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# Parametrization of Coadjoint Orbits of $\mathbb{R}^n \rtimes \mathbb{R}$

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**Abstract.** We give an algorithm for an explicit construction of quantizable canonical coordinates on the coadjoint orbits of across entire specific layers and an explicit description for the cross-section of the type I Lie group $\mathbb{R}^n \rtimes \mathbb{R}$.
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## 1. Introduction

We begin by setting some notations which will be used throughout the paper. Let $G = \mathbb{R}^n \rtimes \mathbb{R}$ be the connected, simply connected, type I Lie group with Lie algebra $\mathfrak{g}$. Put $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{a}$, where $\mathfrak{n} = \mathbb{R}^n$ is a $n$ dimensional abelian ideal and $\mathfrak{a} = \mathbb{R}$ is a one dimensional subalgebra of $\mathfrak{g}$. Let $\mathfrak{g}^*$ be the linear dual of $\mathfrak{g}$. We denote the complexification of $\mathfrak{g}$ by $\mathfrak{s}$.

In [2], one equips the complexification of the Lie algebra of any exponential Lie group $G$ with an "adaptable basis" $(Z_1, \ldots, Z_n)$. In order to describe explicitly the structure of the coadjoint action for such an exponential Lie group $G$, it is algorithmically built in [2], starting from the adaptable basis,

(1) an (ultrafine) layering $\mathcal{L} = \{\Omega\}$; each ultrafine layer $\Omega$ in $\mathcal{L}$ is $G$-invariant and all the orbits $\mathcal{O}$ in $\Omega$ are isomorphic,

(2) a family of cross-sections for each ultrafine layer $\Omega$ with an analytic cross-section mapping, and

(3) a family of analytical functions, $p_i, q_i$, defined on an open neighborhood of $\Omega$ and whose restrictions to any orbit $\mathcal{O}$ in $\Omega$ gives canonical coordinates for $\mathcal{O}$. These functions are called adaptable coordinates.

In this paper we will be concerned with a type I group $G = \mathbb{R}^n \rtimes \mathbb{R}$ not necessarily exponential, that is the set of purely imaginary roots for $\mathfrak{g}$ can be not empty.

In [1], coadjoint orbits of this kind of groups are classified and in this paper we will focus on the construction of layering, cross-sections and canonical coordinates. To do with, we choose a "suitable basis" $(Z_1, \ldots, Z_n, Z_{n+1} = H)$ for $\mathfrak{s}$, then "suitable layers" $\Omega_{\mathbf{e},\Psi}$ are defined for which we explicit the description of

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the cross-section and the construction of canonical coordinates. We finally prove these coordinates are quantizable.

The paper is organized as follows. In Section 2 we recall some results of linear algebra and matrix reduction and we construct a "suitable basis" in $\mathfrak{s}$. Then we examine the stratification and the "fine" stratification used in [5] and [8]. In Section 3 we describe explicitly the parametrization of a single coadjoint orbit. In Section 4 we complete the stratification procedure of $\mathfrak{g}^*$ and we describe the cross-section and the cross-section mapping. Finally in Section 4, we construct the canonical coordinates and we prove that they are quantizable.

## 2. Stratification of $\mathfrak{g}^*$

### 2.1. Preliminaries.

Let us begin this section by some results of linear algebra and matrix reduction. One chooses $H$ to be an element in $\mathfrak{g}\setminus\mathfrak{n}$ and consider the restriction of the adjoint action of $H$ on $\mathfrak{n}_{\mathbb{C}}$. Put $A = \mathrm{ad}_H|_{\mathfrak{n}_{\mathbb{C}}}$, then we have the following.

(i) If $\alpha$ is an eigenvalue of $A$, then $\overline{\alpha}$ is also an eigenvalue with the same multiplicity $m(\alpha)$ in the characteristic polynomial $C(X) = \det(A - XI_n)$.

(ii) Let $\alpha$ be an eigenvalue of $A$ and set $(Z_1, \ldots, Z_{m(\alpha)})$ be a basis for the characteristic subspace $F(\alpha) = \mathrm{Ker}(A - \alpha I_n)^{m(\alpha)}$, then $F(\overline{\alpha}) = \overline{F(\alpha)}$ and $(\overline{Z_1}, \ldots, \overline{Z_{m(\alpha)}})$ is a basis in $F(\overline{\alpha})$.

An eigenvalue $\alpha$ of $A$ will thus be denoted $\alpha = 0$ or $\alpha = \lambda$ or $\alpha = \lambda \pm i\omega$ or $\alpha = \pm i\omega$ where $\lambda$ is in $\mathbb{R}^* = \mathbb{R}\setminus\{0\}$ and $\omega > 0$. The collection of eigenvalues is the spectrum $\mathrm{Sp}(A)$ of $A$.

**Remark 2.1.** The group $G$ is exponential if and only if $\mathrm{Sp}(A) \cap i\mathbb{R}^* = \emptyset$.

Let $\{\pm i\omega_1, \ldots, \pm i\omega_s\}$ be the set of purely imaginary eigenvalues of $A$ ($\omega_j > 0$), since $G$ is of type I, then there exists $c \in \mathbb{R}^*$ such that:

$$\forall j = 1, \ldots, s,\ \omega_j = ca_j, \quad \text{with} \quad a_j \quad \text{rational}\ .$$

Denote $\mathbb{Z}^+$ the set of strictly positive integral numbers. Thus there is $q \in \mathbb{Z}^+$ and $p_r \in \mathbb{Z}^+$ such that

$$\forall j = 1, \ldots, s,\ \omega_j = c\frac{p_j}{q},$$

and then, changing $H$ by $\frac{q}{c}H$, we can suppose that we have

$$\{\omega_1, \ldots, \omega_s\} \subset \mathbb{Z}^+.$$

Now decompose $\mathfrak{n}_{\mathbb{C}}$ into the direct sum of the characteristic subspaces $F(\alpha)$ for $A$.

$$\mathfrak{n}_{\mathbb{C}} = \bigoplus_{k=1}^{r} F(\alpha_k),$$

where $\mathrm{Sp}(A) = \{\alpha_1, \ldots, \alpha_r\}$. Recall that the matrix $J$ of $A$ in this decomposition has the form

$$J = \begin{pmatrix} J(\alpha_1) & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & J(\alpha_r) \end{pmatrix}$$

where each block $J(\alpha_k)$ is the matrix of the endomorphism $A|_{F(\alpha_k)}$ and for each $k = 1, \dots, r$, there is a Jordan-Hölder basis for $F(\alpha_k)$, on this basis, the matrix $J(\alpha_k)$ has the form

$$J(\alpha_k) = \begin{pmatrix} J_1^{(k)} & 0 & \dots & 0 \\ 0 & J_2^{(k)} & 0 & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & J_{r_k}^{(k)} \end{pmatrix}$$

with

$$J_j^{(k)} = \begin{pmatrix} \alpha_k & 1 & \dots & 0 \\ & \ddots & \ddots & \\ 0 & & \alpha_k & 1 \\ 0 & \dots & & \alpha_k \end{pmatrix} \in M_{m_j^{(k)}}(\mathbb{C}).$$

With these notations, the multiplicity of $\alpha_k$ as a root of the characteristic polynomial $C_A$ is $m(\alpha_k) = \sum_{j=1}^{r_k} m_j^{(k)}$, the multiplicity of $\alpha_k$ as a root of the minimal polynomial $M_A$ of $A$ is $m'(\alpha_k) = \max\{m_j^{(k)},\ j = 1, \dots, r_k\}$ and $r_k$ is the dimensionality of the eigenspace $E(\alpha_k)$ associated to the eigenvalue $\alpha_k$. Let us denote the Jordan-Hölder basis for $F(\alpha_k)$ by

$$\mathcal{B}(\alpha_k) = \big((W_{1,1}^{(k)}, \dots, W_{m_1^{(k)},1}^{(k)}), \dots, (W_{1,r_k}^{(k)}, \dots, W_{m_{r_k}^{(k)},r_k}^{(k)})\big).$$

Then

$$\mathrm{ad}_H(W_{1,j}^{(k)}) = \alpha_k W_{1,j}^{(k)}, \quad j = 1, \dots, r_k$$

and, if $m_j^{(k)} > 1$,

$$\mathrm{ad}_H(W_{i,j}^{(k)}) = \alpha_k W_{i,j}^{(k)} + W_{i-1,j}^{(k)}, \quad \text{with} \quad i > 1, j = 1, \dots, r_k.$$

Consider now the dual basis

$$\big(((W_{1,1}^{(k)})^*, \dots, (W_{m_1^{(k)},1}^{(k)})^*), \dots, ((W_{1,r_k}^{(k)})^*, \dots, (W_{m_{r_k}^{(k)},r_k}^{(k)})^*)\big),$$

then the matrix of $\mathrm{ad}_H^*|_{\mathfrak{n}_{\mathbb{C}}^*}$ on this basis is $-\,{}^tA$, we have the following relations: if $m_j^{(k)} > 1$, and $i < m_j^{(k)}$,

$$\mathrm{ad}_H^*((W_{i,j}^{(k)})^*) = -\alpha_k (W_{i,j}^{(k)})^* - (W_{i+1,j}^{(k)})^*,$$

and

$$\mathrm{ad}_H^*((W_{m_j^{(k)},j}^{(k)})^*) = -\alpha_k (W_{m_j^{(k)},j}^{(k)})^*.$$

**Remark 2.2.** Recall that since $\mathrm{ad}\, H$ is a real endomorphism then we can construct a real basis in $\mathfrak{n}$ as follows. If $\alpha_k \in \mathbb{R}$ we can choose the Jordan Hölder basis $(Z_1, \dots, Z_{m(\alpha_k)})$ for $A|_{F(\alpha_k)}$ in $\mathfrak{n}$, if $\Im(\alpha_k) = \omega_k > 0$, then

$$(Z_1, \dots, Z_{m(\alpha_k)}, \overline{Z_1}, \dots, \overline{Z_{m(\alpha_k)}})$$

is a Jordan-Hölder basis for $A|_{F(\alpha_K) \oplus F(\overline{\alpha_k})}$.

Put $Z_1 = U_1 + iV_1, \dots, Z_{m(\alpha_k)} = U_{m(\alpha_k)} + iV_{m(\alpha_k)}$ where $U_j, V_j \in \mathfrak{g}$. Now we replace this basis by the real basis

$$(U_1, V_1, \dots, U_{m(\alpha_k)}, V_{m(\alpha_k)})$$

and we finally get a basis for $\mathfrak{n}$ on which the matrix of $A$ has the following form

$$S = \begin{pmatrix} S(\alpha_1) & 0 & \dots & 0 \\ 0 & S(\alpha_2) & \dots & 0 \\ & & \ddots & \\ 0 & & \dots & S(\alpha_s) \end{pmatrix}$$

where $S(\alpha_k) = J(\alpha_k)$, if $\alpha_k$ is real, while if $\Im(\alpha_k) = \omega_k > 0$,

$$S(\alpha_k) = \begin{pmatrix} S_1^{(k)} & 0 & \dots & 0 \\ 0 & S_2^{(k)} & 0 & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & S_{r_k}^{(k)} \end{pmatrix}$$

where

$$S_j^{(k)} = \begin{pmatrix} A_k & I_2 & \dots & 0 \\ & \ddots & \ddots & \\ 0 & \dots & A_k & I_2 \\ 0 & \dots & & A_k \end{pmatrix} \quad \text{and} \quad A_k = \begin{pmatrix} \Re(\alpha_k) & \omega_k \\ -\omega_k & \Re(\alpha_k) \end{pmatrix}.$$

Note that each $S(\alpha_k)$ is a $(2m(\alpha_k)) \times (2m(\alpha_k))$ real matrix and $S_j^{(k)}$ is a $(2m_j^{(k)}) \times (2m_j^{(k)})$ real matrix. This reduction is known as the Schur reduction.

**2.2. Suitable basis.**

In this step, we shall define and construct a suitable basis for $\mathfrak{s} = \mathfrak{g}_{\mathbb{C}}$.

Fix $H$ in $\mathfrak{g} \setminus \mathfrak{n}$ with spectrum:

$$\mathrm{Sp}(\mathrm{ad}_H) = \mathrm{Sp}^1(\mathrm{ad}_H) \cup \mathrm{Sp}^2(\mathrm{ad}_H) \cup \mathrm{Sp}^3(\mathrm{ad}_H), \tag{1}$$

where

$$\mathrm{Sp}^1(\mathrm{ad}_H) = \{\alpha_0 = 0, \lambda_1, \dots, \alpha_a = \lambda_a\},$$
$$\mathrm{Sp}^2(\mathrm{ad}_H) = \{\alpha_{a+1} = \lambda_{a+1} + i\omega_{a+1}, \overline{\alpha_{a+1}}, \dots, \alpha_{b-1} = \lambda_{a+s} + i\omega_{a+s}, \alpha_b = \overline{\alpha_{b-1}}\},$$
$$\mathrm{Sp}^3(\mathrm{ad}_H) = \{\alpha_{b+1} = i\omega_{b+1}, -i\omega_{b+1}, \dots, i\omega_{b+t}, \alpha_r = -i\omega_{b+t}\}$$

with $\lambda_i \in \mathbb{R}^*, \omega_j \in \mathbb{R}^+$ $(1 \le j \le s)$ and $\omega_j \in \mathbb{Z}^+$ $(s < j)$. Then $\mathfrak{n}_{\mathbb{C}} = \mathfrak{n}^1 \oplus \mathfrak{n}^2 \oplus \mathfrak{n}^3$, where

$$\mathfrak{n}^1 = F(0) \oplus \bigoplus_{1 \le k \le a} F(\lambda_k),$$
$$\mathfrak{n}^2 = \bigoplus_{k=a+2j+1 \le b} \left(F(\lambda_k + i\omega_k) \oplus F(\lambda_k - i\omega_k)\right),$$
$$\mathfrak{n}^3 = \bigoplus_{k=b+2j+1 \le r} \left(F(i\omega_k) \oplus F(-i\omega_k)\right).$$

A suitable basis contains an union of basis for these subspaces.

For $F(\lambda_k) \subset \mathfrak{n}^1$ and eventually $F(0)$, we choose a real basis $(Z_1, \dots, Z_{m(\lambda_k)})$ for which the matrix of $\mathrm{ad}_{H_{|F(\lambda_k)}}$ has a normal Jordan form $J(\lambda_k)$. The suitable basis for $\mathfrak{n}^1$ is the concatenation of these basis (for instance, we can order the eigenvalues $\lambda_k \neq 0$ following the natural ordering in $\mathbb{R}$).

For $F(\lambda_k + i\omega_k) \oplus F(\lambda_k - i\omega_k) \subset \mathfrak{n}^2$, we choose first a complex basis $(Z_1, \dots, Z_{m(\lambda_k+i\omega_k)})$ for $F(\lambda_k + i\omega_k)$, for which the matrix of $\mathrm{ad}_{H_{|F(\lambda_k+i\omega_k)}}$ has a normal Jordan form $J(\lambda_k + i\omega_k)$, then we get the basis:

$$(Z_1, \overline{Z_1}, \dots, Z_{m(\lambda_k+i\omega_k)}, \overline{Z_{m(\lambda_k+i\omega_k)}}).$$

The suitable basis for $\mathfrak{n}^2$ is the concatenation of these basis.

For $F(i\omega_k) \oplus F(-i\omega_k) \subset \mathfrak{n}^3$, we choose first a complex basis $(Z_1, \dots, Z_{m(i\omega_k)})$ for $F(+i\omega_k)$, for which the matrix of $\mathrm{ad}_{H_{|F(i\omega_k)}}$ has a normal Jordan form $J(i\omega_k)$, then we get the basis:

$$(Z_1, \overline{Z_1}, \dots, Z_{m(\lambda_k+i\omega_k)}, \overline{Z_{m(\lambda_k+i\omega_k)}}).$$

But now, the suitable basis for $\mathfrak{n}^3$ is not a simple concatenation: after performing the union of these basis, we change the ordering by putting at the end the last vector of each Jordan block, recall these vectors were denoted $W^{(k)}_{m_j^{(k)},j}$. For simplicity, with the definition of $a$, $b$ and $t$ in 1, put $c = n - \sum_{j=1}^{t} r_{b+j}$. Thus the end of our suitable basis is

$$(Z_{c+1}, \dots, Z_n) = \left( W^{(b+1)}_{m_1^{(b+1)},1}, \overline{W^{(b+1)}_{m_1^{(b+1)},1}}, \dots, W^{(b+t)}_{m_{r_{b+t}}^{(b+t)}, r_{b+t}}, \overline{W^{(b+t)}_{m_{r_{b+t}}^{(b+t)}, r_{b+t}}} \right).$$

**Definition 2.3.** We call suitable basis any basis $(Z_1, \dots, Z_n, Z_{n+1})$ for $\mathfrak{s}$ such that the purely imaginary roots of $\mathfrak{s}$ take integral values on $Z_{n+1} = H \in \mathfrak{g} \setminus \mathfrak{n}$ and $(Z_1, \dots, Z_n)$ is a basis for $\mathfrak{n}_{\mathbb{C}}$ obtained through the preceding procedure, by putting the matrix of $\mathrm{ad}_H$ in a normal Jordan form and choosing a good ordering on vectors.

**Remark 2.4.** Let us remark that if the set of purely imaginary roots of $\mathfrak{g}$ is empty, then a suitable basis is a good basis (see [6]) and even an adaptable basis in the sense of [2] for $\mathfrak{s}$.

From now on we fix once and for all a suitable basis $(Z_1, \cdots, Z_{n+1})$ for $\mathfrak{s}$.

**2.3. Primary stratification.**

Staring with our suitable basis $(Z_1, \dots, Z_{n+1} = H)$ for $\mathfrak{s}$, we apply the stratification procedure of [6] to $\mathfrak{g}^*$. Consider the flag of ideals in $\mathfrak{s}$:

$$\mathfrak{s}_j = \mathrm{span}\{Z_1, \dots, Z_j\}, \quad \text{and} \quad \mathfrak{s}_0 = \{0\}.$$

We identify an element $\ell$ belonging to the complex dual $\mathfrak{s}^*$ with the $(n+1)-$tuple $(\ell_1, \ell_2, \dots, \ell_{n+1})$ where $\ell_j = \ell(Z_j)$, and we set $Z_j^* = (0, 0, \dots, 0, 1, 0, \dots, 0)$ (where 1 is in the $j$-th position). We embed $\mathfrak{g}^*$ in $\mathfrak{s}^*$ in the natural way so that $\mathfrak{g}^* = \{\ell \in \mathfrak{s}^*,\ \ell(\overline{Z}) = \overline{\ell(Z)}\}$. Let $\mu_k : G \to \mathbb{C}^* = \mathbb{C} \setminus \{0\}$ be defined by $gZ_k^* = \mu_k(g) Z_k^* \mod \mathfrak{s}_k^{\perp}$, and let $\alpha_k : \mathfrak{g} \to \mathbb{C}$ be the differential of $\mu_k$.

To each $\ell \in \mathfrak{g}^*$ there is associated a set $\mathbf{e}(\ell) \subset \{1, 2, \dots, n+1\}$ of "jump indices" defined by

$$\mathbf{e}(\ell) = \{1 \leq j \leq n+1,\ \mathfrak{s}_j \not\subset \mathfrak{s}_{j-1} + \mathfrak{s}(\ell)\}.$$

It is easily seen that

$$\mathbf{e}(\ell) = \{j,\ \mathfrak{s}_j^\ell \neq \mathfrak{s}_{j-1}^\ell\} = \{j,\ Z_j \notin \mathfrak{s}_{j-1} + \mathfrak{s}^\ell\}.$$

One can see that if $\mathbf{e}(\ell) \neq \emptyset$ then $\mathbf{e}(\ell) = \{k, n+1\}$ with $k < n+1$. In fact the index $k$ is given by the following equation

$$k = \min\{1 \leq j \leq n+1,\ \mathfrak{s}_j \not\subset \mathfrak{s}^\ell\}.$$

Put:

$$\mathfrak{h}(\ell) = \mathfrak{s}_k^\ell = \mathfrak{n}_{\mathbb{C}}.$$

The subalgebra $\mathfrak{h}(\ell)$ is the Vergne polarization associated with the sequence $\{\mathfrak{s}_j\}$: $\mathfrak{h}(\ell) = \sum_j \mathfrak{s}_j(\ell)$, and $n+1$ is the unique index $j$ such that $\mathfrak{s}_j \not\subset \mathfrak{h}(\ell)$.

For a subset $\mathbf{e}$ of $\{1, \dots, n+1\}$, the set $\Omega_{\mathbf{e}} = \{\ell \in \mathfrak{g}^*,\ \mathbf{e}(\ell) = \mathbf{e}\}$ is $G$-invariant, and the collection of non-empty $\Omega_{\mathbf{e}}$ is a stratification of $\mathfrak{g}^*$, which we shall call the "coarse stratification" of $\mathfrak{g}^*$. The $\Omega_{\mathbf{e}}$ are determined by polynomials as follows:

**Lemma 2.5.** *[5],[8] If* $\mathbf{e} = \{k, n+1\}$*, we have*

$$\Omega_{\mathbf{e}} = \{\ell \in \mathfrak{g}^*,\ \langle \ell, [H, Z_k] \rangle \neq 0 \quad \text{and for all} \quad j < k,\ \ \langle \ell, [H, Z_j] \rangle = 0\}.$$

Each $\Omega_{\mathbf{e}}$ is a $G$-invariant algebraic set, the collection $\{\Omega_{\mathbf{e}}\}$ constitutes a partition of $\mathfrak{g}^*$, and for each $\mathbf{e} = \{k, n+1\}$ and $\mathbf{e}' = \{k', n+1\}$, the set $\bigcup_{\mathbf{e}',\ k' \geq k} \Omega_{\mathbf{e}'}$ is a Zariski-open subset of $\mathfrak{g}^*$ (see [5], [8]). The collection of non-empty $\{\Omega_{\mathbf{e}}\}$ is refereed to herein as the "primary stratification" of $\mathfrak{g}^*$ ("coarse stratification" in [5] and [6]). As the name suggests, this partition is too coarse for some purposes, and we will see in the sequel the refinement of this stratification.

## 3. Parametrizing an orbit

Set $\Omega = \Omega_{\mathbf{e}}$ a layer in which the dimensional orbits is 2 with $\mathbf{e} = \{k, n+1\}$ and let $\ell \in \Omega_{\mathbf{e}}$. We denote by $\mathcal{O}_\ell$ the orbit of $\ell$. Set

$$\mathcal{U} = \{\ell \in \mathfrak{s}^*,\ \langle \ell, [H, Z_k] \rangle \neq 0\}.$$

Note that $Z_k$ may be in $\mathfrak{s} \setminus \mathfrak{g}$ and in this situation write $Z_k = Y_1 + iY_2$. For $\epsilon = 1, 2$ put

$$\mathcal{U}^\epsilon = \{\ell \in \mathcal{U},\ \langle \ell, [H, Y_\epsilon] \rangle \neq 0\},$$

and

$$\Omega^\epsilon = \Omega \cap \mathcal{U}^\epsilon.$$

Put $Q^\epsilon : \mathbb{R}^2 \times \mathcal{U}^\epsilon \to \mathfrak{s}^*$ be defined by

$$Q^\epsilon(t,\ell) = \exp t_1 H \exp t_2 Y_\epsilon \ell,$$

then if $k \le c$, the map $t \mapsto Q^\epsilon(t,\ell)$ is a diffeomorphism between $\mathbb{R}^2$ and $\mathcal{O}_\ell$, for each $\ell \in \Omega_{\mathbf{e}}$. If $k > c$, this mapping is only surjective. Then we have functions $Q_j^\epsilon : \mathbb{R}^2 \times \mathcal{U}^\epsilon \to \mathfrak{s}^*$ whose restriction to $\mathbb{R}^2 \times \Omega^\epsilon$ generates each orbit:

$$\mathcal{O}_\ell = \{Q^\epsilon(t,\ell),\ t = (t_1,t_2) \in \mathbb{R}^2\}.$$

More precisely, write $Q^\epsilon(t,\ell) = \sum_{j=1}^{n+1} Q_j^\epsilon(t,\ell) Z_j^*$, then the functions $Q_j^\epsilon$ are as follows.

**Proposition 3.1.** *Let $\ell \in \Omega_{\mathbf{e}}$ with $\mathbf{e} = \{k, n+1\}$. Then we have:*

$$Q_{n+1}^\epsilon(t,\ell) = \ell_{n+1} + t_2\langle \ell, [H, Y_\epsilon]\rangle = Q_{n+1}^\epsilon(t_2,\ell).$$

*and*

**Case 1**: $1 \le k \le c$

*(i)* $\forall j = 1,\dots,k-1$, $Q_j^\epsilon(t,\ell) = \ell_j$.

*(ii) If $\alpha_k = 0$, then we have*

$$Q_k^\epsilon(t,\ell) = \ell_k - t_1\ell_{k-1}.$$

*(iii) If $\alpha_k \neq 0$, then we have*

$$Q_k^\epsilon(t,\ell) = \ell_k + \frac{e^{-t_1\alpha_k} - 1}{\alpha_k}\langle \ell, [H, Z_k]\rangle = e^{-t_1\alpha_k}\ell_k$$

*(iv) Finally for $j$ such that $k < j < n+1$, then*

$$Q_j^\epsilon(t,\ell) = e^{-t_1\alpha_j}\left(\ell_j + P_j(t_1, \ell_{k-1}, \dots, \ell_{j-1})\right),$$

*where $P_j$ is linear on $\ell_{k-1},\dots,\ell_{j-1}$ and a polynomial in $t_1$ with degree less then $j-k$ with $P_j(0,\ell_{k-1},\dots,\ell_{j-1}) = 0$.*

**Case 2**: $c+1 \le k \le n$

*In this case $t \mapsto Q^\epsilon(t,\ell)$ is no more a diffeomorphism between $\mathbb{R}^2$ and $\mathcal{O}_\ell$, for each $\ell \in \Omega_{\mathbf{e}}$. But nevertheless we have*

$$\mathcal{O}_\ell = \{Q^\epsilon(t,\ell),\ t = (t_1,t_2) \in \mathbb{R}^2\}.$$

*The functions $Q_j^\epsilon(t,\ell)$ are given as follows:*

$$Q_j^\epsilon(t,\ell) = \ell_j \quad \forall j, \quad j \le k-1,$$

*and*

$$Q_j^\epsilon(t,\ell) = e^{-it_1\omega_j}\ell_j \quad \forall j, \quad k \le j \le n.$$

*In any case, we shall write:*

$$Q^\epsilon(t,\ell) = Q_{n+1}^\epsilon(t_2,\ell)Z_{n+1}^* + \mathcal{Q}(t_1,\ell),$$

*where $\mathcal{Q}$ is an analytic (real) function defined on $\mathbb{R} \times \mathcal{U}$, with values in $\mathfrak{n}^*$.*

**Proof.** The value of $Q^\epsilon_{n+1}$ is usual and easy to compute. For the other coordinates, we get:
**Case 1**: $1 \leq k \leq c$
In this case we have the two following subcases.
**Case 1-1**: $\alpha_k \notin i\mathbb{R}^*$ $(k \leq b)$
The map $Q^\epsilon$ is a diffeomorphism between $\mathbb{R}^2$ and $\mathcal{O}_\ell$ and the expression of $Q^\epsilon_j$ are as in [6], [9] and [8].
**Case 1-2**: $\alpha_k \in i\mathbb{R}^*$ $(b < k \leq c)$
In this case we have the same expressions for $Q^\epsilon_j$ as in the preceding case and the map $Q^\epsilon$ is still a diffeomorphism between $\mathbb{R}^2$ and $\mathcal{O}_\ell$, this is consequence of the fact that $Z^*_k$ is not an eigenvector of $\operatorname{ad}^*_H$. In this case $\operatorname{ad}^*_H Z^*_k = -i\omega_k Z^*_k - Z^*_h$, where $h = k+2$ or $h > c$. In any case, we have

$$Q^\epsilon_k(t,\ell) = e^{-it_1\omega_k}\ell_k \quad \text{and} \quad Q^\epsilon_h(t,\ell) = e^{-it_1\omega_k}(\ell_h - t_1\ell_k).$$

Since $\ell_k \neq 0$, this formula proves that $Q^\epsilon$ is still a global diffeomorphism from $\mathbb{R}^2$ to $\mathcal{O}_\ell$.
**Case 2**: $c+1 \leq k \leq n$
In this case $Z^*_k$ is an eigenvector of $\operatorname{ad}^*_H$. The map $Q^\epsilon$ is not a diffeomorphism between $\mathbb{R}^2$ and $\mathcal{O}_\ell$ ($Q^\epsilon$ is not one to one). In this case the orbit $\mathcal{O}_\ell$ is diffeomorphic to $\mathbb{R} \times S^1$ where

$$S^1 = \{z \in \mathbb{C},\ |z| = 1\},$$

since the non constant coordinates in $\mathfrak{n}^*$ correspond to $c < j \leq n$, and they have the form $Q^\epsilon_j(t,\ell) = e^{-it_1\omega_j}\ell_j$. ■

It is clear from the preceding that for each $j$, $Q^\epsilon_j$ is analytic (real) on $\mathbb{C}^2 \times \mathcal{U}_\epsilon$.
Given an $\ell \in \Omega_{\mathbf{e}}$ with $\mathbf{e} = \{k, n+1\}$ such that $c < k$ and looking for the coordinates $(Q_j(t,\ell)^\epsilon)_{k \leq j \leq n}$, we see that if we need to construct a family of diffeomorphisms between some collection of $\mathcal{O}_\ell$ and the two dimensional manifold $\mathbb{R} \times S^1$ then we must refine our primary stratification.

## 4. Suitable layering and cross-section

### 4.1. Suitable layering.

As usual, we are concerned with $G = \mathbb{R}^n \rtimes \mathbb{R}$ not exponential that is the set of purely imaginary roots of $\mathfrak{g}$ is non empty set. More precisely, keeping our preceding notations, we are interested in refining a primary layer $\Omega_{\mathbf{e}}$, with $\mathbf{e} = \{k, n+1\}$ with $k > c$ as a disjoint union of sublayers.

Recall that our suitable basis $(Z_1, \ldots, Z_n, Z_{n+1})$ in $\mathfrak{s}$ verifies $Z_{n+1} = H$ and the eigenvectors of $\operatorname{ad}^*_H$ corresponding to purely imaginary eigenvalues are ordered at the end of the dual basis: they are $(Z^*_{c+1}, \ldots, Z^*_n)$. Recall that we have also

$$Z_{c+2} = \overline{Z_{c+1}}, \ldots, Z_n = \overline{Z_{n-1}}.$$

Put

$$J = \{c+1, c+3, \ldots, n-1\}.$$

Let $\mathbf{e} = \{k, n+1\}$, with $c < k < n+1$. For each $\ell \in \Omega_{\mathbf{e}}$, we associate

$$\Psi(\ell) = \{j \in J,\ k \leq j \quad \text{and} \quad \ell_j \neq 0\}.$$

For any other layer, if $\ell \in \Omega_{\mathbf{e}}$, $\mathbf{e} = \emptyset$ or $\mathbf{e} = \{k, n+1\}$ with $k \leq c$, we put $\Psi(\ell) = \emptyset$.

Then we have the following

**Lemma 4.1.** *For any $\ell \in \Omega_{\mathbf{e}}$, and for any $s \in G$, we have*

$$\Psi(s\ell) = \Psi(\ell).$$

**Proof.** Let $\ell \in \Omega_{\mathbf{e}}$. Let us remark that in fact $\Psi(\ell) = \Psi(\ell|_{\mathfrak{n}_{\mathbb{C}}})$, then if $s \in \exp(\mathfrak{n})$, it is easily seen that

$$\Psi(\ell) = \Psi(s\ell).$$

Now, since $\ell \in \Omega_{\mathbf{e}}$, then $\langle \ell, [H, Z_{k'}] \rangle = 0$ holds for any $k' < k$ and, for $t \in \mathbb{R}$ and $j \in J$,

$$\langle Ad^*(exptH)\ell, Z_j \rangle = e^{-it\omega_j}\ell_j.$$

and thus the conclusion holds. ■

We now complete the definition of the suitable layering $\mathcal{P}$. For any set $\mathbf{e} \subset \{1, \ldots, n+1\}$ and any set $\Psi \subset J$, set

$$\Omega_{\mathbf{e},\Psi} = \{\ell,\ \ell \in \Omega_{\mathbf{e}} \quad \text{such that} \quad \Psi(\ell) = \Psi\}.$$

Then $\Omega_{\mathbf{e},\Psi}$ is an algebraic subset of $\Omega_{\mathbf{e}}$. Let

$$\mathcal{P} = \{\Omega_{\mathbf{e},\Psi} \neq \emptyset,\ \mathbf{e} \subset \{1, \ldots, n+1\} \quad \text{and} \quad \Psi \subset J\};$$

each $\Omega \in \mathcal{P}$ is an invariant subset of $\mathfrak{g}^*$. The layering $\mathcal{P}$ will be called suitable layering of $\mathfrak{g}^*$ defined from the suitable basis $(Z_1, \ldots, Z_{n+1})$. $\mathcal{P}$ has a partial ordering given as follows:

$$\Omega_{\mathbf{e},\Psi} \ll \Omega_{\mathbf{e}',\Psi'}$$

if and only if

$$\begin{cases} |\mathbf{e}| > |\mathbf{e}'|, \\ \text{or} \\ \mathbf{e} = \{k, n+1\}, \quad \mathbf{e}' = \{k', n+1\} \quad \text{and} \quad k \leq c, \quad k < k', \\ \text{or} \\ \mathbf{e} = \{k, n+1\}, \quad \mathbf{e}' = \{k', n+1\}, \quad \{k, k'\} \subset \{c+1, \ldots, n\} \qquad \text{and} \quad \Psi' \subset \Psi. \end{cases}$$

**4.2. Cross-section.**
Now let us consider $\Omega_{\mathbf{e},\Psi}$ with $\mathbf{e} = \{k, n+1\}$ and $k > c$. Then $\Psi \neq \emptyset$, we call $\omega_\Psi$ the greatest common divisor $(GCD)$ of $\{\omega_j,\ j \in \Psi\}$, and

$$\Gamma_\Psi = \frac{2\pi}{\omega_\Psi}\mathbb{Z} = \bigcap_{j \in \Psi} \frac{2\pi}{\omega_j}\mathbb{Z}.$$

We call the number $\frac{2\pi}{\omega_\Psi}$ the period of each $\ell \in \Omega_{\mathbf{e},\Psi}$.

Since $\omega_\Psi = GCD(\omega_j)$ then there exist $a_j \in \mathbb{Z}$ $(j \in \Psi)$ such that

$$\omega_\Psi = \sum_{j\in\Psi} a_j \omega_j .$$

Fix these numbers $a_j$ and define the function

$$u_\Psi(\ell) = \prod_{j\in\Psi} \ell_j^{a_j} .$$

**Lemma 4.2.** *The rational function $u_\Psi$ is regular and semi-invariant on $\Omega_{\mathbf{e},\Psi}$ with*

$$u_\Psi(Ad^*(\exp tH\ell)) = e^{-i\omega_\Psi t} u_\Psi(\ell).$$

**Proof.** Note that

$$u_\Psi(\ell) = u_\Psi(\ell|_{\mathfrak{n}}),$$

and thus we can easily see that for any $s \in \exp(\mathfrak{n}) = \mathbb{R}^n$ we have

$$u_\Psi(s\ell) = u_\Psi(\ell).$$

Now, for $t \in \mathbb{R}$, we have

$$u_\Psi(Ad^*(\exp tH\ell) = \prod_{j\in\Psi} e^{-i\omega_j a_j t} \ell_j^{a_j} = e^{-it\sum_{j\in\Psi}\omega_j a_j} u_\Psi(\ell) = e^{-it\omega_\Psi} u_\Psi(\ell). \qquad \blacksquare$$

We can now describe a cross-section for the $G$ action on any layer in $\mathcal{P}$ with the following

**Theorem 4.3.** *Let $G$ be the type I semi-direct product $\mathbb{R}^n \rtimes \mathbb{R}$ and $\mathfrak{g}$ its Lie algebra. Choose a suitable basis for the complexification $\mathfrak{s}$ of $\mathfrak{g}$ as in Section 2. Let $\Omega_{\mathbf{e},\Psi}$ be a layer belonging to the resulting suitable stratification of $\mathfrak{g}^*$.*

*If $\mathbf{e} = \emptyset$, then the cross-section $\Sigma_{\mathbf{e},\Psi}$ for $\Omega_{\mathbf{e},\Psi}$ is $\Sigma_{\mathbf{e},\Psi} = \Omega_{\mathbf{e},\Psi}$.*

*If $\mathbf{e} = \{k, n+1\}$, and $\alpha_k = 0$, then the cross-section $\Sigma_{\mathbf{e},\Psi}$ for $\Omega_{\mathbf{e},\Psi}$ is the set*

$$\Sigma_{\mathbf{e},\boldsymbol{\Psi}} = \{\ell \in \Omega_{\mathbf{e},\Psi},\ \ell_k = \ell_{n+1} = 0\}.$$

*If $\mathbf{e} = \{k, n+1\}$, and $\alpha_k \notin i\mathbb{R}$, then the cross-section $\Sigma_{\mathbf{e},\Psi}$ for $\Omega_{\mathbf{e},\Psi}$ is the set*

$$\Sigma_{\mathbf{e},\boldsymbol{\Psi}} = \{\ell \in \Omega_{\mathbf{e},\Psi},\ |\ell_k| = 1, \quad \textit{and} \quad \ell_{n+1} = 0\}.$$

*If $\mathbf{e} = \{k, n+1\}$, $\alpha_k \in i\mathbb{R}^*$, and $k \le c$ then the cross-section $\Sigma_{\mathbf{e},\Psi}$ for $\Omega_{\mathbf{e},\Psi}$ is the set*

$$\Sigma_{\mathbf{e},\boldsymbol{\Psi}} = \left\{\ell \in \Omega_{\mathbf{e},\Psi},\ \Re(\ell_k\overline{\ell_h}) = 0, \quad \textit{and} \quad \ell_{n+1} = 0\right\},$$

*where $h$ is defined by $\mathrm{ad}_H^*(Z_k^*) = -\alpha_k Z_k^* - Z_h^*$.*

*If $\mathbf{e} = \{k, n+1\}$, and $k > c$ then the cross-section $\Sigma_{\mathbf{e},\Psi}$ for $\Omega_{\mathbf{e},\Psi}$ is the set*

$$\Sigma_{\mathbf{e},\boldsymbol{\Psi}} = \{\ell \in \Omega_{\mathbf{e},\Psi},\ u_\Psi(\ell) > 0, \quad \textit{and} \quad \ell_{n+1} = 0\}.$$

**Proof.** Let $\ell \in \Omega_{\mathbf{e},\Psi}$. If $\mathbf{e} = \emptyset$ or $\mathbf{e} = \{k, n+1\}$ with $\alpha_k \notin i\mathbb{R}^*$, then $\Psi = \emptyset$ since each orbit in $\Omega_{\mathbf{e},\Psi}$ is isomorphic to its projection on $(\mathfrak{n}_2 \oplus \mathbb{R}H)^*$, we can conclude from [6] for the cross-section, in fact $\Omega_{\mathbf{e},\Psi} = \Omega_{\mathbf{e}}$. There remain two cases.
**Case 1**: $\mathbf{e} = \{k, n+1\}$ with $k > c$, or $\Psi \neq \emptyset$.
Let $\mathcal{O} \subset \Omega_{\mathbf{e},\boldsymbol{\Psi}}$ a coadjoint orbit in $\Omega_{\mathbf{e},\boldsymbol{\Psi}}$. Suppose $\ell$ and $\ell' \in \mathcal{O} \cap \Sigma_{\mathbf{e},\boldsymbol{\Psi}}$. First, since $k$ is the first jump index, we have $\ell_j = \ell'_j$ for $j < k$. Moreover, by definition of $\Gamma_\Psi$, the stabilizer of $\ell$ (and $\ell'$) contains $\exp \Gamma_\Psi H$.

If $t \in \Gamma_\Psi$, for all $j \in \Psi$, there is $c_j \in \mathbb{Z}$ such that $t\omega_j = 2c_j\pi$ thus $(\exp(tH)\ell)_j = e^{-it\omega_j}\ell_j = \ell_j$. If $b < j \leq n$ and $j \notin \Psi$, $\ell_j = 0$ and $(\exp(tH)\ell)_j = e^{-it\omega_j}\ell_j = 0$. Finally, $(\exp(tH)\ell)_{n+1} = \ell_{n+1} = 0$:

$$\exp(tH) \in G(\ell), \ \forall t \in \Gamma_\Psi.$$

Now put

$$\ell' = Ad^*(\exp(sX))Ad^*(\exp(tH))\ell \quad \text{with} \quad t,\ s \in \mathbb{R} \quad \text{and} \quad X \in \mathfrak{n}.$$

Then, we get

$$u_\Psi(\ell') = e^{-it\omega_\Psi}u_\Psi(\ell) > 0, \quad \text{and} \quad u_\Psi(\ell) > 0.$$

Thus $t\omega_\Psi \in 2\pi\mathbb{Z}$, $t$ belongs to $\Gamma_\Psi$, and $\ell' = Ad^*(\exp(sX))\ell$. Thus $\ell'_j = \ell_j$ for all $j \leq n$. Now using $\ell_{n+1} = \ell'_{n+1} = 0$ we get $\ell = \ell'$.
**Case 2**: $\alpha_k \in i\mathbb{R}^*$ and $k \leq c$.
Let $\mathcal{O} \subset \Omega_{\mathbf{e},\boldsymbol{\Psi}}$ be a coadjoint orbit in $\Omega_{\mathbf{e},\boldsymbol{\Psi}}$. Suppose $\ell$ and $\ell' \in \mathcal{O} \cap \Sigma_{\mathbf{e},\boldsymbol{\Psi}}$. Put

$$\ell' = Ad^*(\exp(sX))Ad^*(\exp(tH))\ell \quad \text{with} \quad t,\ s \in \mathbb{R} \quad \text{and} \quad X \in \mathfrak{n}.$$

Then

$$\ell'_k = \langle \ell', Z_k \rangle = \langle \ell, e^{-t\,\mathrm{ad}_H} Z_k \rangle = e^{-it\omega_k}\ell_k$$

and

$$\ell'_h = \langle \ell, e^{-t\,\mathrm{ad}_H} Z_h \rangle = e^{-it\omega_k}(-t\ell_k + \ell_h).$$

Thus

$$\Re(\ell'_k\overline{\ell'_h}) = -t|\ell_k|^2 + \Re(\ell_k\overline{\ell_h}),$$

but since

$$\Re(\ell_k\overline{\ell_h}) = 0, \quad \text{and} \quad \Re(\ell'_k\overline{\ell'_h}) = 0,$$

then $t = 0$ and $\ell' = Ad^*(\exp sX\ell)$. Now using $\ell_{n+1} = \ell'_{n+1} = 0$, then with the same arguments of the first case we get $\ell = \ell'$. ■

**4.3. Cross-section mapping.**
We use the notations of Section 3 First, if $\mathbf{e} = \{k, n+1\}$, $\mathcal{U} = \{\ell,\ \langle \ell, [H, Z_k] \rangle \neq 0\}$. The step here is to define a function $\sigma$ defined on an open neighborhood $\mathcal{U}_{\mathbf{e},\Psi}$ of $\Omega_{\mathbf{e},\Psi}$, $\sigma : \mathcal{U}_{\mathbf{e},\Psi} \to \mathfrak{s}^*$ which satisfy the following

(i) $\sigma$ is analytic on $\mathcal{U}_{\mathbf{e},\Psi}$.

(ii) $\sigma(\Omega_{\mathbf{e},\Psi}) = \Sigma_{\mathbf{e},\Psi}$.

If $\mathbf{e} = \emptyset$, we juste use $\mathcal{U}_{\mathbf{e},\Psi} = \mathfrak{g}^*$ and $\sigma = id_{\mathfrak{g}^*}$.

Suppose now $\mathbf{e} \neq \emptyset$. In our very simple situation, we can summarize the methods of construction of $\sigma$ used in [6] (when $\alpha_k \notin i\mathbb{R}^*$) in an easy way. In fact, we do not need to perform a complete substitution, that is to find $t(\ell) =$

$(t_1(\ell), t_2(\ell))$ such that $Q^\epsilon(t(\ell),\ell)$ is in the cross-section. Indeed, we saw, that if $\mathbf{e} \neq \emptyset$, the points $\ell^*$ in the cross-section satisfy $\ell^*_{n+1} = 0$. Therefore, for our mapping $\sigma$,

$$\sigma(\ell) = \sum_{j=1}^{n} \sigma_j(\ell) Z_j^*$$

holds. So we just look to the real number $t_1(\ell)$ such that $\mathcal{Q}(t_1(\ell),\ell)$ belongs to the cross-section. Let us put $\mathcal{Q}(t_1,\ell) = \sum_{j=1}^n \mathcal{Q}_j(t_1,\ell)Z_j^*$.
Define now the substitution function $t_1(\ell)$ on $\Omega_{\mathbf{e},\Psi}$, so that $\sigma$ is given as follows:

$$\sigma(\ell) = \mathcal{Q}(t_1(\ell),\ell) = \sum_{j=1}^{n} \mathcal{Q}_j(t_1(\ell),\ell) Z_j^* \in \Sigma_{\mathbf{e},\Psi}.$$

In order to compute this substitution function $t_1(\ell)$, we examine $\mathcal{Q}_k$. We have the following cases
**Case 1**: $\alpha_k = 0$
Here we have $\ell_{k-1} = -\langle \ell, [H, Z_k] \rangle \neq 0$ and

$$\mathcal{Q}_k(t_1,\ell) = \ell_k - t_1 \ell_{k-1}.$$

The substitution function is such that $\mathcal{Q}_k(t_1(\ell),\ell) = 0$ then we get $t_1(\ell) = \dfrac{\ell_k}{\ell_{k-1}}$.

**Case 2**: $\alpha_k \in \mathbb{C} \setminus i\mathbb{R}$ $(0 < k \leq b)$
In this case we have $\ell_k = e^{t_1\alpha_k}\langle \ell, [H, Z_k] \rangle \neq 0$ and

$$\mathcal{Q}_k(t_1,\ell) = e^{-t_1\alpha_k}\ell_k.$$

The $t_1(\ell)$ substitution function is such that $|\mathcal{Q}_k(t_1(\ell),\ell)| = 1$. Then we get

$$t_1(\ell) = \frac{\ln|\ell_k|}{\Re(\alpha_k)}.$$

Finally, in any case, we can write

$$\sigma(\ell) = \sum_{j=1}^{n} \mathcal{Q}_j(t_1(\ell),\ell) Z_j^*.$$

It is clear that the substitution function is analytic on $\mathcal{U}_{\mathbf{e},\Psi} = \mathcal{U}$ and then the mapping $\sigma$ is analytic at each point $\ell$ in $\mathcal{U}$.

We are now in a position to construct the mapping $\sigma$ in the case when $\alpha_k \in i\mathbb{R}^*$. Like in the preceding cases we make the substitution function $t_1(\ell)$. We have the following cases.
**Case 3**: $\alpha_k \in i\mathbb{R}^*$ and $k \notin \Psi$ $(b < k \leq c)$
In this case, as above, $\ell_k \neq 0$ and the equation for $t_1(\ell)$ is

$$\Re\left(\mathcal{Q}_k(t_1(\ell),\ell)\overline{\mathcal{Q}_h(t_1(\ell),\ell)}\right) = -t_1|\ell_k|^2 + \Re(\ell_k\overline{\ell_h}) = 0.$$

So we get the following substitution

$$t_1(\ell) = \frac{\Re(\ell_k\overline{\ell_h})}{|\ell_k|^2}.$$

The substitution function $t_1(\ell)$ and the cross-section function $\sigma(\ell) = \mathcal{Q}(t_1(\ell), \ell)$ are analytic real on $\mathcal{U}_{\mathbf{e},\Psi} = \mathcal{U}$.

In each of the preceding case, the orbit of $\ell \in \Omega_{\mathbf{e},\Psi}$ is diffeormorphic with $\mathbb{R}^2$. In fact, we got a global system of coordinates on the coadjoint orbit $\mathcal{O}$ in $\Omega_{\mathbf{e},\Psi}$ with the functions $t_1$ and $\ell_{n+1}$.

**Case 4**: $\alpha_k \in i\mathbb{R}^*$ and $k \in \Psi$ $(c < k)$

In this case, we have to restrict ourselves to the open neighborhood $\mathcal{U}_{\mathbf{e},\Psi}$ of $\Omega_{\mathbf{e},\Psi}$ defined by:

$$\mathcal{U}_{\mathbf{e},\Psi} = \{\ell \in \mathfrak{g}^*,\ \ell_j \neq 0,\ \forall j \in \Psi\}.$$

On $\mathcal{U}_{\mathbf{e},\Psi}$, we get $u_\Psi(\ell) \neq 0$ and the following equation to define $t_1(\ell)$:

$$u_\Psi\left(\mathcal{Q}(t_1(\ell), \ell\right) = e^{-i\omega_\Psi t_1(\ell)} u_\Psi(\ell) > 0.$$

This equation can be written as

$$e^{-i\omega_\Psi t_1(\ell)} = \frac{|u_\Psi(\ell)|}{u_\Psi(\ell)}.$$

Put $b_j = \frac{\omega_j}{\omega_\Psi} \in \mathbb{Z}$. The generalized substitution function $\zeta(\ell) = e^{-i\omega_\Psi t_1(\ell)}$ and the cross-section function:

$$\sigma(\ell) = \mathcal{Q}(t_1(\ell), \ell) = \sum_{j\in\Psi} e^{-i\omega_j t_1(\ell)} \ell_j Z_j^* = \sum_{j\in\Psi} \zeta(\ell)^{b_j} \ell_j Z_j^*$$

are analytic real on $\mathcal{U}_{\mathbf{e},\Psi}$. Moreover the 1-form $dt_1$ is well defined on $\mathcal{U}_{\mathbf{e},\Psi}$ by:

$$dt_1 = \frac{i}{\omega_\Psi} \frac{d\zeta}{\zeta}.$$

This defines $t_1(\ell)$ on $\mathcal{U}_{\mathbf{e},\Psi}$, only modulo $\frac{2\pi}{\omega_\Psi}\mathbb{Z}$. If we want to have a local analytic function, for any $\ell_0$, we choose a branch of the logarithm on $\mathbb{C}$ that is analytic on $Arg(z) \in ]Arg(u_\Psi(\ell_0)) - \pi, Arg(u_\Psi(\ell_0)) + \pi[$ then we obtain a local analytic expression for $t_1(\ell)$ defined for $\ell \in \mathcal{U}_{\mathbf{e},\Psi}$ such that $Arg(u_\Psi(\ell))$ belongs to $]Arg(u_\Psi(\ell_0)) - \pi, Arg(u_\Psi(\ell_0)) + \pi[$.

Since the orbit is diffeomorphic to a cylinder, we can see any smooth function on $\mathcal{O}$ as a smooth function $f$ in the two variables $(t_1, \ell_{n+1})$ and periodic in the first variable:

$$f\left(t_1 + \frac{2\pi}{\omega_\Psi}, \ell_{n+1}\right) = f(t_1, \ell_{n+1}), \qquad \forall t_1, \ell_{n+1} \in \mathbb{R}^2.$$

This allows us to define the differential operator $\dfrac{\partial}{\partial t_1}$ as the well-defined projection of the corresponding operator on $\mathbb{R}^2$.

## 5. Construction of canonical coordinates

### 5.1. Canonical coordinates.

In this section, we are looking for the structure of symplectic manifold for coadjoint orbits in the dual $\mathfrak{g}^*$ of $\mathfrak{g}$. Let us recall how is this structure, defined by the Kirillov-Kostant-Souriau 2-form $\omega$.

Fix $\ell \in \mathfrak{g}^*$. We identify the tangent space $T_\ell(\mathfrak{g}^*)$ with $\mathfrak{g}^*$ in the canonical way. Complexifications are naturally identified: $T_\ell(\mathfrak{g}^*)_\mathbb{C}$ is identified with $\mathfrak{s}^*$, and the complex dual space of $T_\ell(\mathfrak{g}^*)_\mathbb{C}$ is identified with $\mathfrak{s}$. For $X \in \mathfrak{g}$, let $\xi_\ell^X$ denote the tangent vector at $\ell$ defined by $\ell[X, \cdot\,]$; we also have

$$\xi_\ell^X f = \frac{d}{dt}\bigg|_{t=0} f(\exp(-tX)\ell)$$

where $f$ is any smooth function defined in a neighborhood of $\ell$. Let $\mathcal{O}$ be the coadjoint orbit through $\ell$ and note that $T_\ell(\mathcal{O}) = \xi_\ell^{\mathfrak{g}} = \{\ell[X, \cdot\,],\ X \in \mathfrak{g}\}$. Of course $X \mapsto \xi_\ell^X$ extends to the complexification $\mathfrak{s}$ and the image of $\mathfrak{s}$ under $\xi_\ell$ is the complexification $T_\ell(\mathcal{O})_\mathbb{C}$ of $T_\ell(\mathcal{O})$. The real Kirillov-Kostant-Souriau 2-form $\omega$ is thus:

$$\omega_\ell(\xi_\ell^X, \xi_\ell^Y) = \langle \ell, [X, Y] \rangle \qquad \forall X, Y \in \mathfrak{g}.$$

For each $X \in \mathfrak{s}$, denote the resulting vector field on $\mathfrak{g}^*$ by $\xi^X$; recall that $X \mapsto \xi^X$ is a Lie algebra homomorphism.

For any open set $\mathcal{U}$ of $\mathfrak{g}^*$ let $\mathcal{E}(\mathcal{U})$ be the space of all smooth complex valued functions on $\mathcal{U}$ and $\mathcal{V}(\mathcal{U})$ the space of all smooth vector fields on $\mathcal{U}$. For $\phi \in \mathcal{E}(\mathcal{U})$ and $\ell \in \mathcal{U}$, let $X_\ell^\phi$ be the element of $\mathfrak{s}$ identified with $d\phi(\ell)$. Each function $\phi \in \mathcal{E}(\mathcal{U})$ gives rise to the Hamiltonian vector field $\xi^\phi$ defined on $\mathcal{U}$ by $\xi_\ell^\phi = \xi_\ell^{X_\ell^\phi}$. Recall that if $\tilde{X}$ denotes the coordinate function $\ell \mapsto \langle \ell, X \rangle$ on $\mathfrak{g}^*$ for $X \in \mathfrak{s}$, then $\xi^{\tilde{X}} = \xi^X$.

The Poisson bracket on $\mathcal{E}(U)$ is defined by

$$\{\phi, \psi\} = \langle \ell, [X_\ell^\phi, X_\ell^\psi] \rangle = \omega_\ell(\xi_\ell^\phi, \xi_\ell^\psi),$$

and one has $\xi^{\{\phi,\psi\}} = [\xi^\phi, \xi^\psi]$ and thus $\{\tilde{X}, \tilde{Y}\} = \widetilde{[X, Y]}$ for any $X$, $Y$ in $\mathfrak{s}$.

In the context of this paper, the non trivial orbits $\mathcal{O}$ are

(1) either diffeomorphic to a 2 dimensional plane and we shall say that 2 functions $p$ and $q$, defined on $\mathcal{O}$ are canonical coordinates if they have values in $\mathbb{R}$, if the mapping $\ell \mapsto (q(\ell), p(\ell))$ is a global diffeomorphism between $\mathcal{O}$ and $\mathbb{R}^2$ and if, through this diffeomorphism, the 2 form $\omega$ is simply $dq \wedge dp$.

(2) or diffeomorphic to a 2 dimensional cylinder, in this case, we shall say that 2 functions $q$ and $p$, defined on $\mathcal{O}$ are canonical coordinates if $q$ has values in $\mathbb{R}/\Gamma$, where $\Gamma$ is a discret subgroup of $\mathbb{R}$ (then, as above, $dq$ is a well-defined 1-form on $\mathcal{O}$) and $p$ in $\mathbb{R}$, if the mapping $\ell \mapsto (q(\ell), p(\ell))$ is a global diffeomorphism between $\mathcal{O}$ and $\mathbb{R}/\Gamma \times \mathbb{R}$ and if, through this diffeomorphism, the 2-form $\omega$ is simply $dq \wedge dp$.

Now fix a layer $\Omega_{\mathbf{e},\Psi}$ whose dimensional orbits are 2. We want in this section build couple of functions $(q, p)$, defined and analytic on an open neighborhood of $\Omega_{\mathbf{e},\Psi}$ and such that the restrictions of $q$ and $p$ to any orbit $\mathcal{O}$ in $\Omega_{\mathbf{e},\Psi}$ are canonical coordinates for $\mathcal{O}$.

Recall we defined on the open neighborhood $\mathcal{U}_{\mathbf{e},\Psi}$ of $\Omega_{\mathbf{e},\Psi}$ the functions $\sigma$, $t_1$ and $\ell_{n+1}$ and we have:

$$\sigma(\ell) = \mathcal{Q}(t_1(\ell), \ell) = \exp(t_1(\ell)H) \sum_{j=1}^n \ell_j Z_j^*, \quad \sum_{j=1}^n \ell_j Z_j^* = \exp(-t_1(\ell)H)\sigma(\ell).$$

Thus, for $j$ from 1 to $n$,we can see the function $\tilde{Z}_j$ on $\Omega_{\mathbf{e},\Psi}$ as a function $f_j$ of $t_1 \in \mathbb{R}$ and $\sigma^* \in \Sigma_{\mathbf{e},\Psi}$, smooth in $t_1$:

$$\tilde{Z}_j(\ell) = f_j(t_1(\ell), \sigma(\ell)), \qquad \text{with} \quad f_j(t_1, \sigma^*) = \tilde{Z}_j(\exp(-t_1 H)\sigma^*)$$

(in the case $c < k$, these functions are $\frac{2\pi}{\omega_\Psi}$-periodic). Of course, the last coordinate is the function $\tilde{Z}_{n+1} = \ell_{n+1}$.

Put now $q(\ell) = t_1(\ell)$ and $p(\ell) = \ell_{n+1}$. Consider the 2-form $\beta = dq \wedge dp$ on any orbit $\mathcal{O}$ in $\Omega_{\mathbf{e},\Psi}$, $\beta$ is a symplectic form and the Poisson bracket associated to $\beta$ is just:

$$\{\phi, \psi\}_\beta = \frac{\partial \phi}{\partial p}\frac{\partial \psi}{\partial q} - \frac{\partial \phi}{\partial q}\frac{\partial \psi}{\partial p}.$$

For the linear functions $\tilde{Z}_j$, we get then

$$\{\tilde{Z}_i, \tilde{Z}_j\}_\beta = 0, \quad \forall i, j \leq n,$$

and

$$\{\tilde{Z}_{n+1}, \tilde{Z}_j\}_\beta(\ell) = \left(\frac{\partial f_j}{\partial t_1}\right)(t_1(\ell), \sigma(\ell)), \quad \forall j \leq n.$$

But for the Poisson bracket coming from the Kirillov-Kostant-Souriau form $\omega$, we saw that:

$$\{\tilde{Z}_i, \tilde{Z}_j\}(\ell) = \widetilde{[Z_i, Z_j]}(\ell) = 0 \quad \forall i, j \leq n,$$

and

$$\{\tilde{Z}_{n+1}, \tilde{Z}_j\}(\ell) = \widetilde{[H, Z_j]}(\ell) = \xi_\ell^H \tilde{Z}_j = \frac{d}{ds}|_{s=0}\tilde{Z}_j(\exp - sH\ell).$$

But, with our notations, $\tilde{Z}_j(\ell) = \tilde{Z}_j(\exp - t_1(\ell)H\sigma(\ell)) = f_j(t_1, \sigma(\ell))$, thus

$$\{\tilde{Z}_{n+1}, \tilde{Z}_j\}(\ell) = \frac{d}{ds}|_{s=0}\tilde{Z}_j(\exp - sH \exp - t_1 H\sigma(\ell)) = \left(\frac{\partial f_j}{\partial t_1}\right)(t_1(\ell), \sigma(\ell)).$$

These relations prove that, for any couple of linear functions on $\mathcal{O}$, $\{\phi, \psi\}_\beta = \{\phi, \psi\}$. Since Poisson bracket are biderivations, the same holds for any smooth functions in the variables $\ell_i$, $1 \leq i \leq n+1$. Since, for symplectic manifolds, Poisson bracket characterizes the 2-form, the forms $\omega$ and $\beta$ do coincide and we have:

**Theorem 5.1.** *Fix a layer $\Omega_{\mathbf{e},\Psi}$ whose dimensional orbits are 2. Let $\mathbf{e} = \{k, n+1\}$. Let $q : \Omega_{\mathbf{e},\Psi} \to \mathbb{R}$ or $q : \Omega_{\mathbf{e},\Psi} \to \mathbb{R}/\frac{2\pi}{\omega_\Psi}\mathbb{Z}$ be the unique function for which*

$$\exp q(\ell)H\ell \in \Sigma_{\mathbf{e},\Psi}$$

*holds for each $\ell$ in $\Omega_{\mathbf{e},\Psi}$. Then $(q(\ell), p(\ell) = \ell_{n+1})$ are common canonical coordinates for all the orbits in $\Omega_{\mathbf{e},\Psi}$. More precisely,*

*(i) If $\alpha_k = 0$, then*

$$q(\ell) = \frac{\ell_k}{\ell_{k-1}} \quad \textit{and} \quad p(\ell) = \ell_{n+1}.$$

*(ii) If $k \leq b$ and $\alpha_k \neq 0$, then*

$$q(\ell) = \frac{\ln|\ell_k|}{\Re(\alpha_k)} \quad \textit{and} \quad p(\ell) = \ell_{n+1}.$$

*(iii) If $b < k \leq c$ ($k \notin \Psi$), then $q(\ell) \in \mathbb{R}$,*

$$q(\ell) = \frac{\Re(\ell_k \overline{\ell_h})}{|\ell_k|^2} \quad \textit{and} \quad p(\ell) = \ell_{n+1}.$$

*(iv) If $c < k$ ($k \in \Psi$), then $q(\ell)$ belongs to $\mathbb{R}/\Gamma_\Psi$ and*

$$e^{i\omega_\Psi q(\ell)} = \frac{u_\Psi(\ell)}{|u_\psi(\ell)|} \quad \textit{and} \quad p(\ell) = \ell_{n+1}.$$

**5.2. Quantizable canonical coordinates.**
We fix, as we have throughout, a suitable basis $\{Z_1, Z_2, \ldots, Z_{n+1}\}$ for $\mathfrak{s}$ and a corresponding suitable layering $\Omega_{\mathbf{e},\Psi}$ in $\mathfrak{g}^*$ with $\mathbf{e} = \{k, n+1\}$. Recall that we have defined an open neighborhood $\mathcal{U}_{\mathbf{e},\Psi}$ for $\Omega_{\mathbf{e},\Psi}$.

Denote by $\mathcal{E}(\Omega_{\mathbf{e},\Psi})$ the space of complex-valued functions on $\Omega_{\mathbf{e},\Psi}$ that are restrictions of functions in $\mathcal{E}(\mathcal{U}_{\mathbf{e},\Psi})$. Similarly we define $\mathcal{V}(\Omega_{\mathbf{e},\Psi})$.

Recall that we have the complex Vergne polarizations $\mathfrak{h}(\ell) = \mathfrak{n}_{\mathbb{C}}$, ($\ell \in \Omega_{\mathbf{e},\Psi}$) naturally associated with the layer $\Omega_{\mathbf{e},\Psi}$. In particular, the mapping $\ell \mapsto \mathfrak{h}(\ell)$ is constant in a sense that is evident. For each $\ell \in \Omega_{\mathbf{e},\Psi}$, put

$$\mathcal{F}(\ell) = \{\xi_\ell^Y,\ Y \in \mathfrak{h}(\ell)\} \subset T_\ell(\mathfrak{g}^*)_{\mathbb{C}}.$$

Since $s\mathfrak{h}(\ell) = \mathfrak{h}(s\ell)$ holds for all $\ell \in \Omega_{\mathbf{e},\Psi}$, $s \in G$, it follows that for each coadjoint orbit $\mathcal{O}$ in $\Omega_{\mathbf{e},\Psi}$, $\mathcal{F}|_{\mathcal{O}}$ is a $G$-invariant complex (geometric) polarization of the symplectic manifold $(\mathcal{O}, \omega)$.

Set

$$\mathcal{V}^0(\Omega_{\mathbf{e},\Psi}) = \{\xi \in \mathcal{V}(\Omega_{\mathbf{e},\Psi}),\ \xi_\ell \in \mathcal{F}(\ell) \quad \text{holds for all} \quad \ell \in \Omega_{\mathbf{e},\Psi}\}$$

and

$$\mathcal{E}^0(\Omega_{\mathbf{e},\Psi}) = \{\phi \in \mathcal{E}(\Omega_{\mathbf{e},\Psi}) \mid \xi^\phi \in \mathcal{V}^0(\Omega_{\mathbf{e},\Psi})\}.$$

For any orbit $\mathcal{O}$ in $\Omega_{\mathbf{e},\Psi}$, we define $\mathcal{V}^0(\mathcal{O})$ as in [7] and it is clear that for any $\phi \in \mathcal{E}(\Omega_{\mathbf{e},\Psi})$, we have $\phi \in \mathcal{E}^0(\Omega_{\mathbf{e},\Psi})$ if and only if $\phi \in \mathcal{E}^0(\mathcal{O})$ holds for all orbits $\mathcal{O}$ in $\Omega_{\mathbf{e},\Psi}$. It is easily seen that

$$\mathcal{E}^0(\Omega_{\mathbf{e},\Psi}) = \{\phi \in \mathcal{E}(\Omega_{\mathbf{e},\Psi}),\ \xi(\phi) = 0 \quad \text{holds for all} \quad \xi \in \mathcal{V}^0(\Omega_{\mathbf{e},\Psi})\}.$$

We shall call a function in $\mathcal{E}^0(\Omega_{\mathbf{e},\Psi})$ a polarized function.

Similarly, we define $\mathcal{E}^1(\Omega_{\mathbf{e},\Psi})$ as the space of function $\psi \in \mathcal{E}(\Omega_{\mathbf{e},\Psi})$ such that $\{\phi, \psi\}$ is a polarized function, or which is equivalent, the space of function $\psi \in \mathcal{E}(\Omega_{\mathbf{e},\Psi})$ such that $\xi(\psi)$ is in $\mathcal{E}^0(\Omega_{\mathbf{e},\Psi})$, for all $\xi \in \mathcal{V}^0(\Omega_{\mathbf{e},\Psi})$.

We shall call a function in $\mathcal{E}^1(\Omega_{\mathbf{e},\Psi})$ a quantizable function.

**Theorem 5.2.** *Let $G$ be a type I semi-direct product $\mathbb{R}^n \rtimes \mathbb{R}$ and fix a suitable basis $(Z_1, \dots, Z_n, Z_{n+1} = H)$ for the complexification $\mathfrak{s}$ of the Lie algebra $\mathfrak{g}$ of $G$. Let $\mathcal{P}$ be the suitable layering corresponding to this basis. Let $\Omega_{\mathbf{e},\Psi}$ be a layer with $\mathbf{e} = \{k, n+1\}$ and $\Sigma_{\mathbf{e},\Psi}$ be the corresponding cross-section. Then our explicit construction for a system of coordinates $(p, q)$ for any orbit $\mathcal{O}$ in $\Omega_{\mathbf{e},\Psi}$ satisfies the following:*
*(i) $p$ and $q$ can be extended on analytic functions on the open subset $\mathcal{U}_{\mathbf{e},\Psi}$ containing $\Omega_{\mathbf{e},\Psi}$.*
*(ii) the coordinate $q$ is polarized on $\Omega_{\mathbf{e},\Psi}$: $q \in \mathcal{E}^0(\Omega_{\mathbf{e},\psi})$.*
*(iii) the coordinate $p$ is quantizable on $\Omega_{\mathbf{e},\Psi}$: $p \in \mathcal{E}^1(\Omega_{\mathbf{e},\psi})$.*
*(iv) the coordinates $(q, p)$ are canonical, that is*

$$dq \wedge dp\big(\xi_\ell^X, \xi_\ell^Y\big) = \langle \ell, [X, Y] \rangle$$

*for all $X, Y \in \mathfrak{s}$, $\ell \in \Omega_{\mathbf{e},\Psi}$.*

**Proof.** We proved that $(p, q)$ is a canonical system of coordinates for any orbit $\mathcal{O}$ in $\Omega_{\mathbf{e},\Psi}$. Suppose the orbit is running through $\sigma^* \in \Sigma_{\mathbf{e},\Psi}$. Then, for any $X$ in $\mathfrak{h}(\ell) = \mathfrak{n}$, we saw that the restriction of $\tilde{X}$ to $\Omega_{\mathbf{e},\Psi}$ is a function $f(q, \sigma^*)$ of the variables $q$ and $\sigma^*$ only. Then, for any smooth function on $\Omega_{\mathbf{e},\Psi}$,

$$\xi^X \phi = \{\tilde{X}, \phi\} = -\frac{\partial f}{\partial q}(q, \sigma^*)\frac{\partial}{\partial p}.$$

The function $q$ is thus polarized, since $\xi^X q = 0$ for any $X$ in $\mathfrak{n}$ implies $\xi q = 0$ for any $\xi$ in $\mathcal{V}^0(\Omega_{\mathbf{e},\Psi})$. Thus any function $\phi(q, \sigma^*)$ is polarized also.

Similarly, the function $p$ is quantizable, since $\xi^X p = -\dfrac{\partial f}{\partial q}(q, \sigma^*) \in \mathcal{E}^0(\Omega_{\mathbf{e},\Psi})$ for any $X$ in $\mathfrak{n}$ implies $\xi p$ is polarized for any $\xi$ in $\mathcal{V}^0(\Omega_{\mathbf{e},\Psi})$. ■

Finally we have the following.

**Proposition 5.3.** *Let $G$ be a type I Lie group of the form $\mathbb{R}^n \rtimes \mathbb{R}$ and $\mathfrak{g}$ its Lie algebra. Choose a suitable basis $(Z_1, \dots, Z_{n+1})$ for $\mathfrak{s} = \mathfrak{g}_{\mathbb{C}}$. Let $\Omega_{\mathbf{e},\boldsymbol{\Psi}}$ be a layer in the corresponding layering with $\mathbf{e} = \{k, n+1\}$, set $(p, q)$ the canonical coordinates built in Theorem 5.1 and $\sigma$ the cross-section mapping built in Section 4. Put $M = \mathbb{R}$ if $k \notin \Psi$ and $M = \mathbb{R}/\Gamma_\Psi$ if $k \in \Psi$, then the following occurs*

*(i) The map*

$$\begin{array}{lll} P : \Omega_{\mathbf{e},\boldsymbol{\Psi}} & \longrightarrow & \Sigma_{\mathbf{e},\boldsymbol{\Psi}} \times \mathbb{R} \times M \\ \ell & \longmapsto & \big(\sigma(\ell), p(\ell), q(\ell)\big) \end{array}$$

*is a bijection and a global parametrization of $\Omega_{\mathbf{e},\boldsymbol{\Psi}}$ in the sense of ([4] Théorème 1.6).*

*(ii) The Vergne geometrical polarization is given by*

$$\mathcal{F}(\ell) = \mathbb{C}\ \text{-}span\{(\partial_p)_\ell\ \}.$$

*iii) For each orbit $\mathcal{O}$ in $\Omega_{\mathbf{e},\Psi}$ and for each $X \in \mathfrak{g}$, $X = \sum_{k=1}^{n+1} x_k Z_k$, the function $\tilde{X}$ has the form*

$$\tilde{X}(\ell) = \langle \ell, X \rangle = x_{n+1} p(\ell) + \sum_{i=1}^{n} x_i f_i(q(\ell), \sigma(\ell)), \quad \ell \in \Omega_{\mathbf{e},\Psi}$$

*where for each $i$ the function $f_i(q,\sigma^*)$ is real analytic in the variable $q$.*

## 6. Examples

### 6.1. Example 1.

Let $\mathfrak{g} = \mathrm{span}_{\mathbb{R}}\{X_1, Y_1, X_2, Y_2, H\}$ where

$$[H,(X_1+iY_1)] = 2i(X_1+iY_1) \quad \text{and} \quad [H,(X_2+iY_2)] = 4i(X_2+iY_2).$$

Choose the suitable basis $(Z_1, Z_2, Z_3, Z_4, H)$ of $\mathfrak{s}$ with

$$Z_1 = X_1+iY_1, \quad Z_2 = X_1-iY_1, \quad Z_3 = X_2+iY_2, \quad Z_4 = X_2-iY_2.$$

The basis coordinates for $\ell \in \mathfrak{g}^*$ are $\ell = (z_1, z_2 = \overline{z_1}, z_3, z_4 = \overline{z_3}, h)$.

1. $\mathbf{e} = \{1,5\}$ and $\Psi = \{1,3\}$, $\Omega = \Omega_{\mathbf{e},\Psi} = \{\ell \in \mathfrak{g}^*, z_1z_3 \neq 0\}$.
In this case, $\omega_\Psi = 2$, $\Gamma_\Psi = \pi\mathbb{Z}$ and we can choose $u_\Psi(\ell) = z_1$, then applying the $q$-function formula of Theorem 5.1, we obtain:

$$e^{2iq(\ell)} = \frac{z_1}{|z_1|}, \qquad p(\ell) = h.$$

and thus

$$q(\ell) = \frac{1}{2i}\ln\frac{z_1}{|z_1|} \mod \pi, \qquad p(\ell) = h$$

We can write coordinates for $\ell \in \Omega$:

$$\ell = (r_1e^{2iq}, r_1e^{-2iq}, r_2e^{i\theta}e^{4iq}, r_2e^{-i\theta}e^{-4iq}, p),$$

with

$$r_1 > 0, r_2 > 0 \quad \text{and} \quad \theta \in \mathbb{R}/2\pi\mathbb{Z}.$$

The cross-section is:

$$\Sigma = \{\ell \in \Omega, \mid \ell = (r_1, r_1, r_2e^{i\theta}, r_2e^{-i\theta}, 0)\} \simeq ]0,+\infty[\times(\mathbb{R}^2 \setminus \{0\}).$$

2. $\mathbf{e} = \{1,5\}$ and $\Psi = \{1\}$, $\Omega = \Omega_{\mathbf{e},\Psi} = \{\ell \in \mathfrak{g}^*, z_1 \neq 0 \quad \text{and} \quad z_3 = 0\}$.
In this case, we have also, $\omega_\Psi = 2$, $u_\Psi(\ell) = z_1$ and $\Gamma_\Psi = \pi\mathbb{Z}$. Applying the $q$-function formula of Theorem 5.1, we obtain the same definition for $q$ and $p$ as in the preceding case, however the coordinates for $\ell \in \Omega$ are:

$$\ell = (re^{2iq}, re^{-2iq}, 0, 0, p), \quad \text{with} \quad r > 0.$$

The cross-section is:

$$\Sigma = \{\ell \in \Omega, \mid \ell = (r, r, 0, 0, 0) \quad \text{with} \quad r > 0\} \simeq ]0,+\infty[.$$

3. $\mathbf{e} = \{3,5\}$ and $\Psi = \{3\}$, $\Omega = \Omega_{\mathbf{e},\Psi} = \{\ell \in \mathfrak{g}^*, z_1 = 0 \quad \text{and} \quad z_3 \neq 0\}$.
In this case, $\omega_\Psi = 4$, $u_\Psi(\ell) = z_3$ and $\Gamma_\Psi = \frac{\pi}{2}\mathbb{Z}$. Then applying the $q$-function formula of Theorem 5.1 we obtain:

$$e^{4iq(\ell)} = \frac{z_3}{|z_3|}, \qquad p(\ell) = h.$$

and so

$$q(\ell) = \frac{1}{4i} \ln \frac{z_3}{|z_3|} \mod \frac{\pi}{2}, \qquad p(\ell) = h.$$

The canonical coordinates for $\ell \in \Omega$ are:

$$\ell = (0, 0, re^{4iq}, re^{-4iq}, p), \quad \text{with} \quad r > 0.$$

The cross-section is:

$$\Sigma = \{\ell \in \Omega, |\ \ell = (0, 0, r, r, 0) \quad \text{with} \quad r > 0\} \simeq ]0, +\infty[.$$

**6.2. Example 2.**

Let $\mathfrak{g} = \text{span}_{\mathbb{R}}\{X_1, Y_1, X_2, Y_2, H\}$ where

$$[H, (X_1 + iY_1)] = i(X_1 + iY_1) \quad \text{and} \quad [H, (X_2 + iY_2)] = i(X_2 + iY_2) + X_1 + iY_1.$$

Choose the suitable basis $(Z_1, Z_2, Z_3, Z_4, H)$ of $\mathfrak{s}$ with

$$Z_1 = X_1 + iY_1, \quad Z_2 = X_1 - iY_1, \quad Z_3 = X_2 + iY_2, \quad Z_4 = X_2 - iY_2.$$

Again we use the basis coordinates $\ell = (z_1, z_2 = \overline{z_1}, z_3, z_4 = \overline{z_3}, h)$. Here there are:

1. $\Omega = \Omega_{\mathbf{e}}$ with $\mathbf{e} = \{1, 5\}$. Applying the $q$-function formula of Theorem 5.1, we obtain

$$q(\ell) = \frac{\Re\big(z_1 \overline{z_3}\big)}{|z_1|^2}, \qquad p(\ell) = h.$$

Thus the coordinates for $\ell \in \Omega$ can be written as:

$$\ell = (re^{i(\theta+q)}, re^{-i(\theta+q)}, e^{i(\theta+q)}(ia + rq), e^{-i(\theta+q)}(-ia + rq), p),$$

with

$$r > 0,\ \theta \in \mathbb{R}/2\pi\mathbb{Z} \ \text{and} \quad a \in \mathbb{R}.$$

The cross-section is

$$\begin{aligned}\Sigma &= \{\ell \in \Omega, |\ell = (re^{i\theta}, re^{-i\theta}, iare^{i\theta}, -iare^{-i\theta}, 0), \quad \text{with } re^{i\theta} \neq 0, a \in \mathbb{R}\} \\ &\simeq (\mathbb{R}^2 \setminus \{0\}) \times \mathbb{R}.\end{aligned}$$

2. $\mathbf{e} = \{3, 5\}$, then $\Omega = \Omega_{\mathbf{e},\mathbf{\Psi}} = \{\ell \in \mathfrak{g}^*,\ z_1 = 0,\ z_3 \neq 0\}$ where $\Psi = \{3\}$. In this case $\omega_\Psi = 1$ and $\Gamma_\Psi = 2\pi\mathbb{Z}$. Again applying the $q$-function formula of Theorem 5.1, we get:

$$e^{iq(\ell)} = \frac{z_3}{|z_3|}, \qquad p(\ell) = h,$$

and then

$$q(\ell) = -i \ln \frac{z_3}{|z_3|} \mod 2\pi, \qquad p(\ell) = h,$$

so that $\ell$ is just

$$\ell = \big(0, 0, re^{iq}, re^{-iq}, p\big) \quad \text{with} \quad r > 0.$$

The cross-section is:

$$\Sigma = \{\ell \in \Omega, |\ \ell = (0, 0, r, r, 0) \quad \text{with} \quad r > 0\} \simeq ]0, +\infty[.$$

**6.3. Example 3.**
Let $\mathfrak{g} = \text{span}_{\mathbb{R}}\{X_1, Y_1, X_2, Y_2, H\}$ where

$$[H, (X_1 + iY_1)] = 2i(X_1 + iY_1) \quad \text{and} \quad [H, (X_2 + iY_2)] = 3i(X_2 + iY_2).$$

Choose the suitable basis $(Z_1, Z_2, Z_3, Z_4, H)$ of $\mathfrak{s}$ with

$$Z_1 = X_1 + iY_1, \quad Z_2 = X_1 - iY_1, \quad Z_3 = X_2 + iY_2, \quad Z_4 = X_2 - iY_2.$$

The basis coordinates for $\ell \in \mathfrak{g}^*$ are $\ell = (z_1, z_2 = \overline{z_1}, z_3, z_4 = \overline{z_3}, h)$.

1. $\mathbf{e} = \{1, 5\}$ and $\Psi = \{1, 3\}$, $\Omega = \Omega_{\mathbf{e},\Psi} = \{\ell \in \mathfrak{g}^*, z_1 z_3 \neq 0\}$.
In this case, $\omega_\Psi = 1$, $\Gamma_\Psi = 2\pi\mathbb{Z}$ and we can choose $u_\Psi(\ell) = \dfrac{z_1^2}{z_3}$, then applying the $q$-function formula of Theorem 5.1, we obtain:

$$e^{iq(\ell)} = \frac{\frac{z_1^2}{z_3}}{|\frac{z_1^2}{z_3}|}, \qquad p(\ell) = h,$$

so that

$$q(\ell) = -i \ln\left(\frac{z_1^2 |z_3|}{z_3 |z_1|^2}\right) \mod 2\pi, \qquad p(\ell) = h.$$

We can write coordinates for $\ell \in \Omega$:

$$\ell = (r_1 e^{i(\theta+2q)}, r_1 e^{-i(\theta+2q)}, r_2 e^{i(2\theta+3q)}, r_2 e^{-i(2\theta+3q)}, p),$$

with

$$r_1 > 0, \quad r_2 > 0 \quad \text{and} \quad \theta \in \mathbb{R}/2\pi\mathbb{Z}.$$

The cross-section is:

$$\Sigma = \{\ell \in \Omega, |\ \ell = (r_1 e^{i\theta}, r_1 e^{-i\theta}, r_2 e^{2i\theta}, r_2 e^{-2i\theta}, 0)\} \simeq (\mathbb{R}^2 \setminus \{0\}) \times ]0, +\infty[.$$

2. $\mathbf{e} = \{1, 5\}$ and $\Psi = \{1\}$, $\Omega = \Omega_{\mathbf{e},\Psi} = \{\ell \in \mathfrak{g}^*, z_1 \neq 0 \quad \text{and} \quad z_3 = 0\}$.
In this case, $\omega_\Psi = 2$, $u_\Psi(\ell) = z_1$ and $\Gamma_\Psi = \pi\mathbb{Z}$. Applying the $q$-function formula of Theorem 5.1, we obtain

$$e^{2iq(\ell)} = \frac{z_1}{|z_1|}, \qquad p(\ell) = h,$$

and thus

$$q(\ell) = \frac{1}{2i} \ln \frac{z_1}{|z_1|} \mod \pi, \qquad p(\ell) = h.$$

However the coordinates for $\ell \in \Omega$ are:

$$\ell = (re^{2iq}, re^{-2iq}, 0, 0, p), \quad \text{with} \quad r > 0.$$

The cross-section is:

$$\Sigma = \{\ell \in \Omega, |\ \ell = (r, r, 0, 0, 0) \quad \text{with} \quad r > 0\} \simeq ]0, +\infty[.$$

3. $\mathbf{e} = \{3, 5\}$ and $\Omega = \Omega_{\mathbf{e},\Psi} = \{\ell \in \mathfrak{g}^*, z_1 = 0 \quad \text{and} \quad z_3 \neq 0\}$ with $\Psi = \{3\}$.

In this case, $\omega_\Psi = 3$, $\Gamma_\Psi = \frac{2\pi}{3}\mathbb{Z}$ and $u_\Psi(\ell) = z_3$. Applying the $q$-function formula of Theorem 5.1, we obtain:

$$e^{3iq(\ell)} = \frac{z_3}{|z_3|}, \qquad p(\ell) = h,$$

and thus

$$q(\ell) = \frac{1}{3i} \ln \frac{z_3}{|z_3|} \mod \frac{2\pi}{3}, \qquad p(\ell) = h.$$

We can write coordinates for $\ell \in \Omega$:

$$\ell = (0, 0, re^{3iq}, re^{-3iq}, p), \quad \text{with} \quad r > 0.$$

The cross-section is:

$$\Sigma = \{\ell \in \Omega, |\ \ell = (0, 0, r, r, 0) \quad \text{with} \quad r > 0\} \simeq ]0, +\infty[.$$

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