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Generalized Gelfand Pairs Associated to Heisenberg Type Groups

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Abstract. Let $\mathfrak{N} = \mathfrak{Z} \oplus V$ be the Lie algebra corresponding to a group of Heisenberg type N. Assume that V is an irreducible Clifford module. In this article we determine the generalized Gelfand pairs (K, N), where K is the group of automorphisms of $\mathfrak{N}$ that preserve V.
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1. Introduction

Let $\mathfrak{N}$ be a real two-step nilpotent Lie algebra, endowed with an inner product $\langle \, , \rangle$. Let $\mathfrak{Z}$ denote the centre of $\mathfrak{N}$ and let V be its orthogonal complement.

For $z \in \mathfrak{Z}$, define the linear map $J_z : V \to V$ by

$$\langle J_z v, w\rangle = \langle z, [v, w]\rangle$$

for all $v, w \in V$, where $[\, ,]$ denotes the bracket in $\mathfrak{N}$. We say that $\mathfrak{N}$ is a Lie algebra of Heisenberg type (or of type H) if, for all $z \in \mathfrak{Z}$ with $|z| = 1$, J_z is an orthogonal transformation on V (cf. [6]).

A connected and simply connected Lie group N is of Heisenberg type if its Lie algebra is of type H. Since for $|z| = 1$, J_z is both orthogonal and skew-symmetric,

$$J_z^2 = -Id\,.$$

So by linearity and polarization we have for $z, w \in \mathfrak{Z}$

$$J_z J_w + J_w J_z = -2\langle z, w\rangle Id\,. \tag{1}$$

Let $m := \dim \mathfrak{Z}$ and let $C(m)$ be the Clifford algebra $C(\mathfrak{Z}, -|\;|^2)$. Then by (1) the action J of $\mathfrak{Z}$ on V extends to a representation of $C(m)$. The notions *irreducible* and *isotypic*, when attributed to V, refer to its Clifford module structure.

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Let us denote by $\mathrm{Aut}(\mathfrak{N})$ the automorphisms group of $\mathfrak{N}$. The subgroup $\mathrm{Aut}_V(\mathfrak{N})$ of automorphims that preserve V is essentially the semidirect product of two subgroups U and $\mathrm{Clif}(m)$, which we describe as follows:

$$U = \{g \in \mathrm{Aut}_V(\mathfrak{N}) : g_{|\mathfrak{z}} = Id\}\,.$$

Warning: $U \cap \mathrm{O}(\mathfrak{N})$ corresponds to the subgroup denoted by U in [8] and [11].

For a unit vector $z \in \mathfrak{z}$, let $\rho_z : \mathfrak{z} \to \mathfrak{z}$ denote the reflection through the hyperplane orthogonal to z and let $\mathrm{Pin}(m)$ be the subgroup of $\mathrm{Aut}_V(\mathfrak{N})$ generated by $\{(-\rho_z, J_z) : z \in \mathfrak{z}, |z| = 1\}$. We also denote by $\mathrm{Spin}(m)$ the subgroup generated by the even products $(\rho_z\rho_w, J_zJ_w)$, with $|z| = |w| = 1$, and finally by $\mathrm{Clif}(m)$ the subgroup generated by $\{(-|z|^2\rho_z, J_z),\ z \neq 0\}$. It can be proved that U is a classical group that commutes with $\mathrm{Spin}(m)$. Moreover $U \cap \mathrm{Clif}(m)$ has at most four elements and $\mathrm{Aut}_V(\mathfrak{N}) = U\mathrm{Clif}(m)$ or $[\mathrm{Aut}_V(\mathfrak{N}) : U\mathrm{Clif}(m)] = 2$, depending on the congruence of $m \equiv (8)$. A precise result is given in [12]. Given a compact subgroup K of a Lie group G, we recall that the pair (G, K) is called *a Gelfand pair* if for each irreducible, unitary representation π of G, the space of K-fixed vectors is at most one dimensional.

For a compact subgroup $K \subseteq \mathrm{Aut}(\mathfrak{N})$ we consider the semidirect product $G = K\mathrm{N}$. One says that (K, N) is a Gelfand pair if $(K\mathrm{N}, K)$ is a Gelfand pair. Equivalently if the convolution algebra $L^1_K(\mathrm{N})$ of K-invariant integrable functions on N is commutative.

Let $A(\mathrm{N})$ be the group of orthogonal automorphisms of N. In [11], there is a classification of the groups N for which $(A(\mathrm{N}), \mathrm{N})$ is a Gelfand pair. Also in [8] it was raised the question of when (K, N) is a Gelfand pair, for some specific subgroups K of $A(\mathrm{N})$.

The notion of Gelfand pair was extended to non compact, unimodular subgroups K of a unimodular Lie group: the pair (K, N) is a *generalized Gelfand pair* if for each irreducible, unitary representation π of $K\mathrm{N}$, the space of K-fixed distribution vectors is at most one dimensional. For surveys, see [14] and [16]. In [9] the authors considered the cases (K, H_n) where H_n is the Heisenberg group $2n+1$ dimensional and K is a subgroup of $\mathrm{U}(p,q) \subset \mathrm{Aut}(H_n)$, $p + q = n$.

Our aim here is to determine the generalized Gelfand pairs (K, N) where N is an *irreducible* group of Heisenberg type and $K = \mathrm{Spin}(m) \times U$. For a list of such groups U see the beginning of section 3.

The results obtained here jointly with those in [11] and in [3] allow us to state the following:

Theorem *Let* N *be an irreducible group of Heisenberg type. Then* (K, N) *is a generalized Gelfand pair if and only if* $1 \leq m \leq 9$.

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2. Preliminary results

We recall some definitions and results which will be used in the following. The irreducible, unitary, representations of a group of Heisenberg type N are described in [8]. They are:

* Infinite -dimensional representations, parametrized by the non zero elements of the centre $\mathfrak{z}$: for $0 \neq a \in \mathfrak{z}$, $|a| = 1$, the corresponding representation π_a is realized on the Fock space $\mathcal{F}_a$ of entire functions on (V, J_a).

* Unitary characters, $\chi_v(z, w) = e^{i\langle w,v\rangle}$, defined for each $v \in V$.

For $K \subset \mathrm{Aut(N)}$ we now recall the construction of unitary, irreducible representations of $K\mathrm{N}$, according to Mackey's theory [10]. For each $\pi \in \widehat{\mathrm{N}}$ and $k \in K$, let π_k be the representation

$$\pi_k(n) = \pi(kn)$$

and let K_π be the stabilizer of π, that is, $K_\pi = \{k \in K : \pi_k \simeq \pi\}$. For $k \in K_\pi$, we can choose an intertwining operator $\omega_\pi(k)$ of π, in such a way that the map $k \to \omega_\pi(k)$ is a *projective representation* of K_π, that is,

$$\omega_\pi(k_1k_2) = \sigma(k_1, k_2)\omega_\pi(k_1)\omega_\pi(k_2)\,.$$

ω_π is called the *intertwining representation of* π and σ the *multiplier* for the projective representation ω_π.

Denote by $\widehat{K_\pi^\sigma}$ the set of (equivalence classes) irreducible, unitary projective representations of K_π with multiplier σ.

If $\rho \in \widehat{K_\pi^\sigma}$ then

$$\Theta(k, n) = \rho(k) \otimes \omega_\pi(k)\pi(n)$$

is an irreducible representation of $K_\pi\mathrm{N}$. Moreover the induced representation $\mathrm{Ind}_{K_\pi N}^{KN}(\Theta)$ is an irreducible representation for $K\mathrm{N}$ and, by considering all $\pi \in \widehat{\mathrm{N}}$ and $\rho \in \widehat{K_\pi^\sigma}$, one obtains all equivalence classes of irreducible representations of $K\mathrm{N}$.

¿From now on we will consider the cases $K = \mathrm{Spin}(m)U$. As it is stated in Prop 3.1 in [9], the representations of $K\mathrm{N}$ coming from characters of N give rise to irreducible unitary representations of KV. Since V is an abelian group, (K, V) is a generalized Gelfand pair and so the space of distribution vectors fixed by K is at most one dimensional. Then, in order to determine when (K, N) is a generalized Gelfand pair, it is enough to consider only those representations of $K\mathrm{N}$ associated to π_a, for $a \in \mathfrak{z}$.

For $a \in \mathfrak{z}$, let ω_a be the *intertwining representation* of $\mathcal{F}_a$, also called metaplectic representation. Let $\mathrm{Spin}_a(m)$ denote the group generated by the operators in the set $\{J_bJ_c : b \perp a \perp c, |b| = |c| = 1\}$.

We observe that

$$K_a := K_{\pi_a} = \mathrm{Spin}_a(m)U\,.$$

Since the elements of $\mathrm{Spin}_a(m)$ are orthogonal transformations which commute with J_a, $K_a \subset \mathrm{Sp}(V, J_a) = \{g \in \mathrm{GL}(V) : g^tJ_ag = J_a\}$. Let N_a be the Heisenberg group with Lie algebra $\mathfrak{z}_a = \mathbb{R}a \oplus V$. Then $\mathrm{Sp}(V, J_a)$ is the group of automorphims of N_a, which fix a.

Given a representation (ρ, V_ρ) of a subgroup H of K, let $C(K; V_\rho)$ denote the space of continuous functions $f : K \to V_\rho$ such that $f(kh) = \rho(h^{-1})f(k)$ for all $k \in K, h \in H$, and $\int_{K/H} |f(x)|^2\,dx < \infty$. Then $\mathrm{Ind}_H^K(V_\rho)$ is the completion of $C(K;V_\rho)$. Moreover, a C^∞-vector of $\mathrm{Ind}_H^K(V_\rho)$ is an infinitely differentiable function $f \in C(K;V_\rho)$ (see[17], page 373). We denote by V_ρ^∞ (resp $V_\rho^{-\infty}$) the space of C^∞-vectors (resp. distribution vectors) and recall that $V_\rho^{-\infty}$ is the antidual space of V_ρ^∞.

Lemma 2.1. *Let f be a C^∞-vector of $\mathrm{Ind}_{K_a}^K(V_\rho)$ such that*

$$\int_{\mathrm{Spin}(m)} f(k)\,dk = 0\,.$$

Let μ be a $\mathrm{Spin}(m)-$invariant distribution vector of $\mathrm{Ind}_{K_a}^K(V_\rho)$. Then $\langle \mu, f\rangle = 0$.

Proof. Indeed,

$$\begin{aligned}
\langle \mu, f\rangle &= \int_{\mathrm{Spin}(m)} \langle \mu, f\rangle\,dk && \text{since } dk \text{ on } \mathrm{Spin}(m) \text{ is normalized.}\\
&= \int_{\mathrm{Spin}(m)} \langle \mu, L_k f\rangle\,dk && \text{by left invariance of } \mu.\\
&= \langle \mu, \int_{\mathrm{Spin}(m)} L_k f\,dk\rangle.
\end{aligned}$$

But for $x = hu \in \mathrm{Spin}(m)U$ we have that

$$\int_{\mathrm{Spin}(m)} L_k f(x) dk = \int_{\mathrm{Spin}(m)} f(kx)\,dk = \rho(u^{-1}) \int_{\mathrm{Spin}(m)} f(kh)\,dk = 0$$

since dk is right invariant. ∎

Theorem 2.2. (K,N) *is a generalized Gelfand pair if and only if* (K_a, N_a) *is a generalized Gelfand pair for each* $a \in \mathfrak{z}$ *(cf. Lemma 2 and Lemma 3 stated in [11]).*

Proof. $\Rightarrow$) For $\lambda \neq 0$, let us denote by $(\mathcal{F}_\lambda, \pi_\lambda)$ the Fock representation of N_a determined by $\pi_\lambda(\exp ta) = e^{i\lambda t}$, and by ω_λ the metaplectic representation of $\mathrm{Sp}(V, J_a)$. Let (ρ, V_ρ) be an irreducible representation of $K_a N_a$ and assume, by contradiction, that T_1, T_2 are distribution vectors of V_ρ, fixed by K_a and linearly independent. Then there exists some $\lambda \neq 0$ such that $\rho = \overline{\gamma} \otimes \pi_\lambda \omega_\lambda$, $\gamma \in \widehat{K_a^{\sigma_\lambda}}$. It is immediate that ρ is irreducible as $K_a\mathrm{N}$-module. We know that $(\pi, H_\pi) := \mathrm{Ind}_{K_a N}^{KN}(V_\rho)$ is an irreducible representation of $K\mathrm{N}$ and that as $K-$module, H_π is the representation induced by the K_a-module $\overline{\gamma} \otimes \omega_\lambda$.

We define $\mu_1, \mu_2 : H_\pi^\infty \to C$ by

$$\langle \mu_j, f\rangle := \langle T_j, \int_{\mathrm{Spin}(m)} f\rangle\,.$$

There exists a surjective morphism $C_c^\infty(K, V_\rho) \to C^\infty(K; V_\rho)$ defined by

$$f \to f_\rho(h) = \int_{K_a} \rho(\xi) f(h\xi)\,d\xi\,.$$

Let T be a non zero distribution vector fixed by K_a and $v \in V_\rho$ such that $\langle T, v\rangle \neq 0$. We choose $\varphi \in C^\infty(S^{m-1})$ and $\psi \in C_c^\infty(U)$ with $\int_{S^{m-1}} \varphi \neq 0 \neq \int_U \psi$ and set $f(ku) = \varphi(k\mathrm{Spin}_a)\psi(u)v$, for $k \in \mathrm{Spin}(m), u \in U$. Then, since $\mathrm{Spin}(m)$ commutes with U,

$$\begin{aligned} f_\rho(ku) &= \int_{K_a} \rho(hu')f(kuhu')\,dhdu' \\ &= \int_{K_a} \varphi(kh\mathrm{Spin}_a)\psi(uu')\rho(hu')v\,dhdu' \\ &= \varphi(k\mathrm{Spin}_a)\int_{\mathrm{Spin}_a\times U} \psi(uu')\rho(hu')v\,dhdu'\,. \end{aligned}$$

Hence

$$f_\rho(ku) = \varphi(k\mathrm{Spin}_a)\chi(u)$$

where $\chi(u) = \int_{\mathrm{Spin}_a\times U} \psi(uu')\rho(hu')v\,dhdu'$.

We observe that since T is a distribution vector fixed by $\rho(K_a)$

$$\langle T, \chi(u)\rangle = \langle T, \chi(e)\rangle \text{ for } u \in U\,.$$

Indeed

$$\begin{aligned} \langle T, \int_{\mathrm{Spin}_a\times U} \psi(uu')\rho(hu')v\,dhdu'\rangle &= \int_{\mathrm{Spin}_a\times U} \psi(uu')\langle T, \rho(hu')v\rangle\,dhdu' \\ &= \langle T, v\rangle(\int_U \psi(uu')\,du') \\ &= \langle T, \chi(e)\rangle\,. \end{aligned}$$

Since v was chosen so that $\langle T, v\rangle \neq 0$, we obtain that $\langle T, \chi(e)\rangle \neq 0$ and so $f_\rho(e) \neq 0$. Thus $f_\rho \neq 0$.

Moreover, since

$$\int_{\mathrm{Spin}(m)} f_\rho(g)\,dg = \int_{\mathrm{Spin}(m)} \varphi(g\mathrm{Spin}_a)\,dg\chi(e) \tag{2}$$

we obtain that

$$\langle T, \int_{\mathrm{Spin}(m)} f_\rho\rangle \neq 0\,. \tag{3}$$

So μ defined by $\langle \mu, f\rangle := \langle T, \int_{\mathrm{Spin}(m)} f\rangle$ is a non zero vector distribution of H_π.

Let us see that μ is $\pi(K)-$invariant. We recall that the action of π on H_π is by left translations. For $u \in U$, $\langle \mu, L_u f\rangle = \langle T, \int_{\mathrm{Spin}(m)} L_u f\rangle$. It follows that $\int_{\mathrm{Spin}(m)} L_u f\,dk = \int f(uk)dk = \int f(ku)\,dk = \rho(u^{-1})\int_{\mathrm{Spin}(m)} f(k)\,dk$, since $\mathrm{Spin}(m)$ commutes with U. So by the U-invariance of T we have $\langle T, \int_{\mathrm{Spin}(m)} L_u f\rangle = \langle \rho_{-\infty}(u)T, \int f\rangle = \langle T, \int f\rangle$. Finally if $h \in \mathrm{Spin}(m)$, $\langle \mu, L_h f\rangle = \langle T, \int_{\mathrm{Spin}(m)} L_h f\rangle = \langle T, \int_{\mathrm{Spin}(m)} f\rangle$ by the left invariance of the integral.

Recall that by assumption there exist T_1 and T_2 distribution vectors of V_ρ fixed by K_a and linearly independent. Replacing T above by T_j and choosing $v_j \neq 0$ such that $\langle T_j, v_j\rangle \neq 0$, the above argument shows that there exist two non zero distribution vectors μ_1 and μ_2, fixed by K. They are linearly independent:

indeed, if $a\mu_1 + b\mu_2 = 0$ then $0 = \langle a\mu_1 + b\mu_2, f\rangle = \langle aT_1 + bT_2, \int_{\mathrm{Spin}(m)} f\rangle$ for all $f \in C^\infty(K;\rho)$. But 3 implies that $aT_1 + bT_2 = 0$ and so $a = b = 0$. This contradicts the fact that (K, N) is a generalized Gelfand pair.

$\Leftarrow$) Let (π, H_π) be an irreducible representation of $K\mathrm{N}$ and assume that there exist two linearly independent distribution vectors μ_1, μ_2 fixed by K. Rephrasing Prop. 3.1 in [9], we know that this representation can not be induced by a character. So, there exist $(a \neq 0) \in \mathfrak{z}$, $(\lambda \neq 0) \in \mathbb{R}$, $\gamma \in \widehat{K_a^{\sigma_\lambda}}$ and $\rho = \overline{\gamma} \otimes \pi_\lambda\omega_\lambda$ such that

$$H_\pi = \mathrm{Ind}_{K_a\mathrm{N}}^{K\mathrm{N}}(V_\rho) = \mathrm{Ind}_{K_a\mathrm{N}_a}^{K\mathrm{N}}(V_\rho)\,.$$

Define $T_j \in V_\rho^{-\infty}$ by the rule

$$\langle T_j, \int_{\mathrm{Spin}(m)} f\rangle := \langle \mu_j, f\rangle\,.$$

By Lemma 2.1 above, T_j is well defined. Let us see that T_j is defined on a dense subset of V_ρ^∞ : choosing $v, \varphi, \psi,$and f_ρ as in the first part of the proof, 2 implies that

$$\int_{\mathrm{Spin}(m)} f_\rho(g)\, dg = (\int_{\mathrm{Spin}(m)} \varphi(g\mathrm{Spin}_a)\, dg)\chi(e)$$

with $\chi(e) = \int \psi(u)\rho(u)v du = \rho(\psi)v$. By the well known Garding lemma (see for example [13], page 11), the assertion follows. It is easy to see that T_i are K_a-invariant and linearly independent. ■

In what follows it is enough to consider $\lambda = 1$ and we set $\omega = \omega_1$. We recall that the metaplectic representation of $\mathrm{Sp}(V, J_a)$ extends to an ordinary representation for its double covering. If H is a subgroup of $\mathrm{Sp}(V, J_a)$, let $p : H^\sigma \to H$ the canonical twofold covering homomorphism.

The following theorem is just a slight modification of Theorem 2.1 in [9].

Theorem 2.3. *Let $H \subset \mathrm{Sp}(V, J_a)$ be a subgroup such that every unitary irreducible representation of H^σ has a character distribution. Then the pair (H, N_a) is a generalized Gelfand pair if and only if the restriction to H of the metaplectic representation ω is multiplicity free (as a projective representation).*

Proof. $\Leftarrow$) Let us consider an irreducible, unitary representation V_ρ of HN_a, and assume that V_ρ has two linearly independent fixed distribution vectors. For $h \in H^\sigma$ define $\rho(h) = \rho(p(h))$. Since $V_\rho = W \otimes \omega$ with W an $H^\sigma-$irreducible module, Theorem 2.1 in [9] asserts that $\overline{W}$ appears twice in the decomposition of ω.

$\Rightarrow$) Assume that W appears in ω more than once as a projective representation. Again by Theorem 2.1 in [9] $V_\rho = \overline{W} \otimes \omega$ is a representation of H^σ, having more than one fixed distribution vector. Since W has the same cocycle as ω, V_ρ is a representation of H with more than one fixed distribution vector. So (H, N_a) can not be a generalized Gelfand pair. ■

3. The main result

According to the results in the previous section we are interested in determining when the restriction of the metaplectic representation $\omega\downarrow_{K_a}^{\mathrm{Sp}(V,J_a)}$ is multiplicity free.

Let N be an irreducible group of type H, the subgroups U of $\mathrm{Aut}_V(\mathfrak{N})$ corresponding to N are :

$$\begin{aligned}
\mathrm{SL}(2,\mathbb{R}),\ldots\ldots\ldots\ldots\mathbf{m} &\equiv \mathbf{1}\,(\mathrm{mod}\,8)\\
\mathrm{SL}(2,\mathbb{C}),\ldots\ldots\ldots\ldots\mathbf{m} &\equiv \mathbf{2}\,(\mathrm{mod}\,8)\\
\mathrm{U}(1,\mathbb{H}),\ldots\ldots\ldots\ldots\mathbf{m} &\equiv \mathbf{3}\,(\mathrm{mod}\,8)\\
\mathrm{GL}(1,\mathbb{H}),\ldots\ldots\ldots\mathbf{m} &\equiv \mathbf{4}\,(\mathrm{mod}\,8)\\
\mathrm{U}(1),\ldots\ldots\ldots\ldots\ldots\mathbf{m} &\equiv \mathbf{5}\,(\mathrm{mod}\,8)\\
\mathrm{O}(1),\ldots\ldots\ldots\ldots\ldots\mathbf{m} &\equiv \mathbf{6}\,(\mathrm{mod}\,8)\\
\mathrm{O}(1),\ldots\ldots\ldots\ldots\ldots\mathbf{m} &\equiv \mathbf{7}\,(\mathrm{mod}\,8)\\
\mathbb{R}^*,\ldots\ldots\ldots\ldots\ldots\ldots\mathbf{m} &\equiv \mathbf{8}\,(\mathrm{mod}\,8)\,.
\end{aligned}$$

The group U is compact when $\mathbf{m}\equiv\ 3,5,6,7(\mathrm{mod}\,8)$. It is shown in [11] that in these cases , with N irreducible, $(\mathrm{Spin}(m)\times U,\mathrm{N})$ is a Gelfand pair if and only if $m=5,6$, or 7.

We will therefore study $\omega\downarrow_{K_a}^{\mathrm{Sp}(V,J_a)}$, the restriction of the metaplectic representation, for $\mathbf{m}\equiv 1,2,4,8\,(\mathrm{mod}\,8)$.

First we observe that the groups appearing in the list satisfy the conditions for H in the above theorem.

Next we will apply a result due to V. Kac. Let H be a compact, connected subgroup of $\mathrm{U}(l)$ and denote by $H_{\mathbb{C}}$ its complexification. Assume that the action of H on $\mathbb{C}^l$ is irreducible. In [5] it is given the precise list of pairs $(H_{\mathbb{C}},\mathbb{C}^l)$, such that the corresponding action of $H_{\mathbb{C}}$ on the polynomial ring $P(\mathbb{C}^l)$ is multiplicity free, (see also [2]).

Moreover, let us denote by T the one dimensional torus and by $P_r(\mathbb{C}^l), r\in\mathbb{N}$, the space of homogeneous polynomials of degree α with $|\alpha|=r$. Then T acts on $P_r(\mathbb{C}^l)$ by e^{irt}. Thus $H_{\mathbb{C}}$ acts without multiplicity on each $P_r(\mathbb{C}^l), r\in\mathbb{N}$, if and only if the action of $H_{\mathbb{C}}\times\mathbb{C}^*$ on $P(\mathbb{C}^l)$ is multiplicity free.

Remark 3.1. We recall that there are two models for the representations of the Heisenberg group. The Fock model realized on the space of holomorphic functions on $(V,J_a\ )$ which are square integrable with respect to the measure $e^{-|z|^2}dz$ and the Schroedinger model realized on $L^2(R^N)$, $N=\frac{\dim V}{2}$. An intertwining operator sends the monomials $z^\alpha=z_1^{i_1}z_2^{i_2}...z_N^{i_l}$ to $h_\alpha(x)=h_{i_1}(x_1)h_{i_2}(x_2)...h_{i_N}(x_N)$ where $h_i(t)=H_i(t)e^{-\frac{t^2}{2}}$ and $H_i(t)$ is the Hermite polynomial of degree i. We also define $H_\alpha(x):=H_{i_1}(x_1)H_{i_2}(x_2)...H_{i_N}(x_N)$.

Write $V=\mathbb{R}^N\oplus J_a\mathbb{R}^N$ and let $\mathrm{SO}(N)=\mathrm{U}(N)\cap\mathrm{GL}(N,\mathbb{R})$. Then the metaplectic action of $\mathrm{SO}(N)$ on $P_r(V)$ corresponds to an action on the space $H^r=span\{h_\alpha:|\alpha|=r\}$. This action preserves the filtration given by the degree, and induces an action on $P_r(\mathbb{R}^N)=span\{x^\alpha:|\alpha|=r\}$. If $H_\alpha(x)=x^\alpha+$ lower degree terms and $k\in\mathrm{SO}(N)$ then $k.x^\alpha=$ highest degree terms of $(k.H_\alpha)$. In fact we obtain the natural action of $\mathrm{SO}(N)$ on $P_r(\mathbb{R}^N)$.

Remark 3.2. The Mellin transform is the Fourier transform adapted to $\mathbb{R}_{>0}$ and it is defined by $Mf(\lambda) = \int_0^\infty f(s)s^{i\lambda}\frac{ds}{s}$. The action of $\mathbb{R}_{>0}$ on $L^2(\mathbb{R}_{>0}, \frac{ds}{s})$ given by $\delta_t f(s) = f(ts)$ decomposes, via the Mellin transform, as

$$L^2(\mathbb{R}_{>0}, \frac{ds}{s}) \simeq \int_{-\infty}^{\infty} F_\lambda \, d\lambda \,.$$

where F_λ is the $\mathbb{C}$-vector space generated by $s^{i\lambda}$ (see [13], page 168.)

We notice that the module generated by $g_r(s) = s^r e^{-s}$, $r \in \mathbb{N}$, is $L^2(\mathbb{R}_{>0}, s^{-1}ds)$. Indeed, by a well known Wiener theorem, it is enough to prove that $Mg_r(s) \neq 0$ for all s, but this holds since $Mg_r(\lambda) = \int s^r e^{-s} s^{i\lambda}\frac{ds}{s} = \Gamma(r-1+i\lambda) \neq 0$, where Γ denotes the gamma function.

Now we will consider the cases, according to the congruences of $m \bmod 8$.

$\mathbf{m} \equiv \mathbf{4} (\bmod 8)$.

In this case $U = \mathrm{GL}(1,\mathbb{H}) = \mathrm{SU}(2) \times \mathbb{R}_{>0}$ and $V = V^+ \oplus V^-$, where V^+ and V^- are real, inequivalent, irreducible $C^+(m)$-modules, $\dim V^\pm = N$. Also V^+ and V^- are real irreducible equivalent $\mathrm{Spin}_a(m)$-modules. So $\mathrm{Spin}_a(m)$ embeds in $\mathrm{SO}(N)$, via the spin representation. $\mathrm{GL}(1,\mathbb{H})$ is embedded in $\mathrm{Sp}(V, J_a)$ as $q \to a_q = (R_q, R_{\overline{q}^{-1}})$ so that $\mathrm{SU}(2)$ acts by right multiplication by q. So, the metaplectic action of $\mathrm{Spin}_a(m) \times \mathrm{SU}(2)$ on $L^2(\mathbb{R}^N)$ is the natural one of $\mathrm{SO}(N)$ and setting $L^2(\mathbb{R}^N, dx) = L^2(S^{N-1}, d\sigma) \otimes L^2(\mathbb{R}_{>0}, r^{n-1}dr)$, we have that the action of $\mathbb{R}_{>0}$ is given by

$$\omega(a_t)f(x) = t^{\frac{N}{2}} f(tx), \; t \in \mathbb{R}_{>0}, x \in \mathbb{R}^N \,.$$

This last action is equivalent to $\delta_t f(s) = f(ts)$ on $L^2(\mathbb{R}_{>0}, \frac{ds}{s})$. Assume that the action of $\mathrm{Spin}_a(m) \times \mathrm{SU}(2)$ is multiplicity free on each $P_r(V)$ and let V_α be an irreducible representation of $\mathrm{Spin}_a(m) \times \mathrm{SU}(2)$ in $P_r(V)$. For $p \in V_\alpha$, we consider the function $p(x)e^{-\frac{|x|^2}{2}} = p(\frac{x}{|x|})|x|^r e^{-\frac{|x|^2}{2}}$. Then $\mathrm{SO}(N)$ acts on $p(\frac{x}{|x|})$ in the natural way and by Remark 3.2, the action of $\mathbb{R}_{>0}$ on $s^r e^{-s}$ generates a space isomorphic to $L^2(\mathbb{R}_{>0}, \frac{ds}{s})$. We conclude that the K_a- module generated by V_α, which is the isotypical component, is $V_\alpha \otimes L^2(\mathbb{R}_{>0}, s^{n-1}ds)$. So

$$\omega\downarrow_{K_a}^{\mathrm{Sp}(V,J_a)} = \bigoplus_\alpha \int_{-\infty}^{\infty} \alpha \otimes e^{i\lambda t}\, dt$$

and the decomposition is multiplicity free.

On the other hand, if $p(x)$ is a homogeneous polynomial of degree k

$$\omega(a_t)p(x)e^{-|x|^2} = t^{\frac{N}{2}+k}p(x)e^{-|x|^2} \,. \tag{4}$$

and the infinitesimal action is given by

$$\omega(t\frac{d}{dt})(p(x)e^{-|x|^2}) = t(k + \frac{N}{2} - 2|x|^2)p(x)e^{-|x|^2} \,. \tag{5}$$

Let W_1, W_2 two equivalent, irreducible $\mathrm{Spin}_a(m) \times \mathrm{SU}(2)$-modules in some $P_r(V)$, and let H_1, H_2 be the $\mathbb{R}_{>0}$ -modules generated by them. Then the above proof shows that H_1 is equivalent to H_2. If the metaplectic action of K is multiplicity free then $H_1 = H_2$. So $H_1 \cap P_r(V) = H_2 \cap P_r(V)$. But 5 implies that $W_1 = W_2$.

Since $\mathbf{m} \equiv \mathbf{4} \pmod 8$, we have that V is a complex irreducible $\mathrm{Spin}_a(m) \times \mathrm{SU}(2)$-module. By looking at Kac list, we know that the action of $\mathrm{Spin}_a(m) \times \mathrm{SU}(2) \times T$ on $P(V)$ is multiplicity free only for $m = 4$. This case corresponds to the action of $\mathrm{GL}(2,\mathbb{C}) \times \mathrm{SL}(2,\mathbb{C})$ on $\mathbb{C}^2 \otimes \mathbb{C}^2$ and the decomposition of $\omega\downarrow_{K_a}^{\mathrm{Sp}(V,J_a)}$ was given in [3].

$\mathbf{m} \equiv \mathbf{0} \pmod 8$.

In this case $U = \mathbb{R}^*$ and the action is given by

$$\omega(a_t)f(x) = |t|^{\frac{N}{2}} f(tx)\,.$$

We observe that $-I \in \mathrm{Spin}_a(m) \cap U$. Thus the action of K_a on $L^2(\mathbb{R}^N)$ is the same action of $\mathrm{Spin}_a(m) \times \mathbb{R}_{>0}$ and we repeat the argument of the above proof to conclude that $\omega\downarrow_{K_a}^{\mathrm{Sp}(V,J_a)}$ is multiplicity free only for $m = 8$. See also [3].

$\mathbf{m} \equiv \mathbf{1} \pmod 8$

The case $m = 1$ corresponds to the classical Heisenberg group. It is well known that $L^2(V)$ decomposes under the metaplectic action of $U \simeq \mathrm{SL}(2,\mathbb{R})$ as a sum of two unequivalent irreducible components corresponding to the even and odd functions respectively. When $m > 1$, $U \simeq \mathrm{SL}(2,\mathbb{R})$ and $K_a \simeq \mathrm{Spin}_a(m) \times \mathrm{SL}(2,\mathbb{R})$. Also, V can be decomposed as $\mathrm{Spin}(m)$-module as an orthogonal direct sum

$$V = V_\Lambda \oplus J_a V_\Lambda$$

where V_Λ is the real spin representation of $\mathrm{Spin}(m), \dim V_\Lambda = N$. Thus, via the spin representation, $\mathrm{Spin}_a(m)$ is embedded in $\mathrm{SO}(N)$ and as $\mathrm{Spin}_a(m)-$modules $V_\Lambda = V_{\Lambda^+} \oplus V_{\Lambda^-}$ where $V_{\Lambda^+}, V_{\Lambda^-}$ are the half spin representations. Thus we have the embeddings

$$\mathrm{Spin}_a(m) \hookrightarrow \mathrm{SO}(\frac{N}{2}) \times \mathrm{SO}(\frac{N}{2}) \hookrightarrow \mathrm{SO}(N)\,.$$

Besides, $\mathrm{SL}(2,\mathbb{R})$ is embedded in $\mathrm{Sp}(V,J_a)$ as $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \to \begin{pmatrix} aI & -bQ \\ cQ & dI \end{pmatrix}$, where $Q = Q^t, QQ^t = I$ (see Prop 5.2 in [7]). It is well known that (see [15], page 443)

$$\omega\downarrow_{\mathrm{SO}(N)\times\mathrm{SL}(2,\mathbb{R})}^{\mathrm{Sp}(V,J_a)} = \bigoplus_k V_{k\Lambda} \otimes D_{l(k)}\,.$$

where $V_{k\Lambda}$ denotes the irreducible representation of $\mathrm{SO}(N)$ on the harmonic polynomials of degree k on V_Λ, and $D_{l(k)}$ is a discrete series representation of $\mathrm{SL}(2,\mathbb{R})$ and $l(k) = \frac{k}{2} + \frac{N}{4}$ denotes the lowest K-type. Also

$$V_{k\Lambda}\downarrow_{\mathrm{SO}(\frac{N}{2})\times\mathrm{SO}(\frac{N}{2})}^{\mathrm{SO}(N)} = \bigoplus_{r,s} V_{r\Lambda^+} \otimes V_{s\Lambda^-}$$

where the sum runs over the integers r, s such that $k - r - s$ is an even, non negative integer (see [15], page 211).

We consider two possibilities for m:

Case $m > 9$.

We have that as $\mathrm{SO}(\frac{N}{2})$-modules, $P_r(V^+) = V_{r\Lambda^+} \oplus V_{(r-2)\Lambda^+} \oplus V_{(r-4)\Lambda^+} \oplus \ldots$ and $P_r(V^-) = V_{r\Lambda^-} \oplus V_{(r-2)\Lambda^-} \oplus V_{(r-4)\Lambda^-} \oplus \ldots$. As $\mathrm{Spin}_{\mathbb{C}}(m-1) \times \mathbb{C}^*$ does not appear in Kac list, we deduce that there exists r for which the action of $\mathrm{Spin}_a(m)$ on $P_r(V^+)$ can not be multiplicity free. Thus there exists an irreducible representation α that appears in $V_{(r-2i)\Lambda^+}$ and in $V_{(r-2j)\Lambda^+}$, for some $i \neq j$. Then $V_\alpha \otimes V_{r\Lambda^-}$ appears in $V_{(r-2i)\Lambda^+} \otimes V_{r\Lambda^-}$ and in $V_{(r-2j)\Lambda^+} \otimes V_{r\Lambda^-}$ concluding that $V_{k\Lambda}\downarrow^{\mathrm{SO}(\frac{N}{2})\times \mathrm{SO}(\frac{N}{2})}_{\mathrm{Spin}_a(m)}$ is not multiplicity free.

Case $m = 9$.

In this case $\omega\downarrow^{\mathrm{Sp}(V,J_a)}_{K_a}$ is multiplicity free and the proof together with the corresponding decomposition is given in [3].

$\mathbf{m \equiv 2}\,(\mathrm{mod}\,8)$

In this case $U \simeq \mathrm{SL}(2,\mathbb{C})$. When $m = 2$, the metaplectic representation of $\mathrm{SL}(2,\mathbb{C})$ splits as a sum of two inequivalent irreducible $\mathrm{SL}(2,\mathbb{C})$-modules. So we can assume $m \geq 10$. Then $K_a \simeq \mathrm{Spin}_a(m) \times \mathrm{SL}(2,\mathbb{C})$ and V decomposes as $\mathrm{Spin}_a(m)-$ module as an orthogonal direct sum

$$V = V_\Lambda \oplus J_a J_b V_\Lambda \oplus J_a V_\Lambda \oplus J_b V_\Lambda\,.$$

where a is orthogonal to b, and V_Λ denotes its real spin representation, $\dim V_\Lambda = \frac{N}{2}$. Thus, $\mathrm{Spin}_a(m)$ is embedded in $\mathrm{SO}(\frac{N}{2})$.

Besides, $\mathrm{SL}(2,\mathbb{C})$ is embedded in $\mathrm{Sp}(V,J_a)$ as $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \to \begin{pmatrix} aI & -bQ \\ cQ & dI \end{pmatrix}$, where a, b, c, d belong to $\mathbb{C} = \{\alpha + \beta J_a J_b \text{ s.t. } \alpha, \beta \in \mathbb{R}\}$ and Q is given by (4.2) in [7].

It is well known that $(\mathrm{O}(\frac{N}{2},\mathbb{C}), \mathrm{SL}(2,\mathbb{C}))$ is a dual pair in $\mathrm{Sp}(V,J_a)$. It follows from [1] that the restriction of ω to $\mathrm{O}(\frac{N}{2},\mathbb{C}) \times \mathrm{SL}(2,\mathbb{C})$ is multiplicity free and decomposes as $\omega\downarrow^{\mathrm{Sp}(V,J_a)}_{\mathrm{O}(\frac{N}{2},\mathbb{C})\times \mathrm{SL}(2,\mathbb{C})} = \int_\oplus P_\lambda(L^2(\mathbb{R}^N))\,d\mu(\lambda)$, where $P_\lambda(L^2(\mathbb{R}^N)) \simeq \pi_\lambda \otimes \pi^\lambda$. Moreover, for this pair, the correspondence between π_λ and π^λ is given explicitly in terms of the lowest K-types. D. Barbasch pointed to us that we can consider π^λ a tempered representation of $\mathrm{SL}(2,\mathbb{C})$ and in that case the restriction to $\mathrm{SO}(\frac{N}{2},\mathbb{R})$ of the corresponding π_λ is not multiplicity free. Indeed let π^λ be a tempered representation of $\mathrm{SL}(2,\mathbb{C})$ then $\pi^k := \pi^\lambda$ is a unitary principal series of $\mathrm{SL}(2,\mathbb{C})$ with lowest K-type, the $k+1-$dimensional irreducible module of $\mathrm{SU}(2)$.

The corresponding $\pi_k := \pi_\lambda$ is the unitary principal series of $\mathrm{O}(\frac{N}{2},\mathbb{C})$ with lowest K-type the irreducible representation of $\mathrm{SO}(\frac{N}{2},\mathbb{R})$ given by the harmonic polynomials on V_Λ of degree k.

We will check that the restriction of π_k to $\mathrm{SO}(\frac{N}{2},\mathbb{R})$ is not multiplicity free. First recall that if $\mathrm{O}(\frac{N}{2},\mathbb{C}) = \mathrm{O}(\frac{N}{2},\mathbb{R})AN$ denotes the Iwasawa decomposition, the commutator M of A in $\mathrm{O}(\frac{N}{2},\mathbb{R})$ is a maximal torus of it. Thus, by Frobenius reciprocity, the multiplicity of the representation with highest weight $2k\Lambda$ in π_k, $[\pi_k : V_{2k\Lambda}]$ is equal to $m_{2k\Lambda}(k\Lambda)$, the multiplicity of the weight $k\Lambda$ in $V_{2k\Lambda}$.

We compute $m_{2k\Lambda}(k\Lambda)$ using Kostant multiplicity formula [4].

Lemma 3.3. *We have for* $k = 2j$

$$m_{2k\Lambda}(k\Lambda) = \binom{\frac{N}{4} + j - 2}{j}$$

and $m_{2k\Lambda}(k\Lambda) = 0$ *otherwise.*

Proof. Let W be the Weyl group, W_1 the stabilizer of Λ, Δ a set of positive roots, $\Delta_1 = \{\alpha \in \Delta : \langle \alpha, \Lambda \rangle = 0\}$ and Π a set of simple roots for Δ. Then by Kostant formula

$$m_{2k\Lambda}(k\Lambda) = \sum_{\sigma \in W} sg(\sigma) K(\rho - \sigma(\rho + 2k\Lambda) + k\Lambda)$$

where $K(\mu)$ is the number of ways in which $-\mu$ can be written as a sum of positive roots. We will show that

$$m_{2k\Lambda}(k\Lambda) = \sum_{\sigma \in W_1} sg(\sigma) K(\rho - \sigma(\rho + 2k\Lambda) + k\Lambda) . \tag{6}$$

Indeed, to prove 6 we will see that

$$\text{if } \sigma \notin W_1, \text{ then } K(\rho - \sigma(\rho + 2k\Lambda) + k\Lambda) = 0 .$$

Or equivalently, that if $\sigma \notin W_1$ then

$$\rho - \sigma(\rho + 2k\Lambda) + k\Lambda = \sum_{\alpha \in \Pi} k_\alpha \alpha \text{ for some } k_\alpha \geq 1 . \tag{7}$$

To see (7) we do induction on $l(\sigma)$: for $l(\sigma) \geq 1$, we write $\sigma = \tau r_\alpha$ with $l(\tau) < l(\sigma)$ and r_α the reflection corresponding to $\alpha \in \Pi$. Then

$$\begin{aligned}
\rho - \sigma(\rho + 2k\Lambda) + k\Lambda &= \rho - \tau\rho + \tau\rho - \tau r_\alpha(\rho + 2k\Lambda) + k\Lambda \\
&= \rho - \tau\rho + \tau(\rho - r_\alpha \rho) - \tau r_\alpha(2k\Lambda) + k\Lambda \\
&= \rho - \tau\rho + \tau(\alpha) - \tau(2k\Lambda - 2\frac{2k\langle \Lambda, \alpha \rangle}{\langle \alpha, \alpha \rangle}\alpha) + k\Lambda \\
&= \rho - \tau(\rho + 2k\Lambda) + k\Lambda + 2\frac{2k\langle \Lambda, \alpha \rangle}{\langle \alpha, \alpha \rangle}\tau(\alpha) + \tau(\alpha) .
\end{aligned}$$

We have two cases:

i) $\tau \notin W_1$. By induction $\rho - \tau(\rho + 2k\Lambda) + k\Lambda = \sum_{\beta \in \Pi} k_\beta \beta$ with some $k_\beta \geq 1$ and also $\tau(\alpha)$ is a positive root.

ii) $\tau \in W_1$. Then $r_\alpha \notin W_1$ and so $\alpha = \alpha_1$ and

$$\begin{aligned}
& \rho - \tau(\rho + 2k\Lambda) + k\Lambda + 2\frac{2k\langle \Lambda, \alpha \rangle}{\langle \alpha, \alpha \rangle}\tau(\alpha) + \tau(\alpha) \\
= \; & \rho - \tau(\rho) - k\Lambda + (2\frac{2k\langle \Lambda, \alpha \rangle}{\langle \alpha, \alpha \rangle} + 1)\tau(\alpha) \\
= \; & \rho - \tau(\rho) - k\Lambda + (2k + 1)\tau(\alpha) .
\end{aligned}$$

Let us see that $k_{\alpha_1} \geq 1$. We have that $\rho - \tau(\rho)$ is a sum of positive roots and since $\tau \in W_1, \tau$ is a product of reflections not involving r_{α_1}. Therefore $\tau(\alpha_1) = \alpha_1 + \sum_{\beta \in \Pi \setminus \alpha_1} k_\beta \beta$, and $\Lambda = \alpha_1 + \sum_{\beta \in \Pi \setminus \alpha_1} k_\beta \beta$. So $k_{\alpha_1} \geq k + 1$ and this proves 7.

Thus

$$m_{2k\Lambda}(k\Lambda) = \sum_{\sigma \in W_1} sg(\sigma) K(\rho - \sigma\rho - k\omega_1) .$$

Now by the proposition in page 317 of [3], with $S = \emptyset, T = \Delta$, and $\lambda = \Lambda$, we have

$$\sum_{\sigma \in W_1} sg(\sigma) K(\rho - \sigma\rho - k\Lambda) = K_{\Delta \setminus \Delta_1}(-k\Lambda)\,.$$

Here $K_S(\mu)$ is the number of ways in which $-\mu$ can be written as a sum of roots in S. In this case $\Delta = \{\epsilon_i \pm \epsilon_j : 1 \leq i < j \leq \frac{N}{4}\}$, $\Lambda = \epsilon_1$ and $\Delta \setminus \Delta_1 = \{\epsilon_1 \pm \epsilon_j : 2 \leq j \leq \frac{N}{4}\}$. It is not difficult to check that for even $k = 2j$,

$$K_{\Delta \setminus \Delta_1}(-k\Lambda) = \binom{\frac{N}{4} + j - 2}{j} \geq 2 \quad \text{for} \quad j \geq 2\,.$$

and when $k = 2j+1$, $K_{\Delta \setminus \Delta_1}(-k\Lambda) = 0$. This completes the proof of the Lemma. ■

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