group actions adapted to the symplectic symmetric setup; it provides a semi-global set of coordinates (global only in the direction of the group orbits) in which the standard momentum map takes a particularly convenient and simple form and in which the conditions of the Local-to-Global Principle can be verified. This normal form has been generalized to the context of the cylinder valued momentum map in [23, 26]. We briefly review this generalization.

In this section we will work on a connected and paracompact symplectic manifold (M, ω) acted properly and symplectically upon by the Lie group G with Lie algebra $\mathfrak{g}$. The first step in the construction of the symplectic slice theorem is the splitting of the Lie algebra $\mathfrak{g}$ of G into three parts. The first summand is defined by

$$\mathfrak{k} := \left\{\xi \in \mathfrak{g} \mid \xi_M(m) \in (\mathfrak{g} \cdot m)^{\omega(m)}\right\}, \tag{5}$$

where $m \in M$ is the point around whose G-orbit we want to construct the symplectic slice. The set $\mathfrak{k}$ is clearly a vector subspace of $\mathfrak{g}$ that contains the Lie algebra $\mathfrak{g}_m$ of the isotropy subgroup G_m of the point $m \in M$. In fact, $\mathfrak{k}$ is a Lie subalgebra of $\mathfrak{g}$. Since the G-action is proper (by hypothesis), the isotropy subgroup G_m is compact and hence there is an Ad_{G_m}-invariant inner product $\langle\cdot, \cdot\rangle_{\mathfrak{g}}$ on $\mathfrak{g}$. We decompose

$$\mathfrak{k} = \mathfrak{g}_m \oplus \mathfrak{m} \quad \text{and} \quad \mathfrak{g} = \mathfrak{g}_m \oplus \mathfrak{m} \oplus \mathfrak{q}, \tag{6}$$

where $\mathfrak{m}$ is the $\langle\cdot, \cdot\rangle_{\mathfrak{g}}$–orthogonal complement of $\mathfrak{g}_m$ in $\mathfrak{k}$ and $\mathfrak{q}$ is the $\langle\cdot, \cdot\rangle_{\mathfrak{g}}$–orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$. The splittings in (6) induce similar ones on the duals

$$\mathfrak{k}^* = \mathfrak{g}_m^* \oplus \mathfrak{m}^* \quad \text{and} \quad \mathfrak{g}^* = \mathfrak{g}_m^* \oplus \mathfrak{m}^* \oplus \mathfrak{q}^*. \tag{7}$$

Each of the spaces in this decomposition should be understood as the set of covectors in $\mathfrak{g}^*$ that can be written as $\langle\xi, \cdot\rangle_{\mathfrak{g}}$, with ξ in the corresponding subspace; e.g., $\mathfrak{q}^* = \{\langle\xi, \cdot\rangle_{\mathfrak{g}} \mid \xi \in \mathfrak{q}\}$. The subspace $\mathfrak{q} \cdot m$ is a symplectic subspace of $(T_m M, \omega(m))$.

Let now $\langle\langle\cdot, \cdot\rangle\rangle$ be a G_m–invariant inner product in $T_m M$ (available by the compactness of G_m). Define V to be the $\langle\langle\cdot, \cdot\rangle\rangle$-orthogonal complement to $\mathfrak{g} \cdot m \cap (\mathfrak{g} \cdot m)^{\omega(m)} = \mathfrak{k} \cdot m$ in $(\mathfrak{g} \cdot m)^{\omega(m)}$:

$$(\mathfrak{g} \cdot m)^{\omega(m)} = \left(\mathfrak{g} \cdot m \cap (\mathfrak{g} \cdot m)^{\omega(m)}\right) \oplus V = \mathfrak{k} \cdot m \oplus V.$$

The subspace V is a symplectic G_m–invariant subspace of $(T_mM, \omega(m))$ such that $V \cap \mathfrak{q}\cdot m = \{0\}$. Any such space V is called a *symplectic normal space* at m. Since the G_m–action on $(V, \omega(m)|_V)$ is linear and symplectic it has a standard associated equivariant momentum map $\mathbf{J}_V : V \to \mathfrak{g}_m^*$ given by $\langle \mathbf{J}_V(v), \eta\rangle = \frac{1}{2}\omega(m)(\eta_V(v), v)$. The proof of the following two results can be found in [23, 24].

Proposition 3.1. **(The symplectic tube)** *Let (M, ω) be a connected paracompact symplectic manifold and G a Lie group acting properly and symplectically on it. Let $m \in M$, V be a symplectic normal space at m, and $\mathfrak{m} \subset \mathfrak{g}$ the subspace introduced in the splitting* (6). *Then there exist G_m–invariant neighborhoods $\mathfrak{m}_r^*$ and V_r of the origin in $\mathfrak{m}^*$ and V, respectively, such that the twisted product*

$$Y_r := G \times_{G_m} (\mathfrak{m}_r^* \times V_r) \tag{8}$$

is a symplectic manifold with the two-form ω_{Y_r} defined by:

$$\begin{aligned}
&\omega_{Y_r}([g,\rho,v])(T_{(g,\rho,v)}\pi(T_eL_g(\xi_1),\alpha_1,u_1), T_{(g,\rho,v)}\pi(T_eL_g(\xi_2),\alpha_2,u_2))\\
&\quad := \langle \alpha_2 + T_v\mathbf{J}_V(u_2), \xi_1\rangle - \langle \alpha_1 + T_v\mathbf{J}_V(u_1), \xi_2\rangle + \langle \rho + \mathbf{J}_V(v), [\xi_1,\xi_2]\rangle\\
&\qquad + \Psi(m)(\xi_1,\xi_2) + \omega(m)(u_1,u_2),
\end{aligned} \tag{9}$$

where $\Psi : M \to Z^2(\mathfrak{g})$ is the Chu map associated to the G-action on (M,ω), $\pi : G \times (\mathfrak{m}_r^* \times V_r) \to G \times_{G_m} (\mathfrak{m}_r^* \times V_r)$ is the projection, $[g,\rho,v] \in Y_r$, $\xi_1,\xi_2 \in \mathfrak{g}$, $\alpha_1,\alpha_2 \in \mathfrak{m}^*$, and $u_1,u_2 \in V$.

The Lie group G acts symplectically on (Y_r, ω_{Y_r}) by $g\cdot[h,\eta,v] := [gh,\eta,v]$, for any $g \in G$ and any $[h,\eta,v] \in Y_r$.

The symplectic manifold (Y_r, ω_{Y_r}) is called a *symplectic tube* of (M,ω) at the point m.

Theorem 3.2. **(The symplectic slice)** *Let (M,ω) be a symplectic manifold and let G a Lie group acting properly and symplectically on M. Let (Y_r, ω_{Y_r}) be the G-symplectic tube at $m \in M$ constructed in Proposition* 3.1. *Then there is a G-invariant neighborhood U of m in M and a G-equivariant symplectomorphism $\phi : U \to Y_r$ satisfying $\phi(m) = [e, 0, 0]$.*

We now provide an expression in the symplectic tube for the cylinder valued momentum map, called the normal form. The proof of the following theorem can be found in [26].

Theorem 3.3. **(Normal form for cylinder valued momentum maps)** *Let (M,ω) be a connected paracompact symplectic manifold acted properly and symplectically upon by the connected Lie group G. Let (Y_r,ω_{Y_r}) be a symplectic tube at $m \in M$ that models a G-invariant neighborhood U of the orbit $G\cdot m$ via the G-equivariant symplectomorphism $\phi : (Y_r, \omega_{Y_r}) \to (U, \omega|_U)$. Let $\mathbf{K} : M \to \mathfrak{g}^*/\overline{\mathcal{H}}$ be a cylinder valued momentum map associated to the G-action on M with non-equivariance one-cocycle $\sigma : G \to \mathfrak{g}^*/\overline{\mathcal{H}}$. Then for any $[g,\rho,v] \in Y_r$ we have*

$$\mathbf{K}(\phi[g,\rho,v]) = \Theta_g\Big(\mathbf{K}(m) + \pi_C(\rho + \mathbf{J}_V(v))\Big) = \Theta_g\left(\mathbf{K}(m)\right) + \pi_C\left(\mathrm{Ad}^*_{g^{-1}}(\rho + \mathbf{J}_V(v))\right)$$

where $\pi_C : \mathfrak{g}^* \to \mathfrak{g}^*/\overline{\mathcal{H}}$ is the projection and $\Theta : G \times \mathfrak{g}^*/\overline{\mathcal{H}} \to \mathfrak{g}^*/\overline{\mathcal{H}}$ is the affine action associated to the non-equivariance one-cocycle σ.

3.3. Closed Hamiltonian holonomies and covering spaces.

We will use in our study of the convexity properties of the cylinder valued momentum map a hypothesis that allows us to naturally endow the target space of this map with all the necessary metric properties. More specifically, *we will assume in all that follows that the Hamiltonian holonomy $\mathcal{H}$ is a closed subgroup of $(\mathfrak{g}^*, +)$.*

To spell out the implications of this hypothesis we introduce some terminology. Let G be a group that acts on a topological space X. This action is called *totally discontinuous* if every point $x \in X$ has a neighborhood U such that $g \cdot U \cap U = \varnothing$ for all $g \in G$ satisfying $g \cdot x \neq x$.

The proof of the following two results can be found in [8], Propositions 3.4.15 and 3.4.16. We recall that if $f : X \to Y$ is a continuous map between two topological spaces and $f_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ is the induced homomorphism of fundamental groups then a covering map $f : X \to Y$ is said to be *regular* if $f_*(\pi_1(X, x_0))$ is a normal subgroup of $\pi_1(Y, y_0)$ or, equivalently, $f_*(\pi_1(X, x_0))$ does not depend on $x_0 \in f^{-1}(y_0)$.

Proposition 3.4. *Let G be a group acting on a topological space X freely and totally discontinuously. Then the projection $\pi_G : X \to X/G$ onto the orbit space is a regular covering map. Moreover, the group of its deck transformations coincides with G.*

Theorem 3.5. *Let $f : X \to Y$ be a regular covering and G its group of deck transformations. Then the length metrics on Y are in one-to-one correspondence with the G-invariant length metrics on X so that for corresponding metrics d_X on X and d_Y on Y, f is a local isometry.*

We analyze the implications of the closedness hypothesis on the Hamiltonian holonomy $\mathcal{H}$.

Proposition 3.6. *Let (M, ω) be a connected paracompact symplectic manifold acted symplectically upon by the Lie algebra $\mathfrak{g}$ with Hamiltonian holonomy $\mathcal{H}$. If $\mathcal{H}$ is closed in $\mathfrak{g}^*$ then:*

(i) *The projection $\pi_C : \mathfrak{g}^* \to \mathfrak{g}^*/\mathcal{H}$ is a smooth regular covering map and hence the Euclidean metric in $\mathfrak{g}^*$ projects naturally to a length metric on $\mathfrak{g}^*/\mathcal{H}$ with respect to which this space is complete and locally compact and the projection π_C is a local isometry.*

(ii) *Suppose that there exists a compact connected Lie group G whose Lie algebra is $\mathfrak{g}$. Identify the positive Weyl chamber $\mathfrak{t}^*_+ \subset \mathfrak{g}^*$ with the orbit space of the coadjoint action of G on $\mathfrak{g}^*$. Then, $\mathfrak{t}^*_+$ is a closed convex subset of $\mathfrak{g}^*$ and hence it has a natural length space structure with respect to which it is complete and locally compact. In addition, $\mathcal{H}$ acts on $\mathfrak{t}^*_+$ in a totally discontinuous fashion and hence the length metric in $\mathfrak{t}^*_+$ projects naturally to a length metric on $\mathfrak{t}^*_+/\mathcal{H}$ with respect to which this space is complete and locally compact and the orbit space projection $\pi_C^+ : \mathfrak{t}^*_+ \to \mathfrak{t}^*_+/\mathcal{H}$ is a local isometry.*

(iii) *In the hypotheses of part* (ii)*, the natural identification of* $\mathfrak{t}_+^*$ *with the orbit space* $\mathfrak{g}^*/G$ *of the coadjoint action induces an identification of* $\mathfrak{t}_+^*/\mathcal{H}$ *with the orbit space* $(\mathfrak{g}^*/\mathcal{H})/G$ *of the* $\mathcal{A}d^*$ *action of* G *on* $\mathfrak{g}^*/\mathcal{H}$ *with respect to which the following diagram commutes:*

$$\begin{array}{ccc} \mathfrak{g}^* & \xrightarrow{\pi_C} & \mathfrak{g}^*/\mathcal{H} \\ {\scriptstyle \pi_G}\big\downarrow & & \big\downarrow{\scriptstyle \pi_G^+} \\ \mathfrak{g}^*/G \simeq \mathfrak{t}_+^* & \xrightarrow{\pi_C^+} & \mathfrak{t}_+^*/\mathcal{H} \simeq (\mathfrak{g}^*/\mathcal{H})/G. \end{array} \tag{10}$$

Proof. Since $\mathcal{H}$ acts on $(\mathfrak{g}^*, +)$ by translations, the Euclidean metric on $\mathfrak{g}^*$ is $\mathcal{H}$-invariant. Additionally, as this action is free and proper, the Slice Theorem guarantees that any point $\mu \in \mathfrak{g}^*$ has an $\mathcal{H}$-invariant neighborhood that is equivariantly diffeomorphic to the product $\mathcal{H} \times U$, with $U \subset \mathfrak{g}^*$ an open neighborhood of zero in $\mathfrak{g}^*$. In this semi-global model the point μ is represented by the element $(0,0)$. Since $\mathcal{H}$ is a closed zero dimensional submanifold of $\mathfrak{g}^*$, it follows that the set $\{0\} \times U$ is an open neighborhood of $(0,0) \equiv \mu$. Moreover, for any $\nu \in \mathcal{H}$ different from zero, we have $\nu \cdot (\{0\} \times U) = \{\nu\} \times U$ and since $(\{\nu\} \times U) \cap (\{0\} \times U) = \varnothing$ we conclude that $\mathcal{H}$ acts totally discontinuously on $\mathfrak{g}^*$. The first part in the statement (i) follows then by Proposition 3.4 and Theorem 3.5. The local compactness of $\mathfrak{g}^*/\mathcal{H}$ is a consequence of the open character of the orbit projection π_C and the local compactness of $\mathfrak{g}^*$ (we recall that every orbit projection is an open map and that if $f : X \to Y$ is an arbitrary open map from a locally compact topological space onto a Hausdorff space Y, then Y is locally compact). The completeness of $\mathfrak{g}^*/\mathcal{H}$ follows from the fact that $\mathcal{H}$ is a discrete subgroup of $(\mathfrak{g}^*, +)$ acting freely on this space by translations; hence the quotient $\mathfrak{g}^*/\mathcal{H}$ is a cylinder with its natural metric structure inherited from $\mathfrak{g}^*$ and it is therefore complete.

As to part (ii) we recall that the positive Weyl chamber $\mathfrak{t}_+^*$ is a closed convex subset of $\mathfrak{g}^*$ and hence a complete and locally compact length metric space (see Definitions 4.4 and 2.8). We now recall that if G is connected, then $\mathcal{H}$ is pointwise fixed by the coadjoint action [25] and hence the G-coadjoint action and the $\mathcal{H}$-action on $\mathfrak{g}^*$ commute, which guarantees that the $\mathcal{H}$-action on $\mathfrak{g}^*$ drops to an $\mathcal{H}$-action on $\mathfrak{g}^*/G \simeq \mathfrak{t}_+^*$. Since this action can be viewed as the restriction to $\mathfrak{t}_+^*$ of the totally discontinuous $\mathcal{H}$-action on $\mathfrak{g}^*$, we conclude that it is also totally discontinuous and hence Proposition 3.4 and Theorem 3.5 apply. Finally, we show that $\mathfrak{t}_+^*/\mathcal{H}$ is complete by showing that it is a closed subset of the complete metric space $\mathfrak{g}^*/\mathcal{H}$. Indeed, let $[\tau]$ be an element in the closure of $\mathfrak{t}_+^*/\mathcal{H}$ in $\mathfrak{g}^*/\mathcal{H}$ and let $\{[\tau_n]\}_{n\in\mathbb{N}}$ be a sequence of elements in $\mathfrak{t}_+^*/\mathcal{H}$ such that $[\tau_n] \to [\tau]$. We will now prove that $[\tau] \in \mathfrak{g}^*/\mathcal{H}$. Since π_C is a covering map, there exists a neighborhood $U_{[\tau]}$ of $[\tau]$ in $\mathfrak{g}^*/\mathcal{H}$ and an open set U in $\mathfrak{g}^*$ such that $\pi_C^{-1}(U_{[\tau]}) = \bigcup_{\nu\in\mathcal{H}}(\nu + U)$, $\pi_C|_{\nu+U}$ is a diffeomorphism onto its image, and $(\nu + U) \cap (\mu + U) = \varnothing$, for any $\nu, \mu \in \mathfrak{g}^*$. Let $N \in \mathbb{N}$ such that $[\tau_n] \in U_{[\tau]}$ for any $n \geq N$ and let $\tau_n := \pi_C^{-1}([\tau_n]) \cap U$, $\tau := \pi_C^{-1}([\tau]) \cap U$, for any $n \geq N$. Since $\mathcal{H}$ acts on $\mathfrak{t}_+^*$ and π_C is a local isometry, the sequence $\{\tau_n\}_{n\geq N}$ lies in $\mathfrak{t}_+^*$ and $\tau_n \to \tau$. Since $\mathfrak{t}_+^*$ is a closed subset of $\mathfrak{g}^*$ it follows that $\tau \in \mathfrak{t}_+^*$ and hence $[\tau] \in \mathfrak{g}^*/\mathcal{H}$, as required.

Regarding part (iii), the identification $\mathfrak{t}_+^*/\mathcal{H} \simeq (\mathfrak{g}^*/\mathcal{H})/G$ is a consequence of the fact that the G-coadjoint action and the $\mathcal{H}$-action on $\mathfrak{g}^*$ commute because G is connected. The rest of the statement is a straightforward diagram chasing exercise. ■

3.4. Convexity properties of the cylinder valued momentum map. The Abelian case.

In this subsection it will be shown that the image of the cylinder valued momentum map associated to a proper Abelian Lie group action is weakly convex. To prove this statement we use the Symplectic Slice Theorem 3.2 to show that this map satisfies the local hypotheses needed to apply the generalization of the Local-to-Global Principle for length spaces (Theorem 2.17). The main step in that direction is taken in the following proposition.

Proposition 3.7. *Let (M,ω) be a connected paracompact symplectic manifold and let G a connected Abelian Lie group acting properly and symplectically on M with closed Hamiltonian holonomy $\mathcal{H}$. Let $\mathbf{K} : M \to \mathfrak{g}^*/\mathcal{H}$ be a cylinder valued momentum map for this action and $m \in M$ arbitrary such that $\mathbf{K}(m) = [\mu] \in \mathfrak{g}^*/\mathcal{H}$. Then there exists an open neighborhood U of m in M and open neighborhoods W and V of $\mu \in \mathfrak{g}^*$ and $[\mu] \in \mathfrak{g}^*/\mathcal{H}$, respectively, such that $\mathbf{K}(U) \subset V$, $\pi_C|_W : W \to V$ is a diffeomorphism, and*

$$\pi_C|_W^{-1} \circ \mathbf{K}|_U = \mathbf{J}_U + c, \tag{11}$$

with $c \in \mathfrak{g}^*$ a constant and $\mathbf{J}_U : U \to \mathfrak{g}^*$ a map that in symplectic slice coordinates around the point m has the expression

$$\mathbf{J}_U([g,\rho,v]) = \rho + \mathbf{J}_V(v) - \langle \mathbb{P}_{\mathfrak{q}}(\exp^{-1}(s([g]))), \cdot\rangle_{\mathfrak{q}}. \tag{12}$$

The neighborhood U has been chosen so that it can be written in slice coordinates as $U \equiv U_e \times_{G_m} (\mathfrak{m}^* \times V_r)$, with U_e an open G_m-invariant neighborhood of e in G small enough so that there exists a local section $s : U_e/G_m \to V_e$ for the projection $G \to G/G_m$. The set V_e is an open neighborhood of $e \in G$ such that $\exp : U_0 \to V_e$ *is a diffeomorphism, for some open neighborhood U_0 of $0 \in \mathfrak{g}^*$. The map $\mathbb{P}_{\mathfrak{q}} : \mathfrak{g} = \mathfrak{g}_m \oplus \mathfrak{m} \oplus \mathfrak{q} \to \mathfrak{q}$ is the projection onto $\mathfrak{q}$ constructed using the splitting in* (6) *and $\langle \cdot,\cdot\rangle_{\mathfrak{q}}$ is the non-degenerate bilinear form on $\mathfrak{q}$ induced by the Chu map at m, that is, for any $\xi,\eta \in \mathfrak{q}$, $\langle \xi,\eta\rangle_{\mathfrak{q}} := \omega(m)(\xi_M(m),\eta_M(m))$.*

Proof. Since $\mathcal{H}$ is closed in $\mathfrak{g}^*$, the projection π_C is a local diffeomorphism (see Proposition 3.6) and hence there exist an open neighborhood V of $\mathbf{K}(m)$ in $\mathfrak{g}^*/\mathcal{H}$ and a neighborhood W of some element in the fiber $\pi_C^{-1}(\mathbf{K}(m)) \subset \mathfrak{g}^*$ such that $\pi_C|_W : W \to V$ is a diffeomorphism. Let U be the connected component containing m of the intersection of $\mathbf{K}^{-1}(V)$ with the domain of a symplectic slice chart around m and shrink it, if necessary, so that the group factor in the slice coordinates has the properties in the statement of the proposition.

We start by noticing that (12) is well defined because for any $h \in G_m$ and any $[g,\rho,v] \in U_e \times_{G_m} (\mathfrak{m}^* \times V_r)$ we have

$$\begin{aligned}
\mathbf{J}_U([gh, h^{-1}\cdot\rho, h^{-1}\cdot v]) &= c + \mathrm{Ad}_h^*\rho + \mathbf{J}_V(h^{-1}\cdot v) - \langle \mathbb{P}_{\mathfrak{q}}(\exp^{-1}(s([gh]))), \cdot\rangle_{\mathfrak{q}} \\
&= c + \mathrm{Ad}_h^*(\rho + \mathbf{J}_V(v)) - \langle \mathbb{P}_{\mathfrak{q}}(\exp^{-1}(s([g]))), \cdot\rangle_{\mathfrak{q}} \\
&= c + \rho + \mathbf{J}_V(v) - \langle \mathbb{P}_{\mathfrak{q}}(\exp^{-1}(s([g]))), \cdot\rangle_{\mathfrak{q}}.
\end{aligned}$$

The last equality follows from the Abelian character of G.

Next, we will show that for any $z \in U$ and any $\xi \in \mathfrak{g}$, the map $\mathbf{J}_U^\xi := \langle \mathbf{J}_U, \xi \rangle$ satisfies

$$X_{\mathbf{J}_U^\xi}(z) = \xi_M(z), \tag{13}$$

with $X_{\mathbf{J}_U^\xi}$ the Hamiltonian vector field associated to the function $\mathbf{J}_U^\xi \in C^\infty(M)$. Since this is a local statement, it suffices to show that

$$\mathbf{i}_{\xi_{Y_r}} \omega_{Y_r}([\exp \zeta, \rho, v]) = \mathbf{d}\mathbf{J}_U^\xi([\exp \zeta, \rho, v]), \tag{14}$$

where $[\exp \zeta, \rho, v]$ is the expression of z in slice coordinates and $\zeta \in \mathfrak{g}$ is chosen so that $s([\exp \zeta]) = \exp \zeta$. We prove (14) by using the expression of ω_{Y_r} in Proposition 3.1. First, notice that

$$\xi_{Y_r}([\exp \zeta, \rho, v]) = T_{(\exp \zeta, \rho, v)} \pi (T_e L_{\exp \zeta}(\xi), 0, 0),$$

where $\pi : G \times (\mathfrak{m}_r^* \times V_r) \to G \times_{G_m} (\mathfrak{m}_r^* \times V_r)$ is the orbit projection. If we let

$$w := T_{(\exp \zeta, \rho, v)} \pi (T_e L_{\exp \zeta}(\eta), \alpha, u) \in T_{[\exp \zeta, \rho, v]} (G \times_{G_m} (\mathfrak{m}_r^* \times V_r))$$

then

$$\begin{aligned}
& \omega_{Y_r}([\exp \zeta, \rho, v])(\xi_{Y_r}([\exp \zeta, \rho, v]), w) \\
= {} & \langle \alpha + T_v \mathbf{J}_V(u), \xi \rangle + \omega(m)(\xi_M(m), \eta_M(m)) \\
= {} & \langle \alpha + T_v \mathbf{J}_V(u), \xi \rangle + \omega(m)((\mathbb{P}_\mathfrak{q} \xi)_M(m), (\mathbb{P}_\mathfrak{q} \eta)_M(m)) \\
= {} & \langle \alpha + T_v \mathbf{J}_V(u), \xi \rangle + \langle \mathbb{P}_\mathfrak{q} \xi, \mathbb{P}_\mathfrak{q} \eta \rangle_\mathfrak{q}.
\end{aligned} \tag{15}$$

On the other hand, by (12), we have

$$\begin{aligned}
& \mathbf{d}\mathbf{J}_U^\xi([\exp \zeta, \rho, v]) \cdot w \\
& \qquad = \langle \alpha + T_v \mathbf{J}_V(u), \xi \rangle - \left.\frac{d}{dt}\right|_{t=0} \langle \mathbb{P}_\mathfrak{q} \exp^{-1} s([\exp \zeta \exp t\eta]), \mathbb{P}_\mathfrak{q} \xi \rangle_\mathfrak{q}.
\end{aligned} \tag{16}$$

In order to compute the second summand of the right hand side, notice that

$$s([\exp \zeta \exp t\eta]) = s([\exp(\zeta + t\eta)]) = \exp(\zeta + t\eta) h(t),$$

with $h(t)$ a curve in G_m such that $h(0) = e$ and $h'(0) = \lambda \in \mathfrak{g}_m$. Consequently,

$$\left.\frac{d}{dt}\right|_{t=0} s([\exp \zeta \exp t\eta]) = T_e L_{\exp \zeta}(\eta + \lambda) = \left.\frac{d}{dt}\right|_{t=0} \exp(\zeta + t(\eta + \lambda))$$

and hence, since $\mathbb{P}_\mathfrak{q} \lambda = 0$, we have

$$\begin{aligned}
& \left.\frac{d}{dt}\right|_{t=0} \langle \mathbb{P}_\mathfrak{q} \exp^{-1} s([\exp \zeta \exp t\eta]), \mathbb{P}_\mathfrak{q} \xi \rangle_\mathfrak{q} \\
& \qquad = \left.\frac{d}{dt}\right|_{t=0} \langle \mathbb{P}_\mathfrak{q}(\zeta + t(\eta + \lambda)), \mathbb{P}_\mathfrak{q} \xi \rangle_\mathfrak{q} = \langle \mathbb{P}_\mathfrak{q} \eta, \mathbb{P}_\mathfrak{q} \xi \rangle_\mathfrak{q}.
\end{aligned} \tag{17}$$

The equalities (15), (16), and (17) show that (13) holds.

With this in mind we will now show that for any $z \in U$

$$T_z(\pi_C|_W^{-1} \circ \mathbf{K}|_U) = T_z\mathbf{J}_U. \tag{18}$$

Indeed, for any $v_z \in T_zM$ and $\rho \in \pi_C^{-1}(V)$ such that $\mathbf{K}(z) = \pi_C(\rho)$,

$$T_z(\pi_C|_W^{-1} \circ \mathbf{K}|_U)(v_z) = \left(T_{\mathbf{K}(z)}\, \pi_C|_W^{-1} \circ T_z\mathbf{K}\right)(v_z) = \left(T_{\mathbf{K}(z)}\, \pi_C|_W^{-1} \circ T_\rho\pi_C\right)(\nu) = \nu,$$

where $\nu \in \mathfrak{g}^*$ is uniquely determined by the expression

$$\langle \nu, \xi\rangle = (\mathbf{i}_{\xi_M}\omega)(z)\,(v_z)\,, \text{ for all } \xi \in \mathfrak{g}. \tag{19}$$

On the other hand, by (13), we can write

$$\langle T_z\mathbf{J}\,(v_z)\,, \xi\rangle = \mathbf{d}\mathbf{J}_U^\xi(z)\,(v_z) = (\mathbf{i}_{\xi_M}\omega)(z)\,(v_z)\,.$$

This, together with (19), shows that (18) holds.

Let $c(t)$ be a smooth curve such that $c(0) = m$ and $c(1) = z$, available by the connectedness of U. Then by (18)

$$\begin{aligned}
\left(\pi_C|_W^{-1} \circ \mathbf{K}\right)(z) \quad - \quad \left(\pi_C|_W^{-1} \circ \mathbf{K}\right)(m) &= \int_0^1 \frac{d}{dt}\left(\pi_C|_W^{-1} \circ \mathbf{K}\right)(c(t))dt \\
&= \int_0^1 T_{c(t)}\left(\pi_C|_W^{-1} \circ \mathbf{K}\right)(\dot{c}(t))\,dt \\
&= \int_0^1 T_{c(t)}\mathbf{J}_U\,(\dot{c}(t))\,dt = \mathbf{J}_U(z) - \mathbf{J}_U(m).
\end{aligned}$$

Since $z \in M$ is arbitrary and $m \in M$ is fixed, the previous equality shows that (11) holds by setting $c = \left(\pi_C|_W^{-1} \circ \mathbf{K}\right)(m) - \mathbf{J}_U(m)$. ■

Theorem 3.8. *Let (M, ω) be a connected paracompact symplectic manifold and let G a connected Abelian Lie group acting properly and symplectically on M with closed Hamiltonian holonomy $\mathcal{H}$. Let $\mathbf{K} : M \to \mathfrak{g}^*/\mathcal{H}$ be a cylinder valued momentum map for this action. If $\mathbf{K}$ is a closed map then the image $\mathbf{K}(M) \subset \mathfrak{g}^*/\mathcal{H}$ is a weakly convex subset of $\mathfrak{g}^*/\mathcal{H}$. We think of $\mathfrak{g}^*/\mathcal{H}$ as a length metric space with the length metric naturally inherited from $\mathfrak{g}^*$ (see Proposition* 3.6*). If $\mathcal{H} = \{0\}$ (so $\mathbf{K} : M \to \mathfrak{g}^*$ is a standard momentum map) then, as $\mathfrak{g}^*$ is uniquely geodesic, we have that $\mathbf{K}(M)$ is convex, $\mathbf{K}$ has connected fibers, and it is open map onto its image.*

Proof. We will establish this result by using the Local-to-Global Principle for length spaces (Theorem 2.17). First, notice that the closedness of the Hamiltonian holonomy implies, by Proposition 3.6, that $\mathfrak{g}^*/\mathcal{H}$ is a complete and locally compact length space and that the projection π_C is a local isometry. Therefore, in order to apply Theorem 2.17 we need to show that $\mathbf{K}$ is locally open onto its image, locally fiber connected, and has local convexity data. Now, by Proposition 3.7 (more specifically by (11)) and Lemma 4.5, it suffices to prove that those three local properties are satisfied by the map $\mathbf{J}_U : U \to \mathfrak{g}^*$.

We start the proof of this by recalling that since the G-action is proper the isotropy subgroup G_m is compact and hence its connected component containing the identity is isomorphic to a torus. Thus, the map $\mathbf{J}_V : V \to \mathfrak{g}_m^*$ is the momentum map of the symplectic representation of a torus on the symplectic vector space V and hence it automatically has (see for instance [16] for a proof) local convexity data and it is locally fiber connected and locally open onto its image. Additionally, if we split $\mathfrak{g}^*$ as $\mathfrak{g}^* = \mathfrak{g}_m^* \oplus \mathfrak{m}^* \oplus \mathfrak{q}^*$ then the map $\mathbf{J}_U - c$ can be decomposed as

$$\mathbf{J}_U([g, \rho, v]) - c = (\mathbf{J}_V(v), \rho, -\langle \mathbb{P}_{\mathfrak{q}}(\exp^{-1}(s([g]))), \cdot \rangle_{\mathfrak{q}}).$$

Each of the three components of the map has local convexity data, is locally open onto its image, and is locally fiber connected. Thus $\mathbf{J}_U$ also has these properties, as required. ■

Remark 3.9. This result implies the weak convexity of the image of *Lie group valued momentum maps* introduced in [20, 11, 1] when the group is Abelian. Let (M, ω) be a symplectic manifold, T^k a torus acting symplectically on (M, ω), and $(\cdot, \cdot)$ an inner product in the Lie algebra $\mathfrak{t}$ of T^k. The map $\mu : M \to T^k$ is called a Lie group valued momentum map if for any $\xi \in \mathfrak{t}$, $\mathbf{i}_{\xi_M}\omega = \mu^*(\theta, \xi)$, where $\theta \in \Omega^1(T^k, \mathfrak{t})$ is the bi-invariant Maurer-Cartan form.

A typical example for this momentum map is provided by the following situation. Take the symplectic manifold $T^2 = S^1 \times S^1$ with symplectic form the standard area form and consider the action of the circle on the first circle of T^2. The S^1-valued momentum map associated to this action is the projection on the second circle of T^2, namely, $\mu(e^{i\varphi_1}, e^{i\varphi_2}) = e^{i\varphi_2}$.

A very simple argument shows that the image of $\mu : M \to T^k$ is a weakly convex subset of T^k. Indeed, by Proposition 3.4 and Remark 3.3 of [1], each point in M has an open simply connected neighborhood $U \subset M$ and a standard momentum map $\Phi : U \to \mathfrak{t}$ ($\mathfrak{t}$ and $\mathfrak{t}^*$ are identified here) such that the restriction $\exp|_{\Phi(U)} : \Phi(U) \to T^k$ is a diffeomorphism onto its image and one has $\mu|_U = \exp \circ \Phi$. This immediately implies that μ has the local properties (LFC), (LOI), and (LCD). Thus the hypotheses of Theorem 2.17 hold and the statement is a direct consequence of this theorem.

Remark 3.10. For a map with values in a metric spaces that is not uniquely geodesic, the convexity property of its image is not related to the connectedness of its fibers. This is in sharp contrast to the situation encountered for standard momentum maps. For instance, if in the example in the remark above we multiply the symplectic form by two, then the associated S^1-valued momentum map is $\mu(e^{i\varphi_1}, e^{i\varphi_2}) = e^{2i\varphi_2}$ which satisfies the hypothesis of Theorem 2.17 and hence has a convex image but does not have connected fibers.

3.5. Convexity properties of the cylinder valued momentum map. The Non-Abelian case.

The study of the convexity properties of the image of the cylinder valued momentum map for non-Abelian groups presents two main complications with respect to its Abelian analog. First, unless additional hypotheses are introduced, there is no convenient local representation for $\mathbf{K}$ (the counterpart of Proposition 3.7) implying the necessary local properties that ensure convexity by application of the Local-to-Global Principle. Second, the entire image is not likely to be convex since, already in the standard momentum map case, one has to intersect with a Weyl chamber to obtain convexity. We will take care of the first problem by working with special actions, namely those that are *tubewise Hamiltonian*. The second problem will be solved, as in the classical case, by intersecting the image of $\mathbf{K}$ with $\mathfrak{t}^*_+/\mathcal{H}$ and taking advantage of the good behavior of the projection $\pi_C^+ : \mathfrak{t}^*_+ \to \mathfrak{t}^*_+/\mathcal{H}$ introduced and discussed in the second part of Proposition 3.6.

Definition 3.11. Let (M, ω) be a symplectic manifold acted symplectically upon by a Lie group G. We say that the G-action on M is *tubewise Hamiltonian at* $m \in M$ if there exists a G-invariant open neighborhood of the orbit $G \cdot m$ such that the restriction of the action to the symplectic manifold $(U, \omega|_U)$ has an associated standard momentum map. The G-action on M is called *tubewise Hamiltonian* if it is tubewise Hamiltonian at any point of M.

Sufficient conditions ensuring that a symplectic action is tubewise Hamiltonian have been given in [23, 22]. For example, here are two useful results.

Proposition 3.12. *Let (M, ω) be a symplectic manifold and let G a Lie group with Lie algebra $\mathfrak{g}$ acting properly and symplectically on M. For $m \in M$ let $Y_r := G \times_{G_m} (\mathfrak{m}^*_r \times V_r)$ be the slice model around the orbit $G \cdot m$ introduced in Proposition 3.1. If the G-equivariant $\mathfrak{g}^*$-valued one form $\gamma \in \Omega^1(G; \mathfrak{g}^*)$ defined by*

$$\langle \gamma(g)\left(T_eL_g(\eta)\right), \xi\rangle := -\omega(m)\left(\left(\mathrm{Ad}_{g^{-1}}\xi\right)_M(m), \eta_M(m)\right) \tag{20}$$

for any $g \in G$ and $\xi, \eta \in \mathfrak{g}$ is exact, then the G-action on Y_r given by $g \cdot [h, \eta, v] := [gh, \eta, v]$, for any $g \in G$ and any $[h, \eta, v] \in Y_r$, has an associated standard momentum map and thus the G-action on (M, ω) is tubewise Hamiltonian at m.

Corollary 3.13. *Let (M, ω) be a symplectic manifold and let G a Lie group with Lie algebra $\mathfrak{g}$ acting properly and symplectically on M. If either $H^1(G) = 0$, or the orbit $G \cdot m$ is isotropic, then the G-action on (M, ω) is tubewise Hamiltonian at m.*

The following result is the analog of Proposition 3.7 in the non-Abelian setup.

Proposition 3.14. *Let (M, ω) be a connected paracompact symplectic manifold and let G a compact connected Lie group acting symplectically on M in a tubewise Hamiltonian fashion with closed Hamiltonian holonomy $\mathcal{H}$. Let $\mathbf{K} : M \to \mathfrak{g}^*/\mathcal{H}$ be a cylinder valued momentum map for this action and $m \in M$ arbitrary such that $\mathbf{K}(m) = [\mu] \in \mathfrak{g}^*/\mathcal{H}$. Then there exist an open neighborhood U of m in M and open neighborhoods W and V of $\mu \in \mathfrak{g}^*$ and $[\mu] \in \mathfrak{g}^*/\mathcal{H}$, respectively, such that $\mathbf{K}(U) \subset V$, $\pi_C|_W : W \to V$ is a diffeomorphism, and*

$$\pi_C|_W^{-1} \circ \mathbf{K}|_U = \mathbf{J}_U + c, \tag{21}$$

where $c \in \mathfrak{g}^*$ is a constant and $\mathbf{J}_U : U \to \mathfrak{g}^*$ is a map that in symplectic slice coordinates around the point m has the expression

$$\mathbf{J}_U([g, \rho, v]) = \mathrm{Ad}^*_{g^{-1}}(\nu + \rho + \mathbf{J}_V(v)), \tag{22}$$

with $\nu \in \mathfrak{g}^*$ a constant.

Proof. Due to the closedness hypothesis on the Hamiltonian holonomy $\mathcal{H}$, the projection π_C is a local diffeomorphism (see Proposition 3.6) and hence there exists an open neighborhood V of $\mathbf{K}(m)$ in $\mathfrak{g}^*/\mathcal{H}$ and a neighborhood W of some element in the fiber $\pi_C^{-1}(\mathbf{K}(m)) \subset \mathfrak{g}^*$ such that $\pi_C|_W : W \to V$ is a diffeomorphism. Let U be the connected component containing m of the intersection of $\mathbf{K}^{-1}(V)$ with the domain of a symplectic slice chart around m. Given that the G-action is by hypothesis tubewise Hamiltonian, the symplectic slice chart can be chosen so that the restriction of the G-action to that chart has a standard associated momentum map $\mathbf{J}_{Y_r}$ that in slice coordinates has, by the Marle-Guillemin-Sternberg normal form [19, 15], the expression

$$\mathbf{J}_{Y_r}([g, \rho, v]) = \mathrm{Ad}^*_{g^{-1}}(\nu + \rho + \mathbf{J}_V(v)) + \sigma(g),$$

with $\nu \in \mathfrak{g}^*$ a constant and $\sigma : G \to \mathfrak{g}^*$ the non-equivariance one-cocycle of $\mathbf{J}_{Y_r}$. Since the group G is compact, $\mathbf{J}_{Y_r}$ can be chosen equivariant and hence with trivial non-equivariance cocycle σ (see [21] for the original source of this result, or [24], Proposition 4.5.19). Let $\mathbf{J}_U$ be the restriction of that equivariant momentum map to U. By the definition of the standard momentum map we have that for any $z \in U$ and any $\xi \in \mathfrak{g}$, the map $\mathbf{J}_U^\xi := \langle \mathbf{J}_U, \xi \rangle$ satisfies $X_{\mathbf{J}_U^\xi}(z) = \xi_M(z)$, with $X_{\mathbf{J}_U^\xi}$ the Hamiltonian vector field associated to the function $\mathbf{J}_U^\xi \in C^\infty(M)$. With this in mind, it suffices to mimic the proof of Proposition 3.7 starting from expression (18) to establish the statement of the proposition. ■

Theorem 3.15. *Let (M, ω) be a connected paracompact symplectic manifold and let G a compact connected Lie group acting symplectically on M in a tubewise Hamiltonian fashion with closed Hamiltonian holonomy $\mathcal{H}$. Let $\mathbf{K} : M \to \mathfrak{g}^*/\mathcal{H}$ be a cylinder valued momentum map for this action. If $\mathbf{K}$ is a closed map then the intersection of the image $\mathbf{K}(M) \subset \mathfrak{g}^*/\mathcal{H}$ with $\mathfrak{t}^*_+/\mathcal{H}$ is a weakly convex subset of $\mathfrak{g}^*/\mathcal{H}$. We think of $\mathfrak{g}^*/\mathcal{H}$ and $\mathfrak{t}^*/\mathcal{H}$ as length metric spaces with the length metric naturally inherited from $\mathfrak{g}^*$ (see Proposition 3.6). If $\mathcal{H} = \{0\}$ (and hence $\mathbf{K}$ is a standard momentum map), then as $\mathfrak{t}^*_+$ is uniquely geodesic, $\mathbf{K}(M) \cap \mathfrak{t}^*_+$ is convex, $\mathbf{K}$ has connected fibers, and it is open onto its image.*

Proof. First of all, notice that the closedness of the Hamiltonian holonomy implies, by Proposition 3.6, that $\mathfrak{g}^*/\mathcal{H}$ and $\mathfrak{t}^*_+/\mathcal{H}$ are complete and locally compact length spaces and that the projections π_C and π_C^+ are local isometries. Moreover, the identification $\mathfrak{t}^*_+/\mathcal{H} \simeq (\mathfrak{g}^*/\mathcal{H})/G$, introduced in Proposition 3.6(iii) and diagram (10) allow us to think of π_C^+ as the restriction of π_C to $\mathfrak{t}^*_+$. Consequently, if $V \subset \mathfrak{g}^*/\mathcal{H}$ is an open set such that $\pi_C|_W : W \to V$ is an isometric diffeomorphism then $\pi_C^+|_{\pi_G(W)} : \pi_G(W) \simeq W \cap \mathfrak{t}^*_+ \to \pi_G^+(V) \simeq (W \cap \mathfrak{t}^*_+)/\mathcal{H}$ is an isometry. Notice that $\pi_G(W) \simeq W \cap \mathfrak{t}^*_+ \subset \mathfrak{t}^*_+$ is an open subset of $\mathfrak{t}^*_+$ since π_G is an open map.

Using the identification $\mathfrak{t}^*_+/\mathcal{H} \simeq (\mathfrak{g}^*/\mathcal{H})/G$ we can study the convexity properties of the intersection $\mathbf{K}(M) \cap (\mathfrak{t}^*_+/\mathcal{H})$ by looking at the convexity properties of the image of the map $k := \pi_G^+ \circ \mathbf{K} : M \to \mathfrak{t}^*_+/\mathcal{H}$. We will do so by applying the Local-to-Global Principle for length spaces (Theorem 2.17) to k, that is, by showing that k is locally open onto its image, locally fiber connected, and has local convexity data. By Proposition 3.14 there exist an open neighborhood U of m in M and an open neighborhood V of $[\mu]$ in $\mathfrak{g}^*/\mathcal{H}$ such that $\mathbf{K}(U) \subset V$, $\pi_C|_W : W \to V$ is a diffeomorphism, and $\pi_C|_W^{-1} \circ \mathbf{K}|_U = \mathbf{J}_U + c$, with $c \in \mathfrak{g}^*$ a constant and $\mathbf{J}_U : U \to \mathfrak{g}^*$ a map that has the expression (22). Applying π_G to both sides of this equality, using the commutativity of diagram (10), and recalling the remarks above, we obtain

$$\left(\pi_C^+\big|_{\pi_G(W)}\right)^{-1} \circ k|_U = j_U + \pi_G(c),$$

with $j_U := \pi_G \circ \mathbf{J}_U$. Two results due to Sjamaar [28, Theorem 6.5] and Knop [18, Theorem 5.1] show that j_U is locally open onto its image, locally fiber connected, and has local convexity data. Consequently, since $\pi_C^+\big|_{\pi_G(W)}$ is an isometry, the maps j_U and $k|_U$ are related by a distance preserving diffeomorphism. More explicitly $k|_U = \lambda \circ j_U$, with $\lambda : \pi_G(W) \subset \mathfrak{t}_+^* \to \mathfrak{t}_+^*/\mathcal{H}$ given by the map $\pi_C^+\big|_{\pi_G(W)}$ composed with the translation by $\pi_G(c)$. Consequently, since j_U is locally open onto its image and locally fiber connected then so is $k|_U$. Additionally, since j_U has local convexity data and λ is a distance preserving map, Lemma 4.5 guarantees that $k|_U$ also has this property. The statement of the theorem follows then as a consequence of Theorem 2.17. ■

Remark 3.16. The classical convexity theorem of Guillemin-Sternberg-Kirwan (see [13, 14], [17]) states that if G is a compact connected Lie group acting symplectically on the compact connected symplectic manifold (M, ω) and this action has an associated standard coadjoint equivariant momentum map $\mathbf{J} : M \to \mathfrak{g}^*$, then $\mathbf{J}(M) \cap \mathfrak{t}_+^*$ is a compact convex polytope (also called polyhedron in the literature; e.g., [5]). Recall that, by definition, a *polyhedron* is an intersection of finitely many closed half spaces. We deduce this result here from Theorem 3.15 and standard results about polyhedra. First, the last statement of Theorem 3.15 guarantees that $\mathbf{J}(M) \cap \mathfrak{t}_+^*$ is a convex subset of $\mathfrak{t}_+^*$ and hence of $\mathfrak{g}^*$, $\mathbf{J}$ has connected fibers, and is a G-open map onto its image. See also [4] where the same result was obtained in a different manner. Second, since M is compact and $\mathfrak{t}_+^*$ is closed in $\mathfrak{g}^*$, it follows that $\mathbf{J}(M) \cap \mathfrak{t}_+^*$ is compact in $\mathfrak{g}^*$. Third, by [28], $\mathbf{J}(M)$ is locally polyhedral at each point. By definition, a convex subset C of a Euclidean space is said to be *locally polyhedral* at a point $x \in C$ if there is a neighborhood of x in C which is a polyhedron. Fourth, a compact convex set in a Euclidean space that is locally polyhedral at each of its points is a polyhedron ([5], TVS II.91, Exercise 24,b). Therefore $\mathbf{J}(M) \cap \mathfrak{t}_+^*$ is a compact convex polyhedron, $\mathbf{J}$ has connected fibers, and is a G-open map onto its image, which is the statement of the Guillemin-Sternberg-Kirwan convexity theorem.

4. Appendix: length spaces and convexity

In this appendix we collect some standard results and we fix the notations that we use when dealing with convexity in the context of length spaces. Most of the quoted statements below and their proofs can be found in [6] and [8].

Let (X, d) be a metric space. We recall that the *length* $l_d(c)$ of a curve $c : [a, b] \to X$ induced by the metric d is

$$l_d(c) := \sup_{\Delta_n} \sum_{i=0}^{n-1} d(c(t_i), c(t_{i+1})),$$

where the supremum is taken over all possible partitions

$$\Delta_n : a = t_0 \leq t_1 \leq \cdots \leq t_n = b$$

of the interval $[a, b] \subset \mathbb{R}$. The length of a curve is a non-negative number or it is infinite. The curve c is said to be *rectifiable* if its length is finite.

Given a metric space (X, d) we can construct a new map $\overline{d} : X \times X \to [0, \infty]$ in the following way:

$$\overline{d}(x, y) := \inf_{R_{x,y}} l_d(\gamma),$$

with $R_{x,y} = \{\text{rectifiable curves joining } x \text{ and } y\}$. If there are no such curves we set $\overline{d}(x, y) := \infty$.

Proposition 4.1. (Bridson and Haefliger [6]) *The map $\overline{d}$ has the following properties:*

(i) *$\overline{d}$ is a metric. We will refer to it as the length metric induced by d.*

(ii) *$\overline{d}(x, y) \geq d(x, y)$ for all $x, y \in X$.*

(iii) *If $c : [a, b] \to X$ is continuous with respect to the topology induced by $\overline{d}$, then it is continuous with respect to the topology induced by d. (The converse is false, in general).*

(iv) *If a map $c : [a, b] \to X$ is a continuous and rectifiable curve in (X, d), then it is a continuous and rectifiable curve in $(X, \overline{d})$.*

(v) *The length of a curve $c : [a, b] \to X$ in $(X, \overline{d})$ is the same as its length in (X, d).*

(vi) *$\overline{\overline{d}} = \overline{d}$.*

The assertion in point (iii) of the above proposition is a consequence of the fact that the topology induced by the metric d is coarser than the topology induced by the metric $\overline{d}$. We say that (X, d) *is a length metric space* (also called *length space*, or *path metric space* or *inner space* in the literature) whenever $\overline{d} = d$. Note that in a length metric space the distance between every pair of points $x, y \in X$ is equal to the infimum of the lengths of rectifiable curves joining them. It is well known that every Riemannian metric on a manifold makes it into a length space. Length metrics share many properties with Riemannian metrics but they can be defined in more general settings. For various properties and characterizations of length metrics see [6], [8], and [12].

A curve $c : [a, b] \to (X, d)$ is called a *shortest path* if its length is minimal among all the curves with the same endpoints. Shortest paths in length spaces are also called *distance minimizers.* If (X, d) is a length space then a curve $c : [a, b] \to X$ is a shortest path if and only if its length is equal to the distance between endpoints, that is, $l_d(c) = d(c(a), c(b))$. Next we introduce the notion of geodesic in length spaces that generalizes the one in Riemannian geometry.

Definition 4.2. Let (X,d) be a length space. A curve $c : I \subset \mathbb{R} \to X$ is called *geodesic* if for every $t \in I$ there exists a subinterval J containing a neighborhood of t in I such that $c|_J$ is a shortest path. In other words, a geodesic is a curve which is locally a distance minimizer. A length space (X,d) is called a *geodesic metric space* if for any two points $x, y \in X$ there exists a shortest path between x and y.

In a length space a shortest path is a geodesic. The extension of the Hopf-Rinow theorem from Riemannian geometry to the case of length metric spaces is due Cohn-Vossen and is a key result needed in the developments of this paper. Its proof can be found in [6] or [8].

Theorem 4.3. (Hopf-Rinow-Cohn-Vossen) *For a locally compact length space (X,d), the following assertions are equivalent:*

(i) *X is complete,*

(ii) *every closed metric ball in X is compact.*

If one of the above assertions holds, then for any two points $x, y \in X$ there exists a shortest path connecting them. In other words, (X,d) is a geodesic metric space.

Having introduced the notion of shortest path we can define the key concept of metric convexity.

Definition 4.4. A subset C in a length metric space (X,d) is said to be *convex* if for any two points $x, y \in C$ there exists a rectifiable shortest path in (X,d) connecting x and y which is entirely contained in C.

Notice that if C is convex in the length metric space (X,d) then (C, d_C) is a length metric space, where $d_C = d|_{C\times C}$.

Lemma 4.5. *Assume that (X,d_X) and (Y,d_Y) are two geodesic metric spaces and $f : X \to Y$ a distance-preserving map. If d_X is finite then the image* $\mathrm{Im}(f) := f(X)$ *is a convex subset of Y.*

Proof. Since f is a distance preserving map it is injective. Let $y_1, y_2 \in f(X)$ and $x_1, x_2 \in X$ be the corresponding preimages. Then there exists a shortest geodesic between x_1 and x_2, namely a curve $c : [a,b] \to X$ satisfying $c(a) = x_1$, $c(b) = x_2$ and $l_{d_X}(c) = d_X(x_1,x_2) < \infty$. Since f is distance preserving we have that $l_{d_X}(c) = l_{d_Y}(f\circ c)$. Consequently $l_{d_Y}(f\circ c) = d_Y(y_1,y_2) < \infty$, which proves that $f\circ c$ is a shortest geodesic connecting y_1 and y_2 entirely contained in $f(X)$. Therefore, $f(X)$ is a convex subset of Y. ■

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