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Survey on the Burnside Ring of Compact Lie Groups

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Abstract. The definition and basic properties of the Burnside ring of compact Lie groups are presented, with emphasis on the analogy with the construction of the Burnside ring of finite groups.
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The Burnside ring of a finite group encodes the "calculus of cosets" of the group. It was defined by Burnside in his work on tables of marks of finite groups [3, page 236]. The Burnside ring of a compact Lie group was defined by tom Dieck in the context of equivariant stable homotopy theory. It can also be described as encoding the "calculus of cosets", provided only certain transitive orbits of G are made visible. Namely, those transitive orbits, G/H, such that H has finite order in its normalizer. Some references are [1] [7] [9] [10] [16] [23, Chapter V].

1. The Burnside ring of a finite group

Let G be a finite group. The Burnside ring of G is the Grothendieck group completion of the semiring of isomorphism classes of finite G-sets. It is denoted by $A(G)$ (other notations are $B(G)$ and $\Omega(G)$). Addition is given by disjoint union, and multiplication by Cartesian product. These operations are well defined on isomorphism classed of G-sets. The Burnside ring of G is isomorphic, as an abelian group, to the free abelian group with generators the isomorphism classes of transitive G-sets. Under this identification, the multiplication of the additive generators is given by the double coset formula. The double coset formula says that the G-set $G/H \times G/K$ is G-isomorphic to the disjoint union of the transitive G-sets $G/(H \cap gKg^{-1})$ where HgK runs over the double coset $H\backslash G/K$.

Let H be a subgroup of G and let X be a G-set. The H-fixed point set X^H is the subset $\{x \in X \mid hx = x, h \in H\}$ of X. The number of elements in X^H, denoted $|X^H|$, only depends on the G-isomorphism class of X and the G-conjugacy class of H. For every conjugacy class of a subgroup H of G the map

$$X \mapsto |X^H|$$

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gives a semiring homomorphism from the semiring of isomorphism classes of finite G-sets to the integers. Hence there is an induced H-fixed point ring homomorphism $\phi_H\colon A(G) \to \mathbb{Z}$. These H-fixed point ring homomorphisms ensemble to give a ring homomorphism

$$\phi\colon A(G) \to \textstyle\prod_{(H)} \mathbb{Z},$$

where the product is over the G-conjugacy classes, (H), of subgroups H of G.

The map ϕ is sometimes called the mark homomorphism. Choose a linear ordering of the conjugacy classes of subgroups of G that respects subconjugacy. The matrix with the $(H),(K)$ entry given by $\phi_K([G/H])$ is called the table of marks, or the mark matrix, of G.

The basic properties of the Burnside ring of a finite group are described nicely in the first chapter of [10] and in [16]. A recent survey is [4].

2. Recollections about compact Lie groups

Compact Lie groups need not be connected, so all finite groups are compact Lie groups. Only closed subgroups of compact Lie groups are considered. The Weyl group W_GH of a subgroup H in G is N_GH/H.

A theorem of Montgomery and Zippin says that for any closed subgroup H of G there is an open neighborhood U of the identity element in G such that all subgroups of HU are subconjugate to H [2, II.5.6] [26].

Let H and K be subgroups of G. The normalizer N_GH acts from the left on $(G/K)^H$. Montgomery and Zippin's theorem implies that the coset $(G/K)^H/N_GH$ is finite [2, II.5.7]. In particular, if W_GH is finite, then $(G/K)^H$ is finite. The Weyl group W_GK acts freely on $(G/K)^H$ from the right by $gK \cdot nK = gnK$, where $gK \in (G/K)^H$ and $nK \in W_GK$. So $|W_GK|$ divides $|(G/K)^H|$. The fixed point space $(G/K)^H$ is nonempty if and only if H is subconjugated to K in G. Hence, if H is subconjugated to K and H has finite Weyl group, then K also has finite Weyl group.

Let J be a subgroup of G and let X be a G-space. A point x in X has orbit type J if the stabilizer $\{g \in G \mid gx = x\}$ of x is conjugate to J. Let $X_{(J)}$ denote the subspace of X consisting of the points of X with orbit type J. Let X_{fin} denote the subspace of X consisting of all points whose stabilizers have finite Weyl group. If H has finite Weyl group and X is a G-space, then $X^H_{\mathrm{fin}} = X^H$ by the results in the previous paragraph. A G-CW-complex is a space built out of n-dimensional G-cells $S^{n-1} \times G/H \to D^n \times G/H$, for subgroups H of G, by gluing them to cells of dimension $(n-1)$ or lower, for $n \geq 0$ (see [24, I.3] for a precise definition). If X is a G-CW-complex, then X_{fin} is a subcomplex of X. This follows because the stabilizer of any point in $D^n \times G/H$ is conjugated to H, and cells whose stabilizers have finite Weyl group can only map to other cells whose stabilizers also have finite Weyl group.

3. Construction of the Burnside ring of a compact Lie group

The basic idea in the definition of the Burnside ring of a compact Lie group G is to consider finite disjoint unions of G-orbits, but ignore those orbits G/H where

H does not have finite Weyl group in G. Denote the G-isomorphism class of a G-space X by $[X]$. The set of G-isomorphism classes of finite disjoint unions of transitive G-spaces, whose orbit types have finite Weyl group, has a structure of an abelian semigroup given by disjoint union.

If G is a compact Lie group, then the Cartesian product of two homogeneous G-spaces G/H and H/K is not isomorphic to a disjoint union of homogeneous G-spaces. So the definition of the product has to be modified. If G is a finite group, then a reformulation of the double coset formula says that

$$[G/H] \cdot [G/K] = \Sigma_{(J)} |(G/H \times G/K)_{(J)}/G| \, [G/J].$$

The following key observation is due to Schwänzl [33].

Lemma 3.1. *Assume that Z is a G-space such that Z^J is a finite subspace of Z. Then $Z_{(J)}/G$ is finite.*

Proof. In fact there is an inequality $|Z_{(J)}/G| \leq |Z^J|$. It suffices to check this for G/gJg^{-1}. In this case the claim is true since $(G/gJg^{-1})/G$ is a point and $(G/gJg^{-1})^J$ is nonempty, for all $g \in G$. ∎

In particular, if X and Y are G-spaces so that X^J and Y^J are finite, then $(X \times Y)_{(J)}/G$ is finite. Hence, if J is a subgroup of G with finite Weyl group, and H and K are subgroups of G, then G/H^J and G/K^J are finite, and therefore $(G/H \times G/K)_{(J)}/G$ is finite.

Illman has proved that the product $G/H \times G/K$ is a finite G-CW-complex [22]. The G-cells of $G/H \times G/K$ with stabilizer a subgroup of G with finite Weyl group are all 0-dimensional by Lemma 3.1. Hence $(G/H \times G/K)_{\text{fin}}$, the subspace of $G/H \times G/K$ obtained by removing the G-cells whose stabilizers do not have finite Weyl group, is a finite disjoint union of homogeneous G-spaces.

Definition 3.2. Define a product as follows

$$[G/H] \cdot [G/K] = [(G/H \times G/K)_{\text{fin}}] = \Sigma_J \, n_J [G/J],$$

where the sum is over the conjugacy classes of subgroups J of G with finite Weyl group, and n_J is the number of elements in the finite set $(G/H \times G/K)_{(J)}/G$ [33]. The sum is finite since $G/H \times G/K$ has only finitely many orbit types. The isomorphism class of the point, $[G/G]$, is the multiplicative unit. The multiplication is clearly commutative and distributive with respect to the addition.

Lemma 3.3. *The multiplication in Definition 3.2 is associative.*

Proof. Consider three subgroups H, J, and K of G all with finite Weyl groups. It suffices to show that $((G/H \times G/J)_{\text{fin}} \times G/K)_{\text{fin}}$ is equivalent to

$$(G/H \times G/J \times G/K)_{\text{fin}}.$$

Let U be a subgroup of G with infinite Weyl group. Then $G/U \times G/K$ consists of G-cells with stabilizers that are subconjugated to U. Hence they all have infinite Weyl groups (see section 2). The result follows. ∎

Definition 3.4. Let G be a compact Lie group. Then the Burnside ring $A(G)$ is the Grothendick construction of the semiring of isomorphism classes of finite disjoint unions of homogeneous G-spaces, G/H, for closed subgroups H of G with finite Weyl group; the sum is given by disjoint union and the multiplication is given by Definition 3.2.

Recall from section 2 that $(G/K)^H$ is finite, whenever H is a subgroup of G with finite Weyl group.

Lemma 3.5. *Let H be a subgroup of G with finite Weyl group. The function $\phi_H \colon A(G) \to \mathbb{Z}$, defined by $\phi_H(G/K) = |(G/K)^H|$ on the free generators $[G/K]$ of $A(G)$, is a ring homomorphism.*

Proof. It suffices to show that ϕ_H is a semiring homomorphism before passing to the Grothendieck construction. The map is well defined, additive and respects both the additive and the multiplicative units. The map respects the multiplication by observing that

$$(G/K \times G/L)^H_{\text{fin}} = (G/K \times G/L)^H$$

since H has finite Weyl group. ■

4. Other definitions of Burnside rings

There is a general categorical approach to Burnside rings [25]. Peter May associates to any symmetric tensor triangulated category (with a skeletally small category of dualizable objects) a Burnside ring. It is a subring of the ring of selfmaps of the unit object. The Burnside ring of a compact Lie group is isomorphic to the Burnside ring associated to the G-equivariant stable homotopy category (see section 16).

The following is a sketch of the main ideas in tom Dieck's original construction of $A(G)$ for compact Lie groups. The general categorical definition of May is modeled on this example. Instead of disjoint unions of homogeneous G-spaces the full subcategory of the stable homotopy category consisting of the dualizable objects is used. A spectrum is dualizable if and only if it is a suspension spectrum, $\Sigma^\infty_G X$, of a retract of finite G-cell complex X. There is a semiring of stable G-homotopy classes of such objects. (Alternatively, start out with all G-spaces of the homotopy type of a retract of a finite G-CW-complex. There is a semiring of G-homotopy types of such spaces; the sum is given by disjoint union, and the product by Cartesian product.) There is a semiring homomorphism into the integers given by Euler characteristic of the (geometric) H-fixed point spectra, for each subgroup H of G. These maps induces ring homomorphisms from the Grothendieck construction of the semiring of stable G-homotopy classes of dualizable objects to the integers. The Burnside ring is the ring obtained by dividing out by the intersection of the kernel ideals of these homomorphisms for all subgroups H of G. Hence a formal difference of two stable G-homotopy types $\Sigma^\infty_G X$ and $\Sigma^\infty_G Y$ is equal to 0 in the Burnside ring if and only if the Euler characteristic of the fixed point spaces X^H and Y^H are equal for all closed subgroups H in G. It is enough to check this for all subgroups H with finite Weyl groups since

$\chi(X^K) = \chi(X^H)$, whenever H/K is a torus. The comparison between the definition of the Burnside ring sketched above and the one given in Definition 3.4 uses the additivity of the Euler characteristic on stable cofiber sequences.

The Burnside ring from the perspective of stable equivariant homotopy theory and geometric topology are surveyed in [24] [27].

5. Maps between Burnside rings

Let G_1 and G_2 be two compact Lie groups. Then there is a natural map

$$p\colon A(G_1) \otimes A(G_2) \to A(G_1 \times G_2)$$

given by sending $[G_1/H_1] \otimes [G_2/H_2]$ to $[G_1 \times G_2/H_1 \times H_2]$. The map p is an injective ring map, however it is not an isomorphisms unless all subgroups of $G_1 \times G_2$, with finite Weyl groups, are of the form $H_1 \times H_2$, for subgroups $H_1 \leq G_1$ and $H_2 \leq G_2$. If G_1 and G_2 are finite groups and $|G_1|$ and $|G_2|$ are relative prime, then p is an isomorphism.

Let G be a finite group. Then there is a map

$$\alpha\colon A(\mathbb{Z}/|G|) \to A(G)$$

such that for any subgroup H of G the composite map $\phi_H \circ \alpha$ is equivalent to $\phi_{\mathbb{Z}/|H|}$ where $\mathbb{Z}/|H|$ is the (unique) cyclic subgroup of $\mathbb{Z}/|G|$ of order $|H|$ [17].

Let H be a subgroup of a compact Lie group G. The induction map $\mathrm{ind}\colon A(H) \to A(G)$ is given by sending a generator $[H/K]$ to $[G/K]$ if K has finite Weyl group in G and to 0 otherwise.

The restriction map between Burnside rings is most easily described when the Burnside ring is defined in terms of equivalence classes of compact G-spaces (see section 4). The restriction map $\mathrm{res}\colon A(G) \to A(H)$ is defined by sending the isomorphism class of a G-space X to $X|H$, X regarded as an H-space via the inclusion $H < G$. Let L be a subgroup of H with finite Weyl group in H, then

$$\phi_L \mathrm{res}\, x = \phi_{L'} x,$$

where L' is an extension in G of L by a torus such that L' has finite Weyl group in G. Let H be a normal subgroup of a finite group G. The restriction map $A(G) \to A(H)$ is give by sending the isomorphism class of a G-set G/L to

$$\frac{|G||H \cap L|}{|H||L|}\,[H/H \cap L],$$

for any subgroup L of G.

The restriction map $A(G) \to A(1) \cong \mathbb{Z}$ is called the augmentation map. The kernel of this map is called the augmentation ideal. The augmentation map is given by sending $[G/L]$ to $|(G/L)^T|$ where T is a maximal torus in G.

6. Examples of Burnside rings of abelian groups

The only compact Lie groups with no proper subgroups with finite Weyl groups are the trivial group and the tori:

$$A(1) \cong \mathbb{Z}$$

$$A((S^1)^n) \cong \mathbb{Z},$$

for any $n \geq 1$. If G is a compact abelian Lie group, then G is isomorphic to the cartesian product of a torus and a finite abelian group. Hence if G is a compact abelian Lie group, then $A(G) \cong A(G/G^\circ)$, where G° is the unit component of G and G/G° is the group of components of G. A finite abelian group is isomorphic to the cartesian product of p-groups. Hence to calculate the Burnside ring of compact abelian Lie groups it suffices to calculate the Burnside ring of finite abelian p-groups, for primes p. They are of the form $G = \mathbb{Z}/p^{n_1} \times \cdots \times \mathbb{Z}/p^{n_m}$ where $m \geq 1$ and $n_i \geq 1$.

The multiplication in the Burnside ring (double coset formula) is particularly simple for finite abelian groups. It is given by

$$[G/K] \cdot [G/L] \cong \frac{|G|\,|K \cap L|}{|K|\,|L|}\,[G/K \cap L],$$

for subgroups K and L of G. While it is easy to find the isomorphism classes of subgroups of G, it is more involved to keep track of all subgroups of G, and their intersections. In the rest of this section bookkeeping of the subgroups is described in the special case of cyclic p-groups $\mathbb{Z}/p^n$ and elementary p-groups $(\mathbb{Z}/p)^n$.

There is an isomorphisms

$$A(\mathbb{Z}/p) \cong \mathbb{Z}[x]/(x^2 - px).$$

More generally,

$$A(\mathbb{Z}/p^n) \cong \mathbb{Z}[a_1, \ldots, a_n]/a_i a_j = p^i a_j \quad (\text{for } j \geq i),$$

where a_i denotes the coset of $\mathbb{Z}/p^n$ by the subgroup $\mathbb{Z}/p^{n-i}$, for $i = 1, \ldots n$.

The Burnside ring $A((\mathbb{Z}/p)^2)$ is isomorphic to

$$\mathbb{Z}[a_0, \ldots, a_p, b]/a_i b = pb, b^2 = p^2 b, a_i^2 = pa_i, a_i a_j = b \text{ for } i \neq j,$$

where b is the coset $(\mathbb{Z}/p)^2$ and a_i is the coset of $(\mathbb{Z}/p)^2$ by the subgroup generated by the element $(1, i)$ for $0 \leq i < p$ and the subgroup generated by $(0, 1)$ when $i = p$.

To bookkeep the subgroups of $(\mathbb{Z}/p)^n$ and their intersections the tactic is to associate to any subgroup H of $(\mathbb{Z}/p)^n$ a distinguished set of generators of H. This gives a systematic description of the subgroups of $(\mathbb{Z}/p)^n$ and their intersections.

Fix n and let H be a subgroup of $(\mathbb{Z}/p)^n$. There is a tuple of integers (with $m \leq n$)

$$n \geq i_1 > i_2 > \cdots > i_m \geq 1$$

and elements $\alpha_1, \ldots, \alpha_m \in (\mathbb{Z}/p)^n$ such that $\alpha_k^{i_k} = 1$, $\alpha_k^j = 0$ for $j > i_k$, and $\alpha_k^{i_l} = 0$ whenever $l \neq k$. The superscript j denotes the jth coordinate in $(\mathbb{Z}/p)^n$. The subgroup H is the subgroup of $(\mathbb{Z}/p)^n$ generated by the elements $\alpha_1, \ldots, \alpha_m$. The elements are linearly independent and H is isomorphic to $(\mathbb{Z}/p)^m$. There is exactly one such set of distinguished generators that generates H. It is illustrative to write the generators in the form of an $m \times n$-matrix with values in $\mathbb{Z}/p$. The following is an example when the rank of the subgroup is $m = 3$

$$\begin{pmatrix} 0 & \cdots & 0 & 1 & * & \cdots & * & 0 & * & \cdots & * & 0 & * & \cdots \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 1 & * & \cdots & * & 0 & * & \cdots \\ 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & 1 & * & \cdots \end{pmatrix}$$

The intersection of a subgroup given by

$$n \geq i_1 > i_2 > \cdots > i_m \geq 1$$

and elements $\alpha_1, \ldots, \alpha_m \in (\mathbb{Z}/p)^n$ by another subgroup given by

$$n \geq i'_1 > i'_2 > \cdots > i'_{m'} \geq 1$$

and elements $\alpha'_1, \ldots, \alpha'_{m'} \in (\mathbb{Z}/p)^n$ is the subgroup given by the generators

$$n \geq j_1 > j_2 > \cdots > j_s \geq 1$$

and elements $\beta_1, \ldots, \beta_s \in (\mathbb{Z}/p)^n$ such that the set of tuples $\{(j_k, \beta_k)\}_{k=1,\ldots,s}$ equals the intersection of the sets of tuples $\{(i_k, \alpha_k)\}_{k=1,\ldots,m}$ and $\{(i'_k, \alpha'_k)\}_{k=1,\ldots,m'}$.

7. Examples of Burnside rings of nonabelian groups

If H and K are subgroups of G with finite Weyl groups, and H is normal in G, then

$$[G/H] \cdot [G/K] = \frac{|G/H|\,|(G/K)^{K\cap H}|}{|W_G(K \cap H)|}\,[G/K \cap H],$$

if $K \cap H$ has finite Weyl group in G, and $[G/H] \cdot [G/K] = 0$ if $K \cap H$ does not have finite Weyl group in G.

Let G be the permutation group Σ_3. It is isomorphic to the semidirect product $\mathbb{Z}/3 \rtimes \mathbb{Z}/2$. The isomorphism classes of transitive G-sets are: $[G/G]$, $a = [G/\mathbb{Z}/2]$, $b = [G/\mathbb{Z}/3]$, and $c = [G/1]$. The Burnside ring $A(G)$ is isomorphic to the polynomial ring $\mathbb{Z}[a, b, c]/\sim$, where the relations are $a^2 = a + c$, $b^2 = 2b$, $c^2 = 6c$, $ab = c$, $ac = 3c$, and $bc = 2c$.

Let G be the nontrivial semidirect product $S^1 \rtimes \mathbb{Z}/2$ (also known as $\mathrm{O}(2)$). The subgroups of G with finite Weyl groups are G, $S^1 \rtimes 0$, and $\mathbb{Z}/n \rtimes \mathbb{Z}/2$ for $n \geq 1$. The normalizers are G, G, and $\mathbb{Z}/2n \rtimes \mathbb{Z}/2$ for $n \geq 1$, respectively. Let y denote the element $[G/S^1 \rtimes 0]$, and let x_n denote $[G/(\mathbb{Z}/n \rtimes \mathbb{Z}/2)]$, for $n \geq 1$. Let $\sim$ be the relations generated by $y \cdot y = 2y$, $x_n \cdot x_m = 2x_{(n,m)}$, for $m, n \geq 1$, and $x_n \cdot y = 0$, for $n \geq 1$, where (m, n) is the greatest common divisor of m and n. Then there is an isomorphism

$$A(S^1 \rtimes \mathbb{Z}/2) \cong \mathbb{Z}[y, x_1, x_2, x_3, \ldots] / \sim .$$

The Burnside ring of $\mathrm{SU}(3)$ is described in great detail in [10, 5.14].

Computer programs facilitate the calculation of the table of marks and the Burnside ring for many groups. See for example [28].

8. The mark homomorphism $\phi : \mathbf{A(G)} \to \mathbf{C(G)}$

Let $\mathcal{C}G$ be the space of closed subgroups of G with the Hausdorff topology from G. This is a compact metric space. Let ΨG be the quotient space of $\mathcal{C}$ obtained by identifying G-conjugate subgroups of G. The space ΨG is countable, hence it is a totally disconnected space [8].

Let ΦG be the subspace of ΨG consisting of conjugacy classes of closed subgroups of G with finite Weyl group. By Montgomery and Zippin's theorem the complement $\Psi G - \Phi G$ is open. So ΦG is a closed subspace of ΨG, hence a compact space.

There is a continuous retract map $\omega \colon \Psi G \to \Phi G$ given by sending (H) to the conjugacy class of a largest possible extension of H by a torus [21, 1.2]. This extension is unique up to conjugation. The following result gives a useful description of ω [21, 2.2].

Lemma 8.1. *The conjugacy class $\omega(H)$ is the conjugacy class of the subgroup generated by (H) and a maximal torus in the centralizer $C_G H$.*

Let $C(G)$ denote the ring of continuous functions from ΦG to the integers $\mathbb{Z}$. Recall Lemma 3.5.

Lemma 8.2. *The homomorphisms ϕ_H, for $H \leq G$, ensemble to give a ring homomorphism*

$$\phi \colon A(G) \to C(G).$$

Proof. It suffices to show that the map $(H) \mapsto |(G/K)^H|$ is continuous for each (K) in ΦG. Let (H_i) be a sequence converging to H. Montgomery and Zippin's theorem implies that there is no loss of generality in assuming that $H_i < H$. As an H-manifold, $(G/K)|H$, has finitely many isotropy types [22]. The fixed point space $(G/K)^H$ is equal to $(G/K)^{H_i}$ whenever H_i is not subconjugated, in H, to any of the isotropy groups of $(G/K)|H$ that are properly contained in H.

Since $(G/K)|H$ has finitely many orbit types there is an $\epsilon > 0$ such that the distance, in the metric space ΨG, between (H) and each of the G-conjugacy classes of the H-isotropy groups of $(G/K)|H$, different from (H), are all greater or equal to ϵ. Hence if the distance between (H) and (H_i) is less than ϵ, then $|(G/K)^H| = |(G/K)^{H_i}|$. ∎

Lemma 8.3. *The mark homomorphism ϕ is an injective ring map.*

Proof. Assume that

$$\textstyle\sum_{i=1}^n q_i \, \phi([G/H_i]) = 0$$

where all $q_i \in \mathbb{Z}$ (or in $\mathbb{Q}$) are nonzero. Let (H_m) be a maximal conjugacy class among $(H_1), \ldots, (H_n)$. The function evaluated at (H_m) is $q_m |W_G H_m| = 0$. This gives a contradiction. ∎

If G is a finite group and ϕ is a surjective map, then G is the trivial group.

9. The mark homomorphism ϕ is a rational isomorphism

The topology on ΦG can be described in a way which makes it clear that the functions $\phi([G/H])$, for $(H) \in \Phi G$, generate $C(G) \otimes_{\mathbb{Z}} \mathbb{Q}$ as a $\mathbb{Q}$-module.

Lemma 9.1. *The topology on ΦG is the smallest topology such that*

$$V(K) = \{(H) \in \Psi G \mid (H) \leq (K)\}$$

is both an open and a closed subset of ΦG for all $(K) \in \Phi G$.

Proof. The definition of the Hausdorff topology shows that $V(K)$ is closed. By Montgomery and Zippin's theorem $V(K)$ is also open for all $(K) \in \Phi G$.

Since ΦG is a countable metric space it has a basis for the topology consisting of open and closed sets. (For each element x the function $d(x, -)$ is a continuous function to $\mathbb{R}$.) Let (K) be in ΦG. Let U be an open and closed neighborhood of (K) in ΦG. Since $V(K)$ is open and closed there is no loss of generality assuming that U lies inside $V(K)$. The set $V(K) - U$ is open and closed. The collection of sets $V(H)$, for all (H) in $V(K) - U$, is an open covering of the closed subspace $V(K) - U$ of ΦG. Since ΦG is a compact space there is a finite set $\{(H_1), (H_2), \ldots, (H_n)\}$ such that each H_i is properly subconjugated to K in G and

$$V(K) - U \subset \cup_{i=1}^{n} V(H_i).$$

Then $V(K) - \cup_{i=1}^{n} V(H_i)$ is an open and closed neighborhood of (K) contained in U. Hence $V(K)$ and its complement $\Phi G - V(K)$, for $(K) \in \Phi G$, generates the topology on ΦG. ∎

Proposition 9.2. *The map*

$$\phi \otimes_{\mathbb{Z}} \mathbb{Q} \colon A(G) \otimes_{\mathbb{Z}} \mathbb{Q} \to C(G) \otimes_{\mathbb{Z}} \mathbb{Q}$$

is an isomorphism.

Proof. It suffices to prove that the characteristic function on each of the open and closed subsets $V(H)$ for $(H) \in \Phi G$ are in the rational image of ϕ. Since $W_G H$ acts freely from the right on $(G/H)^K$ the values of $\phi([G/H])(K) = |G/H^K|$ are multiples of $|W_G H|$.

Let $V(H, n)$ be $\{(K) \mid |G/H^K| \geq n|W_G H|\}$. Then $V(H, 1)$ equals $V(H)$. Define subsets

$$U(H, n) = \{(K) \mid |G/H^K| = n|W_G H|\} = V(H, n) - V(H, n+1).$$

Since $\phi([G/H])$ is a continuous function on a compact set, only finitely many of the $U(H, n)$ are nonempty. By considering linear combinations of different powers of $\phi([G/H])$ the characteristic function on $U(H, n)$ is in the rational image of ϕ, for each n. Hence so is the characteristic function on $V(H) = \cup_{n \geq 1} U(H, n)$. ∎

10. An alternative additive basis for $\mathbf{C(G)}$

Assume that H has a finite Weyl group. Since W_GH acts freely on $(G/H)^K$ it follows that $\phi_K([G/H])$ is divisible by $|W_GH|$ for all (K). Hence the function

$$a_H = \frac{1}{|W_GH|}\phi([G/H])$$

is in $C(G)$. It is not possible to divide any further since $a_H(H) = 1$.

Proposition 10.1. *The elements a_H, for $(H) \in \Phi G$, are linearly independent and generates $C(G)$ as an abelian group.*

Proof. The elements a_H, for $(H) \in \Phi G$, are linearly independent in $C(G)$ by the proof of Lemma 8.3.

Any element f in $C(G)$ is in the rational image of ϕ by Proposition 9.2. It suffices to show that if

$$f = \sum_{i=1}^{n} q_i\, a_{H_i}$$

is in $C(G)$, where $q_i \in \mathbb{Q}$, then $q_i \in \mathbb{Z}$, for all i. Assume that (H_k) is maximal among the (H_i) with $q_i \notin \mathbb{Z}$. Then $f(H_k) = q_k + \sum_j q_j a_{H_j}(H_k)$ where the sum is over subgroups H_j that properly contains a conjugate of H_k. This gives a contradiction, so all q_i are integers. ■

11. Congruence relations describing the image $\phi(\mathbf{A(G)})$ in $\mathbf{C(G)}$

It is possible to describe the image $\phi(A(G))$ in $C(G)$ by a set of congruence relations. There is one congruence relation for each element in ΦG.

Lemma 11.1. *Let G be a finite group. Then*

$$\sum_{g\in G} |X^g| \equiv 0 \ mod\ |G|$$

for all finite G-sets X.

Proof. Note that $\Sigma_{g\in G} |(G/H)^g|$ equals $\Sigma_{kH} |kHk^{-1}| = |G|$ by rearranging the summation. This implies that $\Sigma_{g\in G} |X^g| = |G||X/G|$. ■

Proposition 11.2. *Let G be a compact Lie group. An element $f \in C(G)$ is in the image of ϕ: $A(G) \to C(G)$ if and only if, for each $(H) \in \Phi G$, it satisfies the following congruence relation*

$$\sum_C n_{C/H} f(C) \equiv 0 \ mod\ |W_GH|$$

where the sum is over all C such that $H \lhd C$ and C/H is a cyclic group; $n_{C/H}$ is the number of generators of the cyclic subgroup C/H.

Proof. For each $(H) \in \Phi G$ apply Lemma 11.1 to $N_G H/H$ acting on the finite $N_G H/H$-sets $(G/L)^H$ for any subgroup L of G with finite Weyl group. The congruence relation for H is

$$\sum_C n_{C/H}\phi_C([G/L]) \equiv 0 \bmod |W_G H|$$

for any $[G/L]$ where the sum is over all $H \lhd C$ such that C/H is a cyclic group and $n_{C/H}$ is the number of different generators of the cyclic subgroup C/H. Hence any element in the image of the mark homomorphism satisfies these relations.

Any element in $C(G)$ can be written as a sum $\Sigma_K m_K a_K$ where m_K are integers and all but finitely many of them are zero. Assume the element $\Sigma_K m_K a_K$ satisfies all the congruence relations. If the integer m_K is divisible by $|W_G K|$ for all K, then $\Sigma_K m_K a_K$ is in the image of ϕ since $m_K a_K = \frac{m_K}{|W_G K|}\phi([G/K])$. Assume that this is not the case and let (H) be maximal for which m_H is not divisible by $|W_G H|$. Since $m_K a_K$ satisfies all the congruence relations for all K properly containing a conjugate of H, the congruence relation for (H) gives that $m_H \equiv 0 \bmod |W_G H|$. This is a contradiction. Hence $|W_G K|$ must divide m_K for all K. ∎

It is a theorem of tom Dieck that there is a greatest common divisor of $|W_G H|$ for all subgroups H of G [8]. Let $|G|$ denote the greatest common divisor of $|W_G H|$ for all subgroups H of G with finite Weyl group. If G is a finite group, then $|G|$ equals the number of elements in G. The congruence relations gives the following.

Corollary 11.3. *There is an inclusion $|G|C(G) \subset \phi(A(G))$.*

For certain purposes the number $|G|$ serves as the order of the compact Lie group. A generalization of the Artin induction theorem to compact Lie groups make use of a whole family of different orders of compact Lie groups [19]. These orders all reduce to the number of elements in G when G is a finite group.

12. The prime ideal spectrum of A(G)

The ring map $\phi\colon A(G) \to C(G)$ is an integral extension since $C(G)$ is additively generated by idempotent elements. Hence by the going up theorem in commutative algebra the map

$$\operatorname{spec}\phi\colon \operatorname{spec} C(G) \to \operatorname{spec} A(G)$$

is surjective. That is, all the prime ideals of $A(G)$ are of the form $\phi^{-1}(P)$ for some prime ideal P in $C(G)$.

The prime ideals in $C(G)$ are all obtained by applying the following standard lemma to $X = \Phi G$ and $R = \mathbb{Z}$.

Lemma 12.1. *Let X be a totally disconnected compact Hausdorff space, and let R be a ring. Then there is an homeomorphism*

$$q\colon X \times \mathit{spec}\, R \to \mathit{spec}\, C(X,R)$$

given by sending (x,P) to the prime ideal

$$\{f \in C(X,R) \mid f(x) \in P\}.$$

Proof. (Outline of a proof.) Let c be an ideal in $C(X, R)$. Let

$$I(c, x) = \{f(x) \mid f \in c\}$$

be the ideal of all the values of the functions in c at x. If J is a prime ideal, then there is a unique x_J such that $1 \notin I(c, x_J)$. The map sending J to $(x_J, I(c, x_J))$ is an inverse to q. The two maps are continuous. ∎

The prime ideals of $A(G)$ are all of the form

$$q(H, 0) = \{x \in A(G) \mid \phi_H(x) = 0\}$$
$$q(H, p) = \{x \in A(G) \mid \phi_H(x) \in p\mathbb{Z}\}$$

for p any prime number and (H) in ΦG. This follows by pulling back the prime ideals of $C(G)$ along ϕ.

Lemma 12.2. *The Burnside ring $A(G)$ has krull dimension 1 for all compact Lie groups G. The maximal ideals of $A(G)$ are $q(H, p)$ for $(H) \in \Phi G$ and primes p. The minimal prime ideals of $A(G)$ are $q(H, 0)$ for $(H) \in \Phi G$. There is an inclusion $q(H, 0) \subset q(K, p)$ if and only if $q(H, p) = q(K, p)$.*

Proof. The quotients of $A(G)$ by its ideals are $A(G)/q(H, p) \cong \mathbb{Z}/p$ and $A(G)/q(H, 0) \cong \mathbb{Z}$. The isomorphisms are induced by ϕ_H. The ideal $q(H, p)$ is maximal since $A(G)/q(H, p)$ is a field. There are no proper containments among the prime ideas of the form $q(H, 0)$ because a quotient of $\mathbb{Z}$ by a nonzero ideal is finite. Hence $q(H, 0)$ is a minimal prime ideal.

There is an inclusion $q(H, 0) \subset q(H, p)$ for $(H) \in \Phi G$ and p a prime number. Hence the equality $q(H, p) = q(K, p)$ implies that $q(H, 0) \subset q(K, p)$. Conversely, assume that $q(H, p)$ and $q(K, p)$ are not equal. Since they are maximal ideals, there is then an element $x \in A(G)$ such that $x \in q(H, p)$ and $x \notin q(K, p)$. Define the element $y = x - \phi_H(x) \cdot 1$. Then $y \in q(H, 0)$ and $y \notin q(K, p)$. So $q(H, 0)$ is not a subset of $q(K, p)$. ∎

If $K \triangleleft H$ is a normal subgroup with a p-group quotient, then $q(K, p) = q(H, p)$. (This follows since for any finite $\mathbb{Z}/p$-set X, the difference $|X| - |X^{\mathbb{Z}/p}|$ is divisible by p.) Given a closed subgroup H of G. Let H'_p be the minimal normal subgroup of H such that H/H'_p is a finite p-group. The group H'_p might not have finite Weyl group as a subgroup of G. The conjugacy class $(H_p) = \omega(H'_p)$ is called the p-perfection of H in G (See section 8).

Lemma 12.3. *Choose a representative H_p for the conjugacy class (H_p). There is an extension H^p of H_p such that H^p/H_p is a finite p-group and $|W_G H^p|$ is relative prime to p. The conjugacy class (H^p) only depends on (H). There are identities*

$$((H_p)^p) = (H^p) \text{ and } ((H^p)_p) = (H_p),$$

and

$$q(H, p) = q(H^p, p) = q(H_p, p),$$

for $(H) \in \Phi G$ and prime numbers p.

Proof. Let H^p be the preimage of a Sylow p-subgroup of $N_G H_p/H_p$ in $N_G H_p$ (unique up to conjugation). Assume the order of the Weyl group $W_G H^p$ is not relative prime to p. Then there are two proper normal extensions

$$H_p \lhd H^p \lhd K$$

both of whose quotients are p-groups. Then H_p must be normal in K since p-groups are solvable. This gives a contradiction since H^p/H_p was a maximal p-group in $N_H H_p/H_p$. The equality $(H_p)^p = (H^p)$ follows by the definition of H^p. There is an inclusion $(H^p)'_p \leq H_p$. The normal subgroup $(H^p)'_p \cap H'_p$ of H must equal H'_p since

$$H'_p/(H^p)'_p \cap H'_p \cong H_p/(H^p)'_p$$

is a p-group. Hence $(H^p)'_p$ equals H_p, and so $(H^p)_p = H_p$. ■

Example 12.4. Consider the nontrivial semidirect product $S^1 \rtimes \mathbb{Z}/2$, and $p = 2$. The subgroup $H = 0 \rtimes \mathbb{Z}/2$ has finite Weyl group. In this case $H_2 = S^1 \rtimes 0$ and $H^2 = S^1 \rtimes \mathbb{Z}/2$.

Lemma 12.5. *There is an equality $q(H, 0) = q(K, 0)$ if and only if $(H) = (K)$. There is an equality $q(H, p) = q(K, q)$ if and only if $p = q$ and $(H^p) = (K^p)$.*

Proof. Note that $\phi([G/H])(H) = |W_G H|$ and $\phi([G/H])(K) = 0$, whenever K is not subconjugated to H. A closed subgroup of G can not be properly subconjugated to itself. Hence if $(H) \neq (K)$ in ΦG, then $q(H, 0) \neq q(K, 0)$.

By Lemma 12.3 there are identifications $q(H, p) = q(H^p, p)$ and $q(K, p) = q(K^p, p)$. Since $A(G)/q(H, p) \cong \mathbb{Z}/p$, the equality $q(H, p) = q(K, q)$ implies that $p = q$. Both $(G/H^p)^{K^p}$ and $(G/K^p)^{H^p}$ are nonempty since the Weyl groups of H^p and K^p are not divisible by p. Hence (H^p) and (K^p) must be equal. ■

Lemma 12.6. *Let $H \leq J \leq K$ be subgroups of G. If $\omega(H)$ is equal to $\omega(K)$, then $\omega(H) = \omega(J) = \omega(K)$.*

Proof. The conjugacy class $\omega(H)$ is the conjugacy class of HT_H where T_H is a maximal torus in $C_G H$. Similarly for J and K. Since $C_G(K) \leq C_G(J) \leq C_G H$ the maximal tori can be chosen such that $T_K \leq T_J \leq T_H$. The torus T_H is a maximal torus in $C_G(HT_H)$ and T_K is a maximal torus in $C_G(HT_K)$. The assumption that HT_H and KT_K are conjugate subgroups in G gives that that T_H is subconjugated to T_K, hence $T_H = T_K$. Since $T_K \leq T_J \leq T_H$ this implies that $T_H = T_J = T_K$. The result now follows since $HT_H = JT_J = KT_K$. ■

The next result was first proved by Bauer and May [1].

Proposition 12.7. *Let $H \leq J \leq K$ be subgroups of G. If $q(H, p)$ is equal to $q(K, p)$, then $q(H, p) = q(J, p) = q(K, p)$.*

Proof. Assume that H is a subgroup of J. Let J_p' be the smallest normal subgroup of J such that J/J_p' is a p-group. Then $J_p' \cap H$ is a normal subgroup of H such that $H/J_p' \cap H \leq J/J_p'$ is a p-group. Hence $H_p' \leq J_p' \leq K_p'$ and Lemma 12.6 gives the result. ■

The space ΦG has only finitely many elements if and only if the Weyl group of a maximal torus T of G acts trivially on T [10, 5.10.8]. Hence $A(G)$ is a Noetherian ring if and only if the Weyl group of a maximal torus T of G acts trivially on T.

13. Idempotent and unit elements in A(G)

The idempotent elements in $A(G)$, and the idempotent elements in $A(G)$ with a set of primes inverted, have been completely described [9] [16] [21] [36]. For example, when G is a finite group, then $A(G)$ has no idempotent elements different from 0 and 1 if and only if G is a solvable group [10, 5.11.4]. This fact was emphasized in [16].

Let π be a collection of prime numbers. A group H is π-perfect if the group of components of H does not have a nontrivial solvable quotient π-group. Let $\Phi_\pi(G)$ be the subspace of ΦG consisting of conjugacy classes of π-perfect subgroups of G with finite Weyl group.

Let X be a topological space, and let $\Pi_0(X)$ denote the space of components (with the quotient topology from X). Let R be a ring, and let $R_{(\pi)}$ denote the localization of R obtained by inverting all primes not in the set π. There is a map $\beta\colon \Phi_\pi(G) \to \Pi_0(\operatorname{spec} A(G)_{(\pi)})$ defined by sending (H) to the component of the prime ideal $q(H,0)$. The following was proved in [21, Proposition 3.3].

Proposition 13.1. *The map*

$$\beta\colon \Phi_\pi(G) \to \Pi_0(\mathit{spec}\, A(G)_{(\pi)})$$

is a homeomorphism.

There is a close connection between idempotent elements in $A(G)$ and unit elements in $A(G)$. The following is a consequence of the embedding of $A(G)$ into $C(G)$. If e is an idempotent element in $A(G)$, then $2e-1$ is a unit element in $A(G)$. If u is a unit element in $A(G)$, then $\frac{\phi(u)+1}{2}$ is an idempotent element in $C(G)$. If G is a compact Lie group and $|G|$ is odd, then $\frac{u+1}{2}$ satisfies the congruence relations of Proposition 11.2, because both u and 1 satisfy the relations and $|W_G H|$ is not divisible by 2 for any $(H) \in \Phi G$. Proposition 11.2 gives that $\frac{u+1}{2}$ is in $A(G)$. Hence there is a bijection between idempotent elements and unit elements in $A(G)$ when $|G|$ is odd.

There is a homomorphism $R(G;\mathbb{R}) \to A(G)^\times$ given by sending a real representation V to the function $(-1)^{\dim V^H}$ [10, 5.5.9]. Tornehave has proved that this map is surjective when G is a finite 2-group [35].

14. A map from the Burnside ring to the representation ring

Let k be a field.

There is a canonical map

$$A(G) \to R(G;k)$$

from the Burnside ring of G to the representation ring of G with coefficients in k. The map is given by sending G/H to the alternating sum of the G-representations $H^i(G/H;k)$ [7]. Here $H^i(G/H;k)$ is the i-th singular cohomology of G/H with coefficients in k endowed with a left G-action induced by the left G-action on G/H. The map $A(G) \to R(G;k)$ is neither injective nor surjective in general. However if P is a finite p-group, then $A(P) \to R(P;\mathbb{Q})$ is surjective [31].

The composition $A(G) \to R(G;\mathbb{R}) \to A(G)^\times$ is called the exponential map of the Burnside ring. This map is surjective if G is a 2-group with no subquotients isomorphic to the dihedral group of order 16 [18].

15. Modules over $A(G)$

Modules over $A(G)$ have been studied by tom Dieck and Petrie [14]. Much attention has been given to invertible modules over $A(G)$ [11] [12] [15]. These are closely related to stable homotopy representations. A homotopy representation is a retract of a finite G-CW complex X such that X^H is homotopy equivalent to $S^{n(H)}$ for some integer $n(H)$, for each subgroup H in G. A stable homotopy representation is a suspension spectrum of a homotopy representation. Stable homotopy representation are exactly the invertible objects in the stable equivariant homotopy category [12] [20].

The finite groups G such that there are only a finite number of finitely generated indecomposable $A(G)$-modules (which are free over the integers) are characterized by Reichenbach in [29].

16. The Burnside ring in equivariant stable homotopy theory

Let X be a finite G-CW-complex. The (categorial) Euler characteristic of $\Sigma^\infty_G X$ turns out to be the stable homotopy class

$$\Sigma^\infty_G S^0 \xrightarrow{\tau} \Sigma^\infty_G X_+ \xrightarrow{c} \Sigma^\infty_G S^0$$

where the first map is the transfer map and the last map is the collapse map that sends X to a point [24, Chapter XVII]. This defines a homomorphism

$$\chi\colon A(G) \to \pi_0^G(\Sigma^\infty_G S^0) = \pi_0^G(S^0).$$

There is a degree homomorphism

$$d\colon \pi_0^G(S^0) \to C(G)$$

which sends a stable self map $h\colon \Sigma^\infty_G S^0 \to \Sigma^\infty_G S^0$ to the continuous function that map (K) to the degree of (the geometric fixed points) h^K as a stable self map of $\Sigma^\infty S^0$. The composite $d \circ \chi$ equals ϕ.

Theorem 16.1. *The Euler characteristic map*

$$\chi\colon A(G) \to \pi_0^G(S^0)$$

is an isomorphism.

The injectivity of χ follows from the injectivity of ϕ. The surjectivity of χ requires a more careful understanding of $\pi_0^G(S^0)$. It suffices to show that $\pi_0^G(S^0)$ is additively generated by maps of the form $\Sigma_G^\infty S^0 \xrightarrow{\eta} \Sigma_G^\infty X_+ \xrightarrow{c} \Sigma_G^\infty S^0$. This follows from the spectrum level Segal–tom Dieck splitting theorem [23, IV.9.3].

Another proof consists of using equivariant obstruction theory to show that d is injective and to show that every function in the image of d satisfies the congruence relations of Proposition 11.2. Since d is injective the map χ must then be surjective. The details are given in [10, Chapter 8].

A variation of Theorem 16.1 is the Segal conjecture, proved by Carlsson [5]. Let G be a finite group and let EG be a free contractible G-space. The stable G-homotopy classes of maps from $\Sigma_G^\infty EG_+$ to $\Sigma_G^\infty S^0$ is isomorphic to the completion of the Burnside ring at its augmentation ideal. This is a deep result of importance in homotopy theory.

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