Journal of Lie Theory
Volume **18** (2008) 193–203

The Role of Real Characters in the Pontryagin Duality of Topological Abelian Groups

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Communicated by K. H. Hofmann

Abstract. We study the role that real characters play in the context of Pontryagin duality of topological abelian groups. Among other problems, we show how the existence of enough real characters to separate points is related with the density of the arc–component of the dual group. This is exploited to study the arc–component of groups of continuous maps into $\mathbb{T}$.
Mathematics Subject Classification 2000: 22A05, 22D35.
Key Words and Phrases: Pontryagin duality, real character, one-parameter subgroup, arc–component, exponential function.

1. Introduction and preliminaries.

A real character of a topological abelian group G is a continuous homomorphism of G into the additive group $\mathbb{R}$ of real numbers. It is said that a topological abelian group *has enough real characters,* if any nonzero element of the group is sent to a real number different from zero by some real character. Groups with enough real characters have been studied, during the last fifty years, under different points of view.

In the class of locally compact abelian (LCA) groups, groups with enough real characters are well known. They were first characterized in 1948 by Mackey ([12]), as the groups of the form $\mathbb{R}^n \oplus D$, where n is a nonnegative integer and D is a discrete torsion-free group. From the viewpoint of duality we recognize them as those locally compact abelian groups whose Pontryagin dual is connected (see [7, 24.35]).

The main result in this paper (Theorem 2.14) gives further insight into the relationship between connectedness properties of the Pontryagin dual of a topological abelian group G and the existence of real characters on G. Namely, we prove that whenever a maximally almost periodic abelian group G is a k–space, it has enough real characters if and only if the arc–component of the identity in its Pontryagin dual is pointwise dense.

* The authors acknowledge the financial aid received from BFM2006-03036 and FEDER Funds

ISSN 0949–5932 / $2.50

There is a natural correspondence between real characters and maximal subsemigroups of the group which has been observed and exploited with different aims, both for abelian and nonabelian groups (see [5] [6], [8, Ch. 5], [9], [11], [15], [16]). For each real character f on G, the set $\{x \in G : f(x) \geq 0\}$ is a maximal subsemigroup of G which determines f up to a positive multiple (see [6]). Conversely, Tsitristskii obtained in [15] that a closed maximal proper subsemigroup S of a group G can be written as $\{x \in G : f(x) \geq 0\}$ for some real character f on G if and only if $G = S \cup -S$ (this result had already been established for discrete G by Hoffman and Singer in [9]). In [15], *weakly convex subsets* of G are defined as those which can be obtained as an intersection of a family of translates of such semigroups, and it is proved that a topological abelian group G has enough real characters whenever it has a basis of neighborhoods of zero formed by sets with weakly convex closure. This result comes as a natural generalization of the known fact that locally convex spaces have enough continuous linear functionals to separate their points (a subset of a topological vector space is weakly convex in this sense if and only if it is closed and convex).

Any topological abelian group which admits an injective real character has trivially enough real characters. In general, a group has enough real characters if and only if the intersection of the kernels of all its real characters is trivial. Diem and Wright [5] gave an intrinsic description of the subgroups of a topological abelian group G that can be obtained as such kernels. They called these subgroups *planar*, by analogy with hyperplanes of topological vector spaces. They also found out that the intersection of all planar subgroups of a group G coincides with the *radical* of G (which we shall define below among the preliminaries). The radical had been defined by Wright himself in [16] as a tool for establishing structure results for LCA groups in an intrinsic way.

From a categorical point of view, a characterization of groups with enough real characters can be obtained from the adjoint functor theorem. The forgetful functor from the category of locally convex topological vector spaces to that of topological abelian groups has a left adjoint. Hence, for each topological abelian group G there exists a locally convex vector space E_G and a continuous morphism $i_G : G \to E_G$ such that for any locally convex vector space V and continuous homomorphism $f : G \to V$ there exists a continuous linear map $f' : E_G \to V$ such that $f = f' \circ i_G$. If we consider the particular case of $V = \mathbb{R}$, we can deduce that G has enough real characters if and only if i_G is injective (see 7.34 in [10]).

In particular, a topological abelian group with enough real characters can be continuously embedded into a locally convex space, and this fact lets these groups enjoy some of the most important properties of real locally convex vector spaces. For instance, one can easily generalize the classical Schauder-Tychonoff fixed point theorem in the following sense: if G is a topological abelian group with enough real characters and K is a compact subset of G with $K + K = 2K$, then any continuous map $f : K \to K$ has a fixed point.

Preliminaries.

The following version of the Hahn-Banach Theorem for abelian groups was proved by Hayes:

Proposition 1.1. (Hahn-Banach [6]) *Let G be an abelian group, ρ a subadditive functional on G and f_0 an additive functional on a subgroup H of G, such* *that $f_0(x) \leq \rho(x)$ for every $x \in H$. Then f_0 can be extended to an additive functional f on G with $f(x) \leq \rho(x)$ for all $x \in G$.*

We recall the notion of radical of a group following Wright (see [16]). A semigroup B of G is called 0*–proper* if $0 \notin B$, and *residual* if there exists a maximal 0–proper open semigroup M of G such that B is the complement in G of $M \cup -M$. The intersection of all residual subgroups is called *radical* of the group and denoted $\mathrm{rad}(G)$. When G does not have any maximal 0–proper open semigroup we say that $\mathrm{rad}(G) = G$ and call G a *radical group.* If $\mathrm{rad}(G) = \{e_G\}$ we say that G is *radical-free.* It is interesting to note that the radical of an LCA group coincides with the union of all its compact subgroups. In particular, the radical of a discrete subgroup is its torsion subgroup.

Proposition 1.2. (Diem and Wright [5]) *The radical of any topological abelian group G is the set of elements annihilated by every real character of the group.*

It is commonly understood that a *character* (as opposed to a *real character*) on an abelian topological group G is a continuous homomorphism from G into the unit circle $\mathbb{T}$ of the complex plane, endowed with the Euclidean topology. Under multiplication, the continuous characters on G constitute the dual group $\mathrm{Hom}(G, \mathbb{T})$. This group can be endowed with the compact open topology τ_{co} and then it becomes a Hausdorff topological group, usually denoted by $\widehat{G}$.

The duality theorem of Pontryagin-van Kampen states that a locally compact abelian group G is topologically isomorphic to its bidual group $\widehat{\widehat{G}}$ by means of the natural evaluation mapping. One important point in the proof of this theorem is a classical result by Peter and Weyl saying that a locally compact abelian group G has *enough characters* (to separate the points of G), that is, the natural evaluation mapping from the group G into $\widehat{\widehat{G}}$ is injective. Hausdorff topological abelian groups with enough characters are named *maximally almost periodic groups or* MAP *groups.* It is clear that if a topological group has enough real characters, then it is a MAP group, but every compact abelian group is a MAP group without nonconstant real characters.

Definition 1.3. Let $p \colon \mathbb{R} \to \mathbb{T}$ be defined by $p(t) = e^{2\pi i t}$ and G a topological abelian group. Let $p' := \mathrm{Hom}(G, p) : \mathrm{Hom}(G, \mathbb{R}) \to \mathrm{Hom}(G, \mathbb{T}) = \widehat{G}$ be defined by $p'(f) = p \circ f$. We say that a character $\varphi \colon G \to \mathbb{T}$ can be *lifted to a real character*, if φ is in the image of p'. We denote by $\widehat{G}_{\mathrm{lift}}$ the subgroup of $\widehat{G}$ formed by the characters that can be lifted to a real character.

Convention. All topological groups considered in this paper are assumed to be Hausdorff.

2. Real characters and dual group properties.

In the following statement we characterize in three different ways the elements of a topological abelian group which can be separated from zero by a continuous real character.

Lemma 2.1. *Let G be a topological abelian group and $e_G \neq x \in G$. The following assertions are equivalent:*

a) *There exists a real character* f *such that* $f(x) \neq 0$, *that is,* $x \notin \mathrm{rad}(G)$.

b) *There exists a character* $\varphi \in \widehat{G}_{\mathrm{lift}}$ *such that* $\varphi(x) \neq 1$.

c) *There exists a nonnegative subadditive symmetric and continuous functional* ρ *on* G *with* $\inf_{n\in\mathbb{N}} \frac{\rho(nx)}{n} > 0$.

Proof. a)$\Rightarrow$ b): Suppose that for every $\varphi \in G^{\wedge}_{\mathrm{lift}}$ we have $\varphi(x) = 1$. For each $x \in G$ consider the map $\mathrm{ev}_x\colon \mathrm{Hom}(G,\mathbb{R}) \to \mathbb{R}$, $\mathrm{ev}_x(f) = f(x)$. We claim that $\mathrm{ev}_x(\mathrm{Hom}(G,\mathbb{R})) \subset \mathbb{Z}$; indeed, for each $f \in \mathrm{Hom}(G,\mathbb{R})$, $p\circ f \in G^{\wedge}_{\mathrm{lift}}$. So $(p\circ f)(x) = 1$, and this implies that $f(x) \in \mathbb{Z}$, establishing the claim. The fact that ev_x is linear gives us that $\mathrm{ev}_x \equiv 0$ and hence for all $f \in \mathrm{Hom}(G,\mathbb{R})$, $f(x) = \mathrm{ev}_x(f) = 0$.

b)$\Rightarrow$ a): Fix $x \neq e_G$. Given a character $\varphi \in G^{\wedge}_{\mathrm{lift}}$ with $\varphi(x) \neq 1$ take f such that $p\circ f = \varphi$. Then $(p\circ f)(x) = \varphi(x) \neq 1$, and hence $f(x) \neq 0$.

a)$\Rightarrow$ c): Fix a continuous real character f with $f(x) \neq 0$. The functional ρ defined by $\rho(y) = |f(y)|$ has the desired properties.

c)$\Rightarrow$ a): Let ρ be as in c), and $\varepsilon = \inf_{n\in\mathbb{N}} \frac{\rho(nx)}{n}$. Consider the subgroup $H = \langle x\rangle$, and the continuous group homomorphism $f_0 : H \to \mathbb{R}$ given by $f_0(nx) = n\varepsilon$, for every $n \in \mathbb{Z}$ (note that (c) implies in particular that H is free over x). By Proposition 1.1, there exists an additive functional $f : G \to \mathbb{R}$ such that $|f| \leq \rho$ on G and $f(y) = f_0(y)$ for every $y \in H$. In particular $f(x) = \varepsilon \neq 0$. The functional f is continuous and hence is a real character. ∎

For an abelian topological group G the *annihilator* of a subset $A \subset G$ is $A^{\triangleright} = \{\varphi \in \widehat{G} : \varphi(A) = \{1\}\} = \bigcap_{a\in A}\{\varphi \in \widehat{G} : \varphi(a) = 1\}$. Then $A^{\triangleright}$ is a subgroup that is closed in $\mathrm{Hom}(G,\mathbb{T})$ in the topology of pointwise convergence and thus in $\widehat{G}$, that is, with respect to the finer compact open topology. The *inverse annihilator* of a subset $B \subset \mathrm{Hom}(G,\mathbb{T})$ is $B^{\triangleleft} = \{x \in G : (\forall \varphi \in B)\, \varphi(x) = 1\} = \bigcap_{\varphi\in B} \ker\varphi$. Then $B^{\triangleleft}$ is a closed subgroup of G.

We can rewrite the equivalence a) $\Leftrightarrow$ b) in Lemma 2.1 in terms of the radical as follows.

Proposition 2.2. $(\widehat{G}_{\mathrm{lift}})^{\triangleleft} = \mathrm{rad}(G)$.

The characterization of those groups G for which $\widehat{G}_{\mathrm{lift}} = \widehat{G}$ is not an easy question, even for discrete groups. It is clear that if G is a discrete free group, $\widehat{G}_{\mathrm{lift}} = \widehat{G}$; however, the statement that a discrete group satisfying this equality is necessarily free is undecidable in ZFC. (Shelah's Independence Theorem, see [10, A1.70]).

Given two topological spaces X and Y, let $C(X,Y)$ denote the space of continuous functions endowed with the compact-open topology.

Lemma 2.3. ([4, 10 §3, n^o 4, Cor. 2]) *Let* X, Y *and* Z *be topological spaces such that* Y *is locally compact. Then there is a natural homeomorphism*

$$\alpha_{X,Y,Z} : C(X\times Y, Z) \to C(X, C(Y,Z))$$

In the following $C_0(X,Y)$ will stand for the set of continuous pointed maps between the topological spaces X and Y, with fixed base points. We endow this space with the compact open topology. In the case of topological groups A,B we also consider the topological group $C_0(A,B)$ with the compact open topology.

In order to accommodate a concept that will be used prominently we formulate a definition:

Definition 2.4. A topological group G will be called *productive* if the image of the mapping $\beta_{G,\mathbb{I},\mathbb{T}} : C_0(\mathbb{I}, C_0(G,\mathbb{T})) \to \mathbb{T}^{G\times\mathbb{I}}$, $\beta_{G,\mathbb{I},\mathbb{T}}(f)(g,r) = f(r)(g)$, is contained in $C_0(G\times\mathbb{I},\mathbb{T})$.

By Lemma 2.3, every locally compact group is productive. This remains true for topological groups whose underlying space is a k-space.

Proposition 2.5. *For a productive topological abelian group G, the arc–component $\widehat{G}_a$ of $\widehat{G}$ coincides with $\widehat{G}_{\text{lift}}$:*

$$\widehat{G}_a = \widehat{G}_{\text{lift}}.$$

Proof. Let us denote by G' the vector space $\operatorname{Hom}(G,\mathbb{R})$ of all real characters of G endowed with the compact-open topology. As a continuous image of the topological vector space G', the group $\widehat{G}_{\text{lift}}$ is arc–connected. Now, in order to prove that $\widehat{G}_{\text{lift}}$ contains $\widehat{G}_a$, it is enough to show that the mapping

$$C_0(\mathbb{I},p')\colon C_0(\mathbb{I},G') \to C_0(\mathbb{I},\widehat{G}), \quad C_0(\mathbb{I},p')(f)(t) = p'(f(t))$$

is surjective, since for a character φ in the arc–component of $\widehat{G}$ we can take an arc φ_t in $\widehat{G}$ from the identity to φ; this arc can be lifted to an arc $\overline{\varphi}_t$ in G' whose final point lifts φ.

Consider the following commutative diagram, where the mappings are defined in the natural way:

$$\begin{array}{ccc}
\operatorname{Hom}(G,C_0(\mathbb{I},\mathbb{R})) & \xrightarrow{\delta_{G,\mathbb{I},\mathbb{R}}} & C_0(\mathbb{I},G') \\
{\scriptstyle \operatorname{Hom}(G,C_0(\mathbb{I},p))}\big\downarrow & & \big\downarrow{\scriptstyle C_0(\mathbb{I},p')} \\
\operatorname{Hom}(G,C_0(\mathbb{I},\mathbb{T})) & \xrightarrow[\delta_{G,\mathbb{I},\mathbb{T}}]{} & C_0(\mathbb{I},\widehat{G})
\end{array}$$

Now since $\mathbb{I}$ is simply connected, $C_0(\mathbb{I},p) : C_0(\mathbb{I},\mathbb{R}) \to C_0(\mathbb{I},\mathbb{T})$ is an isomorphism of topological abelian groups, which implies that $\operatorname{Hom}(G,C_0(\mathbb{I},p))$ is an isomorphism. Let us see that $\delta_{G,\mathbb{I},\mathbb{T}}$ is also bijective. Since G is productive we have in particular that $\beta_{G,\mathbb{I},\mathbb{T}}(C_0(\mathbb{I},\widehat{G}))$ is a subspace L of $C_0(G\times\mathbb{I},\mathbb{T})$.

Since $\mathbb{I}$ is locally compact, we can consider the natural homeomorphism $\alpha_{G,\mathbb{I},\mathbb{T}}$ from Lemma 2.3. We observe that $\alpha_{G,\mathbb{I},\mathbb{T}}(L) = \operatorname{Hom}(G,C_0(\mathbb{I},\mathbb{T}))$. The mapping $\delta_{G,\mathbb{I},\mathbb{T}}$ is the inverse of $\alpha_{G,\mathbb{I},\mathbb{T}} \circ \beta_{G,\mathbb{I},\mathbb{T}}|_{C_0(\mathbb{I},\widehat{G})}$ on its range $\operatorname{Hom}(G,C_0(\mathbb{I},\mathbb{T}))$; hence it is bijective. ■

Corollary 2.6. *Let G be a productive topological abelian group and $\widehat{G}_a$ the arc–component of the neutral element of $\widehat{G}$. Then the following conclusions hold:*

a) $(\widehat{G}_a)^{\lhd} = \mathrm{rad}(G)$.

b) *G has enough real characters if and only if* $\{e_G\} = (\widehat{G}_a)^{\lhd}$.

c) $G/(\widehat{G}_a)^{\lhd}$ *is radical-free.*

Proof. Part a): By 2.5, $\widehat{G}_a = \widehat{G}_{\mathrm{lift}}$, and the statement follows from Proposition 2.2. Subsequently, parts b) and c) are direct consequences of a). ∎

Remark 2.7. We observe some properties of the radical in the case of a locally compact abelian group G.

a) It is known that G is radical-free (i.e., has enough real characters) if and only if its dual is connected.

b) If G is a radical group, $\widehat{G}$ is totally disconnected. For a proof note $(\widehat{G}_a)^{\lhd} = \mathrm{rad}(G) = G$. Also, the arc–component is dense in the connected component of $\widehat{G}$ (Theorem 7.71 in [10]), and so the latter connected component is trivial.

Remark 2.8. One recent result about the arc–component of a compact group is that the dual of the arc–component of a compact group is discrete ([3]).

For a topological group G the evaluation map $\alpha_G : G \to \widehat{\widehat{G}}$ is defined by $\alpha_G(x)(\varphi) = \varphi(x)$ for $x \in G$ and $\varphi \in G^{\wedge}$. Given a topological group G, we can define the Lie algebra of G as $\mathfrak{L}(G) = \mathrm{Hom}(\mathbb{R}, G)$. The exponential function of G, $\exp_G : \mathfrak{L}(G) \to G$ of G is defined by $\exp X = X(1)$. So the elements of $\mathrm{im}\exp_G$ are those lying on one-parameter subgroups.

For the ease of formulation we propose a definition:

Definition 2.9. An abelian topological groups will be called *hemireflexive* iff $\alpha_G \colon G \to \widehat{\widehat{G}}$ is continuous.

If the underlying space of a topological abelian group is a k-space, then it is hemireflexive.

Lemma 2.10. *If G is hemireflexive, then* $\widehat{G}_{\mathrm{lift}} = \mathrm{im}\exp_{\widehat{G}}$.

Proof. If A and B are topological groups with continuous α_A and α_B, then the existence of an adjoint of the Pontryagin dual functor yields a group isomorphism

$$\psi_{A,B} : \mathrm{Hom}(A, \widehat{B}) \to \mathrm{Hom}(B, \widehat{A})$$

(See [10], Proposition 7.11(iii)). Now we conclude from the following commutative diagram

$$\begin{array}{ccccccc}
\mathrm{Hom}(G,\mathbb{R}) & \xrightarrow{\cong} & \mathrm{Hom}(G,\widehat{\mathbb{R}}) & \xrightarrow{\psi_{G,\mathbb{R}}} & \mathrm{Hom}(\mathbb{R},\widehat{G}) & = & \mathfrak{L}(\widehat{G}) \\
{\scriptstyle p'}\downarrow & & \downarrow & & \downarrow & & \downarrow{\scriptstyle \exp_{\widehat{G}}} \\
\widehat{G} & = & \mathrm{Hom}(G,\mathbb{T}) & \xrightarrow[\psi_{G,\mathbb{Z}}]{} & \mathrm{Hom}(\mathbb{Z},\widehat{G}) & \xrightarrow[\cong]{} & \widehat{G}
\end{array}$$

that the set of characters that can be lifted (the image of p') coincides with the image of $\exp_{G^{\wedge}}$. ∎

Proposition 2.11. *If G is hemireflexive and productive, then $\widehat{G}_a = \operatorname{im}\exp_{\widehat{G}}$.*

Proof. This is a consequence of Proposition 2.5 and Lemma 2.10. ∎

Remark 2.12. The hypotheses of the previous proposition are satisfied for groups which are k–spaces, and even $k_{\mathbb{R}}$–spaces. Nickolas proved in [13] that if a group G is a k–space, then $\operatorname{im}\exp_{\widehat{G}}$ coincides with $\widehat{G}_a$.

Given a topological abelian group G, we denote by τ_p the topology of pointwise convergence on $\operatorname{Hom}(G,\mathbb{T})$. The dual group with this topology will be denoted by $(\widehat{G},\tau_p)$. Recall that a subbasic neighborhood of the identity in $(\widehat{G},\tau_p)$, for a given element $x \in G$ and a given identity neighborhood W of $\mathbb{T}$ can be written as $V_{x,W} = \{\psi \in \operatorname{Hom}(G,\mathbb{T}) : \psi(x) \in W\}$.

Proposition 2.13. *For any abelian topological group and any subgroup L of $\widehat{G}$, its closure $\overline{L}^{\tau_p}$ in $(\widehat{G},\tau_p)$ coincides with $(L^{\triangleleft})^{\triangleright}$.*

Proof. a) $(L^{\triangleleft})^{\triangleright} \supseteq \overline{L}^{\tau_p}$: Trivially, $L \subseteq (L^{\triangleleft})^{\triangleright}$. As we noted following Lemma 2.1, every annihilator in $\widehat{G}$ is τ_p-closed. Hence $\overline{L}^{\tau_p} \subseteq (L^{\triangleleft})^{\triangleright}$.
b) $(L^{\triangleleft})^{\triangleright} \subseteq \overline{L}^{\tau_p}$: For a proof we will take a χ that is not in $\overline{L}^{\tau_p}$ and show that there exists an $x \in L^{\triangleleft}$ with $\chi(x) \neq 1$.

The closure $\overline{L}^{\tau_p}$ is the intersection of all subsets LV where V ranges through the identity neighborhoods and then $\overline{L}^{\tau_p}$ is indeed the intersection of all $LV_{y,W}$ over all $y \in G$, W ranging through the identity neighborhoods of $\mathbb{T}$. Accordingly, there is a $y \in G$ and a W such that $\chi \notin LV_{y,W}$. The set $L(y) = \{\varphi(y) : \varphi \in L\} \subset \mathbb{T}$ is a subgroup of $\mathbb{T}$. We claim that $L(y) \cap \chi(y)W^{-1} = \emptyset$; suppose that this is not the case. Then there is a $\varphi \in L$ and a $w \in W$ such that $\varphi(y) = \chi(y)w^{-1}$. This means that $(\varphi^{-1}\chi)(y) = w \in W$, and thus that $\varphi^{-1}\chi \in V_{y,W}$ and so $\chi \in \varphi V_{y,W} \subseteq LV_{y,W}$ contradicting the choice of y and W and proving our claim. Therefore $L(y)$ is a closed proper subgroup of $\mathbb{T}$ and thus is cyclic, of order n, say. Define $\mu_n(s) = s^n$ on $\mathbb{T}$. Let $x = n \cdot y$ in G. Then $L(x) = \mu_n(L(y)) = \{1\}$, whence $x \in L^{\triangleleft}$. Moreover, $\chi(x) = \chi(y)^n \neq 1$ since, firstly, $\chi(y) \notin L(y)$ and secondly, $L(y) = \ker\mu_n$ is the unique subgroup in $\mathbb{T}$ of all elements whose exponent divides n. The proof is complete. ∎

Theorem 2.14. *Let G be a productive abelian topological group and consider the following conditions:*

a) *G has enough real characters.*

b) *$\widehat{G}_a$ is dense in $(\widehat{G},\tau_p)$.*

Then a)⇒b). *If G is an* MAP *group, then both conditions are equivalent.*

Proof. a) ⇒ b): Since G has enough real characters, by Corollary 2.6b) we have $(\widehat{G}_a)^{\triangleleft} = \{e_G\}$. By Proposition 2.13, the pointwise closure of $\widehat{G}_a$ is $\{e_G\}^{\triangleright} = \widehat{G}$.

b) ⇒ a): Again by Proposition 2.13, $\widehat{G} = ((\widehat{G}_a)^{\triangleleft})^{\triangleright}$. Let $x \neq e_G$. Since G is an MAP group, there exists $\chi \in \widehat{G}$ with $\chi(x) \neq 1$, thus $x \notin (\widehat{G}_a)^{\triangleleft}$. Hence $(\widehat{G}_a)^{\triangleleft} = \{e_G\}$ and by Corollary 2.6b), G has enough real characters. ∎

Contrary to what happens in the locally compact case, for a group G which is merely a k–space, it is not possible to replace the topology of pointwise convergence with the compact-open topology in the preceding theorem. We will see an example in the next section which illustrates this difference.

Corollary 2.15. *Let G be a group which is also a k–space and has enough real characters, then $\widehat{G}_a$ is dense in $(\widehat{G}, \tau_p)$.*

3. The arc–component of $C(X,\mathbb{T})$ and $C_0(X,\mathbb{T})$.

Let X be a completely regular space, with a base point x_0. In this section we describe the arc–component of $C(X,\mathbb{T})$ and $C_0(X,\mathbb{T})$, under mild conditions on X. We then apply Theorem 2.14 to prove that these components are pointwise dense in the respective spaces.

Proposition 3.1. *a) If the mapping $\beta_{X,\mathbb{I},\mathbb{T}} : C_0(\mathbb{I}, C_0(X,\mathbb{T})) \to \mathbb{T}^{X\times\mathbb{I}}$ is such that $\operatorname{im}\beta_{X,\mathbb{I},\mathbb{T}} \subset C_0(X\times\mathbb{I},\mathbb{T})$, then $C_0(X,\mathbb{T})_a = \{e^{2\pi i f}; f \in C_0(X,\mathbb{R})\}$.*

b) If the mapping $\beta_{X,\mathbb{I},\mathbb{T}} : C(\mathbb{I}, C(X,\mathbb{T})) \to \mathbb{T}^{X\times\mathbb{I}}$ is such that $\operatorname{im}\beta_{X,\mathbb{I},\mathbb{T}} \subset C(X\times\mathbb{I},\mathbb{T})$, then $C(X,\mathbb{T})_a = \{e^{2\pi i f}; f \in C(X,\mathbb{R})\}$.

Proof. Part a) can be proved in the same way as Proposition 2.5, with pointed continuous mappings playing the role of continuous homomorphisms.

Part b) is proved in [14] for X a k-space. Alternatively, it can be obtained as a consequence of the Homotopy Lifting Theorem. We include the proof for the sake of completeness.

The set $\{e^{2\pi i f}; f \in C(X,\mathbb{R})\}$ is arc–connected, as a continuous image of the topological vector space $C(X,\mathbb{R})$. Let us see that

$$C(X,\mathbb{T})_a \subset \{e^{2\pi i f}; f \in C(X,\mathbb{R})\}.$$

Take $g \in C(X,\mathbb{T})_a$ and $\alpha : \mathbb{I} \to C(X,\mathbb{T})$ continuous and such that $\alpha(0) = e_{C(X,\mathbb{T})}$ and $\alpha(1) = g$. Since $\alpha \in C(\mathbb{I}, C(X,\mathbb{T}))$, $F := \beta_{X,\mathbb{I},\mathbb{T}}(\alpha) \in C(X\times\mathbb{I},\mathbb{T})$. We have the following commutative diagram:

$$\begin{array}{ccc} X & \xrightarrow{\ 0\ } & \mathbb{R} \\ {\scriptstyle i}\big\downarrow & & \big\downarrow{\scriptstyle p} \\ X\times\mathbb{I} & \xrightarrow[F]{} & \mathbb{T} \end{array}$$

By the Homotopy Lifting Theorem we obtain a continuous mapping $F' : X\times\mathbb{I} \to \mathbb{R}$ such that $p(F'(x,t)) = F(x,t)$ for every $x \in X$ and $t \in \mathbb{I}$. In particular $p(F'(x,1)) = F(x,1) = \alpha(1)(x) = g(x)$ for every $x \in X$. Therefore $f : X \to \mathbb{R}$ defined by $f(x) = F'(x,1)$ is a lifting of g. ■

We will denote by $A_G(X)$ the Graev free topological abelian group over the pointed space X (with base point x_0 identified with the neutral element of $A_G(X)$) and $A_M(X)$ the Markov free topological abelian group over the space X. Then $A_M(X)$ is isomorphic to $A_G(X \cup \{x_0\})$, where $X \cup \{x_0\}$ is X with an isolated point adjoined which is to be considered the base point of $X \cup \{x_0\}$. In the following, whenever it is not needed to distinguish between them, we will use the notation $A(X)$ to refer to both $A_M(X)$ and $A_G(X)$. Algebraically $\mathrm{Hom}(A_M(X), \mathbb{T}) \cong C(X, \mathbb{T})$ and $\mathrm{Hom}(A_G(X), \mathbb{T}) \cong C_0(X, \mathbb{T})$.

We have $A_M(X) \cong A_G(X) \oplus \mathbb{Z}$ topologically and algebraically. It is known that $A_M(X)$, and then also $A_G(X)$, is a k–space provided that the space X satisfies one of the following additional conditions:

i) X is a k_ω–space.

ii) X is metrizable, locally compact and the set of non isolated points of X is separable or compact (see [1]).

With this fact in mind we state the following property for the groups $C(X, \mathbb{T})$ and $C_0(X, \mathbb{T})$ endowed with the compact open topology.

Proposition 3.2. *If X is a pointed k_ω–space, or a pointed a metrizable locally compact space and the set of non isolated points of X is separable or compact, then the arc–components of $C(X, \mathbb{T})$, respectively, of $C_0(X, \mathbb{T})$ are dense subgroups with respect to the topology of pointwise convergence.*

Proof. We start by proving that $A_M(X)$ (and then $A_G(X)$) has enough real characters. Let $x = k_1x_1 + k_2x_2 + \cdots + k_nx_n$ be the reduced expression of a nonzero element of $A_M(X)$ (thus, k_i is a nonzero integer for every $i \in \{1, \ldots, n\}$ and the points x_i are pairwise different). Since X is completely regular, there exists a continuous $f : X \to \mathbb{R}$ with

$$f(x_1) = 1,\ f(x_2) = f(x_3) = \cdots = f(x_n) = 0$$

We can extend f to a continuous real character Φ by the universal property of $A_M(X)$. Then $\Phi(x) = k_1\Phi(x_1) + k_2\Phi(x_2) + \cdots + k_n\Phi(x_n) = k_1 \neq 0$.

Since $A(X)$ is a k–space, by Theorem 2.14, we get that $(\widehat{A(X)})_a$ is dense in $(\widehat{A(X)}, \tau_p)$.

Since X is paracompact, it follows that the restriction $i\colon \widehat{A_M(X)} \to C(X, \mathbb{T})$ is a topological isomorphism (see 15.1 in [2]). Similarly $i\colon \widehat{A_G(X)} \to C_0(X, \mathbb{T})$ is a topological isomorphism. Let σ_p denote the topology of pointwise convergence on $C(X, \mathbb{T})$. Then $i\colon (A_M(X), \tau_p) \to (C(X, \mathbb{T}), \sigma_p)$ is continuous since $\chi \mapsto i(\chi)(x) = \chi(x)$ is continuous for each $x \in X$. Therefore

$$\overline{C(X, \mathbb{T})_a}^{\sigma_p} = \overline{i\left((\widehat{A_M(X)})_a\right)}^{\sigma_p} \supseteq i\left(\overline{(\widehat{A_M(X)})_a}^{\tau_p}\right) = i\left(\widehat{A_M(X)}\right) = C(X, \mathbb{T}).$$

We thus have that $C(X, \mathbb{T})_a$ is dense in $(C(X, \mathbb{T}), \sigma_p)$. ■

Corollary 3.3. *Under the hypothesis of* Proposition 3.2*, every function in $C(X, \mathbb{T})$, respectively, $C_0(X, \mathbb{T})$, can be pointwise approximated by functions with a continuous logarithm.*

Proof. This is a consequence of Proposition 3.2 and Proposition 3.1. ∎

For compact spaces X, where the groups $C(X,\mathbb{R})$ and $C_0(X,\mathbb{T})$ have the topology of uniform convergence, we encounter a rather clear situation. The following relevant information has been taken from [10], Proposition 8.50ff. We denote by $H^1(X,\mathbb{Z})$ the first Čech cohomology group of X over the integers.

Proposition 3.4. *For a compact pointed space X, we have the following conclusions:*

(i) *There is an exact sequence*

$$0 \to C_0(X,\mathbb{Z}) \to C_0(X,\mathbb{R}) \stackrel{C_0(X,p)}{\longrightarrow} C_0(X,\mathbb{T}) \to H^1(X,\mathbb{Z}) \to 0$$

where $\operatorname{im} C_0(X,p) = C_0(X,\mathbb{T})_a$ *is the arc component of* 0 *in* $C_0(X,\mathbb{T})$.

(ii) *If X is connected, then $C_0(X,\mathbb{Z}) = 0$ and*

$$C_0(X,\mathbb{T}) \cong C(X,\mathbb{R}) \oplus H^1(X,\mathbb{Z}),$$

with a discrete torsion free abelian group $H^1(X,\mathbb{Z})$.

(iii) *The arc component $C_0(X,\mathbb{T})_a$ is dense in $C_0(X,\mathbb{T})$ if and only if $H^1(X,\mathbb{Z}) = 0$.*

(iv) *For any torsion free abelian group A there is a compact space X such that $H^1(X,\mathbb{R}) \cong A$, for instance, the space X underlying the character group $\widehat{A}$ satisfies this condition.* ∎

The isomorphism $\widehat{A_G(X)} \cong C_0(X,\mathbb{T})$ yields the following corollary.

Corollary 3.5. *For a compact connected pointed space X, the topological abelian group $A = A_G(X)$ is a k-space and $\widehat{A} = \widehat{A}_a \oplus H^1(X,\mathbb{Z})$. On the one hand, the arc component $\widehat{A}_a$ is dense in $\widehat{A}$ if and only if $H^1(X,\mathbb{Z}) = 0$. On the other hand, $\widehat{A}_a$ is always dense in $(\widehat{A},\tau_p)$.* ∎

Example 3.6. Let X be the 1-sphere, i.e., the space underlying $\mathbb{T}$. Then $H^1(X,\mathbb{Z}) \cong \mathbb{Z}$. The group $A = A_G(X)$ is defined on a k-space. The subgroup $\widehat{A}_a$ of $\widehat{A}$ is closed and proper while $\widehat{A}_a$ is dense in $(\widehat{A},\tau_p)$.

Acknowledgment: The authors are especially indebted to Prof. K. H. Hofmann. His thorough reading and suggestions have deeply improved some of the results and the overall quality of this article, exceeding by far his obligations as an editor. In particular he has guided us through the earlier work on real characters that we used in the Introduction, suggested the present exposition of the results preceding our main theorem and provided the examples on $C(X,\mathbb{T})$ for a compact X which close the paper.

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Received July 2, 2007
and in final form December 21, 2007