

Closedness of the Tangent Spaces to the Orbits of Proper Actions

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Abstract. In this note we show that for any proper action of a Banach–Lie group G on a Banach manifold M , the corresponding tangent maps $\mathfrak{g} \rightarrow T_x(M)$ have closed range for each $x \in M$, i.e., the tangent spaces of the orbits are closed. As a consequence, for each free proper action on a Hilbert manifold, the quotient M/G carries a natural manifold structure.

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1. Introduction

It is a classical result of Palais that a proper smooth action of a non-compact finite-dimensional Lie group G on a smooth manifold M behaves in many respects like the action of a compact Lie group. In particular, all orbits are closed submanifolds and the action of G on a suitable neighborhood of an orbit can be described nicely in terms of the action of the compact stabilizer group on a slice ([Pa61]). Since properness of a group action is a purely topological concept, it also applies to smooth actions of Banach–Lie groups on Banach manifolds. However, one cannot expect results as strong as Palais' in this context. One reason is that a closed subspace of a Banach space need not have a closed complement, $c_0(\mathbb{N}) \subseteq \ell^\infty(\mathbb{N})$ being a simple example. Since the translation action of the closed subspace $c_0(\mathbb{N})$ on $\ell^\infty(\mathbb{N})$ is proper, properness does not imply anything on the existence of closed complements. In particular, slices need not exist for proper smooth actions.

Since the stabilizer subgroups G_x are compact for a proper action, they are in particular finite-dimensional Lie subgroups, so that the quotient G/G_x carries a natural manifold structure and the smooth injection $G/G_x \rightarrow Gx$ is a topological embedding. In this sense all orbits of a proper action carry a natural manifold structure. The main concern of this note is to show that this manifold structure is, at least in a very weak sense, compatible with that of the surrounding manifold M , namely the tangent space of Gx , i.e., the image of the tangent map $T_e(G) \rightarrow T_x(M)$ of the orbit map, is a closed subspace of $T_x(M)$. As a consequence, Palais' theory carries immediately over to proper actions on Hilbert

manifolds. In the general case our result eliminates at least one of the assumptions, usually stated in addition to properness, for a slice theorem or a quotient theorem to hold (cf. [Bo89a, Ch. 3, §1, no. 5, Prop. 10]).

2. Tangent maps of proper actions

Let G be a Banach–Lie group with Lie algebra $\mathfrak{g} = \mathbf{L}(G)$ and neutral element e . In the following M will be a smooth Banach manifold with a smooth action $\Phi : G \times M \rightarrow M$ of G . The diffeomorphism of M defined by $g \in G$ will be denoted $\Phi_g : M \rightarrow M$ and $\Phi(g, x)$ will simply be written $g \cdot x$ or gx . For each $x \in M$ we denote the *orbit map* by $\Phi^x : G \rightarrow M$, $g \mapsto gx$.

To each $\xi \in \mathfrak{g}$ we associate the corresponding vector field ξ_M on M , defined by $\xi_M(x) = \frac{d}{dt} \Big|_{t=0} \Phi_{\exp(t\xi)}(x)$ and recall that the linear map $\mathfrak{g} \rightarrow \mathcal{V}(M)$, $\xi \mapsto -\xi_M$ is a morphism of Lie algebras.

The action is said to be *proper* if the smooth map $\Theta : G \times M \rightarrow M \times M$, $(g, x) \mapsto (gx, x)$, is proper, i.e., a closed map with compact fibers

$$\Theta^{-1}(x, y) = \{(g, y) \in G \times M : gy = x\}.$$

If G and M are finite-dimensional, i.e., locally compact, properness of the action is equivalent to inverse images of compact subsets $K \subseteq G \times M$ under Θ being compact.

In this note we show the following theorem:

Theorem 2.1. *If the action of G on M is proper, then the image of the bounded linear map*

$$\Psi_x = T_e \Phi^x : \mathfrak{g} \rightarrow T_x M, \quad \xi \mapsto \xi_M(x)$$

is closed for all $x \in M$.

For the proof we need the following three lemmas. With the first lemma we are able to describe the kernel of the map $T_e \Phi^x$ as the Lie algebra of the isotropy group of x , for an arbitrary $x \in M$.

Lemma 2.2. *For each $x \in M$ the following assertions hold:*

- (a) *The isotropy subgroup $G_x := \{g \in G \mid gx = x\}$ is a compact Lie subgroup of G . In particular, its Lie algebra $\mathfrak{g}_x = \{\xi \in \mathfrak{g} \mid \xi_M(x) = 0\}$ is finite-dimensional, hence complemented. The quotient space G/G_x carries a natural manifold structure such that for each closed complement \mathfrak{h}_x of \mathfrak{g}_x in \mathfrak{g} the map $\mathfrak{h}_x \rightarrow G/G_x$, $\xi \mapsto \exp(\xi)G_x$ is a local diffeomorphism in 0.*
- (b) *The orbit map $\Phi^x : G \rightarrow M$ induces a topological embedding $G/G_x \rightarrow M$ onto a closed subset.*

Proof. (a) It follows from the properness of the action that the set $\Theta^{-1}(x, x) = G_x \times \{x\}$ is compact. Hence G_x is a compact subgroup of G and therefore a finite-dimensional Lie subgroup (cf. [Ne06, Thm. IV.3.16]; [GN08] and [HM98, Thm. 5.28]) whose Lie algebra is

$$\mathfrak{g}_x = \{\xi \in \mathfrak{g} : \exp(\mathbb{R}\xi) \subseteq G_x\} = \{\xi \in \mathfrak{g} : \xi_M(x) = 0\} = \ker(\Psi_x).$$

Since the finite-dimensional subspace \mathfrak{g}_x of \mathfrak{g} has a closed complement, G_x is a split Lie subgroup (called a Lie subgroup in [Bo89a]), so that the remainder of (a) follows from [Bo89a, Ch. 3, §1.6, Prop. 11].

(b) This is a general property of proper group actions ([Bo89b, Ch. 3, §4.2, Prop. 4]). ■

The second lemma is a well-known fact in functional analysis and can be found for example in [RY08, Lemma 4.47].

Lemma 2.3. *Let X and Y be Banach spaces and $T: X \rightarrow Y$ be continuous linear. If there exists an $\alpha > 0$ such that $\|Tx\| \geq \alpha\|x\|$ for all $x \in X$, then $\text{Im}(T)$ is closed.*

The third lemma we will need can be found in [Bo51, IV,§1, no. 4]:

Lemma 2.4. *Let I be an interval and W an open subset of a normed space E . Let $k: I \rightarrow \mathbb{R}_{>0}$ be a real regulated function and $f: I \times W \rightarrow E$ a function satisfying $\|f(t, x_1) - f(t, x_2)\| \leq k(t)\|x_1 - x_2\|$ for all $t \in I$ and $x_1, x_2 \in W$. Let $u_i: I \rightarrow W$, $i = 1, 2$, be two differentiable function which are approximate solutions of $\dot{c}(t) = f(t, c(t))$ in the sense that $\|\dot{u}_i(t) - f(t, u_i(t))\| \leq \varepsilon_i$ for all $t \in I$ and $i = 1, 2$. Then we have for $t \in I$ with $t \geq t_0$:*

$$\|u_1(t) - u_2(t)\| \leq \|u_1(t_0) - u_2(t_0)\|\Lambda(t) + (\varepsilon_1 + \varepsilon_2)\Pi(t)$$

where

$$\Lambda(t) = 1 + \int_{t_0}^t k(s) \exp\left(\int_s^t k(\tau) d\tau\right) ds$$

and

$$\Pi(t) = t - t_0 + \int_{t_0}^t (s - t_0)k(s) \exp\left(\int_s^t k(\tau) d\tau\right) ds.$$

Proof. [of Theorem 2.1] Let $\mathfrak{h}_x \subseteq \mathfrak{g}$ be a closed complement of \mathfrak{g}_x (see Lemma 2.2(a)). There exists an open neighborhood V of $0 \in \mathfrak{h}_x$ such that the map $\gamma: \mathfrak{h}_x \rightarrow G/G_x, \xi \mapsto \exp(\xi)G_x$, is a diffeomorphism onto an open subset of G/G_x .

Choose $r > 0$ such that the closed ball $\overline{B_r(0)}$ is contained in V . Since the orbit map Φ^x defines an embedding of G/G_x into M , for each sequence (ξ_n) in \mathfrak{h}_x with $\|\xi_n\| = r$ for all $n \in \mathbb{N}$, the sequence $(\exp(\xi_n) \cdot x)$ cannot converge to $x \in M$ because $\exp(\xi_n)G_x$ does not converge to G_x in G/G_x (Lemma 2.2(b)).

For any $x \in M$, the image of Ψ_x is $\Psi_x(\mathfrak{h}_x)$ since $\mathfrak{g}_x = \ker \Psi_x$ is its kernel (Lemma 2.2(a)). Since \mathfrak{h}_x is a closed subspace of \mathfrak{g} , it is a Banach space and we consider the injective map

$$A_x := \Psi_x|_{\mathfrak{h}_x}: \mathfrak{h}_x \rightarrow T_x M, \quad \xi \mapsto \xi_M(x).$$

If $\Psi_x(\mathfrak{g}) = A_x(\mathfrak{h}_x)$ is not closed in $T_x M$, then with the aid of Lemma 2.3 we find for each $n \in \mathbb{N}$ an element $\xi_n \in \mathfrak{h}_x$ such that $\|A_x(\xi_n)\| < \frac{1}{n}\|\xi_n\|$, and hence

$$\left\| A_x \left(\frac{\xi_n}{\|\xi_n\|} \right) \right\| < \frac{1}{n}.$$

Thus, setting $\eta_n := \frac{r\xi_n}{\|\xi_n\|}$ we get a sequence (η_n) in \mathfrak{g} such that $\|\eta_n\| = r$ for all n and

$$\lim_{n \rightarrow \infty} A_x(\eta_n) = 0.$$

Let $\phi : U \rightarrow W \subseteq E$ be a local chart for M centered on the point x (that is, $\phi(x) = 0$) and $\xi_W := \phi_*(\xi_M|_U)$ denote the corresponding smooth vector field on the open subset W of E . For two Banach spaces X and Y we write $\mathcal{B}(X, Y)$ for the space of continuous linear operators $X \rightarrow Y$. Then we obtain a smooth map

$$\Gamma : W \rightarrow \mathcal{B}(\mathfrak{g}, \mathcal{B}(E)), \quad \Gamma_z(\xi) := T_z(\xi_W)$$

(that can be understood as the smooth map $z \mapsto (\xi \mapsto D_1F(z, \xi))$, where $F(z, \xi) = D_1\Phi(e, z)\xi$). Since Γ is smooth, it is in particular continuous in 0, so that we find for each $\varepsilon > 0$ a $\delta > 0$ with $\Gamma(B_\delta(0)) \subseteq B_\varepsilon(\Gamma_0)$. Replacing W by $B_\delta(0)$, we get for each $z \in W$:

$$\|\Gamma_z\| \leq \|\Gamma_0\| + \varepsilon =: L.$$

Setting $D := L \cdot r$, this yields for all x_1 and $x_2 \in W$ and all $\xi \in \mathfrak{g}$ with $\|\xi\| \leq r$:

$$\begin{aligned} \|\xi_W(x_1) - \xi_W(x_2)\| &\leq \sup_{z \in W} \|T_z \xi_W\| \cdot \|x_1 - x_2\| \leq \sup_{z \in W} (\|\Gamma_z\| \cdot \|\xi\|) \cdot \|x_1 - x_2\| \\ &\leq L \cdot \|\xi\| \cdot \|x_1 - x_2\| \leq L \cdot r \cdot \|x_1 - x_2\| = D \cdot \|x_1 - x_2\|. \end{aligned}$$

We now shrink r such that the map

$$\xi \mapsto \phi(\exp(\xi) \cdot x)$$

maps the closure of the ball $B_r(0) \subseteq \mathfrak{g}$ into the ball $W = B_\delta(0)$. Then for each $\xi \in \mathfrak{g}$ with $\|\xi\| \leq r$ the curve $\gamma^\xi : [0, 1] \rightarrow W, t \mapsto \phi(\exp t\xi \cdot x)$ is part of the integral curve of the vector field ξ_W through $0 = \phi(x)$ (thus, γ^ξ can be considered as an approximate solution of $\dot{c}(t) = \xi_W(c(t))$ for $\varepsilon_1 = 0$). Define $\gamma(t) = 0$ for all $t \in [0, 1]$ and note that γ is a approximate solution of $\dot{c}(t) = \xi_W(c(t))$ for $\varepsilon_2 := \|\xi_W(0)\|$. Applying Lemma 2.4 to $I = [0, 1]$, $t_0 = 0$, $f : I \times W \rightarrow E$, $f(z, t) = \xi_W(z)$, the constant function $k = D$ and the curves γ and γ^ξ , we get

$$\|\gamma^\xi(t) - 0\| \leq \|0 - 0\| \cdot \Lambda(t) + \|\xi_W(0)\| \cdot \Pi(t) = \|\xi_W(0)\| \cdot \Pi(t)$$

with $\Pi(t) = t + \left[-\frac{Ds+1}{D} \exp((t-s)D)\right]_0^t = \frac{1}{D}(\exp(Dt) - 1)$. Hence, we have

$$\|\phi(\exp(\xi) \cdot x)\| = \|\gamma^\xi(1)\| \leq \Pi(1) \cdot \|\xi_W(0)\|.$$

Thus, since Π doesn't depend on ξ , we get for all $n \in \mathbb{N}$:

$$\|\phi(\exp(\eta_n) \cdot x)\| \leq \Pi(1) \cdot \|(\eta_n)_W(0)\|$$

and because we have $\lim_{n \rightarrow \infty} (\eta_n)_W(0) = 0$, we obtain $\lim_{n \rightarrow \infty} \exp(\eta_n) \cdot x = x$. We have seen above that this is a contradiction to the properness of the action. \blacksquare

Combining [Bo89a, Ch. 3, §1, no. 5, Prop. 10] with Theorem 2.1, we immediately obtain:

Corollary 2.5. *If M is a Hilbert manifold and $\Phi : G \times M \rightarrow M$ a smooth action of a Banach-Lie group which is free and proper, then the set M/G of G -orbits in M carries a unique Hilbert manifold structure for which the quotient map $M \rightarrow M/G$ is a submersion.*

Remark 2.6. In [AN07] it is shown that for any smooth action $G \times M \rightarrow M$ of a Banach–Lie group and $x \in M$, the stabilizer subgroup G_x is a (not necessarily split) Lie subgroup of G if the map $\Psi_x: \mathfrak{g} \rightarrow T_x(M)$ has closed range. This result demonstrates the interesting interplay between properties of smooth actions and closedness conditions for the maps Ψ_x . For proper actions, stabilizers are compact, hence even split Lie subgroups (Lemma 2.2).

It is an open problem whether stabilizer subgroups G_x of smooth actions are always Lie subgroups. That they need not be split is an immediate consequence of the fact that for each non-split closed subspace F of a Banach space $G = X$, the subspace F is the stabilizer group G_x of any point x in the quotient space $M = X/F$ with respect to the induced action.

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