Journal of Lie Theory
Volume **18** (2008) 433–443

Linear Maps Preserving Fibers

Gerald W. Schwarz*

Communicated by F. Knop

Abstract. Let $G \subset \mathrm{GL}(V)$ be a complex reductive group where $\dim V < \infty$, and let $\pi\colon V \to V/\!\!/G$ be the categorical quotient. Let $\mathcal{N} := \pi^{-1}\pi(0)$ be the null cone of V, let H_0 be the subgroup of $\mathrm{GL}(V)$ which preserves the ideal $\mathcal{I}$ of $\mathcal{N}$ and let H be a Levi subgroup of H_0 containing G. We determine the identity component of H. In many cases we show that $H = H_0$. For adjoint representations we have $H = H_0$ and we determine H completely. We also investigate the subgroup G_F of $\mathrm{GL}(V)$ preserving a fiber F of π when V is an irreducible cofree G-module.
Mathematics Subject Index 2000: 20G20, 22E46, 22E60.
Keywords and phrases: Invariants, null cone, cofree representations.

1. Introduction

Our base field is $\mathbb{C}$, the field of complex numbers. Let V be a finite dimensional G-module where $G \subset \mathrm{GL}(V)$ is reductive. Let R denote $\mathbb{C}[V]$. We have the categorical quotient $\pi\colon V \to V/\!\!/G$ dual to the inclusion $R^G \subset R$. Let $\mathcal{N}_G := \pi^{-1}\pi(0)$ (or just $\mathcal{N}$) denote the null cone. Let $G_0 = \{g \in \mathrm{GL}(V) \mid f \circ g = f$ for all $f \in R^G\}$. Let H_0 denote the subgroup of $\mathrm{GL}(V)$ which preserves $\mathcal{N}_G$ schematically. Equivalently, H_0 is the group preserving the ideal $\mathcal{I} = R^G_+R$ where R^G_+ is the ideal of invariants vanishing at 0. Let G_1 be a Levi factor of G_0 containing G and let H denote a Levi factor of H_0 containing G_1. We show that $H^0 \subset G_1\,\mathrm{GL}(V)^{G_1}$, hence that $H^0 \subset G_1\,\mathrm{GL}(V)^G$. In many cases H_0 and G_0 are reductive, for example, if V is irreducible. In the case that $V = \mathfrak{g}$ is a semisimple Lie algebra and G its adjoint group we show that $H = H_0 = (\mathbb{C}^*)^r \,\mathrm{Aut}(\mathfrak{g})$ where r is the number of simple ideals in $\mathfrak{g}$. We also obtain information about the subgroup of $\mathrm{GL}(\mathfrak{g})$ preserving a fiber of π (other than the zero fiber). We have similar resuts in the case that V is a cofree G-module. Our results generalize those of Botta, Pierce and Watkins [1] and Watkins [12] for the case $\mathfrak{g} = \mathfrak{sl}_n$. Finally, we show that if $G \subset G' \subset \mathrm{GL}(V)$ where G' is connected reductive such that π and $\pi'\colon V \to V/\!\!/G'$ have a common fiber, then $R^G = R^{G'}$.

We thank M. Raïs for his help and for the questions and conjectures in his work [5] which led to this paper.

* Partially supported by NSA Grant H98230-06-1-0023

ISSN 0949–5932 / \$2.50

2. Equal fibers

Let $G \subset G' \subset \mathrm{GL}(V)$ be reductive where G' is connected. We have quotient mappings $\pi\colon V \to V/\!\!/G$ and $\pi'\colon V \to V/\!\!/G'$. Let $\rho\colon V/\!\!/G \to V/\!\!/G'$ denote the canonical map.

Theorem 2.1. *Suppose that there is a fiber F of π which is also a fiber of π' (as sets). Then $R^G = R^{G'}$.*

Proof. The hypothesis implies that there is a point $z' \in X' := V/\!\!/G'$ such that $\rho^{-1}(z')$ is a point in $X := V/\!\!/G$. Since ρ is surjective, the minimal dimension of any irreducible component of a fiber is the difference in the dimensions of X and X', so we have that $\dim X = \dim X'$. Then there is a nonempty open subset U of X' such that the fiber of ρ over any point of U is finite. But for $z' \in X'$, the fiber $(\pi')^{-1}(z')$ is connected since G' is connected. Hence the fiber $\rho^{-1}(z') = \pi((\pi')^{-1}(z'))$ is connected. It follows that $\rho\colon \rho^{-1}(U) \to U$ is 1-1 and onto, hence birational. Thus ρ is an isomorphism [3, II.3.4] ∎

Remark 2.2. Solomon [10, 11] has classified many of the pairs of groups $G \subset G' \subset \mathrm{GL}(V)$ with the same invariants, including the case where V is irreducible. Often, $R^G = R^{G'}$ forces $G = G'$. Suppose that (V, G) is *generic*, i.e., it has trivial principal isotropy groups and the complement of the set of principal orbits has codimension two in V. Then $R^G = R^{G'}$ implies that $G = G'$ [9].

3. Groups preserving the ideal of $\mathcal{N}$

Let V be a G-module. We assume that G is a Levi subgroup of G_0. Let H be a Levi subgroup of H_0 containing G. Our aim is to show that H^0 is generated by $\mathrm{GL}(V)^G$ and G^0.

Proposition 3.1. *Let V, G and H be as above. Then G is normal in H.*

Proof. Let $p_1, \dots, p_r$ be a set of minimal homogeneous generators of R^G. Let $d_1 < d_2 < \cdots < d_s$ be the distinct degrees of the p_i. Then clearly H preserves the span W_1 of the p_i of degree d_1. Assuming that $s > 1$, let W_2' be the span of the p_i of degree d_2. Then H stabilizes $W_0 := R_{d_2-d_1}W_1$ and H stabilizes $W := W_2' + W_0 = \mathcal{I} \cap R_{d_2}$ where R_d for $d \in \mathbb{N}$ denotes the elements of R homogeneous of degree d. Note that $W_2' \cap W_0 = W_2' \cap R^G \cdot W_1 = 0$. Since H is reductive, there is an H-stable subspace W_2 of W complementary to W_0. Since G acts trivially on W_2', it acts trivially on W/W_0 and on W_2. Continuing in this way we obtain H-modules $W_1, \dots, W_s$ consisting of G-invariant functions such that $W' := W_1 + \cdots + W_s$ generates R^G. Clearly G is the kernel of the action of H on W'. ∎

Corollary 3.2. *Suppose that H_0 is reductive. Then G_0 is reductive and normal in H_0.*

Since G^0 is reductive, H^0 acts on G^0 by inner automorphisms. Hence $H^0 = H_1G^0$ where $H_1 := Z_H(G^0)^0$ is the connected centralizer of G^0 in H.

Lemma 3.3. *Let $g \in G$. Then there is a homomorphism $\theta\colon H_1 \to Z(G^0)$ such that $ghg^{-1} = \theta(h)h$, $h \in H_1$.*

Proof. Let $h \in H_1$. Since conjugation by h preserves the connected components of G there is an element $\theta(h) \in G^0$ such that $hg^{-1}h^{-1} = g^{-1}\theta(h)$. Let $h_1 \in H_1$. Then

$$g^{-1}\theta(h_1h) = h_1hg^{-1}h^{-1}h_1^{-1} = h_1g^{-1}\theta(h)h_1^{-1} = h_1g^{-1}h_1^{-1}\theta(h) = g^{-1}\theta(h_1)\theta(h).$$

Thus θ is a homomorphism. From $hg^{-1}h^{-1} = g^{-1}\theta(h)$ it follows that $ghg^{-1} = \theta(h)h$. Since h centralizes G^0, so does ghg^{-1}, and we see that $\theta(h)$ centralizes G^0. Thus $\theta(h) \in Z(G^0)$. ∎

Corollary 3.4. *Suppose that $G = G_0$ and that G_0 is normal in H_0. Then H_0 is reductive.*

Proof. As above, we have $(H_0)^0 = H_2G^0$ where $H_2 \subset H_0$ is connected and centralizes G^0, and H_0 is reductive if and only if H_2 is reductive. Let R be the unipotent radical of H_2. Corresponding to each $g \in G$ there is a homomorphism $\theta\colon H_2 \to Z(G^0)$, and since R is unipotent, $\theta(R) = \{e\}$. Thus $R \subset \mathrm{GL}(V)^G$ where $\mathrm{GL}(V)^G$ is obviously in H_2. Thus R is trivial and H_0 is reductive. ∎

Write $H^0 = H_s^0G_s^0T$ where H_s^0 (resp. G_s^0) is the semisimple part of H_1 (resp. G^0) and $T := Z(H^0)^0 \subset H_1$ is a torus. Set $T_0 := Z(G^0)^0$.

Corollary 3.5. *The group H_s^0 is contained in $\mathrm{GL}(V)^G$.*

Theorem 3.6. *Let V, G and H be as above. Then $H^0 = \mathrm{GL}(V)^GG^0$.*

Proof. Write $H^0 = H_s^0G_s^0T$ as above and set $F := G/G^0$. Then F normalizes T and by Lemma 3.3, F acts trivially on T/T_0. Thus T^F projects onto T/T_0. Choose a torus S in $(T^F)^0$ complementary to $(T^F \cap T_0)^0$. Then $H^0 = H_s^0SG^0$ where H_s^0S lies in $\mathrm{GL}(V)^G$. ∎

Remark 3.7. Write $V = \bigoplus_{i=1}^r m_iV_i$ where the V_i are irreducible and pairwise non-isomorphic and m_iV_i denotes the direct sum of m_i copies of V_i. Then the theorem shows that $H^0 = G^0\prod_{i=1}^r \mathrm{GL}(m_i)$.

Example 3.8. Let $\{e\} \neq G \subset \mathrm{GL}(V)$ be finite. Then $\mathcal{N}_G$, as a set, is just the origin, and it is preserved by $\mathrm{GL}(V)$. Thus it is essential in Theorem 3.6 that H preserve $\mathcal{N}_G$ schematically.

Corollary 3.9. *Suppose that $V = \bigoplus_{i=1}^r V_i$ where the V_i are irreducible, non-trivial and pairwise non-isomorphic. Let $H' \subset \mathrm{GL}(V)$ be semisimple. Then the following are equivalent:*

(1) $H' \subset H_0$.

(2) $H' \subset G_0$.

Proposition 3.10. *Suppose that V is an irreducible G-module. Then G_0 and H_0 are reductive and $H^0 = \mathbb{C}^* G^0$.*

Proof. The fixed points of the unipotent radical R of G_0 are a G_0-stable nonzero subspace of V. Thus R acts trivially on V, i.e., $R = 0$. Hence G_0 is reductive. Similarly, H_0 is reductive. ∎

Corollary 3.11. *Suppose that $V = mW$ where W is an irreducible G-module. Then H_0 is reductive.*

Proof. The group H contains $G \times \mathrm{GL}(m)$ which acts irreducibly on $V \simeq W \otimes \mathbb{C}^m$. Thus H_0 is reductive. ∎

In the remainder of this section, we do not assume that G is a Levi subgroup of G_0.

Corollary 3.12. *Let $G \subset \mathrm{GL}(W)$ and let $V = pW \oplus qW^*$ where $2 \le p \le q$ and the G-modules W and W^* are irreducible and non isomorphic. Then*

(1) G_0 and H_0 are reductive.

(2) $G_0 \subset \mathrm{GL}(W)$.

(3) $H^0 = \mathrm{GL}(p)\,\mathrm{GL}(q)(G_0)^0$.

Proof. First we consider the case that $G = \mathrm{GL}(W)$. Then Example 4.3 below shows that $G_0 = \mathrm{GL}(W)$ and that $(H_0)^0 = \mathrm{GL}(p)\,\mathrm{GL}(q)\,\mathrm{GL}(W)$. Now the invariants of $\mathrm{GL}(W)$ are generated by those of degree 2 and the degree 2 invariants of G and of $\mathrm{GL}(W)$ are the same. Thus G_0 must be a subgroup of $\mathrm{GL}(W)$ and $(H_0)^0$ must be a subgroup of $\mathrm{GL}(p)\,\mathrm{GL}(q)\,\mathrm{GL}(W)$ containing $\mathrm{GL}(p)\,\mathrm{GL}(q)$. Hence $(H_0)^0 = \mathrm{GL}(p)\,\mathrm{GL}(q) H_1$ where $H_1 \subset \mathrm{GL}(W)$. Note that GH_1 is a finite extension of H_1. Since W is an irreducible G-module and G_0 and GH_1 contain G, both G_0 and H_1 (hence $(H_0)^0$) are reductive and we have (1) and (2). Theorem 3.6 gives (3). ∎

Lemma 3.13. *Suppose that $V^G = (0)$ and let $V = \bigoplus_{i=1}^r m_i V_i$ be the isotypic decomposition of V where the V_i are pairwise non-isomorphic G-modules. Suppose that $\mathfrak{h}_0(m_i V_i) \subset m_i V_i$ for all i. Then H_0 is reductive.*

Proof. For any i, $G(H_0)^0$ is a finite extension of $(H_0)^0$ which contains the product $G \prod_i \mathrm{GL}(m_i)$. The latter group acts irreducibly on $m_i V_i$, hence the image of $G(H_0)^0$ in $\mathrm{GL}(m_i V_i)$ is reductive for all i. It follows that $(H_0)^0$ is reductive, hence that H_0 is reductive. ∎

Corollary 3.14. *Suppose that V_i is an irreducible nontrivial G_i-module where G_i is reductive and $\mathbb{C}[V_i]^{G_i} \neq \mathbb{C}$, $i = 1, \dots, r$. Let $V := \bigoplus_i m_i V_i$ with the canonical action of $G := G_1 \times \cdots \times G_r$ where $m_i \ge 1$ for all i. Then H_0 is reductive.*

Proof. Suppose that $\mathfrak{h}_0$ is not contained in $\bigoplus_i \mathrm{End}(m_iV_i)$. Since $\mathfrak{h}_0$ is H-stable, it must contain one of the irreducible $G_i \times \mathrm{GL}(m_i) \times G_j \times \mathrm{GL}(m_j)$-modules $\mathrm{Hom}(m_iV_i, m_jV_j)$, $i \neq j$. Without loss of generality suppose that $\mathfrak{h}_0 \supset \mathrm{Hom}(m_2V_2, m_1V_1)$. Let $f \in \mathcal{O}(m_1V_1)^{G_1}$ be a nonconstant homogeneous invariant of minimal degree $d \geq 2$. Let $\varphi \in \mathrm{Hom}(m_2V_2, m_1V_1)$. Then φ sends f to the function $h(v_1, v_2) := df(v_1)(\varphi(v_2))$ where $v_i \in m_iV_i$, $i = 1, 2$. Clearly there is a φ such that $h \neq 0$. Thus h is a nonzero element of bidegree $(d-1, 1)$ in $\mathbb{C}[m_1V_1 \oplus m_2V_2]$. But by the minimality of d and the fact that no nonzero invariant in $\mathbb{C}[m_2V_2]$ has degree 1, there is no element of $\mathcal{I}$ of this bidegree. Hence $\mathrm{Hom}(m_2V_2, m_1V_1)$ does not preserve $\mathcal{I}$, a contradiction. Thus $\mathfrak{h}_0$ is contained in $\bigoplus_i \mathrm{End}(m_iV_i)$ and one can apply Lemma 3.13. ■

Corollary 3.15. *Suppose that $G \subset \mathrm{GL}(V)$ is a finite group generated by pseudoreflections. Then H_0 is reductive.*

Proof. We have that $V = \bigoplus V_i$ and $G = \prod G_i$ where $G_i \subset \mathrm{GL}(V_i)$ is an irreducible group generated by pseudoreflections. Now apply Corollary 3.14. ■

Proposition 3.16. *Suppose that V is an orthogonal representation of G where $V^G = (0)$. Then H_0 is reductive.*

Proof. We have an isotypic decomposition $V = \bigoplus_i m_iV_i \bigoplus n_j(W_j \oplus W_j^*)$ where the V_i are irreducible nontrivial orthogonal representations of G and the W_j are irreducible nonorthogonal representations of G. Note that for each i there is a quadratic invariant $p_i \in \mathbb{C}[m_iV_i]^G$ and for each j a quadratic invariant (a contraction) $q_j \in \mathbb{C}[n_j(W_j \oplus W_j^*)]^G$. Suppose that $\mathfrak{h}_0$ is not contained in $\bigoplus_i \mathrm{End}(m_iV_i) \bigoplus_j \mathrm{End}(n_j(W_j \oplus W_j^*))$. For example, suppose that there is a nonzero element φ of $\mathfrak{h}_0$ whose restriction to m_2V_2 has nonzero projection to m_1V_1. Then we have the function $h(v_1, v_2) := dp_1(v_1)(\varphi(v_2))$ for $v_1 \in m_1V_1$ and $v_2 \in m_2V_2$. As before, the actions of G and the $\mathrm{GL}(m_i)$ guarantee that we can assume that $h \neq 0$. Now the bidegree of h is $(1, 1)$ and $h \in \mathcal{I}$. However, there are no nonconstant invariants of bidegree (a, b) in $\mathbb{C}[m_1V_1 \oplus m_2V_2]$ for $a \leq 1$ and $b \leq 1$. Thus h cannot lie in $\mathcal{I}$. One similarly gets contradictions for all the possible ways that $\mathfrak{h}_0 \not\subset \bigoplus_i \mathrm{End}(m_iV_i) \bigoplus_j \mathrm{End}(n_j(W_j \oplus W_j^*))$ can occur. Finally, note that the normalizer N of the image of G in $\mathrm{GL}(n_j(W_j \oplus W_j^*))$ contains an element interchanging the copies of W_j and W_j^*. Thus N acts irreducibly and we can now apply the argument of Lemma 3.13. ■

Corollary 3.17. *If G is any one of the following groups, then H_0 is reductive for any representation V of G with $V^G = (0)$.*

(1) $\mathrm{SO}(n)$, $n \geq 3$.

(2) G_2, F_4, E_8.

(3) B_{4n+3} *and* B_{4n+4}, $n \geq 0$.

(4) D_{4n}, $n \geq 1$.

4. Some examples and a conjecture

We give examples where G_0 is not reductive and we give examples where G_0 is reductive but H_0 is not.

Example 4.1. Let V and W be G-modules such that $\mathcal{O}(V \oplus W)^G = \mathcal{O}(V)^G$. Then $\operatorname{Hom}(V, W)$ is contained in the radical of $\mathfrak{g}_0$ so that G_0 and H_0 are not reductive. A concrete example is given by $G = \mathrm{SL}_4$ and $V \oplus W = \wedge^2\mathbb{C}^4 \oplus \mathbb{C}^4$ with the obvious G action.

Example 4.2. Let W be an irreducible G-module where $W^G = (0)$ and $\mathcal{O}(W)^G \neq \mathbb{C}$. Let $V = W \oplus \mathbb{C}$ where G acts trivially on $\mathbb{C}$. Then $\mathfrak{g}_0 \subset \mathfrak{gl}(W)$ while $\operatorname{Hom}(\mathbb{C}, W)$ is contained in the Lie algebra of the radical of H_0.

Example 4.3. Let $1 \leq p \leq q$ and consider the $G = \mathrm{GL}(W)$ representation on $V = pW \oplus qW^*$ where $W = \mathbb{C}^n$, $n \geq 1$. (See Corollary 3.12.) By classical invariant theory, the G-invariants are just the contractions of elements of the copies of W with elements of the copies of W^*. Let U denote $W \oplus W^* \simeq \mathbb{C}^{2n}$.

Three cases arise:

Case 1: $p = q = 1$. Then our invariant is the bilinear form $(\ ,\ )$ corresponding to the matrix $J := \left(\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right) \subset \mathrm{GL}(2n)$, i.e., $(x, y) = x^t J y$, x, $y \in U$. Thus $G_0 = \mathrm{O}(2n)$ and $H_0 = \mathbb{C}^* G_0$.

Case 2: $p = 1$, $q > 1$. Then H_0 contains a copy of $\mathrm{GL}(q)$ and the action of H_0 on the invariants is a representation $H_0 \to \mathrm{GL}(q)$ whose kernel is G_0. Thus $H_0 = \mathrm{GL}(q) G_0$. A matrix computation shows that $G_0 = \mathrm{GL}(W) \ltimes (\wedge^2(W^*) \otimes \mathbb{C}^q)$. If $x \in W$ and $y_1, \dots, y_q \in W^*$, then the unipotent radical of G_0 sends $(x, y_1, \dots, y_q)$ to $(x, y_1 + B_1 x, \dots, y_q + B_q x)$ where for each j, B_j is a skew symmetric matrix, $B_j \in \wedge^2(W^*) \subset \operatorname{Hom}(W, W^*)$.

Case 3: $p \geq 2$. We show that $G_0 = \mathrm{GL}(W)$, that $H_0 = H$ and that $H^0 = \mathrm{GL}(p)\,\mathrm{GL}(q)\,\mathrm{GL}(W)$. We also determine H. First suppose that $p = q = 2$. Then G_0 preserves the inner products on $2U$, i.e., G_0 is a subgroup of $\mathrm{O}(2n)$. Moreover, G_0 preserves the skew product on $2U$ sending x, y to $x^t K y$ where $K = \left(\begin{smallmatrix} 0 & I \\ -I & 0 \end{smallmatrix}\right)$. Hence G_0 lies in the intersection of $\mathrm{O}(2n)$ and $\mathrm{Sp}(2n)$ which is the copy of $\mathrm{GL}(W)$ acting on U by the matrices $\left(\begin{smallmatrix} A & 0 \\ 0 & {}^tA^{-1} \end{smallmatrix}\right)$, $A \in \mathrm{GL}(W)$. Clearly, as long as $2 \leq p \leq q$ we must have that $G_0 = G = \mathrm{GL}(W)$. We have a representation $\varphi\colon H_0 \to \mathrm{GL}(pq)$ given by the action of H_0 on the pq generators of the invariants. The kernel of φ is $G_0 = G$. Thus H_0 is reductive. By Theorem 3.6 we have $H^0 = \mathrm{GL}(p)\,\mathrm{GL}(q)\,\mathrm{GL}(W)$. Let $h \in H$. If h stabilizes pW and qW^*, then h induces an automorphism of $\mathrm{GL}(W)$ which is trivial on $\mathbb{C}^* I$ and must be inner on $\mathrm{SL}(W)$. Hence modulo an element of $\mathrm{SL}(W)$, h lies in the centralizer of $\mathrm{GL}(W)$, which is $\mathrm{GL}(p)\,\mathrm{GL}(q)$. Hence $h \in H^0$. The only other possibility is that h interchanges the copies of pW and qW^*. This can only happen if $p = q$. Thus H is connected if $p \neq q$ and H/H^0 has order two if $p = q$.

Example 4.4. Let $G = \mathbb{Z}/4\mathbb{Z} \subset \mathbb{C}^*$ and let $V = \mathbb{C}^2$ where $\xi(a, b) = (\xi^2 a, \xi b)$ for $(a, b) \in \mathbb{C}^2$, $\xi \in G$. Since G is finite, $G_0 = G$. Let x and y be the usual coordinate functions on V. Then the invariants are generated by x^2, xy^2 and y^4. Consider the element $\varphi \in \operatorname{End}(V)$ which sends (a, b) to $(0, a)$ for a, $b \in \mathbb{C}$.

Then φ acts on $\mathbb{C}[V]$ by the derivation $x\partial/\partial y$. This derivation preserves $\mathcal{I}$ and it follows that φ is a basis of the Lie algebra of the unipotent radical of H_0.

Example 4.5. Let $G = \mathbb{C}^*$ and let V be the $p+q+r$ dimensional representation with weights -1 of multiplicity p, 1 of multiplicity q and 2 of multiplicity r where $p,\ q,\ r \in \mathbb{N}$ and $pqr \neq 0$. If x_i, y_j and z_k are corresponding coordinate functions, then the invariants are generated by the monomials x_iy_j and $x_ix_{i'}z_k$. We have $G_0 = G$ while the radical of H_0 has Lie algebra spanned by the linear mappings corresponding to the derivations $y_j\partial/\partial z_k$.

Example 4.6. Let $V \oplus W = S^2(\mathbb{C}^n) \oplus \mathbb{C}^n$ with the obvious action of $G = \mathrm{SL}_n$, $n \geq 2$. Then using classical invariant theory [6] one computes that the invariants have homogeneous generators p and q of bidegrees $(n,0)$ and $(n-1,2)$, respectively. Now $\mathrm{Hom}(V,W)$ contains a copy of W^* where $\xi \in W^*$ sends $v \in V$ to $i_\xi(v) \in W$ (contraction). Then this copy of W^* acts on $\mathbb{C}[V \oplus W]$ sending a polynomial $f(v,w)$ into $df(v,w)(0,i_\xi(v))$, $v \in V$, $w \in W$. This action annihilates p and sends q to a subspace of $\mathcal{O}(V \oplus W)$ of bidegree $(n,1)$ transforming under G as W^*. But the only way to get a copy of W^* in this bidegree is to multiply p times the copy of W^* in degree 1 in $\mathcal{O}(V \oplus W)$. Thus $\mathcal{I}$ is preserved. It is now easy to establish that the unipotent radical of H_0 has Lie algebra the copy of W^* in $\mathrm{Hom}(V,W)$.

Conjecture 4.7. If G is semisimple and V is generic (see 2.2) with $V^G = (0)$, then H_0 is reductive.

5. Cofree Representations

Recall that V is *cofree* if R is a free module over R^G. Equivalently, R^G is a polynomial ring and $\pi\colon V \to V/\!\!/G$ is equidimensional [8, 17.29]. If $p_1,\dots,p_d$ are minimal homogeneous generators of R^G, then we can identify π with the polynomial map $p = (p_1,\dots,p_d)\colon V \to \mathbb{C}^d$. Cofreeness is equivalent to the fact that the p_i form a regular sequence in $\mathbb{C}[V]$. See [7] for the classification of cofree representations of the simple algebraic groups and [4] for the classification of irreducible cofree representations of semisimple algebraic groups.

We say that $G' \subset \mathrm{GL}(V)$ stabilizes a fiber F of π if G' preserves F schematically, i.e., preserves the ideal I_F of F.

Proposition 5.1. *Suppose that G is reductive and V is a cofree G-module. If $G' \subset \mathrm{GL}(V)$ stabilizes a fiber of $\pi\colon V \to V/\!\!/G$, then G' stabilizes $\mathcal{N}_G$.*

Proof. Let F be a fiber of π. Then there are constants c_i, $i = 1,\dots,d$, such that I_F is the ideal generated by $p_i - c_i$, $i = 1,\dots,d$. Let $0 \neq f \in I_F$ and let $\operatorname{gr} f$ denote the nonzero homogeneous part of f of largest degree. Then the elements $\operatorname{gr} f$ for $0 \neq f \in I_F$ generate a homogeneous ideal I which obviously contains $\mathcal{I}$. We show that $I \subset \mathcal{I}$ so that $I = \mathcal{I}$. If G' preserves I_F, it preserves $I = \mathcal{I}$, and we have the proposition.

Let d_i be the degree of p_i, $i = 1,\dots,d$. Let $0 \neq f \in I_F$ where $\operatorname{gr} f$ is homogeneous of degree r. We have $f = \sum a_i(p_i - c_i)$ where $a_1,\dots,a_n \in R$. Let

$s = \max_i\{\deg a_i + d_i\}$. Let a_i' denote the homogeneous part of a_i of degree $s - d_i$. If $s > r$, then we must have that $\sum_i a_i' p_i = 0$. Since the p_i are a regular sequence, this relation is generated by the Koszul relations $p_j p_i - p_i p_j = 0$, $1 \le i < j \le d$. Hence there are $b_{ij} \in R$, $b_{ij} = -b_{ji}$, such that

$$\sum_i a_i'(p_i - c_i) = \sum_{i \ne j} b_{ij}(p_j(p_i - c_i) - p_i(p_j - c_j)) = \sum_{i \ne j} b_{ij}(c_j(p_i - c_i) - c_i(p_j - c_j))$$

where for fixed i, $\deg \sum_{j \ne i} b_{ij} c_j < s - d_i$. Thus we may replace each a_i by a polynomial of degree less than $s - d_i$ without changing f. Continuing inductively we reduce to the situation that $\deg a_i \le r - d_i$ for all i. Let a_i' denote the homogeneous degree $r - d_i$ term in a_i, $i = 1, \ldots, d$. Then $\operatorname{gr} f = \sum_i a_i' p_i \in \mathcal{I}$. ∎

Example 5.2. Let $G = \mathbb{C}^*$ and $V = \mathbb{C}^3$ with coordinate functions x, y and z corresponding to weights -1, 1 and 2. The fiber defined by $xy = 1$ and $x^2 z = 0$ is the fiber defined by $xy = 1$ and $z = 0$, and it has a symmetry which interchanges x and y. However, this is not a symmetry of the ideal generated by the invariants. Thus Proposition 5.1 does not hold in case the representation is not cofree.

Remark 5.3. Let F be a principal fiber of π where V is cofree. Then $d\pi$ has rank $d = \dim V/\!\!/G$ on F so that F is smooth. It follows that G' preserves I_F if and only if G' preserves the set F.

Corollary 5.4. *Let $V = \bigoplus_{i=1}^r V_i$ where the V_i are pairwise non-isomorphic nontrivial G-modules and V is cofree. Suppose that $G \subset G' \subset \mathrm{GL}(V)$ where G' is connected semisimple. Then the following are equivalent.*

(1) $R^G = R^{G'}$.

(2) G' preserves a fiber of $\pi \colon V \to V/\!\!/G$.

(3) G' preserves $\mathcal{N}_G$.

Proof. Use Corollary 3.9 and Proposition 5.1. ∎

Corollary 5.5. *Let V be an irreducible nontrivial cofree G-module with $R^G \ne \mathbb{C}$. Let $F \ne \mathcal{N}$ be a fiber of $\pi \colon V \to V/\!\!/G$ and let G_F be the subgroup of $\mathrm{GL}(V)$ stabilizing F. Then*

(1) $G_0 \subset G_F \subset H_0$ are reductive.

(2) $H^0 = \mathbb{C}^*(G_0)^0$.

(3) G_F/G_0 is finite.

Proof. Parts (1) and (2) are clear. Since $F \ne \mathcal{N}$, it is only stabilized by a finite subgroup of $\mathbb{C}^*$, hence we have (3). ∎

It would be nice to find an example of an irreducible module V of a semisimple group G with $G = (G_0)^0$ such that the subgroup of $\mathrm{GL}(V)$ fixing a fiber F of $\pi \colon V \to V/\!\!/G$, $F \ne \mathcal{N}_G$, has dimension bigger than $\dim G$.

Remark 5.6. Let V be an irreducible nontrivial cofree representation of a simple algebraic group G such that $R^G \neq \mathbb{C}$. The cases for which $G \neq (G_0)^0$ are as follows (we use the numbering and notation of [6]).

(1) $(\varphi_3, \mathsf{B}_3)$.

(2) $(\varphi_4, \mathsf{B}_4)$.

(3) $(\varphi_5, \mathsf{B}_5)$.

(4) $(\varphi_1, \mathsf{G}_2)$.

6. The adjoint case

Let $\mathfrak{g}$ be a simple Lie algebra. Choose a Cartan subalgebra $\mathfrak{t}$ of $\mathfrak{g}$ and a base Π of the root system. Choose $x_\alpha \in \mathfrak{g}_\alpha$ and $y_\alpha \in \mathfrak{g}_{-\alpha}$, $\alpha \in \Pi$, such that $(x_\alpha,\ y_\alpha,\ [x_\alpha, y_\alpha])$ is a standard $\mathfrak{sl}_2$ triple. Then there is a unique order 2 automorphism ψ of $\mathfrak{g}$ which is -1 on $\mathfrak{t}$ and sends x_α to $-y_\alpha$, $\alpha \in \Pi$.

Now let $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_r$ where the $\mathfrak{g}_i$ are simple ideals of the Lie algebra $\mathfrak{g}$. Let $\psi_i \in \operatorname{Aut}(\mathfrak{g}_i)$ be as above. Let G denote the adjoint group of $\mathfrak{g}$ and let G_0, H_0 and H be as in the introduction.

Theorem 6.1. *We have that $H = (\mathbb{C}^*)^r \operatorname{Aut}(\mathfrak{g})$ and that $G_0 \simeq (\mathbb{Z}/2\mathbb{Z})^r G$ where the ith copy of $\mathbb{Z}/2\mathbb{Z}$ is generated by $-\psi_i$.*

Proof. By Dixmier [2], $(G_0)^0 = G$, and using Corollary 3.14 we obtain that $H_0 = H$ where $H^0 = (\mathbb{C}^*)^r G$. Hence if $\varphi \in H$, we obtain an automorphism σ of $\mathfrak{g} \simeq \operatorname{ad}\mathfrak{g} \subset \mathfrak{h} \subset \mathfrak{gl}(\mathfrak{g})$ where $\operatorname{ad}(\sigma(X)) = \varphi \circ \operatorname{ad} X \circ \varphi^{-1}$, $X \in \mathfrak{g}$. Clearly $\operatorname{Aut}(\mathfrak{g}) \subset H$, so replacing φ by $\varphi \circ \sigma^{-1}$ we can arrange that $\varphi \circ \operatorname{ad} X \circ \varphi^{-1} = \operatorname{ad} X$ for all $X \in \mathfrak{g}$. Then by Schur's lemma, $\varphi \in (\mathbb{C}^*)^r \subset H^0$ so that $H = (\mathbb{C}^*)^r \operatorname{Aut}(\mathfrak{g})$. If we start with $\varphi \in G_0$, then since φ induces the identity on $\mathbb{C}[\mathfrak{g}]^G$, so does σ, and it follows from Schur's lemma that φ is a product $\prod_i \lambda_i \sigma_i$ where, for all i, $\sigma_i \colon \mathfrak{g}_i \to \mathfrak{g}_i$ is an automorphism and $\lambda_i \in \mathbb{C}^*$ acts via multiplication on $\mathfrak{g}_i$. But φ has to preserve the invariants of degree 2 of each $\mathfrak{g}_i$, hence $\lambda_i = \pm 1$ for all i. Now [9, Theorem 2.5] shows that, for each i, $\lambda_i \sigma_i \in G_i$ or $\lambda_i \sigma_i \in (-\psi_i) G_i \neq G_i$, where G_i is the adjoint group of $\mathfrak{g}_i$. Hence $G_0 \simeq (\mathbb{Z}/2\mathbb{Z})^r G$. ■

Corollary 6.2. (See [1]). *Let $\mathfrak{g} = \mathfrak{sl}_n$. Then H is generated by G, $\mathbb{C}^*$ and transposition.*

Proof. In the case of $\mathfrak{sl}_n$ with the usual choice of $\mathfrak{t}$ and Π, the automorphism ψ is $X \mapsto -X^t$, $X \in \mathfrak{sl}_n$. Then ψ generates the group of outer automorphisms of $\mathfrak{sl}_n$ (which is the trivial group for $n = 2$). Hence H is generated by G, $\mathbb{C}^*$ and transposition. ■

Corollary 6.3. (See[12]). *Let G_F be the subgroup of* $\mathrm{GL}(\mathfrak{gl}_n)$ *which preserves the* $G := \mathrm{PGL}(n)$*-orbit F of an element x_0 of $\mathfrak{gl}_n$ which has nonzero trace and distinct eigenvalues. Then G_F is generated by G and transposition.*

Proof. The condition on x_0 implies that F is a smooth fiber of the quotient mapping (see Remark 5.3). Write $x_0 = \mu I + y_0$ where $\mu \in \mathbb{C}^*$, $y_0 \in \mathfrak{sl}_n$ and I is the $n \times n$ identity matrix. Then F is just $\mu I + F_1$ where $F_1 = G \cdot y_0$. We may write an element of G_F as $\left(\begin{smallmatrix} 1 & 0 \\ c & \lambda g \end{smallmatrix}\right)$ where $c \in \mathfrak{sl}_n$, $g \in \mathrm{GL}(\mathfrak{sl}_n)$ is a linear mapping preserving the schematic null cone of $\mathfrak{sl}_n$ and $\lambda \in \mathbb{C}^*$ (use 3.10 and 5.1). Then g is in G or g is an element of G composed with transposition. Applying the inverse of g we obtain an element h of the form $y \mapsto \lambda y + c$, $y \in F_1$. We need to show that $c = 0$. Suppose not. Let $g \in G$ such that $gc \neq c$. Then $h^{-1}ghg^{-1}(y) = y + c'$, $0 \neq c' \in \mathfrak{sl}_n$, $y \in F_1$. Thus $F_1 = F_1 + c'$. It follows that for any invariant polynomial p on $\mathfrak{sl}_n$, $p(y + nc') = p(y)$ for all $y \in F_1$ and $n \in \mathbb{Z}$. Thus $dp(y)(c') = 0$ for any $y \in F_1$. But the covectors $dp(y)$ for $y \in F_1$ span $(\mathfrak{sl}_n)^*$. Thus $c' = 0$, a contradiction. ■

References

[1] Botta, P., S. Pierce, and W. Watkins, *Linear transformations that preserve the nilpotent matrices*, Pac. J. Math. **104** (1983), 39–46.

[2] Dixmier, J., *Champs de vecteurs adjoints sur les groupes et algèbres de Lie semi-simples*, J. Reine Angew. Math. **309** (1979), 183–190.

[3] Kraft, H., „Geometrische Methoden in der Invariantentheorie“, Aspekte der Mathematik, vol. **D1** Vieweg Verlag, Braunschweig/Wiesbaden, 1985.

[4] Littelmann, P., *Koreguläre und äquidimensionale Darstellungen*, J. of Alg. **123** (1989), 193–222.

[5] Raïs, M., *Notes sur la notion d'invariant caractéristique*, http://arxiv.org /abs/0707.0782.

[6] Schwarz, G. W., *Representations of simple Lie groups with regular rings of invariants*, Invent. Math. **49** (1978), 167–191.

[7] —, *Representations of simple Lie groups with a free module of covariants*, Invent. Math. **50** (1978), 1–12.

[8] —, *Lifting smooth homotopies of orbit spaces*, IHES Publ. Math. **51** (1980), 37–135.

[9] —, *Linear maps preserving invariants*, http://arxiv.org/abs/0708.2890.

[10] Solomon, S., *Irreducible linear group-subgroup pairs with the same invariants*, J. Lie Theory **15** (2005), 105–123.

[11] —, *Orthogonal linear group-subgroup pairs with the same invariants*, J. Alg. **299** (2006), 623–647.

[12] Watkins, W., *Linear transformations that preserve a similarity class of matrices*, Lin. Mult. Alg. **11** (1982), 19–22.

Gerald W. Schwarz
Department of Mathematics
Brandeis University
MS 050, PO Box 549110
Waltham, MA 02454-9110
schwarz@brandeis.edu

Received November 13, 2007
and in final form April 8, 2008