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A Lévy-Khinchin formula for the space of infinite dimensional Hermitian matrices

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Abstract. We give a Lévy-Kinchin formula for continuous negative definite functions defined on the space of Hilbert-Schmidt and hermitian matrices which are invariant under the action by conjugation of the infinite dimensional unitary group.
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1. Introduction

A complex valued function ψ defined on a real vector space V is said to be negative definite if $\psi(0) \geq 0$, $\psi(-\xi) = \overline{\psi(\xi)}$, and for all $\xi_1, \dots, \xi_n \in V$, $c_1, \dots, c_n \in \mathbb{C}$ with $\sum_{i=1}^{n} c_i = 0$,

$$\sum_{i,j=1}^{n} c_i \overline{c_j} \psi(\xi_i - \xi_j) \leq 0.$$

Observe that, if φ is a positive definite function, then $\psi(\xi) = \varphi(0) - \varphi(\xi)$ is negative definite. The next theorem, due to Schoenberg, is basic for the study of negative definite functions.

Theorem 1.1. *Let ψ be a function. The function ψ is negative definite if and only if $\psi(0) \geq 0$, and for all $t \geq 0$, $e^{-t\psi}$ is positive definite.*

See [2].

The set of negative definite functions is a convex cone. If the vector space V is finite dimensional, it is known that the continuous negative definite functions admit an integral representation. It is the Lévy-Khinchin formula.

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In the case of $V = \mathbb{R}^{(\infty)}$, the space of infinite real sequences with finitely many non zero terms, Schoenberg gave an integral representation of $O(\infty)$-invariant negative definite continuous functions. See [16].

We will give an analogous result in the case of the space of Hilbert-Schmidt infinite Hermitian matrices, with the action by conjugation of the infinite dimensional unitary group. This is the main result of this paper. Its proof is inspired from the one of Proposition 5.13 in the book [1].

Let V_∞ be the space of infinite Hermitian matrices with entries in $\mathbb{F} = \mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$ (the quaternion field), $V(\infty)$ the subspace of Hermitian matrices with finitely many non zero entries:

$$V(\infty) = \bigcup_{n=1}^{\infty} V_n, \quad V_n = \mathrm{Herm}(n, \mathbb{F}),$$

equipped with the inductive limit topology, and V_∞^2 the space of Hilbert-Schmidt infinite Hermitian matrices, with the topology defined by the Hilbert-Schmidt norm

$$|||\xi||| = \left(\sum_{i,j=1}^{\infty} |\xi_{ij}|^2 \right)^{\frac{1}{2}}.$$

We define K_∞ as the group of infinite unitary matrices $[u_{ij}]_{ij=1}^{\infty}$ with a finite number of entries such that $u_{ij} \neq \delta_{ij}$. It is the inductive limit of the unitary groups:

$$K_\infty = \bigcup_{n=1}^{\infty} K_n, \quad K_n = U(n, \mathbb{F}).$$

The group K_∞ acts on $V(\infty)$, V_∞, and V_∞^2 by conjugation.

We denote by $\mathcal{M}$ the set of probability measures on V_∞ which are invariant under K_∞, and by $\mathcal{P}$ the set of continuous K_∞-invariant, positive definite functions φ on $V(\infty)$, with $\varphi(0) = 1$. By the Bochner theorem, the Fourier transform is one-to-one from the set $\mathcal{M}$ onto the set $\mathcal{P}$. The extreme points of the convex set $\mathcal{M}$ are the ergodic measures. See [15], Proposition 10.4.

The extreme points of the convex set $\mathcal{P}$ are multiplicative functions, it means functions of the form

$$\varphi(\xi) = \det \Phi(\xi).$$

where Φ is a continuous function on $\mathbb{R}$, with $\Phi(0) = 1$. See [12] Theorem 23.8 or [8] Theorem V.2.1. Furthermore the Fourier transform maps extreme points of $\mathcal{M}$ to extreme points of $\mathcal{P}$.

In the case of $\mathbb{F} = \mathbb{C}$, Olshanski and Vershik determined the extreme points of the convex set $\mathcal{P}$. See Theorem 2.1 and Theorem 2.9 in [13]. In the general case ($\mathbb{F} = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$) this is done in the thesis of M. Bouali.

Theorem 1.2. *The extreme points of the convex set $\mathcal{P}$ are the functions*

$$\varphi_\omega = \det \Pi_\omega,$$

where Π_ω is the Pólya function

$$\Pi_\omega(\lambda) = e^{i\beta\lambda} e^{-\frac{\gamma}{d}\lambda^2} \prod_{k=1}^{\infty} \frac{e^{-i\alpha_k\lambda}}{(1 - i\frac{2}{d}\alpha_k\lambda)^{\frac{d}{2}}},$$

and $\omega = (\alpha, \beta, \gamma)$, $\alpha = (\alpha_1, \alpha_2, \ldots) \in \ell^2(\mathbb{N})$, $\alpha_j \in \mathbb{R}$, $\beta \in \mathbb{R}$ and $\gamma \geq 0$.
Then we obtain a parametrization of the set $\mathrm{ext}(\mathcal{P})$:

$$\Omega \to \mathrm{ext}(\mathcal{P}), \quad \omega \mapsto \det \Pi_\omega,$$

with

$$\Omega = \{\omega = (\alpha, \gamma, \beta) \mid \beta \in \mathbb{R},\ \gamma \geq 0,\ \alpha_k \in \mathbb{R},\ \sum_{k=1}^{\infty} \alpha_k^2 < \infty\}.$$

By a result of Olshanski and Borodin (see [4], theorem 9.1) there is an integral representation of the K_∞-invariant positive definite continuous functions on the space $V(\infty)$. This is the invariant Bochner theorem which is stated below.

We denote by φ_ω the generalized Pólya function defined on $V(\infty)$ by

$$\varphi_\omega(\xi) = \det \Pi_\omega(\xi) = e^{i\beta \mathrm{tr}(\xi)} e^{-\frac{\gamma}{d}\mathrm{tr}(\xi^2)} \prod_{k=1}^{\infty} \frac{e^{-i\alpha_k \mathrm{tr}(\xi)}}{\det(1 - i\frac{2}{d}\alpha_k\xi)^{\frac{d}{2}}}.$$

Theorem 1.3. *Let φ be a K_∞-invariant positive definite continuous function on the space $V(\infty)$, with $\varphi(0) = 1$. Then there exists a unique probability measure μ on Ω such that, for all $\xi \in V(\infty)$,*

$$\varphi(\xi) = \int_\Omega \varphi_\omega(\xi)\mu(d\omega).$$

For establishing our result we will use this theorem and some other preliminary results. To the parameters $\alpha = (\alpha_1, \alpha_2, \ldots)$ and γ we associate the positive bounded measure σ on $\mathbb{R}$ given by

$$\sigma = \gamma\delta_0 + \sum_{k=1}^{\infty} \alpha_k^2 \delta_{\alpha_k}.$$

Then the set Ω becomes a subset of $\mathcal{M}_b(\mathbb{R}) \times \mathbb{R}$, where $\mathcal{M}_b(\mathbb{R})$ is the set of positive bounded measure on $\mathbb{R}$. We consider on the set Ω the topology which is induced by the one of $\mathcal{M}_b(\mathbb{R}) \times \mathbb{R}$, where $\mathcal{M}_b(\mathbb{R})$ is equipped with the weak topology. We denote by Ω_0 the following closed subset of Ω:

$$\Omega_0 = \{\omega \in \Omega \mid \beta = 0\}.$$

2. K_∞-invariant positive definite functions on V_∞^2

Proposition 2.1. *The function φ_ω, which is defined on $V(\infty)$, extends as a continuous function on the Hilbert space V_∞^2, if and only if $\omega \in \Omega_0$.*

It is easy to prove the only if part of this proposition by using Proposition 6.3.2 of [11].

By using the invariant Bochner theorem and Proposition 1.1 we obtain:

Proposition 2.2. *Let φ be a K_∞-invariant positive definite continuous function on $V(\infty)$. Then φ extends as a continuous function on V_∞^2 if and only if the measure associated to the function φ by the invariant Bochner theorem is concentrated on Ω_0.*

Proof. a) Assume that the measure μ is concentrated on Ω_0. Then, by Proposition 2.1, for ω in the support of μ, the function φ_ω is continuous on V_∞^2. By the dominated convergence theorem it follows that φ is continuous on V_∞^2.

b) Assume now that the function φ is continuous on V_∞^2. We can consider the parameters σ and β as functions of ω and write $\beta = \beta(\omega)$, $\sigma = \sigma_\omega$:

$$\varphi_\omega(\xi) = e^{i\beta(\omega)\mathrm{tr}(\xi)}\varphi_{\sigma_\omega}(\xi).$$

For $t \in \mathbb{R}$ consider the following sequence of diagonal matrices in $V(\infty)$:

$$\xi_0^{(n)} = \mathrm{diag}(\underbrace{\frac{t}{n},\ldots,\frac{t}{n}}_{n-\text{times}},0,\ldots).$$

Then

$$\mathrm{tr}((\xi_0^{(n)})^2) = \frac{t^2}{n},$$

and

$$\mathrm{tr}(\xi_0^{(n)}) = t.$$

The sequence $\xi_0^{(n)}$ has limit 0 in the Hilbert-Schmidt norm.

Furthermore

$$\varphi(\xi_0^{(n)}) = \int_\Omega e^{i\beta(\omega)t}\varphi_{\sigma_\omega}(\xi_0^{(n)})\mu(d\omega),$$

and

(1) $\lim\limits_{n\to\infty}\varphi_{\sigma_\omega}(\xi_0^{(n)}) = \varphi_{\sigma_\omega}(0) = 1$, because φ_{σ_ω} is continuous on V_∞^2 (see Proposition 2.1)

(2) $|e^{i\beta(\omega)t}\varphi_{\sigma_\omega}(\xi_0^{(n)})| \leq 1,$.

(3) μ is a probability measure.

By the dominated convergence theorem we obtain

$$\lim_{n\to\infty}\int_{\Omega} e^{i\beta(\omega)t}\varphi_{\sigma_\omega}(\xi_0^{(n)})\mu(d\omega)=\int_{\Omega} e^{i\beta(\omega)t}\mu(d\omega).$$

The function φ is continuous on V_∞^2; therefore

$$\lim_{n\to\infty}\varphi(\xi_0^{(n)})=\varphi(0)=1,$$

and for all $t\in\mathbb{R}$

$$\int_{\Omega} e^{i\beta(\omega)t}\mu(d\omega)=1.$$

This proves that

$$\beta(\omega)=0 \qquad \mu-\text{p.p},$$

which means that the measure μ is concentrated on Ω_0. ■

3. K_∞-invariant negative definite continuous functions on V_∞^2

3.1. Lévy-Khinchin integral representation

Here is our main result:

Theorem 3.1. *Let ψ be a K_∞-invariant and continuous function on V_∞^2. Then ψ is negative definite if and only if it admits the following integral representation*

$$\psi(\xi)=A_0+A_1\mathrm{tr}(\xi^2)+\int_{\Omega_0\setminus\{0\}}(1-\varphi_\omega(\xi))\,\mu(d\omega).$$

where A_0, A_1 are positive constants and μ is a positive measure on $\Omega_0\setminus\{0\}$ such that

$$\int_{\Omega_0\setminus\{0\}}\frac{||\omega||^2}{1+||\omega||^2}\mu(d\omega)<\infty,$$

where

$$||\omega||^2=\gamma+\sum_{k=1}^{\infty}\alpha_k^2.$$

The constants A_0, A_1 and the measure μ are unique.

Let us establish first a preliminary result.

Lemma 3.2. *Put*

$$\varphi_\omega(\xi) = 1 - \frac{1}{d}||\omega||^2 \mathrm{tr}(\xi^2) + R(\omega, \xi).$$

1) For every $\xi \in V(\infty)$,

$$\lim_{\omega \to 0} \frac{R(\omega, \xi)}{||\omega||^2} = 0$$

2) For every $\rho > 0$, there exists $C > 0$, $\varepsilon > 0$, such that, if $||\omega|| \leq \varepsilon$ and $|||\xi||| \leq \rho$, then

$$|1 - \varphi_\omega(\xi)| \leq C||\omega||^2.$$

Proof. 1) If $\xi = 0$, it clearly holds. We assume that $\xi \neq 0$. For $\omega \in \Omega_0$ in a neighborhood of zero

$$\varphi_\omega(\xi) = \exp\left(-\frac{\gamma}{d}\mathrm{tr}(\xi^2) - \sum_{k=1}^{\infty}\left[i\alpha_k \mathrm{tr}(\xi) + \frac{d}{2}\mathrm{tr}\log\left(1 - i\frac{2}{d}\alpha_k\xi\right)\right]\right).$$

In fact, for small $z \in \mathbb{C}$,

$$\det(1 + z\xi) = \exp(\mathrm{tr}\log(1 + z\xi)).$$

We use the power expansion of the logarithm: for $|z| < 1$,

$$\log(1 - z) = -\sum_{m=1}^{\infty} \frac{z^m}{m}.$$

Then for $\xi \in V(\infty)$ and ω in a neighborhood of zero,

$$\varphi_\omega(\xi) = \exp\left(-\frac{\gamma}{d}\mathrm{tr}(\xi^2) - \sum_{k=1}^{\infty}\left[i\alpha_k \mathrm{tr}(\xi) + \frac{d}{2}\mathrm{tr}\left(\sum_{m=1}^{\infty}\frac{(-i\frac{2}{d})^m}{m}\alpha_k^m\xi^m\right)\right]\right). \quad (3.1)$$

Observe that, for $m \geq 2$,

$$|p_m(\alpha)| \leq \sum_{k=1}^{\infty} |\alpha_k|^m \leq ||\omega||^m, \quad (3.2)$$

where $p_m(\alpha) = \sum_{k=1}^{\infty} \alpha_k^m$.

Recall that $||\omega||^2 = \gamma + p_2(\alpha)$. Then if ω is small enough and $||\omega|| < \frac{d}{2|||\xi|||}$,

$$\sum_{m\geq 3}\sum_{k\geq 1}\frac{1}{m}\left(\frac{2|\alpha_k|}{d}\right)^m |||\xi|||^m \leq \frac{1}{3}\sum_{m\geq 3}\left(\frac{2||\omega||\,|||\xi|||}{d}\right)^m$$

$$= \frac{1}{3}\left(\frac{2||\omega||\,|||\xi|||}{d}\right)^3 \frac{(1 - 2||\omega||\,|||\xi|||)}{d}.$$

We can permute summations in (3.1):

$$\varphi_\omega(\xi) = \exp\left(-\frac{1}{d}||\omega||^2\mathrm{tr}(\xi^2) - \frac{d}{2}\sum_{m=3}^{\infty}\frac{(-i\frac{2}{d})^m}{m}p_m(\alpha)\mathrm{tr}(\xi^m)\right). \tag{3.3}$$

Put

$$f(\omega,\xi) = -\frac{1}{d}||\omega||^2\mathrm{tr}(\xi^2) - \frac{d}{2}\sum_{m=3}^{\infty}\frac{(-i\frac{2}{d})^m}{m}p_m(\alpha)\mathrm{tr}(\xi^m),$$

and

$$g(\omega,\xi) = -\frac{d}{2}\sum_{m=3}^{\infty}\frac{(-i\frac{2}{d})^m}{m}p_m(\alpha)\mathrm{tr}(\xi^m).$$

In the sequel we will write $f(\omega,\xi)$ instead of $f(\gamma,\alpha;\xi)$. By using the power expansion of the exponential, (3.3) becomes

$$\varphi_\omega(\xi) = 1 - \frac{1}{d}||\omega||^2\mathrm{tr}(\xi^2) + g(\omega,\xi) + \sum_{n=2}^{\infty}\frac{(f(\omega,\xi))^n}{n!}. \tag{3.4}$$

By (3.2) we have

$$|g(\omega,\xi)| \le \frac{d}{2}\sum_{m=3}^{\infty}\frac{(\frac{2}{d})^{m-1}}{m}(|||\xi|||||\omega||)^m.$$

(We have used the inequality $|\mathrm{tr}(\xi^m)| \le |||\xi|||^m$).

Then, if $||\omega|| < \dfrac{d}{2|||\xi|||}$, we obtain

$$|g(\omega,\xi)| \le \frac{2}{3d}\frac{|||\xi|||^3}{1-\frac{2}{d}|||\xi|||||\omega||}||\omega||^3. \tag{3.5}$$

From the expression of f and (3.5) we deduce

$$|f(\omega,\xi)| \le \left(\frac{1}{d}|||\xi|||^2 + \frac{2}{3d}\frac{|||\xi|||^3||\omega||}{1-\frac{2}{d}|||\xi|||||\omega||}\right)||\omega||^2.$$

Put

$$B(\omega,\xi) = \frac{1}{d}|||\xi|||^2 + \frac{2}{3d}\frac{|||\xi|||^3||\omega||}{1-\frac{2}{d}|||\xi|||||\omega||}.$$

Then we have,

$$\left|\sum_{n=2}^{\infty}\frac{(f(\omega,\xi))^n}{n!}\right| \le \sum_{n=2}^{\infty}\frac{(B(\omega,\xi))^n||\omega||^{2n}}{n!} = ||\omega||^3\sum_{n=2}^{\infty}\frac{(B(\omega,\xi))^n||\omega||^{2n-3}}{n!}.$$

If $||\omega|| < \inf\left(\frac{d}{2|||\xi|||},1\right)$ then, for, $n \ge 2$, $||\omega||^{2n-3} \le 1$ and

$$\left|\sum_{n=2}^{\infty}\frac{(f(\omega,\xi))^n}{n!}\right| \le ||\omega||^3\exp(B(\omega,\xi)), \tag{3.6}$$

By making use of (3.4), for ω small enough we obtain

$$R(\omega,\xi) = g(\omega,\xi) + \sum_{n=2}^{\infty} \frac{(f(\omega,\xi))^n}{n!},$$

and, by (3.5) and (3.6), if $||\omega|| < \inf\left(\frac{d}{2|||\xi|||}, 1\right)$ and ω is small enough, we obtain

$$|R(\omega,\xi)| \leq \left(\exp(B(\omega,\xi)) + \frac{2}{3d}\frac{|||\xi|||^3||\omega||}{1-\frac{2}{d}|||\xi|||||\omega||}\right)||\omega||^3. \qquad (3.7)$$

This implies that

$$\lim_{\omega\to 0}\frac{R(\omega,\xi)}{||\omega||^2} = 0.$$

2) Let $\rho > 0$ and $\varepsilon < \inf(\frac{d}{2\rho}, 1)$. If $||\omega|| \leq \varepsilon$ and $|||\xi||| \leq \rho$, then

$$B(\omega,\xi) = \frac{1}{d}|||\xi|||^2 + \frac{2}{3d}\frac{|||\xi|||^3||\omega||}{1-\frac{2}{d}|||\xi|||||\omega||} \leq (\frac{1}{d}\rho^2 + \frac{2}{3d}\frac{\varepsilon\rho^3}{1-\frac{2}{d}\varepsilon\rho}.) = C_1.$$

From (3.7) we deduce

$$|R(\omega,\xi)| \leq C_2||\omega||^2,$$

where $C_2 = \varepsilon(\exp(C_1) + C_1 - \frac{1}{d}\rho^2)$ and,

$$|1-\varphi_\omega(\xi)| \leq \frac{1}{d}|||\xi|||^2||\omega||^2 + |R(\omega,\xi)| \leq C||\omega||^2,$$

where $C = \frac{1}{d}\rho^2 + C_2$.

This completes the proof of the lemma. ■

Proof. Proof of the main theorem

a) By Lemma (3.2) the integral is well defined and by the dominated convergence theorem and Proposition (2.1), a function ψ of this form is continuous, K_∞-invariant and negative definite.

b) Existence of the representation. Let ψ be a continuous, K_∞-invariant negative definite continuous function on V_∞^2. We known that the function $\psi(\xi) - \psi(0)$ is continuous, K_∞-invariant and negative definite. Then we may assume that $\psi(0) = 0$. For $t \geq 0$, the function $e^{-t\psi}$ is K_∞-invariant, positive definite, and continuous on V_∞^2. By the invariant Bochner theorem (Theorem 1.3), and Proposition (2.2), there exists a unique probability measure m_t on Ω_0 such that

$$e^{-t\psi(\xi)} = \int_{\Omega_0} \varphi_\omega(\xi) m_t(d\omega).$$

Taking real and imaginary parts we obtain

$$e^{-t\Re\psi(\xi)}\cos(t\Im\psi(\xi)) = \int_{\Omega_0} \Re\varphi_\omega(\xi)m_t(d\omega), \tag{3.8}$$

$$e^{-t\Re\psi(\xi)}\sin(t\Im\psi(\xi)) = -\int_{\Omega_0} \Im\varphi_\omega(\xi)m_t(d\omega). \tag{3.9}$$

First, as t goes to 0 in (3.8), we will get an integral representation for the function $\Re\psi$. Second by using (3.9), we will obtain the integral representation for the function ψ.

a) We can write (3.8) as follows:

$$\frac{1 - e^{-t\Re\psi(\xi)}\cos(t\Im\psi(\xi))}{t} = \int_{\Omega_0} (1 - \Re\varphi_\omega(\xi))\frac{m_t}{t}(d\omega).$$

It is easy to see that

$$\lim_{t\to 0} \frac{1 - e^{-t\Re\psi(\xi)}\cos(t\Im\psi(\xi))}{t} = \Re\psi(\xi),$$

$$\lim_{t\to +\infty} \frac{1 - e^{-t\Re\psi(\xi)}\cos(t\Im\psi(\xi))}{t} = 0.$$

Furthermore, for ξ fixed, this expression is continuous in t on $]0, +\infty[$, goes to 0 as t goes to infinity. Hence there exists a constant $C(\xi) \geq 0$ such that

$$0 \leq \frac{1 - e^{-t\Re\psi(\xi)}\cos(t\Im\psi(\xi))}{t} \leq C(\xi),$$

or

$$\int_{\Omega_0} (1 - \Re\varphi_\omega(\xi))\frac{m_t}{t}(d\omega) \leq C(\xi).$$

If we take $\xi_0 = \mathrm{diag}(1, 0, 0, \ldots)$, then

$$\int_{\Omega_0} (1 - \Re\varphi_\omega(\xi_0))\frac{m_t}{t}(d\omega) \leq C(\xi_0) = C. \tag{3.10}$$

Recall that $\varphi_\omega(\xi_0) = \Pi_\omega(1) = e^{-\frac{\gamma}{d}} \prod_{k=1}^{\infty} \frac{e^{-i\alpha_k}}{(1 - i\frac{2}{d}\alpha_k)^{\frac{d}{2}}}$.

We define the bounded positive measure κ_t on Ω_0 by

$$\kappa_t = (1 - \Re\varphi_\omega(\xi_0))\frac{m_t}{t}.$$

For $t > 0$, $\kappa_t(\Omega_0) \leq C$. Hence the set $\{\kappa_t \mid t > 0\}$ is compact in the weak topology $\sigma(\mathcal{M}_b(\Omega_0), \mathcal{C}^0(\Omega_0))$ where $\mathcal{M}_b(\Omega_0)$ is the set of positive bounded measure on Ω_0 and $\mathcal{C}^0(\Omega_0)$ is the space of continuous functions on Ω_0 vanishing at infinity. Therefore there exists a sequence t_j in $]0, +\infty[$ going to 0, such that the measure κ_{t_j} converges weakly to a positive bounded measure κ. This means that, for all $f \in \mathcal{C}^0(\Omega_0)$,

$$\lim_{j\to\infty}\int_{\Omega_0} f(\omega)\,\kappa_{t_j}(d\omega) = \int_{\Omega_0} f(\omega)\,\kappa(d\omega),$$

Using the measure κ_{t_j}, we can write (3.8) as follows: $\frac{1-e^{-t\Re\psi(\xi)}\cos(t\Im\psi(\xi))}{t_j} =$

$$\int_{\Omega_0}\left[\frac{1-\Re\varphi_\omega(\xi)}{1-\Re\varphi_\omega(\xi_0)} - 1\right]\kappa_{t_j}(d\omega) + \frac{1-e^{-t\Re\psi(\xi_0)}\cos(t\Im\psi(\xi_0))}{t_j}. \tag{3.11}$$

For $\xi \neq 0$, the function f defined on Ω_0 by

$$f(\omega) = \begin{cases} \frac{1-\Re\varphi_\omega(\xi)}{1-\Re\varphi_\omega(\xi_0)} - 1 & \text{if } \omega \neq 0,\\ \operatorname{tr}(\xi^2) - 1 & \text{if } \omega = 0,\end{cases}$$

belongs to $\mathcal{C}^0(\Omega_0)$. We can see that $\Re\varphi_\omega(\xi_0) = 1$ if and only if $\omega = 0$. This implies that the function is well defined on $\Omega_0\backslash\{0\}$ and by Proposition VI of [8] it is a continuous function on $\Omega_0\backslash\{0\}$. The continuity at 0 follows from Lemma (3.2) because:

$$\lim_{\omega\to 0}\frac{1-\Re\varphi_\omega(\xi)}{1-\Re\varphi_\omega(\xi_0)} = \operatorname{tr}(\xi^2).$$

By the inequality,

$$0 \le |\Re\varphi_\omega(\xi)| \le e^{-\frac{\gamma}{d}\operatorname{tr}(\xi^2)}\prod_{k=1}^{\infty}\det\left(1+\frac{4}{d^2}\alpha_k^2\xi^2\right)^{-\frac{d}{4}}, \tag{3.12}$$

we see that

$$\lim_{\omega\to+\infty}\Re\varphi_\omega(\xi) = 0,$$

hence

$$\lim_{\omega\to\infty} f(\omega) = 0.$$

As j goes to infinity in (3.11), then, for $\xi \neq 0$,

$$\begin{aligned}\Re\psi(\xi) &= \int_{\Omega_0} f(\omega)\,\kappa(d\omega) + \Re\psi(\xi_0)\\ &= (\operatorname{tr}(\xi^2)-1)\kappa(\{0\}) + \int_{\Omega_0\backslash\{0\}}\left(\frac{1-\Re\varphi_\omega(\xi)}{1-\Re\varphi_\omega(\xi_0)} - 1\right)\kappa(d\omega) + \Re\psi(\xi_0).\end{aligned}$$

By the dominated convergence theorem, as ξ goes to 0, we obtain $\kappa(\Omega_0) = \Re\psi(\xi_0)$. Finally

$$\Re\psi(\xi) = \operatorname{tr}(\xi^2)\kappa(\{0\}) + \int_{\Omega_0\backslash\{0\}}\frac{1-\Re\varphi_\omega(\xi)}{1-\Re\varphi_\omega(\xi_0)}\kappa(d\omega)$$

or

$$\Re\psi(\xi) = A_1 \mathrm{tr}(\xi^2) + \int_{\Omega_0\setminus\{0\}} (1 - \Re\varphi_\omega(\xi))\mu(d\omega)$$

where $A_1 = \kappa(\{0\})$ and μ is the measure defined on Ω_0 by

$$\mu = \frac{1}{1 - \Re\varphi_\omega(\xi_0)} \kappa \mid_{\Omega_0\setminus\{0\}} .$$

We will show that

$$\int_{\Omega_0\setminus\{0\}} \frac{||\omega||^2}{1+||\omega||^2}\mu(d\omega) < \infty.$$

The function $\omega \mapsto \dfrac{||\omega||^2}{(1-\Re\varphi_\omega(\xi_0))(1+||\omega||^2)}$ is continuous on $\Omega_0\setminus\{0\}$, has as limit d at 0 by Lemma (3.2), and limit 1 at infinity. So it is a bounded function and

$$\int_{\Omega_0} \frac{||\omega||^2}{1+||\omega||^2}\mu(d\omega) = \int_{\Omega_0} \frac{||\omega||^2}{(1-\Re\varphi_\omega(\xi_0))(1+||\omega||^2)}\kappa(d\omega) < \infty.$$

c) In the same way we prove that the function g defined on Ω_0 by

$$g(\omega) = \begin{cases} \frac{\Im\varphi_\omega(\xi)}{1-\Re\varphi_\omega(\xi_0)} & \text{if } \omega \neq 0, \\ 0 & \text{if } \omega = 0, \end{cases}$$

belongs to $\mathcal{C}^0(\Omega_0)$. Now if j goes to infinity in the relation

$$e^{-t_j\Re\psi(\xi)}\frac{\sin(t_j\Im\psi(x))}{t_j} = -\int_{\Omega_0} \frac{\Im\varphi_\omega(\xi)}{1-\Re\varphi_\omega(\xi_0)}\kappa_{t_j}(d\omega).$$

we obtain

$$\Im\psi(\xi) = -\int_{\Omega_0} \frac{\Im\varphi_\omega(\xi)}{1-\Re\varphi_\omega(\xi_0)}\kappa(d\omega) = -\int_{\Omega_0\setminus\{0\}} \Im\varphi_\omega(\xi)\mu(d\omega),$$

because $\Im\varphi_\omega(0) = 0$.

Finally we have

$$\psi(\xi) = A_1 \mathrm{tr}(\xi^2) + \int_{\Omega_0\setminus\{0\}} (1 - \varphi_\omega(\xi))\mu(d\omega).$$

d) Uniqueness of the representation. Consider the Pólya function Π_ω. It is of class $\mathcal{C}^2$, positive definite on $\mathbb{R}$, $\Pi_\omega(0) = 1$. Therefore it is the Fourier transform of a probability measure ν on $\mathbb{R}$,

$$1 - \Re\Pi_\omega(s) = \int_{\mathbb{R}} (1-\cos(su))\nu(du) \leq \frac{1}{2}s^2 \int_{\mathbb{R}} u^2\nu(du) = -s^2\Pi''_\omega(0),$$

and

$$\varphi_\omega(s\xi_0) = \Pi_\omega(s); \;\; \Pi''_\omega(0) = -\frac{1}{d}||\omega||^2.$$

Then, for $s \in \mathbb{R}$,

$$\frac{1 - \Re\varphi_\omega(s\xi_0)}{s^2} \leq \begin{cases} \frac{1}{d}||\omega||^2 & \text{si } ||\omega|| \leq 1 \\ 2 & \text{si } ||\omega|| \geq 1. \end{cases}$$

Let us write

$$\frac{\psi(s\xi_0)}{s^2} = A_1 + \int_{\Omega_0\backslash\{0\}} \frac{1 - \Re\varphi_\omega(s\xi_0)}{s^2}\mu(d\omega),$$

and use the dominated convergence theorem. Then

$$\lim_{s\to+\infty} \frac{\psi(s\xi_0)}{s^2} = A_1,$$

This proves that the constant A_1 is uniquely determined.

Now we will show that the measure μ is unique: if μ_1 and μ_2 are two positive measures on $\Omega_0\backslash\{0\}$ such that

$$\int_{\Omega_0\backslash\{0\}} \frac{||\omega||^2}{1+||\omega||^2}\mu_i(d\omega) < \infty, \qquad (i = 1, 2), \tag{3.13}$$

and, for all $\xi \in V(\infty)$,

$$\int_{\Omega_0\backslash\{0\}} (1 - \varphi_\omega(\xi))\, \mu_1(d\omega) = \int_{\Omega_0\backslash\{0\}} (1 - \varphi_\omega(\xi))\, \mu_2(d\omega), \tag{3.14}$$

then we will show that $\mu_1 = \mu_2$. This will follow from the following lemma:

Lemma 3.3. *Let μ_1 and μ_2 be two measures on $\Omega_0\backslash\{0\}$ which satisfy* (3.13) *and* (3.14)*. Then, for $\xi, \eta \in V(\infty)$,*

$$\int_{\Omega_0\backslash\{0\}} \varphi_\omega(\eta)(1 - \Re\varphi_\omega(\xi))\, \mu_1(d\omega) = \int_{\Omega_0\backslash\{0\}} \varphi_\omega(\eta)(1 - \Re\varphi_\omega(\xi))\, \mu_2(d\omega).$$

Proof. The function φ_ω is a spherical function for the spherical pair $(K_\infty \ltimes V(\infty), K_\infty)$. It means that, for $\xi, \eta \in V(\infty)$,

$$\lim_{n\to\infty} \int_{K_n} \varphi_\omega(\xi + k\eta k^*)\, dk = \varphi_\omega(\xi)\varphi_\omega(\eta). \tag{3.15}$$

where dk is the normalized Haar measure on the compact group K_n.

First we will show that

$$\lim_{n\to\infty} \int_{K_n} \int_{\Omega_0\backslash\{0\}} (1 - \varphi_\omega(\xi + k\eta k^*))\, \mu_1(d\omega)\, dk = \int_{\Omega_0\backslash\{0\}} (1 - \varphi_\omega(\xi)\varphi_\omega(\eta))\, \mu_1(d\omega).$$

For that we will find a bound for the function

$$(\omega, k) \mapsto (1+||\omega||^2)\frac{1-\varphi_\omega(\xi+k\eta k^*)}{||\omega||^2},$$

and then use the Fubini theorem. Let $R>0$ such that $|||\xi|||+|||\eta|||\le R$, and $0<\varepsilon<\inf(\frac{d}{2R},1)$ (for the choice of ε, see the proof of Lemma 3.2)

(a) If $||\omega||\le\sqrt{\varepsilon}$, by the inequality $|||\xi+k\eta k^*|||\le|||\xi|||+|||\eta|||\le R$, and Lemma 3.7, we obtain

$$(1+||\omega||^2)\frac{|1-\varphi_\omega(\xi+k\eta k^*)|}{||\omega||^2}\le(1+\varepsilon)C,$$

where C is a constant which depends on R and ε.

(b) If $||\omega||\ge\sqrt{\varepsilon}$, then

$$(1+||\omega||^2)\frac{|1-\varphi_\omega(\xi+k\eta k^*)|}{||\omega||^2}\le 2(1+\frac{1}{\varepsilon}).$$

From the boundedness of the measures dk and $\dfrac{||\omega||^2}{1+||\omega||^2}\mu_1(d\omega)$ it follows that the function

$$(\omega, k) \mapsto (1+||\omega||^2)\frac{|1-\varphi_\omega(\xi+k\eta k^*)|}{||\omega||^2},$$

is integrable with respect to the product measure $\dfrac{||\omega||^2}{1+||\omega||^2}\mu_1(d\omega)\times dk$. By the Fubini theorem,

$$\int_{K_n}\int_{\Omega_0\setminus\{0\}}(1-\varphi_\omega(\xi+k\eta k^*))\,\mu_1(d\omega)\,dk$$
$$=\int_{\Omega_0\setminus\{0\}}\int_{K_n}(1-\varphi_\omega(\xi+k\eta k^*))\,dk\mu_1(d\omega).$$

Using inequalities (a) and (b) and the fact that the measure dk is probability, it follows that the function

$$\omega\mapsto\int_{K_n}(1+||\omega||^2)\frac{1-\varphi_\omega(\xi+k\eta k^*)}{||\omega||^2}\,dk,$$

is bounded independently of n and ω. Furthermore the measure $\dfrac{||\omega||^2}{1+||\omega||^2}\mu_1(d\omega)$ is positive and bounded. Then, by the dominated convergence theorem and (3.15),

$$\lim_{n\to\infty}\int_{\Omega_0\setminus\{0\}}\int_{K_n}(1-\varphi_\omega(\xi+k\eta k^*))\,dk\mu_1(d\omega)=\int_{\Omega_0\setminus\{0\}}(1-\varphi_\omega(\xi)\varphi_\omega(\eta))\mu_1(d\omega),$$

and,

$$\lim_{n\to\infty}\int_{K_n}\int_{\Omega_0\setminus\{0\}}(1-\varphi_\omega(\xi+k\eta k^*))\,\mu_1(d\omega)\,dk=\int_{\Omega_0\setminus\{0\}}(1-\varphi_\omega(\xi)\varphi_\omega(\eta))\,\mu_1(d\omega). \tag{3.16}$$

Therefore we can write (3.14) as

$$\int_{\Omega_0\setminus\{0\}}(1-\varphi_\omega(\xi)\varphi_\omega(\eta))\,\mu_1(d\omega)=\int_{\Omega_0\setminus\{0\}}(1-\varphi_\omega(\xi)\varphi_\omega(\eta))\,\mu_2(d\omega). \tag{3.17}$$

Changing ξ in $-\xi$ in (3.17), and observing that $\varphi_\omega(-\xi)=\overline{\varphi_\omega(\xi)}$, we obtain

$$\int_{\Omega_0\setminus\{0\}}(1-\varphi_\omega(\eta)\Re\varphi_\omega(\xi))\,\mu_1(d\omega)=\int_{\Omega_0\setminus\{0\}}(1-\varphi_\omega(\eta)\Re\varphi_\omega(\xi))\,\mu_2(d\omega).$$

By subtracting the real part of (3.14) from (3.16) the statement of the lemma follows. ■

Consider now the K_∞-invariant function $\widetilde{\varphi}$ defined on $V(\infty)$ by

$$\widetilde{\varphi}(\eta)=\int_{\Omega_0\setminus\{0\}}\varphi_\omega(\eta)(1-\Re\varphi_\omega(\xi))\,\mu_1(d\omega).$$

Since $1-\Re\varphi_\omega(\xi)\geq 0$ and $\varphi_\omega(\eta)$ is positive definite, the function $\widetilde{\varphi}$ is positive definite. The function $\widetilde{\varphi}$ is continuous because $\varphi_\omega(\eta)$ is continuous and bounded by a μ_1-integrable function on $\Omega_0\setminus\{0\}$

$$|\varphi_\omega(y)(1-\Re\varphi_\omega(\xi))|\leq 1-\Re\varphi_\omega(\xi).$$

For $\eta\in V(\infty)$,

$$\int_{\Omega_0}\varphi_\omega(\eta)\,\widetilde{\mu_{1,\xi}}(d\omega)=\int_{\Omega_0}\varphi_\omega(\eta)\,\widetilde{\mu_{2,\xi}}(d\omega).$$

where $\widetilde{\mu_{i,\xi}}(d\omega)=(1-\Re\varphi_\omega(\xi))\chi_{\Omega_0\setminus\{0\}}(\omega)\,\mu_1(d\omega)$ $(i=1,2)$ and $\chi_{\Omega_0\setminus\{0\}}$ is the characteristic function of the set $\Omega_0\setminus\{0\}$. By the uniqueness of the integral representation in the invariant Bochner theorem (Theorem 1.3), we obtain

$$\widetilde{\mu_{1,\xi}}=\widetilde{\mu_{2,\xi}},\qquad \text{on } \Omega_0.$$

Therefore the measures μ_1 et μ_2 are equal on $\Omega_0\setminus\{0\}$. ■

Corollary 3.4. *The extremal rays of the cone of K_∞-invariant positive definite continuous functions on V_∞^2, vanishing at 0, are generated by:*

(i) $1-\varphi_\omega(\xi)$, *where* $\omega\in\Omega_0\setminus\{0\}$,

(ii) $\mathrm{tr}(\xi^2)$.

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