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Nonabelian Cohomology of Compact Lie Groups

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Abstract. Given a Lie group G with finitely many components and a compact Lie group A which acts on G by automorphisms, we prove that there always exists an A-invariant maximal compact subgroup K of G, and that for every such K, the natural map $H^1(A,K) \to H^1(A,G)$ is bijective. This generalizes a classical result of Serre and a recent result of the first and third named authors of the current paper.
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1. Introduction

Let a group A act on a Lie group G by automorphisms. Recall that the set of cocycles $Z^1(A,G)$ consists of maps $\gamma : A \to G$ satisfying $\gamma(ab) = \gamma(a)a(\gamma(b))$ for all $a, b \in A$, and that $\gamma_1, \gamma_2 \in Z^1(A,G)$ are cohomologous if for some $g \in G$ we have $\gamma_2(a) = g^{-1}\gamma_1(a)a(g)$ for all $a \in A$. The first nonabelian cohomology $H^1(A,G)$ of A with coefficients in G is, by definition, the set of all cohomologous classes in $Z^1(A,G)$ (c.f. [6]).

Because of its relation with number theory, most studies of this kind of cohomology concentrate on the case that G is also algebraic. For example, a classical result of Serre [6, III.4.5] asserts that if G is a complex reductive algebraic group with a maximal compact subgroup K, and $A \cong \mathbb{Z}/2\mathbb{Z}$ acts on G by the complex conjugation with respect to K, then the natural map $H^1(A,K) \to H^1(A,G)$ is bijective. Recently, the case that G is an arbitrary connected Lie group was considered in [1, 2]. In particular, it was proved that for any finite group A and any connected Lie group G, there exists an A-invariant maximal compact subgroup K of G, and the natural map $H^1(A,K) \to H^1(A,G)$ is bijective ([1, Thm. 3.1]).

The goal of this paper is to generalize the above results to the case that A is an arbitrary compact Lie group and G has finitely many components. In this setting, by a cocycle $\gamma : A \to G$ we always mean a continuous one. Our main theorem is as follows.

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Theorem 1.1. *Let G be a Lie group with finitely many components, and let A be a compact Lie group which acts on G by automorphisms. Then there exists an A-invariant maximal compact subgroup K of G, and for every such K, then natural map $\iota_1 : H^1(A,K) \to H^1(A,G)$ is bijective.*

It should be pointed out that the main difficulty in the proof of Theorem 1.1 lies in the injectivity of the map ι_1. Recall that the proof of the corresponding part for the special case of Theorem 1.1 where A is finite (and G is connected), which is [1, Thm. 3.1], is based on the well-known fact that if K is a maximal compact subgroup of a Lie group G with finitely many components, then there exist $\mathrm{Ad}_G(K)$-invariant subspaces $\mathfrak{p}_1, \dots, \mathfrak{p}_r$ of $\mathscr{L}(G)$ with $\mathscr{L}(G) = \mathscr{L}(K) \oplus \mathfrak{p}_1 \oplus \cdots \oplus \mathfrak{p}_r$ such that the map $K \times \mathfrak{p}_1 \times \cdots \times \mathfrak{p}_r \to G$, $(k, X_1, \dots, X_r) \mapsto k e^{X_1} \cdots e^{X_r}$ is a diffeomorphism (c.f. [4, Thm. XV.3.1]). (Throughout this paper, we denote by $\mathscr{L}$ the functor which takes a Lie group to its Lie algebra, and denote by e^X the image of $X \in \mathscr{L}(G)$ under the exponential map of a Lie group G.) To prove the injectivity of ι_1 in Theorem 1.1, we need the following generalization of this fact: If a compact Lie group A acts on G by automorphisms and K is A-invariant, then the subspaces $\mathfrak{p}_1, \dots, \mathfrak{p}_r$ can be chosen to be A-invariant (for the precise statement, c.f. Lemma 2.3 below). We will prove this result in Section 2. Theorem 1.1 will be proved in Section 3.

2. Some lemmas

In this section we prove Lemmas 2.2 and 2.3 below. Lemma 2.2 ensures the first assertion in Theorem 1.1, and Lemma 2.3 is a crucial tool to prove the injectivity of the map ι_1 in Theorem 1.1. We need the following well-known fact (c.f. [4, Thm. XV.3.1] or [3, Thm. VII.1.2]).

Theorem 2.1. *Let G be a Lie group with finitely many components, and let K be a maximal compact subgroup of G. Then any compact subgroup of G can be conjugated into K by G.*

Lemma 2.2. *Let G and A be as in Theorem 1.1, and let L be an A-invariant compact subgroup of G. Then there exists an A-invariant maximal compact subgroup K of G which contains L.*

Proof. Denote $H = G \rtimes A$, and view G and A as subgroups of H in the natural way. Since $L \rtimes A$ is a compact subgroup of H, there exists a maximal compact subgroup M of H which contains $L \rtimes A$. Let $K = M \cap G$. It is obvious that the compact group K contains L and is A-invariant. We claim that K is a maximal compact subgroup of G. Indeed, if K' is a compact subgroup of G containing K, by Theorem 2.1, there exists $h \in H$ with $hK'h^{-1} \subset M$. But since G is normal in H, we also have $hK'h^{-1} \subset G$. Thus $hK'h^{-1} \subset M \cap G = K$. The facts that $K \subset K'$ and $hK'h^{-1} \subset K$ imply that K' and K have the same dimension and number of connected components. So we must have $K' = K$. This proves that K is maximal compact subgroup in G. ∎

Our proof of the following lemma is motivated by that of [4, Thm. XV.3.1].

Lemma 2.3. *Let G and A be as in Theorem 1.1, and let K be an A-invariant maximal compact subgroup of G. Then there exist linear subspaces $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ of $\mathscr{L}(G)$ which are invariant under both $\mathrm{Ad}_G(K)$ and A such that $\mathscr{L}(G) = \mathscr{L}(K) \oplus \mathfrak{p}_1 \oplus \cdots \oplus \mathfrak{p}_r$, and such that the map $\varphi : K \times \mathfrak{p}_1 \times \cdots \times \mathfrak{p}_r \to G$ defined by*

$$\varphi(k, X_1, \ldots, X_r) = ke^{X_1} \cdots e^{X_r}$$

is a diffeomorphism.

Proof. We may assume that G is noncompact, and prove the lemma by induction on $\dim G$. Firstly, if G is not semisimple, we define a nontrivial A-invariant closed connected normal abelian subgroup C of G as follows. Let $\mathfrak{s}$ be the solvable radical of $\mathscr{L}(G)$, and define $\mathfrak{s}_i$ inductively as $\mathfrak{s}_0 = \mathfrak{s}$ and $\mathfrak{s}_i = [\mathfrak{s}_{i-1}, \mathfrak{s}_{i-1}]$. Since $\mathfrak{s}$ is solvable, there exists d such that $\mathfrak{s}_d \neq 0$ and $\mathfrak{s}_{d+1} = 0$. Then $\mathfrak{s}_d$ is abelian. Let S_d be the connected Lie subgroup of G with Lie algebra $\mathfrak{s}_d$. Then we define C as the closure of S_d in G. It is obvious that C satisfies the required properties.

Now we define an A-invariant closed normal abelian subgroup D of G according to the following three cases.

(1) If G is semisimple, we define $D = Z(G_0)$.

(2) If G is not semisimple and C is a vector group, we define $D = C$.

(3) If G is not semisimple and C is not a vector group, we define D as the unique maximal compact subgroup of C.

Let $G' = G/D$, and let $\pi : G \to G'$ be the quotient homomorphism. Then A acts on G' by automorphisms. By Lemma 2.2, we can choose an A-invariant maximal compact subgroup K' of G' which contains $\pi(K)$. Let $H = \pi^{-1}(K')$, which is an A-invariant subgroup of G. Clearly, $K \subset H$ is a maximal compact subgroup of H.

We first prove the following two claims.

Claim 1. Lemma 2.3 holds for the pair (G', K').

For case (1), G'_0 is semisimple with trivial center. We choose $\mathfrak{p}'$ as the orthogonal complement of $\mathscr{L}(K')$ in $\mathscr{L}(G')$ with respect to the Killing form of $\mathscr{L}(G')$. Since the Killing form is preserved by any automorphism, we see that $\mathfrak{p}'$ is invariant under both $\mathrm{Ad}_{G'}(K')$ and A. It is well-known that $\mathscr{L}(G') = \mathscr{L}(K') \oplus \mathfrak{p}'$, and the Cartan decomposition ensures that the map $\varphi' : K' \times \mathfrak{p}' \to G'$, $\varphi'(k', X') = k'e^{X'}$ is a diffeomorphism (for the case that G' is non-connected, c.f. [3, Prop. VII.2.3]). For the last two cases, we have $\dim G' < \dim G$, and Claim 1 follows from the induction hypothesis. This finishes the verification of Claim 1.

Claim 2. Lemma 2.3 holds for the pair (H, K).

For case (1), it is well-known that $\pi^{-1}(K'_0)$ is connected (c.f. [5, Thm. 6.31(e)]). So H has finitely many components. Since G is noncompact, we have $\dim K' < \dim G'$. So $\dim H < \dim G$, and in this case Claim 2 follows from the induction hypothesis. For case (2), since D is a closed normal vector subgroup of H and H/D is compact, by a theorem of Iwasawa (c.f. [4, Thm. III.2.3 and Lem. XV.3.2] or [3, Thm VII.4.1]), we indeed have $H = D \rtimes K$. If we let $\mathfrak{p} = \mathscr{L}(D)$, then the map $\varphi_0 : K \times \mathfrak{p} \to H$, $\varphi_0(k, X) = ke^X$ is obviously a diffeomorphism. This proves Claim 2 for case (2). For case (3), since D is compact, H is also compact, and there is nothing to prove.

Now we have a surjective A-equivariant homomorphism $\pi : G \to G'$ with kernel D, an A-invariant maximal compact subgroup K' of G' containing $\pi(K)$, subspaces $\mathfrak{p}'_1, \dots, \mathfrak{p}'_m$ of $\mathscr{L}(G')$, and subspaces $\mathfrak{p}_1, \dots, \mathfrak{p}_n$ of $\mathscr{L}(H)$, where $H = \pi^{-1}(K')$, such that

(1) $\mathscr{L}(G') = \mathscr{L}(K') \oplus \mathfrak{p}'_1 \oplus \cdots \oplus \mathfrak{p}'_m$ and $\mathscr{L}(H) = \mathscr{L}(K) \oplus \mathfrak{p}_1 \oplus \cdots \oplus \mathfrak{p}_n$.

(2) Every $\mathfrak{p}'_i$ is invariant under $\mathrm{Ad}_{G'}(K')$ and A, and every $\mathfrak{p}_j$ is invariant under $\mathrm{Ad}_G(K)$ and A.

(3) The maps $\varphi' : K' \times \mathfrak{p}'_1 \times \cdots \times \mathfrak{p}'_m \to G'$ and $\varphi_0 : K \times \mathfrak{p}_1 \times \cdots \times \mathfrak{p}_n \to H$ defined by

$$\varphi'(k', X'_1, \dots, X'_m) = k' e^{X'_1} \cdots e^{X'_m},$$
$$\varphi_0(k, X_1, \dots, X_n) = k e^{X_1} \cdots e^{X_n}$$

are diffeomorphisms.

Note that the compact group $K \rtimes A$ acts linearly on each $(d\pi)^{-1}(\mathfrak{p}'_i)$, and the subspace $\mathscr{L}(D)$ of $(d\pi)^{-1}(\mathfrak{p}'_i)$ is invariant under $K \rtimes A$. So there exists a subspace $\mathfrak{q}_i$ of $(d\pi)^{-1}(\mathfrak{p}'_i)$ which is invariant under $K \rtimes A$ such that $(d\pi)^{-1}(\mathfrak{p}'_i) = \mathscr{L}(D) \oplus \mathfrak{q}_i$. Now the subspaces $\mathfrak{p}_j$ and $\mathfrak{q}_i$ are all invariant under $\mathrm{Ad}_G(K)$ and A, and it is easy to see that $\mathscr{L}(G) = \mathscr{L}(K) \oplus \mathfrak{p}_1 \oplus \cdots \oplus \mathfrak{p}_n \oplus \mathfrak{q}_1 \oplus \cdots \oplus \mathfrak{q}_m$. It remains to prove that the map $\varphi : K \times \mathfrak{p}_1 \times \cdots \times \mathfrak{p}_n \times \mathfrak{q}_1 \times \cdots \times \mathfrak{q}_m \to G$ defined by

$$\varphi(k, X_1, \dots, X_n, Y_1, \dots, Y_m) = k e^{X_1} \cdots e^{X_n} e^{Y_1} \cdots e^{Y_m}$$

is a diffeomorphism. Since φ_0 and φ' are diffeomorphisms, there are smooth maps $\overline{k} : H \to K$, $\overline{X_j} : H \to \mathfrak{p}_j$, $\overline{k'} : G' \to K'$, and $\overline{X'_i} : G' \to \mathfrak{p}'_i$ such that

$$h = \overline{k}(h) e^{\overline{X_1}(h)} \cdots e^{\overline{X_n}(h)},$$
$$g' = \overline{k'}(g') e^{\overline{X'_1}(g')} \cdots e^{\overline{X'_m}(g')}$$

for all $h \in H$ and $g' \in G'$. We define smooth maps $\widetilde{Y_i} : G \to \mathfrak{q}_i$, $\widetilde{h} : G \to H$, $\widetilde{k} : G \to K$, $\widetilde{X_j} : G \to \mathfrak{p}_j$ as

$$\widetilde{Y_i} = (d\pi|_{\mathfrak{q}_i})^{-1} \circ \overline{X'_i} \circ \pi,$$
$$\widetilde{h}(g) = g(e^{\widetilde{Y_1}(g)} \cdots e^{\widetilde{Y_m}(g)})^{-1} \in H,$$
$$\widetilde{k} = \overline{k} \circ \widetilde{h},$$
$$\widetilde{X_j} = \overline{X_j} \circ \widetilde{h}.$$

Let

$$\psi = (\widetilde{k}, \widetilde{X_1}, \ldots, \widetilde{X_n}, \widetilde{Y_1}, \ldots, \widetilde{Y_m}) : G \to K \times \mathfrak{p}_1 \times \cdots \times \mathfrak{p}_n \times \mathfrak{q}_1 \times \cdots \times \mathfrak{q}_m.$$

Then it is straightforward to check that both $\varphi \circ \psi$ and $\psi \circ \varphi$ are the identity maps. Thus φ is a diffeomorphism. The proof of Lemma 2.3 is finished. ■

3. Proof of the main theorem

Now we prove our main Theorem 1.1.

Proof. The first assertion has been proved in Lemma 2.2. Now we prove the surjectivity of $\iota_1 : H^1(A, K) \to H^1(A, G)$. We first recall that the group operations in $G \rtimes A$ are defined as

$$(g, a)(h, b) = (ga(h), ab), \quad (g, a)^{-1} = (a^{-1}(g^{-1}), a^{-1}).$$

We claim that for every A-invariant maximal compact subgroup K of G, $K \rtimes A$ is a maximal compact subgroup of $G \rtimes A$. Indeed, if L is a compact subgroup of $G \rtimes A$ containing $K \rtimes A$, then $L \cap G$ is a compact subgroup of G containing K. This forces $L \cap G = K$. Now if $h = (g, a) \in L$, since $A \subset L$, we have $g = ha^{-1} \in L \cap G = K$. Hence $h \in K \rtimes A$. This proves that $K \rtimes A$ is maximal compact subgroup in $G \rtimes A$. Now let $\gamma : A \to G$ be a cocycle. Then it is easy to check that the map $\widetilde{\gamma} : A \to G \rtimes A$ defined as $\widetilde{\gamma}(a) = (\gamma(a), a)$ is a homomorphism. Since $\widetilde{\gamma}$ is continuous, we see that $\widetilde{\gamma}(A)$ is a compact subgroup of $G \rtimes A$. Hence there exists $(g, b) \in G \rtimes A$ such that $(g, b)^{-1}\widetilde{\gamma}(A)(g, b) \subset K \rtimes A$. This means that $(g, b)^{-1}(\gamma(a), a)(g, b) \in K \rtimes A$ for all $a \in A$. But

$$(g, b)^{-1}(\gamma(a), a)(g, b) = (b^{-1}(g^{-1}\gamma(a)a(g)), b^{-1}ab).$$

So we have $g^{-1}\gamma(a)a(g) \in K$ for all $a \in A$. Hence γ is cohomologous to a cocycle which takes values in K. This proves that $H^1(A, K) \to H^1(A, G)$ is surjective.

To prove that ι_1 is injective, let $\gamma_1, \gamma_2 : A \to K$ be cocycles which are cohomologous under G, i.e., there exists $g \in G$ with $\gamma_2(a) = g^{-1}\gamma_1(a)a(g)$ for all $a \in A$. By Lemma 2.3, there exist linear subspaces $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ of $\mathscr{L}(G)$ which are invariant under $\mathrm{Ad}_G(K)$ and A such that $\mathscr{L}(G) = \mathscr{L}(K) \oplus \mathfrak{p}_1 \oplus \cdots \oplus \mathfrak{p}_r$, and such that the map $\varphi : K \times \mathfrak{p}_1 \times \cdots \times \mathfrak{p}_r \to G$ defined by

$$\varphi(k, X_1, \ldots, X_r) = ke^{X_1} \cdots e^{X_r}$$

is a diffeomorphism. Write g as $g = \varphi(k, X_1, \ldots, X_r)$. Then for any $a \in A$, we compute

$$\begin{aligned}
&\varphi(\gamma_2(a)^{-1}k\gamma_2(a), \mathrm{Ad}(\gamma_2(a)^{-1})(X_1), \ldots, \mathrm{Ad}(\gamma_2(a)^{-1})(X_1))\\
=&\gamma_2(a)^{-1}g\gamma_2(a)\\
=&\gamma_2(a)^{-1}\gamma_1(a)a(g)\\
=&\gamma_2(a)^{-1}\gamma_1(a)a(k)e^{da(X_1)} \cdots e^{da(X_r)}\\
=&\varphi(\gamma_2(a)^{-1}\gamma_1(a)a(k), da(X_1), \ldots, da(X_r)).
\end{aligned}$$

Since φ is injective, we get

$$\gamma_2(a)^{-1}k\gamma_2(a) = \gamma_2(a)^{-1}\gamma_1(a)a(k).$$

This means that $\gamma_2(a) = k^{-1}\gamma_1(a)a(k)$. So γ_1 and γ_2 are cohomologous under K. This proves the injectivity of ι_1. ■

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References

[1] An, J., and Z. Wang, *Nonabelian cohomology with coefficients in Lie groups*, Trans. Amer. Math. Soc. **360** (2008), 3019–3040.

[2] An, J., *Twisted Weyl groups of Lie groups and nonabelian cohomology*, Geom. Dedicata **128** (2007), 167–176.

[3] Borel, A., "Semisimple groups and Riemannian symmetric spaces," Hindustan Book Agency, New Delhi, 1998.

[4] Hochschild, G., "The structure of Lie groups," Holden-Day, San Francisco, 1965.

[5] Knapp, A. W., "Lie groups beyond an introduction," Second Edition, Birkhäuser, Boston, 2002.

[6] Serre, J.-P., "Galois cohomology," Springer-Verlag, Berlin, 1997.

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