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A Converse to the Second Whitehead Lemma

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Abstract. In this paper we state and prove the following version of a converse to the Second Whitehead Lemma: A finite-dimensional Lie algebra over a field of characteristic zero with vanishing second cohomology in any finite-dimensional module must be one of the following: (i) a one-dimensional algebra; (ii) a semisimple algebra; (iii) the direct sum of a semisimple algebra and a one-dimensional algebra.
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Introduction

The classical First and Second Whitehead Lemmata state that the first, respectively second, cohomology group of a finite-dimensional semisimple Lie algebra with coefficients in any finite-dimensional module vanishes. It is natural to ask whether the converse is true. For the first cohomology this is well-known:

Theorem 0.1. (A converse to the First Whitehead Lemma). *A finite-dimensional Lie algebra over a field of characteristic zero such that its first cohomology with coefficients in any finite-dimensional module vanishes, is semisimple.*

Proof. (See, e.g., [7, Chapter 1, Theorem 3.3]). Due to the cohomological interpretation of module extensions, the vanishing of cohomology with coefficients in any finite-dimensional module is equivalent to the complete reducibility of any finite-dimensional module. Assume that a Lie algebra L satisfies this condition and has a nonzero abelian ideal I. Then, in view of the adjoint representation of L, the ideal I must be a direct summand of L. But then I also should satisfy this condition, a contradiction. ∎

What about the second cohomology? Somewhat surprisingly, it seems that as yet this question has not been addressed in the literature. The aim of the present elementary note is to show that, essentially, the converse is true in this case too — the finite dimensional Lie algebras over a field of characteristic zero such that the second cohomology with coefficients in any finite-dimensional module vanishes, are very close to semisimple ones. The proof we propose in this paper is only slightly

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more involved than the proof of Theorem 0.1, in fact, the result readily follows by a proper application of known facts from the literature.

One may look at this question also from a somewhat different angle. Recently, the interesting class of *strongly rigid* Lie algebras was investigated in [1]; these are the Lie algebras whose universal enveloping algebra is rigid. It turns out that for strongly rigid algebras, the second cohomology in the trivial module and in the adjoint module vanishes. Considering these two modules as the most "natural" ones, one may wonder for which Lie algebras the stronger condition – vanishing of the second cohomology with coefficient in any finite-dimensional module – would hold.

What happens in positive characteristic? As was shown independently by Dzhumadildaev [5] and Farnsteiner – Strade [6], for any finite-dimensional Lie algebra over a field of positive characteristic and any degree not greater than the dimension of the algebra, there is a module with non-vanishing cohomology in that degree. (In fact, the low degree cases interesting us here were settled even earlier – for the first degree cohomology by Jacobson [9, Chapter VI, §3, Theorem 2] and for the second degree cohomology – again by Dzhumadildaev [3]). So the answer is trivially void in this case – one of the rare cases when in positive characteristic the answer (but not the proof!) of an assertion turns out to be simpler. The reason for this is the possibility to construct analogues of induced modules with desired cohomological properties by means of various truncated finite-dimensional versions of the universal enveloping algebra. In characteristic zero, such modules would be infinite-dimensional.

Theorem 0.2. (A converse to the Second Whitehead Lemma). *A finite-dimensional Lie algebra over a field of characteristic zero such that its second cohomology with coefficients in any finite-dimensional module vanishes, is one of the following:*

(i) *a one-dimensional algebra;*

(ii) *a semisimple algebra;*

(iii) *the direct sum of a semisimple algebra and a one-dimensional algebra.*

In the first section of this note we accumulate the results from the literature that are needed in the second section to provide our proof.

We use the standard notation. If L is a Lie algebra then $Rad(L)$ denotes the solvable radical of L. If V is an L-module, then $V^L = \{v \in V \mid xv = 0 \text{ for any } x \in L\}$ is the submodule of L-invariant points. Throughout this note, the ground field K is assumed to be of characteristic zero and all algebras and modules are assumed to be finite-dimensional. When considered as a module over a Lie algebra, K is understood as a trivial module.

1. Needed results

Proposition 1.1. (Dixmier). *Let L be a Lie algebra, V be an L-module, and I be an ideal of L of codimension 1. Then for any $n \in \mathbb{N}$ and $x \in L \backslash I$, $H^n(L,V) \simeq H^n(I,V)^x \oplus H^{n-1}(I,V)^x$.*

Proof. This was proved in the equivalent form of a certain long exact sequence in [2, Proposition 1] and stated without proof (in a more general situation when I is a subalgebra of codimension 1) in [4, Proposition 4]. In fact, this is an easy consequence of the Hochschild–Serre spectral sequence.

Let $L = I \oplus Kx$ for some $x \in L$, $x \notin I$ and consider the Hochschild–Serre spectral sequence abutting to $H^*(L,V)$ relative to the ideal I. As $E_2^{pq} = H^p(L/I, H^q(I,V)) \simeq H^p(Kx, H^q(I,V))$, it is nonzero only for $p = 0,1$, in which cases $E_2^{0q} = H^q(I,V)^x$ and $E_2^{1q} \simeq H^q(I,M)/xH^q(I,M) \simeq H^q(I,M)^x$. Hence the spectral sequence stabilizes at E_2, $H^n(L,V) \simeq E_\infty^{0n} \oplus E_\infty^{1,n-1} = E_2^{0n} \oplus E_2^{1,n-1}$, and the result follows. ■

Proposition 1.2. (Dixmier). *If L is a nilpotent Lie algebra of dimension > 1, then $H^2(L,K) \neq 0$.*

Proof. In [2, Théorème 2], a considerably more general result is stated as follows:

$$\dim H^n(L,V) \geq 2 \quad \text{for any} \quad 0 < n \leq \dim L$$

and any finite-dimensional L-module V containing K. It is proved by repetitive applications of Proposition 1.1. ■

Finally, we will need the following simple result which is contained implicitly already in the foundational paper [8], and is explicitly proved, for example, in [10, Lemma 1]:

Proposition 1.3. (Hochschild – Serre). *Let L be a Lie algebra represented as the semidirect sum $L = P \oplus I$ of a subalgebra P and an ideal I, and V be an L-module. Then the Hochschild–Serre spectral sequence abutting to $H^*(L,V)$ relative to the ideal I, stabilizes at the E_2-term.*

2. Proof of Theorem 0.2

Finite-dimensional Lie algebras with the property that their second cohomology with coefficients in any finite-dimensional module vanishes, will be called 2*-trivial*.

Lemma 2.1. *Let L be a 2-trivial Lie algebra represented as the semidirect sum $L = S \oplus I$ of a subalgebra S and an ideal I. Then:*

(i) *S is 2-trivial;*

(ii) *$(H^2(I,K) \otimes V)^S = 0$ for any S-module V.*

Proof. Let V be an L-module and consider the Hochschild–Serre spectral sequence abutting to $H^*(L,V)$ relative to I. By Proposition 1.3, it stabilizes at E_2, hence all E_2 terms vanish. We have $E_2^{20} = H^2(L/I, H^0(I,V)) \simeq H^2(S, V^I)$ and $E_2^{02} = H^0(L/I, H^2(I,V)) \simeq H^2(I,V)^S$.

Choose V as follows: let V be an arbitrary S-module, and I acts on V trivially. Then $E_2^{20} \simeq H^2(S,V)$, $E_2^{02} \simeq (H^2(I,K) \otimes V)^S$, and they vanish for any S-module V, what proves (i) and (ii) respectively. ■

In the particular case where the semidirect sum reduces to the direct sum, (i) shows that every direct summand of a 2-trivial Lie algebra is 2-trivial. Using the Künneth formula the last assertion could be proved also directly, without appealing to Proposition 1.3.

Lemma 2.2. *An ideal of codimension 1 in a* 2*-trivial Lie algebra is a direct summand.*

Proof. Let L be a 2-trivial Lie algebra and I be an ideal of L of codimension 1. Write $L = I \oplus Kx$ for some $x \in L$. Evidently adx is an x-invariant 1-cocycle in $Z^1(I, I)$. As by Proposition 1.1, $H^1(I, I)^x$ embeds into $H^2(L, I) = 0$, this cocycle is a coboundary, i.e. there is $z \in I$ such that $[y, x] = [y, z]$ for any $y \in I$. Replacing x by $x' = x - z$, we get a direct sum decomposition $L = I \oplus Kx'$, $[I, x'] = 0$. ■

Corollary 2.3. *A* 2*-dimensional Lie algebra is not* 2*-trivial.*

Proof. By Lemma 2.2, a 2-trivial 2-dimensional Lie algebra is abelian. But for any abelian Lie algebra L and any $n \in \mathbb{N}$, $H^n(L, K) = C^n(L, K)$, and hence any abelian Lie algebra of dimension > 1 is not 2-trivial. ■

Lemma 2.4. *Let* L *be a* 2*-trivial Lie algebra. Then* $H^2(Rad(L), K) = 0$.

Proof. The assertion is obvious in the case when $L = Rad(L)$ is solvable, so suppose L is not solvable. Let $L = S \oplus Rad(L)$ be a Levi–Malcev decomposition, where S is a semisimple Malcev subalgebra.

By Lemma 2.1(ii), $(H^2(Rad(L), K) \otimes V)^S = 0$ for any S-module V. Assume $H^2(Rad(L), K) \neq 0$ and take $V = H^2(Rad(L), K)^*$. There is a canonical surjection of S-modules $H^2(Rad(L), K) \otimes H^2(Rad(L), K)^* \to K$. Since any extension of S-modules splits, K contained in $H^2(Rad(L), K) \otimes H^2(Rad(L), K)^*$, hence $(H^2(Rad(L), K) \otimes H^2(Rad(L), K)^*)^S \neq 0$, a contradiction. ■

Now we are ready to prove Theorem 0.2.

Let L be a 2-trivial Lie algebra. We shall reason by induction on the dimension of L.

If $[L, L] = L$, then $Rad(L)$ is nilpotent (in fact, this is the consequence of the Levi Theorem which in turn is the consequence of the Second Whitehead Lemma; see [9, Chapter III, §9, Corollary 2]). Note that $Rad(L)$ cannot be one-dimensional, as then $Rad(L)$ is one-dimensional representation of a semisimple Lie algebra, and hence is a trivial representation, what contradicts the condition $[L, L] = L$. If dim $Rad(L) > 1$, then by Proposition 1.2, $H^2(Rad(L), K) \neq 0$, while by Lemma 2.4, $H^2(Rad(L), K) = 0$, a contradiction. Hence $Rad(L) = 0$, i.e. L is semisimple.

Let $[L, L] \neq L$. As any subspace of L containing $[L, L]$ is an ideal of L, L contains an ideal I of codimension 1. By Lemma 2.2, L is the direct sum of I and a one-dimensional algebra, and by Lemma 2.1(i), I is 2-trivial. By induction assumption, I is either one-dimensional, or semisimple, or the direct sum of a semisimple algebra and a one-dimensional algebra. In the first case L is abelian 2-dimensional, a contradiction (Corollary 2.3). In the third case L is the direct sum of a semisimple algebra and a 2-dimensional abelian algebra. By Lemma 2.1(i), both direct summands are 2-trivial, the same contradiction again. Hence the only possible case is when L is the direct sum of a semisimple algebra and a one-dimensional algebra, what concludes the proof.

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