

Asymptotic Harmonic Analysis on the Space of Square Complex Matrices

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Communicated by G. Olshansky

Abstract. This paper is largely of expository nature. We determine the spherical functions of positive type on the space $V_\infty = M(\infty, \mathbb{C})$ relatively to the action of the product group $K_\infty = U(\infty) \times U(\infty)$. The space V_∞ is the inductive limit of the spaces of square complex matrices $V_n = M(n, \mathbb{C})$, and the group K_∞ is the inductive limit of the product groups $K_n = U(n) \times U(n)$, where $U(n)$ is the unitary group.

Mathematical Subject Classification 2000: 22E30 (Primary); 43A35, 43A85, 43A90 (Secondary).

Key Words and Phrases: Square complex matrices, unitary group, inductive limit, function of positive type, spherical function, ergodic measure, generalized Bochner theorem.

1. Introduction

In this paper of expository nature we deal with infinite dimensional harmonic analysis on spherical pairs. We consider the spherical pair (G_∞, K_∞) , which is the inductive limit of the sequence of Gelfand pairs (G_n, K_n) :

$$G_n = K_n \ltimes V_n, \quad K_n = U(n) \times U(n), \quad V_n = M(n, \mathbb{C}),$$

$$G_\infty = K_\infty \ltimes V_\infty, \quad K_\infty = U(\infty) \times U(\infty).$$

Here $V_\infty = M(\infty, \mathbb{C})$ is the space of infinite complex matrices having only a finite number of non-zero entries, and $U(\infty)$ is the group of the infinite unitary matrices (u_{ij}) with complex coefficients such that $u_{ij} = \delta_{ij}$ for $i + j$ large enough. Further, let V^∞ be the space of all square complex matrices of infinite order. The spaces V_∞ and V^∞ are in a natural duality.

Let \mathfrak{P} be the set of K_∞ -biinvariant continuous functions of positive type on G_∞ satisfying $\varphi(0) = 1$. We are interested in the determination of the extreme points φ of this convex set which are indeed the spherical functions of positive type relatively to the pair (G_∞, K_∞) . The group G_∞ is equipped with the inductive

limit topology. The subgroup K_∞ is closed. The homogeneous space G_∞/K_∞ is the vector space

$$V_\infty = \bigcup_{n=1}^{\infty} V_n.$$

The law of the inductive limit group $G_\infty = K_\infty \times V_\infty$ is given by :

$$(u, x)(v, y) = ((u_1v_1, u_2v_2), x + u_1yu_2^*),$$

where

$$u = (u_1, u_2), v = (v_1, v_2) \in K_\infty \text{ and } x, y \in V_\infty.$$

A function φ on G_∞ which is right invariant under K_∞ does not depend on the variable $u \in K_\infty$. Therefore it is possible to see it like a function on V_∞ :

$$\varphi(g) = \varphi((u, x)) = \varphi_0(x).$$

Moreover, if the function φ is K_∞ -biinvariant then the function φ_0 which is defined on V_∞ is K_∞ -invariant, or $U(\infty)$ -biinvariant. Also, the function φ is of positive type on G_∞ if and only if the function φ_0 is of positive type. Let us note by D_∞ the subspace of diagonal matrices in V_∞ . An element of D_∞ can be decomposed as $\text{diag}(a_1, a_2, \dots)$, with $a_1, a_2, \dots \in \mathbb{R}$ and $a_j = 0$ for j large enough. Any matrix $x \in V_\infty$ can be diagonalised as

$$x = u \text{diag}(a_1, a_2, \dots) v^* \quad (u, v \in U(\infty)).$$

Consequently, any K_∞ -invariant function on V_∞ is uniquely determined by its restriction to the subset D_∞ .

Main Theorem The spherical functions of positive type φ relatively to the pair (G_∞, K_∞) are given by :

$$\varphi(\text{diag}(\xi_1, \dots, \xi_n, 0, \dots)) = \Pi(\omega, \xi_1) \dots \Pi(\omega, \xi_n),$$

where

$$\Pi(\omega, \lambda) := e^{-\frac{1}{4}\gamma\lambda^2} \prod_{k=1}^{\infty} \frac{1}{1 + \frac{1}{4}\alpha_k\lambda^2},$$

with

$$\omega = (\alpha, \gamma), \quad \gamma \in \mathbb{R}_+, \quad \alpha_k \in \mathbb{R}_+ \text{ and } \sum_{k=1}^{\infty} \alpha_k < \infty.$$

The main Theorem is due to D. Pickrell (see [13], Proposition 5.14). This result can be derived from a deep function-theoretic result due to Schoenberg [17]. In fact, Pickrell's method consists in the reduction of the case of rectangular matrices to that of Hermitian matrices, and the latter case is handled with the help of Schoenberg's theorem on totally positive functions. For further details about the reduction procedure we refer the reader to [14]. In this paper, we prove the same result (see Theorem 5.6), but the method of proof which we follow is close to the one adopted by G. Olshanski and A. Vershik in [10]. Indeed, any spherical function of positive type on V_∞ is a Fourier transform of an ergodic measure on

V^∞ and thus the starting point is a general approximation theorem for ergodic measures (see [10] Theorem 3.2). When applied to our situation (see section 5), it implies that any ergodic measure ν on V^∞ is the weak limit of a sequence of orbital measures ν_n . This means that any spherical function φ of positive type on V_∞ can be approximated uniformly on compact sets by a sequence of spherical functions $\varphi^{(n)}$ of positive type on V_n . Then the main problem is to determine the asymptotic behaviour of the sequence $\varphi^{(n)}$. This is done by rewriting the corresponding spherical functions $\varphi^{(n)}$ as a series of Schur polynomials and then by analyzing the asymptotics of the coefficients as $n \rightarrow \infty$.

It should be noted that the problem that is solved in the present paper can be translated into the language of representation theory. Indeed, it is a particular case of the following general problem in representation theory of inductive limit groups: Given a (irreducible) unitary representation of an inductive limit group

$$G(\infty) = \varinjlim G(n),$$

study its approximation by (irreducible) unitary representations of the growing subgroups $G(n)$ as $n \rightarrow \infty$. For further asymptotic results and other possible approaches in representation theory of inductive limit groups, see the survey paper [11].

2. Spherical functions of positive type on (G_n, K_n)

In this section, we will determine the spherical functions of positive type relatively to the pair (G_n, K_n) . An explicit formula of these functions was established for the first time in [2]. It was also done in [9] by a method using the Abel transform. It was also obtained in [1], by means of a contraction, starting from the spherical functions on $SU(n, n)/S(U(n) \times U(n))$. The method that we use here is close to the one used by J. Faraut in ([6], sections X-2 and X-3) for the case of finite dimensional Hermitian matrices. This gives a simple and new proof of the result.

We consider the space V_n of complex matrices on which the group K_n acts as follows :

$$T(k) : x \mapsto k.x = uxv^* \quad (u, v \in U(n)).$$

Every $x \in V_n$ admits a polar decomposition $x = u \operatorname{diag}(\lambda_1, \dots, \lambda_n) v^*$, $u, v \in U(n)$, $\lambda_j \in \mathbb{R}$. Hence any function f which is K_n -invariant on V_n depends only on $\lambda = \operatorname{diag}(\lambda_1, \dots, \lambda_n)$:

$$f(x) = F(\lambda_1, \dots, \lambda_n),$$

where F is a function defined on \mathbb{R}^n , invariant under $\mathfrak{S}_n \times \{-1, 1\}^n$.

Let us consider on V_n the Euclidean structure defined by $\langle x, y \rangle = \Re \operatorname{tr}(xy^*)$. The laplacian associated to this structure is :

$$\begin{aligned} \Delta = \sum_{i,j=1}^n \left(\frac{\partial^2}{\partial(\Re x_{jj})^2} + \frac{\partial^2}{\partial(\Im x_{jj})^2} \right) &+ \sum_{j<k}^n \left(\frac{\partial^2}{\partial(\Re x_{jk})^2} + \frac{\partial^2}{\partial(\Re x_{kj})^2} \right) \\ &+ \sum_{j<k}^n \left(\frac{\partial^2}{\partial(\Im x_{jk})^2} + \frac{\partial^2}{\partial(\Im x_{kj})^2} \right). \end{aligned}$$

The laplacian is invariant under the action of K_n in the following sense: if f is a function of class \mathcal{C}^2 , then $\Delta f(T(k)) = \Delta(f(T(k)))$ ($k \in K_n$). Let f be a function of class \mathcal{C}^2 on V_n which is invariant under the action of K_n :

$$f(uxv^*) = f(x) \quad (k = (u, v) \in K_n).$$

The function Δf is also invariant under K_n . This leads to the introduction of the operator L defined by $\Delta f(x) = LF(\lambda_1, \dots, \lambda_n)$. The operator L is called the *radial part* of the laplacian.

Proposition 2.1. (i) *Let f be a K_n -invariant function of class \mathcal{C}^2 . Then*

$$\Delta f(x) = LF(\lambda_1, \dots, \lambda_n),$$

where

$$\begin{aligned} LF = \sum_{i=1}^n \left(\frac{\partial^2 F}{\partial \lambda_i^2} + \frac{1}{\lambda_i} \frac{\partial F}{\partial \lambda_i} \right) &+ 2 \sum_{i < j} \frac{1}{\lambda_i - \lambda_j} \left(\frac{\partial F}{\partial \lambda_i} - \frac{\partial F}{\partial \lambda_j} \right) \\ &+ 2 \sum_{i < j} \frac{1}{\lambda_i + \lambda_j} \left(\frac{\partial F}{\partial \lambda_i} + \frac{\partial F}{\partial \lambda_j} \right). \end{aligned}$$

(ii) *The preceding formula can also be written as*

$$LF = \frac{1}{D(\lambda)} \sum_{i=1}^n \left(\frac{\partial^2}{\partial \lambda_i^2} + \frac{1}{\lambda_i} \frac{\partial}{\partial \lambda_i} \right) (D(\lambda)F(\lambda)),$$

where D is given by

$$D(\lambda) = \prod_{i < j} (\lambda_i^2 - \lambda_j^2).$$

Proof. In order to prove the preceding proposition, we will use the following result (see [6], Lemma IX-2.2):

Lemma : *Let f be a function of class \mathcal{C}^2 on an open set \mathcal{U} of a finite dimensional real vector space \mathcal{V} . Let \mathcal{A} be an endomorphism of \mathcal{V} , $a \in \mathcal{V}$. Let $\epsilon > 0$ be such that, for $|t| < \epsilon$, $\exp t\mathcal{A} \cdot a \in \mathcal{U}$. We assume that, for $|t| < \epsilon$,*

$$f(\exp t\mathcal{A} \cdot a) = f(a).$$

Then

$$\begin{aligned} (Df)_a(\mathcal{A} \cdot a) &= 0, \\ (D^2 f)_a(\mathcal{A} \cdot a, \mathcal{A} \cdot a) + (Df)_a(\mathcal{A}^2 \cdot a) &= 0. \end{aligned}$$

Let \mathcal{U} be an open set in V_n and \mathcal{A} the endomorphism of V_n defined by: $\mathcal{A} \cdot a = Xa + aY^*$ where $X, Y \in V_n$. If the matrices X, Y are skewHermitian, then for every $t \in \mathbb{R}$, the matrices $\exp tX$, $\exp tY$ are unitary and, for every $a \in \mathcal{U}$,

$$f(\exp tXa \exp tY^*) = f(a).$$

We deduce from ([6], lemma IX-2.2) that

$$(Df)_a (Xa + aY^*) = 0,$$

$$(D^2f)_a (Xa + aY^*, Xa + aY^*) + (Df)_a (X^2a + 2XaY^* + a(Y^*)^2) = 0.$$

(a) Let us put $X = Y = E_{jk} - E_{kj}$ ($j \neq k$), $a = \text{diag}(a_1, \dots, a_n)$. We obtain

$$Xa + aY^* = (a_k - a_j)(E_{jk} + E_{kj}),$$

$$X^2a + 2XaY^* + a(Y^*)^2 = 2(a_k - a_j)(E_{jj} - E_{kk}),$$

and hence

$$(a_k - a_j)^2 (D^2f)_a (E_{jk} + E_{kj}, E_{jk} + E_{kj}) + 2(a_k - a_j)(Df)_a (E_{jj} - E_{kk}) = 0,$$

where

$$\frac{\partial^2 f}{\partial(\Re x_{jk})^2}(a) + \frac{\partial^2 f}{\partial(\Re x_{kj})^2}(a) = \frac{2}{(a_j - a_k)} \left(\frac{\partial f}{\partial(\Re x_{jj})}(a) - \frac{\partial f}{\partial(\Re x_{kk})}(a) \right).$$

(b) Let us put $X = i(E_{jk} + E_{kj})$ et $Y = -X$. We get

$$Xa + aY^* = (a_j + a_k)(iE_{jk} + iE_{kj}),$$

$$X^2a + 2XaY^* + a(Y^*)^2 = -2(a_j + a_k)(E_{jj} + E_{kk}),$$

and hence

$$(a_j + a_k)^2 (D^2f)_a (iE_{jk} + iE_{kj}, iE_{jk} + iE_{kj}) - 2(a_j + a_k)(Df)_a (E_{jj} + E_{kk}) = 0,$$

where

$$\frac{\partial^2 f}{\partial(\Im x_{jk})^2}(a) + \frac{\partial^2 f}{\partial(\Im x_{kj})^2}(a) = \frac{2}{(a_j + a_k)} \left(\frac{\partial f}{\partial(\Re x_{jj})}(a) + \frac{\partial f}{\partial(\Re x_{kk})}(a) \right).$$

(c) Let us put $X = iE_{jj}$ and $Y = -X$. We obtain

$$Xa + aY^* = i2a_jE_{jj},$$

$$X^2a + 2XaY^* + a(Y^*)^2 = -4a_jE_{jj},$$

and hence

$$4a_j^2 (D^2f)_a (iE_{jj}, iE_{jj}) - 4a_j (Df)_a (E_{jj}) = 0,$$

where

$$\frac{\partial^2 f}{\partial(\Im x_{jj})^2}(a) = \frac{1}{a_j} \frac{\partial f}{\partial(\Re x_{jj})}(a).$$

Finally,

$$\begin{aligned}\frac{\partial^2 f}{\partial(\Re x_{jj})^2}(a) &= \frac{\partial^2 F}{\partial\lambda_j^2}, \\ \frac{\partial^2 f}{\partial(\Im x_{jj})^2}(a) &= \frac{1}{\lambda_j} \frac{\partial F}{\partial\lambda_j}, \\ \frac{\partial^2 f}{\partial(\Re x_{jk})^2}(a) + \frac{\partial^2 f}{\partial(\Re x_{kj})^2}(a) &= \frac{2}{(\lambda_j - \lambda_k)} \left(\frac{\partial F}{\partial\lambda_j}(a) - \frac{\partial F}{\partial\lambda_k}(a) \right), \\ \frac{\partial^2 f}{\partial(\Im x_{jk})^2}(a) + \frac{\partial^2 f}{\partial(\Im x_{kj})^2}(a) &= \frac{2}{(\lambda_j + \lambda_k)} \left(\frac{\partial F}{\partial\lambda_j}(a) + \frac{\partial F}{\partial\lambda_k}(a) \right).\end{aligned}$$

This proves (i). In order to prove (ii), we have to use the formula :

$$\Delta_0(DF) = \Delta_0 F + 2(\nabla_0 D | \nabla_0 F) + \Delta(D),$$

where Δ_0 is the laplacian and ∇_0 the gradient on \mathbb{R}^n . The polynomial D is harmonic and

$$\sum_{j=1}^n \frac{1}{\lambda_j} \frac{\partial D}{\partial\lambda_j} = 0.$$

We can then conclude that

$$\frac{1}{D} \Delta_0(DF) + \frac{1}{D} \sum_{j=1}^n \frac{1}{\lambda_j} \frac{\partial(DF)}{\partial\lambda_j} = \Delta_0 F + 2 \frac{1}{D} (\nabla_0 D | \nabla_0 F) + \sum_{j=1}^n \frac{1}{\lambda_j} \frac{\partial F}{\partial\lambda_j}.$$

Since

$$\frac{1}{D} \nabla_0 D = \nabla_0 \log |D| = \sum_{j < k} \frac{1}{\lambda_j - \lambda_k} (e_j - e_k) + \sum_{j < k} \frac{1}{\lambda_j + \lambda_k} (e_j + e_k),$$

where (e_1, \dots, e_n) is the canonical basis of \mathbb{R}^n , we obtain

$$\frac{1}{D} (\nabla_0 D | \nabla_0 F) = \sum_{j < k} \frac{1}{\lambda_j - \lambda_k} \left(\frac{\partial F}{\partial\lambda_j} - \frac{\partial F}{\partial\lambda_k} \right) + \sum_{j < k} \frac{1}{\lambda_j + \lambda_k} \left(\frac{\partial F}{\partial\lambda_j} + \frac{\partial F}{\partial\lambda_k} \right). \quad \blacksquare$$

In the preceding Euclidean polar decomposition, the measure m can be written as :

$$\alpha(du)\alpha(dv) \prod_{j < k} (\lambda_j^2 - \lambda_k^2)^2 \prod_{j=1}^n \lambda_j d\lambda_j,$$

where α is the normalized Haar measure of the unitary group $U(n)$. Moreover, one has the following integration formula :

Proposition 2.2. (see [4], Proposition X.3.4) *For every integrable function f on V_n*

$$\int_{V_n} f(x) m(dx) = c_n \int_{U(n) \times U(n)} \int_{\mathbb{R}_+^n} f(u\lambda v) \alpha(du)\alpha(dv) \prod_{j < k} (\lambda_j^2 - \lambda_k^2)^2 \prod_{j=1}^n \lambda_j d\lambda_j.$$

c_n is the following constant :

$$c_n = \frac{2^n \pi^{n^2}}{n! \left(\prod_{j=1}^{n-1} j! \right)^2}.$$

By using the preceding results, the resolution of the Cauchy problem for the heat equation on V_n leads to the evaluation of the orbital integral $\mathcal{I}(x, y)$, which is defined for $x, y \in V_n$ by

$$\mathcal{I}(x, y) = \int_{U(n)} \int_{U(n)} e^{\Re \operatorname{tr}(xuyv^*)} \alpha(du) \alpha(dv).$$

One can remark that the function $\mathcal{I}(x, y)$ is determined by its restriction to the subspace of diagonal matrices because it is invariant under K_n :

$$\mathcal{I}(uxv^*, y) = \mathcal{I}(x, uyv^*) = \mathcal{I}(x, y) \quad (u, v \in U(n)).$$

The Cauchy problem for the heat equation

$$\frac{\partial U}{\partial t} = \Delta U,$$

$$U(0, x) = f(x),$$

where f is a bounded continuous function on V_n , has a unique solution which is

$$U(t, x) = \frac{1}{(4\pi t)^{\frac{N}{2}}} \int_{V_n} e^{-\frac{1}{4t} \|\|x-y\|\|^2} f(y) m(dy) \quad (t > 0, x \in V_n),$$

where $N = 2n^2$ is the dimension of V_n , $\|\| \cdot \|\|$ is the Hilbert-Schmidt norm on V_n and m is the Euclidean measure.

Let us assume that the function f is invariant under the action of K_n . Then the solution U will be also K_n -invariant. Hence we can write

$$f(x) = f_0(\lambda), \quad U(t, x) = U_0(t, \lambda).$$

By using the Weyl integration formula (Proposition 2.2), the solution $U_0(t, \lambda)$ is given by:

$$U_0(t, \lambda) = \int_{\mathbb{R}_+^n} H_0(t, \lambda, \theta) f_0(\theta) D(\theta) \prod_{j=1}^n \theta_j d\theta_j,$$

with

$$\begin{aligned} H_0(t, \lambda, \theta) &= c_n \frac{1}{(4\pi t)^{\frac{N}{2}}} \int_{U(n)} \int_{U(n)} e^{-\frac{1}{4t} \|\|\lambda - u\theta v^*\|\|^2} f(y) \alpha(du) \alpha(dv) \\ &= c_n \frac{1}{(4\pi t)^{\frac{N}{2}}} e^{-\frac{1}{4t} (\|\lambda\|^2 + \|\theta\|^2)} \int_{U(n)} \int_{U(n)} e^{\frac{1}{2t} \Re \operatorname{tr}(\lambda u \theta v^*)} \alpha(du) \alpha(dv) \\ &= c_n \frac{1}{(4\pi t)^{\frac{N}{2}}} e^{-\frac{1}{4t} (\|\lambda\|^2 + \|\theta\|^2)} \mathcal{I}\left(\frac{1}{2t} \lambda, \theta\right). \end{aligned}$$

Theorem 2.3. If $\lambda = \text{diag}(\lambda_1, \dots, \lambda_n)$ and $\theta = \text{diag}(\theta_1, \dots, \theta_n)$, then

$$\mathcal{J}(\lambda, \theta) = 2^{n(n-1)} [1!2! \times \dots \times (n-1)!]^2 \frac{1}{D(\lambda)D(\theta)} \det \left((I_0(\lambda_i \theta_j))_{1 \leq i, j \leq n} \right),$$

where I_0 is the modified Bessel function:

$$I_0(z) = \sum_{k=0}^{\infty} \frac{1}{2^{2k} (k!)^2} z^{2k} \quad (z \in \mathbb{C}).$$

Proof. From the formula that gives the radial part of the laplacian (Proposition 2.1) one can deduce that the function U_0 is a solution of the equation

$$\frac{\partial^2 U_0}{\partial t^2} = \frac{1}{D(\lambda)} \sum_{i=1}^n \left(\frac{\partial^2}{\partial \lambda_i^2} + \frac{1}{\lambda_i} \frac{\partial}{\partial \lambda_i} \right) (D(\lambda)F(\lambda)).$$

We put then

$$V(t, \lambda) = D(\lambda)U_0(t, \lambda), \quad g(\lambda) = D(\lambda)f_0(\lambda).$$

The function V is in consequence a solution of the Cauchy problem

$$\frac{\partial^2 V}{\partial t^2} = \sum_{i=1}^n \left(\frac{\partial^2 V}{\partial \lambda_i^2} + \frac{1}{\lambda_i} \frac{\partial V}{\partial \lambda_i} \right),$$

$$V(0, \lambda) = g(\lambda).$$

Let us assume that the function f belongs to the Schwartz space $\mathcal{S}(V_n)$. It follows, by a result similar to ([6], Lemme X-3.1), we can prove that, for every $T > 0$, the function V is bounded on $[0, T] \times \mathbb{R}_+^n$. Furthermore, for $n = 1$, the preceding problem is equivalent, in cylindrical coordinates, to the Cauchy problem for the heat equation on \mathbb{R}^2 with f radial. The solution of such a problem is expressed using the modified Bessel function I_0 and it is given by (see [6], Chapter IX, exercice 3) :

$$\frac{1}{2t} \int_0^{\infty} e^{-\frac{r^2 + \rho^2}{4t}} g(\rho) I_0\left(\frac{r\rho}{2t}\right) \rho d\rho.$$

In consequence we can conclude that the solution of our problem for an arbitrary n is given by

$$V(t, \lambda) = \frac{1}{(2t)^n} \int_{\mathbb{R}_+^n} e^{-\frac{1}{4t}(\|\lambda\|^2 + \|\theta\|^2)} g(\theta) \prod_{i=1}^n I_0\left(\frac{\lambda_i \theta_i}{2t}\right) \theta_i d\theta_i.$$

As the function g is skewsymmetric this last relation can be written

$$V(t, \lambda) = \frac{1}{(2t)^n} \int_{\mathbb{R}_+^n} e^{-\frac{1}{4t}(\|\lambda\|^2 + \|\theta\|^2)} g(\theta) \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} \varepsilon(\sigma) \prod_{i=1}^n I_0\left(\frac{\lambda_i \theta_{\sigma(i)}}{2t}\right) \theta_i d\theta_i.$$

Hence, for every function $g(\theta) = D(\theta)f_0(\theta)$, where f_0 is a symmetric function in $\mathcal{S}(\mathbb{R}_+^n)$,

$$\begin{aligned} & \int_{\mathbb{R}_+^n} H_0(t, \lambda, \theta)g(\theta) \prod_{i=1}^n \theta_i \, d\theta_i \\ &= \frac{1}{n!(2t)^n} \int_{\mathbb{R}_+^n} e^{-\frac{1}{4t}(\|\lambda\|^2 + \|\theta\|^2)} g(\theta) \det \left(\left(I_0 \left(\frac{\lambda_i \theta_j}{2t} \right) \right)_{1 \leq i, j \leq n} \right) \prod_{i=1}^n \theta_i \, d\theta_i. \end{aligned}$$

This proves that the kernel H_0 is equal to

$$H_0(t, \lambda, \theta) = \frac{1}{n!(2t)^n} \frac{1}{D(\lambda)D(\theta)} e^{-\frac{1}{4t}(\|\lambda\|^2 + \|\theta\|^2)} \det \left(\left(I_0 \left(\frac{\lambda_i \theta_j}{2t} \right) \right)_{1 \leq i, j \leq n} \right).$$

We obtain the result by comparing, for $t = \frac{1}{2}$, the two expressions we obtained for H_0 . ■

For $x \in V_n$ the orbital measure μ_x is defined on V_n by

$$\int_{V_n} f(y)\mu_x(dy) = \int_{U(n)} \int_{U(n)} f(uxv^*)\alpha(du)\alpha(dv),$$

where α is the normalized Haar measure on $U(n)$ and f is a continuous function on V_n . The Fourier transform of μ_x is the following function $\widehat{\mu}_x$:

$$\begin{aligned} \widehat{\mu}_x(\xi) &= \int_{V_n} e^{i\langle \xi, y \rangle} \mu_x(dy) \\ &= \int_{U(n)} \int_{U(n)} e^{i\Re \operatorname{tr}(\xi uxv^*)} \alpha(du)\alpha(dv) \\ &= \mathcal{I}(x, i\xi). \end{aligned}$$

The spherical functions of positive type for the Gelfand pair (G_n, K_n) are the functions $\varphi_x = \widehat{\mu}_x$, ($x \in \mathbb{R}^n$), Fourier transforms of the orbital measures μ_x .

Corollary 2.4. *If $x = \operatorname{diag}(x_1, \dots, x_n)$ and $\xi = \operatorname{diag}(\xi_1, \dots, \xi_n)$,*

$$\varphi_x(\xi) := \widehat{\mu}_x(\xi) = (\delta!)^2 \frac{(-4)^{\frac{n(n-1)}{2}}}{D(x)D(\xi)} \det \left((J_0(x_j \xi_k))_{1 \leq j, k \leq n} \right),$$

where

$$\delta = (\delta_1, \delta_2, \dots, \delta_n) := (n - 1, n - 2, \dots, 0),$$

$$\delta! = \delta_1! \times \dots \times \delta_n!,$$

and J_0 is the classical Bessel function.

with

$$G(\theta, \zeta) = \int_{K^2(m,n)} \int_{K^2(m,n)} f(h_1 a(\theta) h_2, g_1 a(\zeta) g_2) \kappa(dh) \kappa(dg),$$

$\kappa = \beta \otimes \beta$ where β is the normalized Haar measure of $K(m, n)$, and

$$D_{m,n}(\theta) = c_{m,n} \left| \prod_{1 \leq i < j \leq m} (\sin(\theta_i + \theta_j))^2 (\sin(\theta_i - \theta_j))^2 \prod_{i=1}^m (\sin 2\theta_i) (\sin \theta_i)^{2(n-2m)} \right|,$$

where $c_{m,n}$ is a constant such that

$$\int_{[0,\pi]^m} D_{m,n}(\theta) d\theta_1 \dots d\theta_m = 1.$$

Lemma 3.1. *Let X be a compact topological space and μ a positive measure on X such that for every non-null open set B we have $\mu(B) > 0$. Let δ be a continuous positive function on X , which reaches its maximum at a unique point x_0 . Let us put*

$$\frac{1}{c_n} = \int_X \delta(x)^n \mu(dx).$$

Then, if f is a continuous function on $X \times X$,

$$\lim_{n \rightarrow \infty} c_n^2 \int_{X \times X} f(x, y) \delta(x)^n \delta(y)^n \mu(dx) \mu(dy) = f(x_0, x_0).$$

Proof. We apply Lemma 5.4 in [5] to the function $\delta(x, y) = \delta(x)\delta(y)$. ■

Proposition 3.2. *Let f be a continuous function on K_∞ which is $K_m(\infty) \times K_m(\infty)$ -invariant. Then*

$$\begin{aligned} & \lim_{n \rightarrow \infty} \int_{U(n) \times U(n)} f(k_1, k_2) \alpha_n(dk_1) \alpha_n(dk_2) \\ &= \int_{K_m^2} \int_{K_m^2} f(h_1 w_m h_2, g_1 w_m g_2) \alpha_m(dh_1) \alpha_m(dh_2) \alpha_m(dg_1) \alpha_m(dg_2), \end{aligned}$$

where

$$w_m = a\left(\frac{\pi}{2}, \dots, \frac{\pi}{2}\right) = \begin{pmatrix} 0 & -I_m & 0 \\ I_m & 0 & 0 \\ 0 & 0 & I \end{pmatrix}.$$

Proof. By using the integration formula and the invariance under $K_m(\infty) \times K_m(\infty)$, we obtain

$$\int_{U(n) \times U(n)} f(k_1, k_2) \alpha_n(dk_1) \alpha_n(dk_2) = \int_{[0, \pi]^m \times [0, \pi]^m} F(\theta, \zeta) D_{m,n}(\theta) D_{m,n}(\zeta) d\theta d\zeta,$$

with

$$F(\theta, \zeta) = \int_{K_m^2} \int_{K_m^2} f(h_1 a(\theta) h_2, g_1 a(\zeta) g_2) \alpha_m(dh_1) \alpha_m(dh_2) \alpha_m(dg_1) \alpha_m(dg_2).$$

By using Lemma 3.1, for every fixed m , and for every continuous function g on $[0, \pi]^m \times [0, \pi]^m$,

$$\lim_{n \rightarrow \infty} \int_{[0, \pi]^m \times [0, \pi]^m} g(\theta, \zeta) D_{m,n}(\theta) D_{m,n}(\zeta) d\theta d\zeta = g\left(\frac{\pi}{2}, \dots, \frac{\pi}{2}; \frac{\pi}{2}, \dots, \frac{\pi}{2}\right).$$

As a result

$$\begin{aligned} & \lim_{n \rightarrow \infty} \int_{U(n) \times U(n)} f(k_1, k_2) \alpha_n(dk_1) \alpha_n(dk_2) \\ &= \int_{K_m^2} \int_{K_m^2} f(h_1 w_m h_2, g_1 w_m g_2) \alpha_m(dh_1) \alpha_m(dh_2) \alpha_m(dg_1) \alpha_m(dg_2). \quad \blacksquare \end{aligned}$$

Corollary 3.3. Let φ be a K_∞ -invariant continuous function on V_∞ . Then, for

$$x = \text{diag}(a_1, \dots, a_m, 0, \dots) \text{ and } y = \text{diag}(b_1, \dots, b_m, 0, \dots),$$

$$\lim_{n \rightarrow \infty} \int_{U(n) \times U(n)} \varphi(x + k_1 y k_2^*) \alpha_n(dk_1) \alpha_n(dk_2) = \varphi(\text{diag}(a_1, \dots, a_m, b_1, \dots, b_m, 0, \dots)).$$

Proof. The function $(k_1, k_2) \mapsto \varphi(x + k_1 y k_2^*)$ is $K_m(\infty) \times K_m(\infty)$ -invariant. Hence, we can apply Proposition 3.2:

$$\begin{aligned} & \lim_{n \rightarrow \infty} \int_{U(n) \times U(n)} \varphi(x + k_1 y k_2^*) \alpha_n(dk_1) \alpha_n(dk_2) \\ &= \int_{K_m^2} \int_{K_m^2} \varphi(x + h_1 w_m h_2 y g_2^* w_m^{-1} g_1^*) \alpha_m(dh_1) \alpha_m(dh_2) \alpha_m(dg_1) \alpha_m(dg_2). \end{aligned}$$

Finally, we obtain the result by using the fact that

$$x + h_1 w_m h_2 y g_2^* w_m^{-1} g_1^* \in U(\infty) \text{diag}(a_1, \dots, a_m, b_1, \dots, b_m) U(\infty). \quad \blacksquare$$

Theorem 3.4. (Multiplicativity Property) Let $\varphi \in \mathfrak{F}$. The function φ is spherical, if and only if, there exists a continuous function Φ on \mathbb{R} , with $\Phi(0) = 1$ such that

$$\varphi(\text{diag}(a_1, \dots, a_n, 0, \dots)) = \Phi(a_1) \dots \Phi(a_n).$$

Proof. Let $\varphi \in \mathfrak{P}$. If φ is spherical then for

$$x = \text{diag}(a_1, \dots, a_m, 0, \dots), \quad y = \text{diag}(b_1, \dots, b_m, 0, \dots),$$

$$\lim_{n \rightarrow \infty} \int_{U(n) \times U(n)} \varphi(x + k_1 y k_2^*) \alpha_n(dk_1) \alpha_n(dk_2) = \varphi(x) \varphi(y).$$

By Corollary 3.3,

$$\varphi(\text{diag}(a_1, \dots, a_m, 0, \dots)) (\text{diag}(b_1, \dots, b_m, 0, \dots)) = \varphi(\text{diag}(a_1, \dots, a_m, b_1, \dots, b_m, 0, \dots)).$$

By applying Corollary 3.3 many times as necessary, one obtains

$$\varphi(\text{diag}(a_1, \dots, a_n, 0, \dots)) = \Phi(a_1) \dots \Phi(a_n),$$

where

$$\Phi(\lambda) = \varphi(\text{diag}(\lambda, 0, \dots)).$$

Inversely, let us assume that there exists a continuous function Φ on \mathbb{R} such that

$$\varphi(\text{diag}(a_1, \dots, a_n, 0, \dots)) = \Phi(a_1) \dots \Phi(a_n).$$

Then, by Corollary 3.3,

$$\lim_{n \rightarrow \infty} \int_{U(n) \times U(n)} \varphi(x + k_1 y k_2^*) \alpha_n(dk_1) \alpha_n(dk_2) = \varphi(x) \varphi(y).$$

Hence the function φ is spherical. ■

4. Modified Pólya functions : definition and convergence

Definition 4.1. The modified Pólya function of parameter $\omega = (\alpha, \gamma)$ with $\alpha = \{\alpha_j\}_{j \geq 1} \in \ell^1(\mathbb{N})$, $\alpha_j \in \mathbb{R}_+$ and $\gamma \in \mathbb{R}_+$ is defined on \mathbb{R} by :

$$\Pi(\omega, \lambda) := e^{-\frac{1}{4}\gamma\lambda^2} \prod_{j=1}^{\infty} \frac{1}{1 + \frac{1}{4}\alpha_j\lambda^2}.$$

We consider on the set \mathcal{P} of modified Pólya functions the topology of uniform convergence on compact sets of \mathbb{R} . The topological space \mathcal{P} is metrizable and complete. This topology can be expressed in terms of the set of parameters :

$$\Omega = \left\{ \omega = (\alpha, \gamma) \mid \alpha = (\alpha_j)_{j \geq 1}, \alpha_j \geq 0, \sum_{j=1}^{\infty} \alpha_j < \infty, \gamma \geq 0 \right\}.$$

For a continuous function f on \mathbb{R} , we define the function L_f on Ω by

$$L_f(\omega) = \int_{\mathbb{R}} f(t) \sigma_{\omega}(dt) = \gamma f(0) + \sum_{j=1}^{\infty} \alpha_j f(\alpha_j). \tag{1}$$

Let us remark that the moments of the measure σ_ω are given by

$$\mathcal{M}_0(\sigma_\omega) = \int_{\mathbb{R}} \sigma_\omega(dt) = \gamma + \sum_{k=1}^{\infty} \alpha_k = \gamma + p_1(\alpha),$$

and for $m \geq 1$,

$$\mathcal{M}_m(\sigma_\omega) = \int_{\mathbb{R}} t^m \sigma_\omega(dt) = \sum_{k=1}^{\infty} \alpha_k^{m+1} = p_{m+1}(\alpha),$$

where p_m is the Newton power sum function : for $x = (x_1, x_2, \dots) \in \ell^1(\mathbb{N})$ and $m \geq 1$,

$$p_m(x) = \sum_{k=1}^{\infty} x_k^m.$$

We consider on Ω the initial topology associated to the functions L_f . A point $\omega \in \Omega$ is seen as a point configuration, i.e. a permutation of the numbers $\{\alpha_k\}$, γ does not change ω . For λ fixed, the function $\omega \mapsto \Pi(\omega, \lambda)$ is injective and continuous on Ω . This can be seen by looking at the logarithmic derivative of $\Pi(\omega, \lambda)$:

$$\frac{\Pi'(\omega, \lambda)}{\Pi(\omega, \lambda)} = -\frac{1}{2}(\gamma + p_1(\alpha))\lambda + i \sum_{m=2}^{\infty} p_m(\alpha) \left(\frac{i\lambda}{2}\right)^{2m-1}. \quad (2)$$

Lemma 4.2. *Let \mathcal{I} be the set of positive measures μ on \mathbb{R}_+^* such that, for all $a > 0$*

$$\mu([a, +\infty[) \in \mathbb{N}.$$

Then, the following properties hold :

(i) *For all $\mu \in \mathcal{I}$, there exists a sequence of positive reals $\{\alpha_k\}_k$ such that*

$$\mu = \sum_{k=1}^N \delta_{\alpha_k},$$

with $N \leq \infty$. If $N = \infty$, then the sequence $\{\alpha_k\}$ converges to 0.

(ii) *Let μ_n be a sequence of measures in \mathcal{I} . Assume that there exists a measure μ on \mathbb{R}_+^* such that, for every function f in the set $\mathcal{C}_b^0(\mathbb{R}_+^*)$ of bounded continuous functions on \mathbb{R}_+^* , vanishing near 0,*

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}_+^*} f(x) \mu_n(dx) = \int_{\mathbb{R}_+^*} f(x) \mu(dx). \quad (3)$$

Then $\mu \in \mathcal{I}$.

Proof. (i) For $t > 0$, let F be the function defined on \mathbb{R}_+^* by $F(t) = \mu([t, \infty[)$. We can observe that F is integer-valued, decreasing, and left-continuous. In consequence, the set D of discontinuity points of F is countable : $D = \{\alpha_k, k \in I\}$. Furthermore, we can observe that $D = \{t > 0 \mid \mu(\{t\}) \neq 0\}$. We can also remark that the jump at each discontinuity point α_k is an integer m_k . Hence, if the number of discontinuity points N is finite, then the measure μ is given by : $\mu = \sum_{k=1}^N m_k \delta_{\alpha_k}$ and, provided that we repeat the α_k as many times as their multiplicity m_k , we get $\mu = \sum_{k=1}^N \delta_{\alpha_k}$. Finally, if D is not finite, i.e. $N = \infty$, then the sequence $\{\alpha_k\}$ converges necessarily to 0.

(ii) Let F_n be the function defined on \mathbb{R}_+^* by $F_n(t) = \mu_n([t, \infty[)$, $F(t) = \mu([t, \infty[)$, and D the set of discontinuity points of F . For $t \in D^c$ fixed, the positive measures μ_n and μ are bounded on $[t, \infty[$. In addition, we have $\mu(\{t\}) = \mu(\{\infty\}) = 0$. Furthermore, it follows by (3) that μ_n converges weakly to μ on $[t, \infty[$. As a result

$$\lim_{n \rightarrow \infty} F_n(t) = F(t) \quad (t \in D^c).$$

Since $F_n(t)$ is a sequence of integers, it follows that, if $t \in D^c$, then $F(t) \in \mathbb{N}$. On the other hand, because D is countable, its complementary set D^c is dense in \mathbb{R}_+^* . This implies, since F is left-continuous, that $F(t) \in \mathbb{N}$ for all $t > 0$. Hence $\mu \in \mathcal{I}$. ■

Theorem 4.3. Ω is weakly closed in the set of bounded measures on \mathbb{R}_+ .

Proof. Let $\omega^{(n)} = (\alpha^{(n)}, \gamma^{(n)})$ be a sequence in Ω . Assume that there is a bounded positive measure σ on \mathbb{R}_+ such that, for every bounded continuous function f on \mathbb{R}_+ ,

$$\lim_{n \rightarrow \infty} \gamma^{(n)} f(0) + \sum_{k=1}^{\infty} \alpha_k^{(n)} f(\alpha_k^{(n)}) = \int_{\mathbb{R}_+} f(t) \sigma(dt). \tag{4}$$

For every $a > 0$, there is a finite number of $\alpha_k^{(n)}$ which are greater than a . As a result, the measure μ_n on \mathbb{R}_+^* given by

$$\int_{\mathbb{R}_+^*} f(t) \mu_n(dt) = \sum_{k=1}^{\infty} f(\alpha_k^{(n)})$$

belongs to \mathcal{I} . Furthermore, by (4), we can observe that μ_n converges weakly on \mathbb{R}_+^* to $\frac{\tilde{\sigma}}{t}$, where $\tilde{\sigma}$ is the restriction of the measure σ to \mathbb{R}_+^* . By (ii) of Lemma 4.2, there exists a sequence $\alpha = \{\alpha_k\}_k$ such that, for every function $f \in \mathcal{C}_b^0(\mathbb{R}_+^*)$,

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}_+^*} f(t) \mu_n(dt) = \sum_{k=1}^{\infty} f(\alpha_k).$$

Since the set $\mathcal{C}_c(\mathbb{R}_+^*)$ of continuous functions with compact support in \mathbb{R}_+^* is included in $\mathcal{C}_b^0(\mathbb{R}_+^*)$, it follows that $\tilde{\sigma}$ is given by

$$\int_{\mathbb{R}_+^*} f(t) \tilde{\sigma}(dt) = \sum_{k=1}^{\infty} \alpha_k f(\alpha_k).$$

Since the measure σ is bounded, $\sum_{k=1}^{\infty} \alpha_k < \infty$. Finally

$$\sigma = \gamma\delta_0 + \sum_{k=1}^{\infty} \alpha_k \delta_{\alpha_k},$$

with $\gamma = \sigma(\{0\})$. Hence $\sigma = \sigma_\omega$ with $\omega = (\alpha, \gamma) \in \Omega$. ■

A modified Pólya function of parameter ω is the Fourier transform of a probability measure μ_ω . Let \mathfrak{M}_Ω be the set of these measures :

$$\mathfrak{M}_\Omega = \{\mu_\omega | \Pi(\omega, \lambda) = \widehat{\mu_\omega}(\lambda)\}.$$

We consider on \mathfrak{M}_Ω the weak topology of measures. We will prove that the topology defined on Ω is equivalent to the weak topology of \mathfrak{M}_Ω . We will need the following proposition (see [7], Proposition 3.11) :

Proposition 4.4. *Let ψ_n be a sequence of C^∞ functions on \mathbb{R}^d of positive type with $\psi_n(0) = 1$, and ψ an analytic function on a neighborhood of 0. Assume that, for every $\alpha = (\alpha_1, \dots, \alpha_d)$,*

$$\lim_{n \rightarrow \infty} \partial^\alpha \psi_n(0) = \partial^\alpha \psi(0).$$

Then ψ has an analytic extension to \mathbb{R}^d , and ψ_n converges to ψ uniformly on compact sets in \mathbb{R}^d .

Proposition 4.5. *The topology of Ω is equivalent to the topology of \mathfrak{M}_Ω .*

Proof. (i) Assume that $\omega^{(n)}$ converges to ω in the topology of Ω . Then the modified Pólya functions $\Pi(\omega^{(n)}, \lambda)$ and $\Pi(\omega, \lambda)$ are holomorphic in $D(0, R)$ where $\frac{1}{R} = \sup_{m,n} |\alpha_m^{(n)}|$. Therefore, for every $\lambda \in D(0, R)$, their logarithmic derivatives are given by

$$\frac{\Pi'(\omega^{(n)}, \lambda)}{\Pi(\omega^{(n)}, \lambda)} = -\frac{1}{2}(\gamma^{(n)} + p_1(\alpha^{(n)}))\lambda + i \sum_{m=2}^{\infty} p_m(\alpha^{(n)}) \left(\frac{i\lambda}{2}\right)^{2m-1},$$

$$\frac{\Pi'(\omega, \lambda)}{\Pi(\omega, \lambda)} = -\frac{1}{2}(\gamma + p_1(\alpha))\lambda + i \sum_{m=2}^{\infty} p_m(\alpha) \left(\frac{i\lambda}{2}\right)^{2m-1}.$$

For every bounded continuous function f on \mathbb{R}

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}} f(t) \sigma_{\omega^{(n)}}(dt) = \int_{\mathbb{R}} f(t) \sigma_\omega(dt),$$

where $\sigma_{\omega^{(n)}}$ is the bounded positive measure on \mathbb{R} associated to $\omega^{(n)} = (\alpha^{(n)}, \gamma^{(n)})$. Hence, for $f = 1$, the sequence $\gamma^{(n)} + p_1(\alpha^{(n)})$ converges. In consequence, there exists a constant $A > 0$ such that, for every n ,

$$0 \leq \gamma^{(n)} + p_1(\alpha^{(n)}) \leq A.$$

Therefore, the sequence $\{\alpha_m^{(n)}\}$ is bounded by A for every m and every n .
Hence,

$$\text{supp}(\sigma_{\omega^{(n)}}) \subset [-A, A] \text{ and } \text{supp}(\sigma_\omega) \subset [-A, A].$$

It follows that the sequence of measures $\sigma_{\omega^{(n)}}$ converges to σ_ω for every continuous function on \mathbb{R} . We can then deduce that, for every $m \geq 2$,

$$\begin{aligned} \lim_{n \rightarrow \infty} p_m(\alpha^{(n)}) &= \lim_{n \rightarrow \infty} \int_{\mathbb{R}} t^{m-1} \sigma_{\omega^{(n)}}(dt) \\ &= \int_{\mathbb{R}} t^{m-1} \sigma_\omega(dt) = p_m(\alpha), \end{aligned}$$

and

$$p_m(\alpha^{(n)}) \leq (p_1(\alpha^{(n)}))^m \leq A^m.$$

Furthermore, we have

$$\lim_{n \rightarrow \infty} \gamma^{(n)} + p_1(\alpha^{(n)}) = \gamma + p_1(\alpha).$$

In consequence,

$$\lim_{n \rightarrow \infty} \frac{\Pi'(\omega^{(n)}, \lambda)}{\Pi(\omega^{(n)}, \lambda)} = \frac{\Pi'(\omega, \lambda)}{\Pi(\omega, \lambda)},$$

the convergence being uniform on compact sets in $D(0, R)$. Hence, for $|\lambda| < R$,

$$\lim_{n \rightarrow \infty} \Pi(\omega^{(n)}, \lambda) = \Pi(\omega, \lambda),$$

since $\Pi(\omega^{(n)}, 0) = 1$ and $\Pi(\omega, 0) = 1$.

The functions $\Pi(\omega^{(n)}, \lambda)$ and $\Pi(\omega, \lambda)$ being of positive type, by Proposition 4.4, $\Pi(\omega^{(n)}, \lambda)$ converges uniformly on compact sets in \mathbb{R} to $\Pi(\omega, \lambda)$. Finally, by applying the Lévy-Cramer theorem, one can prove that $\mu_{\omega^{(n)}}$ converges weakly to μ_ω .

(ii) Assume that $\mu_{\omega^{(n)}}$ converges weakly to μ_ω . This implies that $\Pi(\omega^{(n)}, \lambda)$ converges uniformly on compact sets in \mathbb{R} to $\Pi(\omega, \lambda)$. Let $\lambda_0 > 0$. Since the modified Pólya function $\Pi(\omega^{(n)}, \lambda)$ is continuous, non-zero on \mathbb{R} and satisfies $\Pi(\omega^{(n)}, 0) = 1$, there exists $M > 0$ such that, for every n , $\Pi(\omega^{(n)}, \lambda_0) \geq M$. In consequence

$$\frac{1}{4} \gamma^{(n)} \lambda_0^2 + \frac{1}{4} \lambda_0^2 \sum_{k=1}^{\infty} \alpha_k^{(n)} \leq e^{\frac{1}{4} \gamma^{(n)} \lambda_0^2} \prod_{k=1}^{\infty} \left(1 + \frac{1}{4} \alpha_k^{(n)} \lambda_0^2 \right) \leq \frac{1}{M}.$$

Therefore

$$-2\Pi''(\omega^{(n)}, 0) = p_1(\alpha^{(n)}) + \gamma^{(n)} \leq \frac{4}{\lambda_0^2 M} =: R.$$

It follows that the function $\Pi(\omega^{(n)}, \lambda)$ is holomorphic for $|\lambda| < R$ and then also in the strip $\Sigma_R = \{x + iy \mid |y| < R\}$ (see [5], Lemma 6.5). Furthermore, for $r < R$, there exists a constant $M(r) > 0$ such that $|\Pi(\omega^{(n)}, \lambda)| \leq M(r)$ for $\lambda \in \Sigma_r$. From the theorem of Montel, it follows that there is a subsequence $\Pi(\omega^{(n_j)}, \lambda)$

which converges uniformly on compact sets in Σ_R . Since the sequence itself converges to $\Pi(\omega, \lambda)$ on \mathbb{R} , we get that $\Pi(\omega^{(n)}, \lambda)$ converges to $\Pi(\omega, \lambda)$ uniformly on compact sets in Σ_R . As a result, the logarithmic derivatives $\frac{\Pi'(\omega^{(n)}, \lambda)}{\Pi(\omega^{(n)}, \lambda)}$ converge uniformly in a neighborhood of 0 which implies the convergence of the coefficients of their Taylor expansions at 0 :

$$\lim_{n \rightarrow \infty} \gamma^{(n)} + p_1(\alpha^{(n)}) = \gamma + p_1(\alpha),$$

$$\lim_{n \rightarrow \infty} p_m(\alpha^{(n)}) = p_m(\alpha), \quad \text{for } m \geq 2.$$

This proves that $\omega^{(n)}$ converges to ω in the topology of Ω . ■

Let us put for every modified Pólya function of parameter $\omega = (\alpha, \gamma)$,

$$\tilde{p}_1(\omega) = p_1(\alpha) + \gamma,$$

and, for every $R \geq 0$,

$$\Omega_R = \{\omega \in \Omega \mid -2\Pi''(\omega, 0) = \tilde{p}_1(\omega) \leq R\}.$$

Corollary 4.6. Ω_R is a compact subset of Ω .

Proof. Since the modified Pólya function is the Fourier transform of a probability measure μ , we have

$$-\Pi''(\omega, 0) = \int_{\mathbb{R}} t^2 \mu_\omega(dt).$$

We also know that the set of probability measures such that $\int_{\mathbb{R}} t^2 \mu_\omega(dt) \leq \frac{R}{2}$ is relatively compact. By using Theorem 4.3 and Proposition 4.5, we can conclude that the set Ω_R is relatively compact in Ω . Moreover, the convergence of a sequence $\omega^{(n)}$ to ω in Ω implies that $\tilde{p}_1(\omega^{(n)})$ converges to $\tilde{p}_1(\omega)$. Hence, the map $\omega \mapsto \tilde{p}_1(\omega)$ is continuous. In consequence, the set Ω_R is closed and relatively compact in Ω , therefore compact. ■

5. Convergence of orbital measures and spherical functions

Let (X, \mathcal{B}) be a measurable space on which a group G acts by measurable transformations. Let ν be a G -invariant probability measure on X . A set $E \in \mathcal{B}$ is said to be G -invariant relatively to ν if, for every $g \in G$, $\nu((gE)\Delta E) = 0$, where Δ is the symmetric difference. The measure ν is said to be ergodic relatively to the action of G if, for every $E \in \mathcal{B}$ which is G -invariant : $\nu(E) = 0$ or 1. If X is locally compact and K a compact group acting on X by homeomorphisms,

then the ergodic measures are exactly the orbital ones. It is the case, if one takes $X = V_n = M(n, \mathbb{C})$ and $K = K_n = U(n) \times U(n)$.

Let V^∞ be the space of infinite complex matrices. It is the dual space of V_∞ . The space V^∞ is defined as the projective limit of the sequence V_n relatively to the orthogonal projections

$$p_{m,n} : V_n \rightarrow V_m \quad (n > m).$$

The space V^∞ can be identified to \mathbb{R}^∞ . The group K_∞ acts on it in the same way as on V_∞ . Let \mathfrak{M} be the set of probability measures on V^∞ which are invariant by K_∞ . The Fourier transform of a measure ν in \mathfrak{M} is the K_∞ -invariant function of positive type φ defined on V_∞ by

$$\varphi(\xi) = \int_{V^\infty} e^{i\langle x, \xi \rangle} \nu(dx).$$

The Fourier transform establishes a bijection from \mathfrak{M} onto \mathfrak{P} and also between their corresponding subsets of extreme points $\text{ext}(\mathfrak{M})$ and $\text{ext}(\mathfrak{P})$. A function $\varphi \in \mathfrak{P}$ is spherical if and only if the measure ν is ergodic.

Let $\lambda^{(n)}$ be a sequence of diagonal matrices

$$\lambda^{(n)} = \text{diag}(\lambda_1^{(n)}, \dots, \lambda_n^{(n)}).$$

We associate to it the sequence of orbital measures $\nu^{(n)}$ such that, for every continuous function f on V^∞ :

$$\int_{V^\infty} f(x) \nu^{(n)}(dx) = \int_{U(n) \times U(n)} f(u \lambda^{(n)} v^*) \alpha_n(du) \alpha_n(dv).$$

The Fourier transform φ_n of $\nu^{(n)}$ is defined on V_∞ by

$$\begin{aligned} \varphi_n(\xi) &= \varphi_n(\xi, \lambda^{(n)}) = \int_{V^\infty} e^{i\langle \xi, x \rangle} \nu^{(n)}(dx) \\ &= \int_{U(n) \times U(n)} e^{i\langle \xi, u \lambda^{(n)} v^* \rangle} \alpha_n(du) \alpha_n(dv). \end{aligned}$$

We calculated the preceding integral in Corollary 2.4. Let us put

$$F(z) = \sum_{k=0}^{\infty} \frac{1}{(k!)^2} z^k, \quad (z \in \mathbb{C}),$$

$$\Xi = (\Xi_1, \Xi_2, \dots, \Xi_n, 0, \dots) := \left(-\frac{\xi_1^2}{4}, -\frac{\xi_2^2}{4}, \dots, -\frac{\xi_n^2}{4}, 0, \dots \right),$$

$$\Lambda = (\Lambda_1, \Lambda_2, \dots, \Lambda_n) := (\lambda_1^{(n)2}, \lambda_2^{(n)2}, \dots, \lambda_n^{(n)2}).$$

For $\xi = \text{diag}(\xi_1, \dots, \xi_n, 0, \dots)$, we can rewrite the spherical function φ_n as

$$\varphi_n(\xi, \lambda^{(n)}) = \Psi(\Lambda, \Xi) = (\delta!)^2 \frac{\det \left(\left(F(\Lambda_i \Xi_j) \right)_{1 \leq i, j \leq n} \right)}{D(\Lambda)D(\Xi)}, \quad (5)$$

where

$$D(\Xi) = \prod_{i < j} (\Xi_i - \Xi_j), \quad D(\Lambda) = \prod_{i < j} (\Lambda_i - \Lambda_j).$$

By a result in ([5], section 2.3), we can write (5) as

$$\varphi_n(\xi, \lambda^{(n)}) = \sum_{m_1 \geq \dots \geq m_n \geq 0} \left(\frac{\delta!}{(\mathbf{m} + \delta)!} \right)^2 s_{\mathbf{m}}(\Lambda) s_{\mathbf{m}}(\Xi). \quad (6)$$

The generating function of the complete symmetric function

$$h_m(x) = \sum_{|\alpha|=m} x^\alpha$$

is given by :

$$H(x, t) := \sum_{m=0}^{\infty} h_m(x) t^m = \prod_{j=1}^n \frac{1}{1 - x_j t}.$$

The logarithmic derivative of $H(x, -\frac{t^2}{4})$ with respect to t is given by :

$$\frac{H'(x, -\frac{t^2}{4})}{H(x, -\frac{t^2}{4})} = - \sum_{j=1}^n \frac{\frac{1}{2} x_j t}{1 + \frac{1}{4} x_j t^2} = i \sum_{m=1}^{\infty} p_m(x) \left(\frac{it}{2} \right)^{2m-1}. \quad (7)$$

Let Γ be the algebra of symmetric functions on

$$\mathbb{C}^{(\infty)} = \{z_1, z_2, \dots\} \mid z_i \in \mathbb{C} \text{ are zero for } i \text{ large enough}.$$

The set $\{s_{\mathbf{m}}\}$ where \mathbf{m} runs over all partitions is a system of linear generators of Γ . Furthermore, the sets $\{h_m\}_{m \geq 1}$ and $\{p_m\}_{m \geq 1}$ are systems of generators of Γ :

$$\Gamma = \mathbb{C}[h_1, h_2, \dots] = \mathbb{C}[p_1, p_2, \dots].$$

Let us consider the algebra morphism :

$$\Gamma \rightarrow \mathcal{C}(\Omega), \quad g \mapsto \tilde{g}, \quad (8)$$

which is uniquely determined by :

$$\tilde{p}_1(\omega) = \gamma + \sum_{j=1}^{\infty} \alpha_j, \quad \tilde{p}_m(\omega) = \sum_{j=1}^{\infty} \alpha_j^m \quad (m \geq 2). \quad (9)$$

The functions \tilde{p}_m are continuous on Ω . This can be shown by taking in (1) $f \equiv 1$ for $m = 1$, or $f(t) = t^{2m-2}$ for $m \geq 2$.

Proposition 5.1. *It holds that*

$$\tilde{H}(\omega, -\frac{\lambda^2}{4}) = \Pi(\omega, \lambda).$$

Furthermore, the Taylor expansion of $\Pi(\omega, \lambda)$ is :

$$\Pi(\omega, \lambda) = \sum_{m=0}^{\infty} \tilde{h}_m(\omega) \left(-\frac{\lambda^2}{4}\right)^m.$$

Here $\tilde{H}(\omega, -\frac{\lambda^2}{4})$ is the image of $H(\cdot, -\frac{\lambda^2}{4})$ under the morphism (8) and $\Pi(\omega, \lambda)$ is the modified Pólya function.

Proof. By (2), (9) and (7), we have

$$\frac{\Pi'(\omega, \lambda)}{\Pi(\omega, \lambda)} = i \sum_{m=1}^{\infty} \tilde{p}_m(\omega) \left(\frac{i\lambda}{2}\right)^{2m-1} = \frac{\tilde{H}'(\omega, -\frac{\lambda^2}{4})}{\tilde{H}(\omega, -\frac{\lambda^2}{4})}.$$

Since $\Pi(\omega, 0) = 1$ and $\tilde{H}(\omega, 0) = 1$, the statement follows. ■

Let us consider now the map

$$T_n : \mathbb{R}^n \rightarrow \Omega; (\lambda_1, \lambda_2, \dots, \lambda_n) \mapsto \omega = (\alpha, \gamma)$$

given by

$$\alpha_j = \left(\frac{\lambda_j}{n}\right)^2, \quad \gamma = 0.$$

Theorem 5.2. *Let $\lambda^{(n)} \in \mathbb{R}^n$ be a sequence such that the following limit exists for the topology of Ω :*

$$\lim_{n \rightarrow \infty} T_n(\lambda^{(n)}) = \omega.$$

Then, for every $g \in \Gamma$, homogeneous of degree m ,

$$\lim_{n \rightarrow \infty} \frac{1}{n^{2m}} g\left((\lambda^{(n)})^2\right) = \tilde{g}(\omega).$$

Proof. It is enough to prove the result for $g = p_m$ since the Newton power sums generate Γ . Let $m = 1$, then

$$p_1(\lambda^{(n)}) = \sum_{j=1}^n \lambda_j^{(n)}, \quad \tilde{p}_1(\omega) = \gamma + \sum_{j=1}^{\infty} \alpha_j.$$

By assumption, for every continuous function f on \mathbb{R} ,

$$\lim_{n \rightarrow \infty} \sum_{k=1}^n \left(\frac{\lambda_k^{(n)}}{n}\right)^2 f\left(\frac{\lambda_k^{(n)}}{n}\right) = \gamma f(0) + \sum_{k=1}^{\infty} \alpha_k f(\alpha_k).$$

In particular, by taking $f \equiv 1$ we get

$$\lim_{n \rightarrow \infty} \sum_{k=1}^n \left(\frac{\lambda_k^{(n)}}{n} \right)^2 = \gamma + \sum_{k=1}^{\infty} \alpha_k,$$

$$\lim_{n \rightarrow \infty} \frac{1}{n^2} p_1 \left((\lambda^{(n)})^2 \right) = \tilde{p}_1(\omega).$$

For $m \geq 2$, by taking $f(t) = t^{2m-2}$, one obtains

$$\lim_{n \rightarrow \infty} \frac{1}{n^{2m}} p_m \left((\lambda^{(n)})^2 \right) = \tilde{p}_m(\omega). \quad \blacksquare$$

Proposition 5.3. For any $\omega \in \Omega$ and $\xi_1, \dots, \xi_k \in \mathbb{C}$, it holds that

$$\sum_{\mathbf{m}: \text{partitions}} \tilde{s}_{\mathbf{m}}(\omega) s_{\mathbf{m}} \left(-\frac{\xi_1^2}{4}, \dots, -\frac{\xi_k^2}{4} \right) = \prod_{j=1}^k \Pi(\omega, \xi_j).$$

Proof. Recall the Cauchy formula (see [5], Proposition 2.5) :

$$\sum_{\mathbf{m}} s_{\mathbf{m}}(x_1, x_2, \dots) s_{\mathbf{m}}(y_1, \dots, y_k) = \prod_{i=1}^{\infty} \prod_{j=1}^k \frac{1}{1 - x_i y_j} = \prod_{j=1}^k H(x, y_j).$$

We apply the morphism (8) to both sides of the preceding equality with $y_j = -\frac{\xi_j^2}{4}$. The result follows by Proposition 5.1. \blacksquare

Theorem 5.4. As in Theorem 5.2, we assume that

$$\lim_{n \rightarrow \infty} T_n(\lambda^{(n)}) = \omega. \quad \blacksquare$$

Then, for a fixed diagonal matrix $\xi = \text{diag}(\xi_1, \dots, \xi_k, 0, \dots)$, the sequence $\varphi_n(\xi, \lambda^{(n)})$ converges uniformly on compact sets in \mathbb{R}^k :

$$\lim_{n \rightarrow \infty} \varphi_n(\xi, \lambda^{(n)}) = \prod_{j=1}^k \Pi(\omega, \xi_j).$$

Proof. Consider first $k = 1$. This case corresponds to a single variable $\xi = (\xi, 0, \dots)$ and the unique non-zero terms in the Schur function expansion are those for which $\mathbf{m} = (m, 0, \dots, 0)$. Hence by (6) we have

$$\varphi_n(\xi, \lambda^{(n)}) = \sum_{m=0}^{\infty} \left(\frac{(n-1)!}{(m+n-1)!} \right)^2 h_m(\lambda^{(n)2}) \left(-\frac{\xi^2}{4} \right)^m.$$

Since

$$\frac{(n-1)!}{(m+n-1)!} \sim \frac{1}{n^m} \quad \text{as } n \rightarrow \infty,$$

by Theorem 5.2

$$\lim_{n \rightarrow \infty} \left(\frac{(n-1)!}{(m+n-1)!} \right)^2 h_m(\lambda^{(n)^2}) = \tilde{h}_m(\omega).$$

Now by applying Proposition 4.4 about the convergence of \mathcal{C}^∞ -functions of positive type on \mathbb{R}^d , we obtain

$$\lim_{n \rightarrow \infty} \varphi_n(\xi, \lambda^{(n)}) = \sum_{m=0}^{\infty} \tilde{h}_m(\omega) \left(-\frac{\xi^2}{4} \right)^m.$$

Finally, by Proposition 5.1,

$$\sum_{m=0}^{\infty} \tilde{h}_m(\omega) \left(-\frac{\lambda^2}{4} \right)^m = \Pi(\omega, \lambda).$$

Now, let us consider the multivariable case $\xi = (\xi_1, \dots, \xi_k, 0, \dots)$ for which $k > 1$. If $m_{k+1} > 0$, then $s_{\mathbf{m}}(\xi_1, \dots, \xi_k, 0, \dots) = 0$. In consequence

$$\varphi_n(\xi, \lambda^{(n)}) = \sum_{m_1 \geq \dots \geq m_k \geq 0} \left(\frac{\delta!}{(\mathbf{m} + \delta)!} \right)^2 s_{\mathbf{m}}(\lambda^{(n)^2}) s_{\mathbf{m}} \left(-\frac{\xi_1^2}{4}, \dots, -\frac{\xi_k^2}{4}, 0, \dots \right).$$

But, for \mathbf{m} fixed, $n \rightarrow \infty$,

$$\frac{\delta!}{(\mathbf{m} + \delta)!} \sim \frac{1}{n^{|\mathbf{m}|}}, \quad |\mathbf{m}| = m_1 + m_2 + \dots$$

Hence by Theorem 5.2

$$\lim_{n \rightarrow \infty} \left(\frac{\delta!}{(\mathbf{m} + \delta)!} \right)^2 s_{\mathbf{m}}(\lambda^{(n)^2}) = \tilde{s}_{\mathbf{m}}(\omega).$$

Similarly, by Proposition 4.4,

$$\lim_{n \rightarrow \infty} \varphi_n(\xi, \lambda^{(n)}) = \sum_{\mathbf{m}: \text{partitions}} \tilde{s}_{\mathbf{m}}(\omega) s_{\mathbf{m}} \left(-\frac{\xi_1^2}{4}, \dots, -\frac{\xi_k^2}{4}, 0, \dots \right).$$

Finally, by Proposition 5.3,

$$\sum_{\mathbf{m}: \text{partitions}} \tilde{s}_{\mathbf{m}}(\omega) s_{\mathbf{m}} \left(-\frac{\xi_1^2}{4}, \dots, -\frac{\xi_k^2}{4}, 0, \dots \right) = \prod_{j=1}^k \Pi(\omega, \xi_j). \quad \blacksquare$$

The preceding theorem shows that the limit of a spherical function of positive type on V_n is a spherical function of positive type on V_∞ which is given as a finite product of modified Pólya functions. In order to prove that all spherical functions of positive type on V_∞ are obtained in the same way, we need to prove the converse of Theorem 5.4. This will be done in Theorem 5.6 using the following lemma :

Lemma 5.5. For every $n \geq 1$, let $\omega^{(n)}$ be the point in Ω associated to an orbital measure $\nu^{(n)}$ that weakly converges to an ergodic measure ν on V^∞ . If

$$\sup_n \left(p_1(\alpha^{(n)}) \right) = R < \infty, \quad (10)$$

then the sequence of parameters $\omega^{(n)}$ converges in Ω .

Proof. By condition (10), the sequence $\omega^{(n)}$ belongs to the compact set Ω_R (see Corollary 4.6). We can then extract a subsequence $(\omega^{(n_k)})_k$ which converges in Ω to $\omega = (\alpha, \gamma)$. Hence, by Theorem 5.4, the Fourier transform $\varphi^{(n_k)}$ of $\nu^{(n_k)}$ uniformly converges on compact sets :

$$\lim_{k \rightarrow \infty} \varphi^{(n_k)}(\text{diag}(\xi, 0, \dots)) = \Pi(\omega, \xi).$$

Therefore, the Fourier transform φ of ν is identically equal to $\Pi(\omega, \cdot)$.

If another sub-sequence such that $\omega^{(n_k')}$ converges in Ω to ω' , then $\varphi \equiv \Pi(\omega', \cdot)$. By uniqueness, we get $\omega = \omega'$, and $\omega^{(n)}$ do have in consequence a unique accumulation point. Therefore, it necessarily converges to ω . ■

Theorem 5.6. The spherical functions of positive type on V_∞ (i.e. the extreme points of \mathfrak{P}) are the functions φ_ω defined, for every $k \geq 1$, by :

$$\varphi_\omega(\text{diag}(\xi_1, \dots, \xi_k, 0, \dots)) = \Pi(\omega, \xi_1) \dots \Pi(\omega, \xi_k),$$

where $\Pi(\omega, \cdot)$ is the modified Pólya function associated to $\omega \in \Omega$.

Proof. (a) The function φ_ω , which is of positive type, is spherical since it is multiplicative (Theorem 3.4). Therefore, it is an extreme point in \mathfrak{P} .

(b) Let $\varphi \in \text{ext}(\mathfrak{P})$. It is the Fourier transform of an ergodic measure ν on V^∞ relatively to the action of K_∞ . By using ([10], Theorem 3.2), the measure ν is the weak limit of $\nu^{(n)}$, where $\nu^{(n)}$ is a sequence of orbital measures relatively to K_n . Hence φ is the uniform limit on compact sets of the sequence φ_n , where φ_n is the Fourier transform of $\nu^{(n)}$. In particular we have

$$\lim_{n \rightarrow \infty} \varphi_n(\text{diag}(\xi, 0, \dots), \lambda^{(n)}) = \varphi(\text{diag}(\xi, 0, \dots)).$$

We will prove that $\omega^{(n)} = T_n(\lambda^{(n)})$ converges in Ω , which gives the converse of Theorem 5.4. Let us suppose now that the condition (10) is not satisfied :

$$\sup_n \left(p_1 \left(\left(\frac{\lambda^{(n)}}{n} \right)^2 \right) \right) = \infty.$$

There exists a positive sequence ϵ_n such that

$$\lim_{n \rightarrow \infty} \epsilon_n = 0 \text{ and } \lim_{n \rightarrow \infty} p_1 \left(\epsilon_n \left(\frac{\lambda^{(n)}}{n} \right)^2 \right) = 1.$$

One can remark that multiplying $(\frac{\lambda^{(n)}}{n})^2$ by ϵ_n is the same as multiplying ξ by ϵ_n . Hence, by Lemma 5.5, there exists $\omega \in \Omega$ such that :

$$\lim_{n \rightarrow \infty} \varphi_n (\text{diag} (\epsilon_n \xi, 0, \dots), \lambda^{(n)}) = \Pi(\omega, \xi).$$

The modified Pólya function $\Pi(\omega, \cdot)$ is not identically equal to 1, because $\omega \neq 0$. Therefore, in a neighborhood $D(0, R)$ of 0, the function $\Pi(\omega, \cdot)$ is not identically equal to 1. But this leads to a contradiction, because, by our assumption,

$$\begin{aligned} & \lim_{n \rightarrow \infty} \varphi_n (\text{diag} (\epsilon_n \xi, 0, \dots), \lambda^{(n)}) \\ &= \lim_{n \rightarrow \infty} \sum_{m=0}^{\infty} \left(\frac{n^m \epsilon_n^m}{n(n+1) \dots (n+m-1)} \right)^2 h_m \left(\left(\frac{\lambda^{(n)}}{n} \right)^2 \right) \left(-\frac{\xi^2}{4} \right)^m = 1, \end{aligned}$$

and then $\Pi(\omega, \cdot) \equiv 1$. Hence, there exists $\omega \in \Omega$ such that : $\varphi(\xi) = \varphi_\omega(\xi)$. ■

Remark 5.7. A measurable space is said to be standard if it is isomorphic to a Borel subset in a polish space which is equipped with the σ -algebra induced by the Borel one. One can prove that the correspondence $\Omega \leftrightarrow \text{ext}(\mathfrak{P})$ is an isomorphism between two standard spaces. This enables us to prove a parameterized version of the generalized Bochner theorem ([15], Theorem 7) : let φ be a K_∞ -invariant continuous function of positive type on V_∞ with $\varphi(e) = 1$. Then, there exists a unique probability measure μ defined on Ω such that, for every $g \in V_\infty$,

$$\varphi(g) = \int_{\Omega} \varphi_\omega(g) \mu(d\omega).$$

Acknowledgement I would like to express my gratitude to Professor Jacques Faraut for numerous suggestions during the preparation of this paper.

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Received June 3, 2008
and in final form July 27, 2008