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Diamonds of Finite Type in Thin Lie Algebras

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Abstract. Borrowing some terminology from pro-p groups, *thin* Lie algebras are $\mathbb{N}$-graded Lie algebras, generated in degree one, of width two and obliquity zero. In particular, their homogeneous components have degree one or two, and they are termed *diamonds* in the latter case. In one of the two main subclasses of thin Lie algebras the earliest diamond after that in degree one occurs in degree $2q-1$, where q is a power of the characteristic. This paper is a contribution to an ongoing classification project of this subclass of thin Lie algebras. Specifically, we prove that the degree of the earliest diamond *of finite type* in such a Lie algebra can only attain certain values, which occur in explicit examples constructed elsewhere.
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1. Introduction

The coclass conjectures formulated in [26] have had a considerable impact on the theory of p-groups and pro-p groups. Several new ideas and techniques developed for their proofs, which were completed in [24] and [32], have become part of the standard toolbox of a p-group theorist. Two of the leaders in these developments suggested and initiated a study of a coclass theory for Lie algebras in [33]. It is natural at a first stage to focus on Lie algebras over a field, which are graded over the positive integers, and which are generated by their homogeneous component of degree one. The results of [33] show that analogues of the coclass conjectures hold for these Lie algebras, provided the underlying field has characteristic zero. For our present purposes, the main assertion of the coclass conjectures (now theorems) is that these Lie algebras are soluble, and this implies some sort of periodicity in their structure (in the infinite-dimensional case).

Examples constructed in [31] show that, in positive characteristic, analogues of the coclass conjectures fail even for the special case of coclass one, namely, graded Lie algebras of maximal class need not be soluble. (In the rest of this discussion we tacitly assume that graded Lie algebras are generated by their

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homogeneous component of degree one, but results on a different assumption were obtained in [33] in characteristic zero and in [19] in positive characteristic.) The examples in [31] are (the positive parts of *twisted*) *loop algebras* of certain non-classical finite-dimensional simple Lie algebras. By their very construction, they do have a periodic structure, despite not being soluble. This is not always the case for graded Lie algebras of maximal class. In fact, the results of [16] show that the majority of graded Lie algebras of maximal class are not periodic. Subsequent work has lead to a classification of graded Lie algebras of maximal class in arbitrary odd [18] and even characteristic [22].

The possibility of a classification of p-groups and pro-p groups according to invariants other than the coclass was suggested in [23]; see also Chapter 12 of [25]. The three invariant suggested are called *width, obliquity* and *rank*. Finite p-groups of width two and obliquity zero, which constitute the simplest non-trivial case, had already been introduced in [7, 6] under the name of *thin* p-groups. As is often the case in the study of p-groups, interesting information about certain thin p-groups could be obtained in [17] by working in the graded Lie algebras associated with their lower central series. In fact, thinness itself can be detected at the graded Lie algebra level, and that work suggested the opportunity of investigating thin (graded) Lie algebras as independent objects. This has then been pursued in several papers, part of which we survey below.

According to [17], a *thin* Lie algebra is a graded Lie algebra $L = \bigoplus_{i=1}^{\infty} L_i$ over a field $\mathbb{F}$, with $\dim(L_1) = 2$ and satisfying the following *covering property*:

$$L_{i+1} = [u, L_1] \text{ for every } 0 \neq u \in L_i, \text{ for all } i \geq 1. \tag{1}$$

It was shown in [17] that the covering property is equivalent to the following analogue for Lie algebras of having *obliquity zero*: every graded ideal I of L is located between two consecutive terms L^i of the lower central series of L, in the sense that $L^i \supseteq I \supseteq L^{i+1}$ for some i. The definition of thin Lie algebra implies at once that L is generated by L_1 as a Lie algebra, and that every homogeneous component has dimension 1 or 2 (whence L has *width* two). We call a *diamond* any homogeneous component of dimension 2, and a *chain* the sequence of one-dimensional homogeneous components between two consecutive diamonds. In particular L_1 is a diamond, the *first diamond*. If that is the only diamond, L has maximal class. However, it is convenient to explicitly exclude graded Lie algebras of maximal class from the definition of thin Lie algebras, and hence we assume that a thin Lie algebra has at least two diamonds.

It is known from [17] and [1] that the second diamond of an infinite-dimensional thin Lie algebra of zero or odd characteristic can only occur in degree 3, 5, q or $2q-1$ where q is a power of the characteristic p, in case this is positive. All these possibilities do occur, and examples can be constructed by taking *loop algebras* of suitable finite-dimensional Lie algebras. In particular, examples of thin Lie algebras with second diamond in degree 3 or 5 can be produced as loop algebras of classical Lie algebras of type A_1 or A_2 (see [17] and [10]) and arise also as the graded Lie algebras associated to the lower central series of certain p-adic analytic pro-p groups of the corresponding types (see [30], and also [23]). It was also proved in [17] that those loop algebras (two isomorphism types in case of A_1)

are the only infinite-dimensional thin Lie algebras with second diamond in degree 3 or 5 (for $p > 3$, and respectively $p > 5$, and with a further assumption in the former case). In particular, although solubility fails, periodicity in the structure holds for thin Lie algebras with second diamond in degree 3 or 5 (under the assumptions mentioned above).

The remaining cases are typically modular, and most known examples involve finite-dimensional simple Lie algebras of Cartan type. The case of ground fields of characteristic two is peculiar in several ways, as even what we mean by *second diamond* needs to be adjusted, see [3] and [28]. Therefore, in this paper we assume the characteristic of the ground field to be odd. So we do also in the present introductory survey, as well as restricting our discussion to infinite-dimensional thin Lie algebras for simplicity. Thin Lie algebras with the second diamond in degree q have been called *of Nottingham type* in [9, 11, 10, 14, 3], because the simplest example is the graded Lie algebra associated with the lower central series of the so-called *Nottingham group*, see [8]. We refer to the Introduction of [3] for an up-to-date overview of our knowledge of this class of thin Lie algebras.

The present paper focuses on thin Lie algebras having second diamond in degree $2q - 1$, which have been called (-1)-algebras in [13]. Each diamond of a (-1)-algebra after the first one can be assigned a *type* taking value in the underlying field plus infinity. The definition of type will be given in Section 2 according to Equations (4) and (5), but for the purpose of the present discussion it should suffice to anticipate that an infinite-dimensional (-1)-algebra is completely described by specifying the degrees in which the diamonds occur, and their types. We should also anticipate the notion of *fake* diamond: diamonds of type zero are really one-dimensional components, but it is convenient to allow them in certain degrees and dub them *fake*. We will add the attribute *genuine* to explicitly exclude the possibility that a diamond may be fake.

Several diamond patterns are possible, according as whether both finite and infinite diamond types occur, or only one of those. The (-1)-algebras with all diamonds of infinite type form the most complex subclass, but are all known. In fact, an explicit bijection was constructed in [13] between them and a certain subclass of the graded Lie algebras of maximal class (namely, those with at most two distinct *two-step centralizers*). That subclass is large enough that all the complexity features of graded Lie algebras of maximal class (see [16]) also occur in (-1)-algebras with all diamonds of infinite type. In particular, the latter algebras need not be periodic, and there are uncountably many of them (even over a finite field). Nevertheless, they inherit a complete classification from the corresponding one in [18] for graded Lie algebras of maximal class, and so they are well understood.

The (-1)-algebras with all diamonds of finite type have also been classified, but there are very few of them. They were constructed in [13] and were shown to be determined by a certain finite-dimensional quotient. More precisely, for each power $q > 1$ of the characteristic there is a unique infinite-dimensional thin Lie algebra with second diamond in degree $2q - 1$ and of finite (nonzero) type. (Different choices for this finite type lead to isomorphic algebras.) Its diamonds occur at regular intervals (provided we introduce some fake diamonds), namely, in all degrees congruent to 1 modulo $q-1$. Moreover, all diamond types are finite and follow an arithmetic progression. An explicit construction for this thin Lie algebras as a loop algebra of a certain finite-dimensional simple Lie algebras is given in [15].

We are left with the case of (-1)-algebras with diamonds of both finite and infinite types, of which we know examples. According to the previous paragraph, the second diamond here must be of infinite type, and a diamond of finite type must occur later. Certain such (-1)-algebras were constructed in [2]. They have diamonds in all degrees congruent to 1 modulo $q-1$, except those congruent to q modulo p^{s+1} (for some $s \geq 1$). If we are willing to accept as diamonds of type zero (that is, fake) the one-dimensional components in degrees congruent to q modulo $p^{s+1}(q-1)$, then there are diamonds in all degrees congruent to 1 modulo $q-1$. The diamonds occur in sequences of $p^s - 1$ diamonds of infinite type separated by single occurrences of diamonds of finite type, and the latter types follow an arithmetic progression.

Extensive machine calculations have suggested that the only (-1)-algebras with diamonds of both finite and infinite types are those constructed in [2] and briefly described in the previous paragraph. The present paper provides one contribution in that direction, as we now explain. We discuss in Section 7 how this result fits in a prospective proof of a classification of (-1)-algebras.

The machine calculations mentioned above suggest that the occurrence of a single diamond of finite type in a (-1)-algebra forces a sort of periodicity and, therefore, causes the algebra to be uniquely determined. The simplest instance of those calculations, where the second diamond is already of finite type, led to the proof of uniqueness obtained in [13] and described earlier.

This case aside, calculations provided evidence that the pattern of diamonds of infinite type leading to the first diamond of finite type (that is, the degrees in which they occur) is very restricted. Specifically, it appeared that all diamonds (of infinite type) preceding the earliest one of finite type must occur at regular distances, and that their number (counting from the second diamond) has to be one less than a power of p. Our main result is that the former assertion implies the latter.

Theorem 1.1. *Let T be an infinite-dimensional (-1)-algebra, in odd characteristic, with diamonds of infinite type in each degree $k(q-1)+1$ for $k = 2, \ldots, a-1$, and a diamond of nonzero finite type in degree $a(q-1)+1$, for some integer $a \geq 3$. Then $a-1$ is a power of p.*

Theorem 1.1 follows at once from the more precise Theorem 5.1, which includes information on finite-dimensional algebras. We prove Theorem 5.1 in Section 6.

An equivalent formulation of Theorem 1.1 is that the earliest genuine diamond of finite type of an infinite-dimensional (-1)-algebra can only occur in a degree of the form $(p^s+1)(q-1)+1$ for some $s \geq 0$, under the assumption that all preceding diamonds (of infinite type and from the second on) occur at the standard distance of $q-1$ degrees apart. These conclusions provide the ingredients for the Lie algebra presentation considered in Theorem 4.1 (quoted from [4]), which identifies a unique thin Lie algebra for each value of the parameter s. Results from [2] then exhibit that thin Lie algebra as a loop algebra of a Block algebra of the appropriate dimension. Putting these results together and extracting the essential information we obtain the following contribution to a classification of (-1) algebras with diamonds of mixed types.

Theorem 1.2. *Let T be an infinite-dimensional (-1)-algebra, in odd characteristic, with at least one diamond of finite type (the opposite case being covered by the results of [13]). Suppose that all diamonds of T up to and including one of finite type occur in all degrees congruent to 1 modulo $q-1$, except q. Then T belongs to a countable family of isomorphism types, each determined by the degree of the earliest diamond of finite type. Furthermore, T is a loop algebra of a Block algebra of dimension one less than a power of p.*

We believe that the assumption that the diamonds preceding the earliest of finite type occur at regular distances as described in Theorem 1.2 is superfluous. If this belief is founded a classification of (-1)-algebras could be completed, as we outline in Section 7. However, a proof of this fact appears to depend on several ingredients which lie beyond the arguments employed in this paper.

This paper is structured as follows. In Section 2 we recall various notions and known facts about (-1)-algebras. In particular, we give a precise definition of *diamond type*, a term which we have used repeatedly but vaguely in this introduction. The possibility of doing this, at least for the (-1)-algebras considered in this paper (full generality is achieved in [28]), is justified by a result established in Section 3. The same result produces also a lower bound $q-1$ for the difference of the degrees of consecutive diamonds. In Section 4 we quote a uniqueness result from [4] which provides a bridge between our Theorems 1.1 and 1.2. Section 5 contains a more precise version of our main result, Theorem 1.1, which provides information for finite-dimensional (-1)-algebras as well. Its proof appears in Section 6, the longest and most technical of the paper. We have tried to provide a clear thread of thoughts through some rather awkward calculations by dividing the proof into several lemmas. However, as in other papers on this and related topics, some such calculations seem unavoidable (cf. the Introduction of [18]). In the concluding Section 7 we outline a plan of a proof for a classification of (-1)-algebras.

In our calculations we have been strongly guided by computations performed with the Australian National University p-Quotient program ([21]).

2. General facts on (-1)-algebras

Throughout the paper $\mathbb{F}$ denotes a field of odd characteristic p, with prime field $\mathbb{F}_p$. We write the Lie bracket in a Lie algebra without a comma and use the left-normed convention for iterated Lie brackets, so that $[vyz] = [[vy]z]$. We also use the shorthand

$$[v\underbrace{z\dots z}_{i}] = [vz^i].$$

The following generalization of the Jacobi identity in a Lie algebra is easily proved by induction:

$$[v[yz^{\lambda}]] = \sum_{i=0}^{\lambda}(-1)^i\binom{\lambda}{i}[vz^iyz^{\lambda-i}]. \tag{2}$$

This identity will be used repeatedly, usually without specific mention. Consequently, it will very often be necessary to evaluate binomial coefficients modulo p as elements of $\mathbb{F}$. This can be done by means of Lucas' Theorem (see [20, p. 271]), which states that

$$\binom{a}{b} \equiv \prod_{i=0}^{n}\binom{a_i}{b_i} \pmod{p},$$

where $a = \sum_{i=0}^{n} a_ip^i$ and $b = \sum_{i=0}^{n} b_ip^i$ are the p-adic representations of the non-negative integers a and b, thus with $0 \leq a_i, b_i < p$.

We now describe some general features of (-1)-algebras and introduce some terminology, referring the reader to [13] for further details and the missing proofs. In particular, we explain how one can assign a *type* to each diamond successive to the first. In this discussion we assume (-1)-algebras to be infinite-dimensional unless differently specified, as this will save us from cumbersome statements. However, all considerations remain valid for finite-dimensional thin Lie algebras of dimension large enough; how large should be clear in each instance. For example, because of the covering property (1) an infinite-dimensional thin Lie algebra is centreless; however, the centre of a finite-dimensional thin algebra coincides with the nonzero homogeneous component of highest degree. We use repeatedly without mention the following more general consequence of the covering property, for a thin Lie algebra L with $L_1 = \langle x, y\rangle$: whenever an element u spans a one-dimensional component different from the nonzero component of highest degree, and $[uy] = 0$, say, then $[ux] \neq 0$.

Let $L = \bigoplus_{i=1}^{\infty} L_i$ be a (-1)-algebra with second diamond in degree $2q-1$, where $q = p^n$ for some $n \geq 1$. As in the theory of graded Lie algebras of maximal class, a role is played by the centralizers in L_1 of homogeneous components, that is,

$$C_{L_1}(L_k) = \{a \in L_1 | [a, b] = 0 \text{ for all } b \in L_k\}.$$

(These are called *two-step centralizers* in [16, 18], a term borrowed from the theory of p-groups of maximal class). Because of the covering property such centralizers are one-dimensional except for those of diamonds or components immediately preceding a diamond, which are trivial. It is proved in [12] that under our assumptions the one-dimensional components between the first and second diamond, except that immediately before the latter, all have the same centralizer in L_1. Hence, we may choose $x, y \in L_1$ such that $L_1 = \langle x, y\rangle$ and

$$C_{L_1}(L_2) = C_{L_1}(L_3) = \cdots = C_{L_1}(L_{2q-3}) = \langle y\rangle. \tag{3}$$

Our assumption that L is infinite-dimensional (or has dimension large enough) implies that the iterated bracket $v_2 = [yx^{2q-3}]$ is a nonzero element in L_{2q-2}. The relations

$$[v_2xy] + [v_2yx] = 0 \quad \text{and} \quad [v_2yy] = 0$$

hold in L_{2q}. We quote from [13] the calculations which prove them, as basic examples of application of the Jacobi identity and its iterate (2):

$$0 = [yx^{2q-4}[yxy]] = [yx^{2q-4}[yx]y] - [yx^{2q-4}y[yx]] = -[yx^{2q-4}xyy] = -[v_2yy],$$

$$\begin{aligned} 0 = [yx^{q-1}[yx^{q-1}]] &= \sum_{i=0}^{q-1}(-1)^i\binom{q-1}{i}[yx^{q-1+i}yx^{q-1-i}] \\ &= -\binom{q-1}{q-2}[yx^{2q-3}yx] + \binom{q-1}{q-1}[yx^{2q-3}xy] \\ &= [v_2yx] + [v_2xy]. \end{aligned}$$

In the latter calculation most summands vanish because $[yx^jy] = 0$ for $j = 0, \ldots, 2q-4$, according to (3). These two calculations are simple instances of two types of calculations which occur many times in this paper. We use the (iterated) Jacobi relation to expand the Lie bracket $[u, v]$, where u and v are left-normed iterated Lie brackets in the generators x and y of L, into a left-normed iterated Lie bracket in x and y. In most cases the relation $[u, v] = 0$ is obtained as a consequence of a known relation $v = 0$, but sometimes we expand the relation $[u, u] = 0$, which is part of the Lie algebra axioms. We prefer the former type of calculation over the latter whenever there is a choice, because the former applies in the more general context of *Leibniz algebras* (which are, roughly speaking, Lie algebras without the law $[zz] = 0$, see [27] for a formal definition).

Because of the covering property we have $L_{2q} = \langle [v_2yx] \rangle$, and so $[v_2xx]$ is a multiple of $[v_2yx]$. If $[v_2xx] = 0$, the second diamond is said to be of *infinite type*. More generally, suppose that L_h is a diamond for some $h > 1$, and suppose that L_{h-1} is one-dimensional. Choose a nonzero element w of L_{h-1}. The diamond L_h is said to be of *infinite type* if

$$[wxy] + [wyx] = 0, \qquad [wxx] = [wyy] = 0. \tag{4}$$

The diamond L_h is said to be of *finite type* λ if

$$[wxy] + [wyx] = 0, \qquad [wyy] = 0, \qquad [wyx] = \lambda[wxx], \tag{5}$$

for some $\lambda \in \mathbb{F}$. The latter definition, unlike that of diamond of infinite type, depends on the choice of $y \in C_{L_1}(L_2)$ and $x \in L_1 \backslash \langle y \rangle$. A natural normalization, which can be achieved as in [13] by replacing y with $\lambda^{-1}y$, is assuming that the first diamond of finite type in order of occurrence (if any) has type 1. In the finite-dimensional case, note that when the nonzero homogeneous component of highest degree is two-dimensional and follows a one-dimensional component, it formally satisfies the definition of a diamond of any type.

Strictly speaking, a diamond of type zero cannot occur, because then $[wy]$ would be a central element of L, thus violating the covering property. However, there are situations (here and in other papers, like [13], [15] and [5]) where we have found natural and convenient to call *fake diamonds* certain one-dimensional components L_h. In these cases one has $L_{h-1} = \langle w \rangle$ with $[wy] = 0$ and $[wxy] = 0$.

Therefore, a fake diamond formally satisfies the relations (5) with $\lambda = 0$, and will also be referred to as a diamond of type zero. However, this description of a fake diamond should not be taken as a formal definition: although "most" one-dimensional components L_h of a (-1)-algebra fit that description (namely, all one-dimensional components L_h such that both L_{h-1} and L_{h+1} have dimension one, see the next paragraph and Proposition 3.1) only some of them will be dubbed fake diamonds, namely, those which fit in a sequence of diamonds at regular distances and with types following an arithmetic progression.

There is another delicate matter to discuss about assigning types to diamonds in a (-1)-algebra. We call (genuine) diamond any two-dimensional homogeneous component L_h of L (besides the fake diamonds introduced in the previous paragraph), but the definition of type we gave earlier only applies if L_{h-1} is one-dimensional and the relations

$$[wxy] + [wyx] = 0, \qquad [wyy] = 0$$

hold, where w is a nonzero element of L_{h-1}. In fact, it is proved in [28] that these conditions hold for each diamond L_h (except for the first diamond L_1) of any (-1)-algebra L of odd characteristic. (They also hold for (-1)-algebras of characteristic two, which are not considered in the present paper, provided $q > 2$, that is, the second diamond is not L_3.) Thus, every diamond except the first can be assigned a type. It is also proved in [28] that every one-dimensional component not immediately preceding a (genuine) diamond is centralized by y. It follows that an arbitrary infinite-dimensional (-1)-algebra L is completely described by specifying the degrees in which the diamonds occur, and their types. In fact, this information allows one to write down the adjoint action of x and y on a graded basis of L.

The assertions quoted from [28] in the previous paragraph are considerably easier to prove in the special case of the (-1)-algebras under consideration in this paper, where the diamonds (from the second on and at least up to the first diamond of finite type) occur at a regular distances of $q-1$ components apart. We prove those assertions in the next section, thus making the present paper independent of the forthcoming results of [28].

3. The minimum distance between consecutive diamonds

An argument in the first part of the proof of Proposition 1 in [13] shows that a (-1)-algebra T cannot have a third (genuine) diamond earlier than in degree $3q-2$. We simplify that argument and extend it to prove that two consecutive diamonds in a (-1)-algebra must be at least $q-1$ components apart, under certain assumptions which we have partly discussed at the end of Section 2.

Proposition 3.1. *Let T be a (-1)-algebra in odd characteristic, with second diamond in degree $2q-1$. Choose generators $x, y \in T_1$ for T, with $[yxy] = 0$ as usual. Let T_m be a diamond of T of infinite or nonzero finite type, for some integer $m \geq 2q-1$.*

1. If $q > 3$ suppose that y centralizes every homogeneous component from

T_{m-q+2} up to T_{m-2}. Then y centralizes every homogeneous component from T_{m+1} up to T_{m+q-3}. In particular, all components $T_{m+1},\dots,T_{m+q-2}$ are at most one-dimensional, and so a diamond after T_m cannot occur in degree lower than $m+q-1$.

2. *If $T_{m+q-2}=\langle w\rangle$ then the relations $[wxy]+[wyx]=0$ and $[wyy]=0$ hold. If $[wy]\neq 0$ and $T_{m+q}\neq\{0\}$ then T_{m+q-1} is a genuine diamond, and can be assigned a type.*

Proof. Let v be a nonzero element in T_{m-1}. Since T_m is a diamond of infinite or finite nonzero type, say λ, the relations

$$[vxy]+[vyx]=0,\qquad [vyy]=0,\qquad \lambda^{-1}[vyx]=[vxx]$$

hold in degree $m+1$, where we read $\infty^{-1}=0$. The actual value of λ will play no role in most of the proof. Recall from Section 2 that in a (-1)-algebra we have $[yx^iy]=0$ for $1\le i\le 2q-4$. Differently from the proof of Proposition 1 in [13], which used several of these relations, here we use the single relation $[yx^{q-2}y]=0$ to prove by induction on h that

$$[vyx^hy]=0\quad\text{for}\quad 1\le h\le q-3.$$

This will imply, inductively, that T_{m+h} is spanned by $[vyx^h]$, for $1\le h\le q-3$, and that it is centralized by y, thus proving assertion (1).

We prove the induction base and step at the same time. Let $1\le h\le q-3$, and in case $h>1$ suppose that the conclusion holds up to $h-1$. We may choose u such that $v=[ux^{q-2-h}]$. Note that $q-2-h>0$, and that $[ux^iy]=0$ for $0\le i<q-2$ with $i\neq q-2-h,q-1-h$.

Expanding

$$\begin{aligned}0&=[u[yx^{q-2}y]]=[u[yx^{q-2}]y]\\&=(-1)^{h+1}\binom{q-2}{h}[vyx^hy]+(-1)^h\binom{q-2}{h-1}[vxyx^{h-1}y]\\&=(-1)^{h+1}\binom{q-1}{h}[vyx^hy]=-[vyx^hy]\end{aligned}$$

we reach the desired conclusion that $[vyx^hy]=0$.

To prove assertion (2), note that $T_{m+q-2}=\langle w\rangle$, where $w=[vxyx^{q-3}]$. The computations

$$0=[vyx^{q-3}[yxy]]=-[vyx^{q-2}yy]=[vxyx^{q-3}yy]=[wyy]$$

and

$$\begin{aligned}0&=[vx[yx^{q-2}y]]=[vx[yx^{q-2}]y]-[vxy[yx^{q-2}]]\\&=\binom{q-2}{0}[vxyx^{q-2}y]-\binom{q-2}{q-2}[vxxx^{q-3}yy]\\&\quad-\binom{q-2}{q-3}[vxyx^{q-3}yx]+\binom{q-2}{q-2}[vxyx^{q-2}y]\\&=2[wxy]+2[wyx]+\lambda^{-1}[wyy]\end{aligned}$$

produce the desired relations. Now suppose that $[wy]$ spans T_{m+q-1}, whence $[wx]=\nu[wy]$ for some scalar ν. Then $[wyy]=0$ and

$$0=[wxy]+[wyx]=\nu[wyy]+[wyx]=[wyx]$$

imply that $[wy]$ is a central element of T, and so $T_{m+q}=\{0\}$ by the covering property. Therefore, if $[wy]\neq 0$ and $T_{m+q}\neq\{0\}$ then T_{m+q-1} is two-dimensional, and because of the relations proved above it can be assigned a type. ∎

Remark 3.2. We have excluded the case $q=3$ in assertion (1) of Proposition 3.1 because then both its hypothesis and conclusion would be void, apart from the final consequence that a diamond after T_m cannot occur in degree lower than $m+q-1$. This last assertion holds when $q=3$ as well, because assuming that T_m has a type implies that T_{m+1} is one-dimensional.

Note that when $q=3$ the (-1)-algebra T of Proposition 3.1 has its second diamond in degree 5. However, the uniqueness result from [17] for such algebras, which we have mentioned in the Introduction, only holds in characteristic greater than five. In fact, several infinite families of pairwise non-isomorphic (-1)-algebras with $q=3$ arise as special cases of various constructions [15, 2].

4. (-1)-algebras with diamonds of finite and infinite type

In this section we describe a particular family of infinite-dimensional (-1)-algebras having diamonds of both finite and infinite type which was identified in [4]. As we said in the Introduction, we believe that all infinite-dimensional (-1)-algebras having diamonds of both finite and infinite type should belong to this family, and our Theorem 1.1 is a step in this direction.

Let L be an infinite-dimensional (-1)-algebra with diamonds of infinite type in all degrees $k(q-1)+1$ for $1<k\leq p^s$, where $s\geq 1$, and with a diamond of finite type $\lambda\in\mathbb{F}^*$ in degree $(p^s+1)(q-1)+1$. By replacing y with $\lambda^{-1}y$ we may assume that $\lambda=1$. According to [4], the algebra L turns out to be uniquely determined by these prescriptions. It has diamonds in all degrees of the form $t(q-1)+1$. If $t\not\equiv 1 \pmod{p^s}$ the corresponding diamond is of infinite type. If $t\equiv 1 \pmod{p^s}$, say $t=rp^s+1$, the corresponding diamond has type r (viewed modulo p), hence an element of the prime field $\mathbb{F}_p$. In particular, when $r\equiv 0 \pmod{p}$ we have a fake diamond. The diamonds of finite type (including the fake ones, which have type zero) follow an arithmetic progression. Furthermore, each one-dimensional homogeneous component not immediately preceding a genuine diamond is centralized by y. This uniqueness result was proved in [4] by showing that if an algebra N is defined by a finite presentation encoding part of the above prescriptions up to degree $(p^s+1)(q-1)+2$ (that is, up to specifying the type of the first diamond of finite type), then the quotient L of N modulo its centre is a thin algebra and has the structure stated above. We quote the precise result from [4].

Theorem 4.1. *Let $q>1$ be a power of the odd prime p. Let $N=\bigoplus_{i=1}^{\infty}N_i$ be the Lie algebra on two generators x and y subject to the following relations, and* *graded by assigning degree* 1 *to* x *and* y*:*

$$
\begin{aligned}
&[yx^iy]=0 && \text{for } 0<i<2q-3 \text{ with } i\neq 2q-p^t-2,\\
&[yx^{2q-p^t-2}yx]=0 && \text{for } 1\leq t\leq n,\\
&[yxyy]=0 && \text{if } q=p=3,\\
&[v_2xxx]=0=[v_2xxy], &&\\
&[v_kxx]=0 && \text{for } 3\leq k\leq p^s \text{ with } k \text{ even},\\
&[v_{p^s+1}yx]=[v_{p^s+1}xx]. &&
\end{aligned}
$$

Here v_k *is defined recursively by* $v_2=[yx^{2q-3}]$*, and* $v_k=[v_{k-1}xyx^{q-3}]$ *for* $k>2$*. Then* $L=N/\mathrm{Z}(N)$ *is a thin algebra and has the diamond structure described above in the text. Consequently, an infinite-dimensional thin algebra generated by the homogeneous elements* x *and* y *and satisfying the above relations is necessarily isomorphic with* L*.*

Note that the elements v_k have degree $k(q-1)$ and span the components just preceding the (possibly fake) diamonds. The last three relations given in Theorem 4.1 specify the types of the diamonds in the degrees $k(q-1)+1$, for $1<k\leq p^s$.

Theorem 4.1 was also quoted as Theorem 3.1 in [2]. In that paper an explicit construction for the Lie algebra N was given as a loop algebra of a central extension of a certain simple Lie algebra (a Block algebra of dimension $p^{n+s+1}-1$, where $q=p^n$). It was also proved in [2] that the central quotient L of N cannot be finitely presented. That result implies that it is not possible to supplement the presentation given in Theorem 4.1 by adding finitely many relations and obtain a presentation of the thin Lie algebra L. In the opposite direction, some of the relations in the presentation turn out to be superfluous, as we discuss in Section 5.

In view of the preceding discussion the last claim of Theorem 4.1 can be rephrased as follows: up to isomorphism $L=N/\mathrm{Z}(N)$ is the unique infinite-dimensional thin algebra with second diamond in degree $2q-1$, diamonds of infinite type in degrees $i(q-1)+1$, for $2\leq i\leq p^s$, and a diamond of type one in degree $(p^s+1)(q-1)+1$.

5. The degree of the first diamond of finite type

In this section we state our main result, which is a more precise version of Theorem 1.1. Consider a (-1)-algebra T with diamonds of infinite type in each degree $k(q-1)+1$ for $k=2,\dots,a-1$, and another diamond in degree $a(q-1)+1$, for some integer $a\geq 3$. We determine for which values of a the diamond in degree $a(q-1)+1$ can be prescribed to be of nonzero finite type (and, hence, of type 1, possibly after adjusting the generators of T as explained in Section 2) without the algebra being forced to have finite dimension. More precisely, we prove the following result.

Theorem 5.1. *Let* T *be a* (-1)*-algebra in odd characteristic with second diamond in degree* $2q-1$ *and with* $T_{a(q-1)+2}\neq 0$*, where* $a\geq 3$*. Suppose that* T

has diamonds of infinite type in each degree $k(q-1)+1$ for $k=2,\dots,a-1$, and a diamond of nonzero finite type in degree $a(q-1)+1$. Then the following assertions hold:

1. *a is even;*
2. *if $a \not\equiv 1 \bmod p$ then $T_{(a+1)(q-1)+3}=0$;*
3. *if $a \equiv 1 \bmod p$, but $a-1$ is not a power of p, then $T_{(a+p^s)(q-1)+2}=0$, where p^s is the highest power of p which divides $a-1$.*

Assertions (2) and (3) of Theorem 5.1 show that T has bounded dimension unless $a-1$ is a power of p, a statement equivalent to Theorem 1.1. Assertion (1) will play a crucial role in the proof of assertion (2) given in Lemma 6.3. However, it also justifies the omission of the relations $[v_k xx]=0$ for k odd in the presentation for a central extension of T given in Theorem 4.1, as we explain below.

Remark 5.2. We point out an additional piece of information which is not made explicit in Theorem 5.1. When a is odd, the assumptions of Theorem 5.1 can all be satisfied except for the requirement that the diamond in degree $a(q-1)+1$ has nonzero finite type. Assuming only that T has a genuine diamond in that degree, it follows that such diamond must have infinite type. This is essentially the content of Lemma 6.2, but also follows from the present formulation: each such diamond can be assigned a type because of Proposition 3.1, and the type must be infinite because of Theorem 5.1.

There is some redundancy in the hypotheses of Theorem 5.1 on the structure of T up to the diamond of finite type, and we point out here what needs to be assumed and what is a consequence. This is of interest when writing an efficient presentation for T, or possibly some algebra which has T as a graded quotient (as in Theorem 4.1), which was an essential task in carrying out the machine calculations which provided evidence for the present results. In particular, this will also shed some light on the presentation given in Theorem 4.1 (for which a complete proof is given in [4]). At the same time we fix some notation in preparation for the proof of Theorem 5.1 in the next section.

Let T be a (-1)-algebra, with second diamond in degree $2q-1$, and choose $x, y \in T_1$ such that $T_1=\langle x,y\rangle$ and $[yxy]=0$. As we already mentioned in Section 2 in the infinite-dimensional case, it follows from [12] that $\langle y\rangle = C_{T_1}(T_i)$ for $2 \le i \le 2q-3$, provided $T_{3q-1}\neq\{0\}$. We assume the dimension (or, equivalently, the nilpotency class) of T to be large enough throughout the argument and set

$$v_1=[yx^{q-2}] \quad \text{and} \quad v_2=[v_1x^{q-1}]=[yx^{2q-3}].$$

Thus, v_1 and v_2 are nonzero elements in degree $q-1$ and $2q-2$. We have seen in Section 2 that

$$[v_2xy]+[v_2yx]=0 \quad \text{and} \quad [v_2yy]=0.$$

We specify the diamond T_{2q-1} to have infinite type by imposing $[v_2xx]=0$.

Define recursively the elements

$$v_k = [v_{k-1}xyx^{q-3}] \quad \text{for} \quad 3 \le k \le a. \tag{6}$$

Thus, v_k has degree $k(q-1)$. We now show how a few carefully chosen relations involving the elements v_k suffice to determine the structure of T up to degree $a(q-1)+2$, satisfying the hypotheses of Theorem 5.1. Let $3 \le k \le a$. Assume recursively that

$$[v_{k-2}xyx^hy] = 0 \quad \text{for} \quad 0 \le h \le q-4$$

(but $[v_1xxx^hy] = 0$ instead in case $k=3$), whence v_{k-1} spans $T_{(k-1)(q-1)}$ by the covering property. Assume also that $T_{(k-1)(q-1)+1}$ is a diamond of infinite type. Then assertion (1) of Proposition 3.1 yields

$$[v_{k-1}xyx^hy] = 0 \quad \text{for} \quad 0 \le h \le q-4.$$

In particular, v_k spans $T_{k(q-1)}$ because of the covering property, and hence $T_{k(q-1)+1}$ is spanned by $[v_kx]$ and $[v_ky]$. Assertion (2) of Proposition 3.1 then implies

$$[v_kxy] + [v_kyx] = 0 \quad \text{and} \quad [v_kyy] = 0.$$

In order to complete one step of our recursive definition of T we need only impose the single relation $[v_kxx] = 0$ if $k < a$, and $\lambda^{-1}[v_ayx] = [v_axx]$ otherwise. Our assumption that the dimension of T is large enough then implies $[v_ky] \ne 0$, otherwise the relations would yield $T_{k(q-1)+2} = \langle [v_kxx], [v_kxy] \rangle = \{0\}$. Then $T_{k(q-1)+1}$ is a diamond, again according to assertion (2) of Proposition 3.1, and has the type which we have specified.

We conclude that if a thin algebra T with second diamond in degree $2q-1$ satisfies $[v_kxx] = 0$ for $2 \le k < a$ and $\lambda^{-1}[v_ayx] = [v_axx]$, where the elements v_k are defined as above, then T has diamonds in all degrees $k(q-1)+1$ for $2 \le k \le a$, all of infinite type with the possible exception of the last, which has type λ. Furthermore, all one-dimensional components of T which do not immediately precede a diamond are centralized by y. It will follow from Lemma 6.2 that the relations $[v_kxx] = 0$ for k odd are actually superfluous and, in fact, they are omitted in the presentation for a central extension of T given in Theorem 4.1.

If the recursive definition of the elements v_k in (6) is extended past v_a, the inductive argument given above carries through for the subsequent diamonds and chains on the sole assumption that, for each k, some relation of the form $\mu^{-1}[v_kyx] = [v_kxx]$, with μ finite nonzero or infinity, is imposed or proved by other means. This is because the specific type of each diamond $T_{k(q-1)+1}$ was immaterial in the argument. We will use this extension of the argument in the proof of Lemma 6.4.

6. Proof of Theorem 5.1

We divide the proof into several lemmas. After the following technical lemma, which describes the adjoint action of v_1 and v_2 on homogeneous elements close to diamonds, we prove assertions (1), (2) and (3) of Theorem 5.1 in Lemmas 6.2, 6.3 and 6.4, respectively.

Lemma 6.1. *Let the notation and assumptions be as in Section 5, and let* $k \geq 2$.

1. *Suppose that* $[v_k yx] = \mu[v_k xx]$, *with* μ *finite nonzero, or infinity, that is, the diamond* $T_{k(q-1)+1}$ *has type* μ. *Then we have*

$$
\begin{aligned}
&[v_k v_1] = v_{k+1}, && \\
&[v_k x v_1] = [v_{k+1} x] + \mu^{-1}[v_{k+1} y], && [v_k y v_1] = [v_{k+1} y], \\
&[v_k xx v_1] = \mu^{-1}[v_{k+1} yx], && [v_k yx v_1] = [v_{k+1} yx].
\end{aligned}
$$

2. *Suppose that* $[v_k xx] = 0$ *and* $[v_{k+1} xx] = 0$, *that is, both diamonds* $T_{k(q-1)+1}$ *and* $T_{(k+1)(q-1)+1}$ *have infinite type. Then we have*

$$[v_k v_2] = 0, \quad [v_k x v_2] = 0, \quad [v_k y v_2] = 0, \quad [v_k yx v_2] = [v_{k+2} xx],$$

and if $q > 3$ *also*

$$[v_k xyx^{q-4} v_1] = [v_{k+1} xyx^{q-4}], \quad [v_k xyx^{q-4} v_2] = 3[v_{k+2} x^{q-2}].$$

Proof. The first of the set of equalities in (1) is

$$[v_k v_1] = [v_k [yx^{q-2}]] = [v_k yx^{q-2}] + 2[v_k xyx^{q-3}] = v_{k+1},$$

and the last one is

$$[v_k yx v_1] = [v_k yx [yx^{q-2}]] = -2[v_k yx^{q-2} yx] - [v_k yx^{q-1} y] = [v_{k+1} yx].$$

The remaining equalities can be proved similarly.

To prove (2) suppose that $[v_k xx] = 0$ and $[v_{k+1} xx] = 0$. We have

$$[v_k v_2] = [v_k [yx^{2q-3}]] = [v_k yx^{2q-3}] + 3[v_k xyx^{2q-4}] = 2[v_{k+1} x^{q-1}] = 0,$$

and

$$[v_k y v_2] = \binom{2q-3}{q-2}[v_{k+1} yxx^{q-2}] - \binom{2q-3}{q-1}[v_{k+1} xyx^{q-2}] = 0,$$

because of the vanishing modulo p of the binomial coefficients involved. The other two equalities are similar. Finally, if $q > 3$ we have

$$[v_k xyx^{q-4} v_1] = -\binom{q-2}{1}[v_{k+1} yxx^{q-4}] + \binom{q-2}{2}[v_{k+1} xyx^{q-4}] = [v_{k+1} xyx^{q-4}],$$

and

$$[v_k xyx^{q-4} v_2] = -\binom{2q-3}{1}[v_{k+1} yxx^{2q-5}] + \binom{2q-3}{2}[v_{k+1} xyx^{2q-5}] = 3[v_{k+2} x^{q-2}],$$

which completes the proof. ∎

According to the very first equality proved in Lemma 6.1, the recursive definition (6) of v_k can now be replaced by the more compact formula $v_k = [v_2 v_1^{k-2}]$. We can use this, together with the other equalities given in Lemma 6.1, to compute the adjoint action of any v_h on homogeneous elements close to diamonds. One instance of this type of calculation occurs in the last part of the proof of the following lemma. More instances will occur in the rest of this section, and will be treated in less detail.

Lemma 6.2. *Under the hypotheses of Theorem 5.1, if a is odd then the relation $[v_a xx] = 0$ holds in degree $a(q-1)+2$.*

Proof. We expand the relation $0 = [u, u]$, where u is a homogeneous element of degree $a(q-1)/2+1$. We distinguish two cases. For the case $a = 3$ we choose $u = [v_1 x^{(q+1)/2}] = [yx^{(3q-3)/2}]$ and we have

$$\begin{aligned} 0 = [u[yx^{(3q-3)/2}]] &= \sum_{i=0}^{(3q-3)/2} (-1)^i \binom{(3q-3)/2}{i} [ux^i yx^{((3q-3)/2)-i}] \\ &= \pm \binom{(3q-3)/2}{(q-3)/2} [v_2 yx^q] = \pm [v_3 xx]. \end{aligned}$$

Now suppose that $a > 3$ and write $a = 2h+1$. We choose $u = [v_h xyx^{(q-3)/2}]$ and we have

$$\begin{aligned} 0 = [u[v_h xyx^{(q-3)/2}]] &= \sum_{i=0}^{(q-3)/2} (-1)^i \binom{(q-3)/2}{i} [ux^i [v_h xy] x^{((q-3)/2)-i}] \\ = \sum_{i=0}^{(q-3)/2} (-1)^i \binom{(q-3)/2}{i} &\left([ux^i [v_h x] yx^{((q-3)/2)-i}] - [ux^i y [v_h x] x^{((q-3)/2)-i}] \right). \end{aligned}$$

The element $[ux^i[v_h x]]$ belongs to $T_{2h(q-1)+((q+3)/2)+i}$, and so is centralized by y unless $i = (q-5)/2, (q-3)/2$. Similarly, the element $[ux^i y]$ vanishes, possibly, when $i = (q-3)/2$. Therefore, we have

$$\begin{aligned} 0 = (-1)^{(q-5)/2}[u, u] &= \binom{(q-3)/2}{(q-5)/2} [v_h xyx^{q-4}[v_h x] yx] - [v_{h+1}[v_h x] y] + [v_{h+1} y [v_h x]] \\ = \binom{(q-3)/2}{(q-5)/2} &[v_h xyx^{q-4} v_h xyx] - \binom{(q-3)/2}{(q-5)/2} [v_{h+1} v_h yx] + \\ &- [v_{h+1} v_h xy] + [v_{h+1} x v_h y] + [v_{h+1} y v_h x] - [v_{h+1} yxv_h] \\ = -[v_{h+1} yxv_h] &= (-1)^{h+1} [v_a xx], \end{aligned}$$

which gives the desired conclusion. The final steps of the above calculation depend on the following relations, which we prove using Lemma 6.1 several times. If $q > 3$

we have

$$
\begin{aligned}
[v_h xyx^{q-4} v_h xyx] &= [v_h xyx^{q-4} [v_2 {v_1}^{h-2}] xyx] \\
&= \sum_{i=0}^{h-2} (-1)^i \binom{h-2}{i} [v_h xyx^{q-4} v_1^i v_2 {v_1}^{h-2-i} xyx] \\
&= \sum_{i=0}^{h-2} (-1)^i \binom{h-2}{i} [v_{h+i} xyx^{q-4} v_2 {v_1}^{h-2-i} xyx] \\
&= 3 \sum_{i=0}^{h-2} (-1)^i \binom{h-2}{i} [v_{h+i+2} x^{q-2} {v_1}^{h-2-i} xyx] = 0,
\end{aligned}
$$

since $h+i+2 \leq 2h$ and $[v_{h+i+2}xx] = 0$. If $q = 3$ this term may be interpreted as $-[v_h y v_h xyx]$ but does not really appear in the earlier calculation, because its coefficient $\binom{(q-3)/2}{(q-5)/2}$ vanishes in this case. Similar calculations show that

$$
[v_{h+1} v_h yx] = 0, \quad [v_{h+1} x v_h y] = 0 \quad \text{and} \quad [v_{h+1} y v_h x] = 0.
$$

Finally, the calculation

$$
\begin{aligned}
[v_{h+1} yx v_h] &= \sum_{i=0}^{h-2} (-1)^i \binom{h-2}{i} [v_{h+i+1} yx v_2 {v_1}^{h-2-i}] \\
&= \sum_{i=0}^{h-2} (-1)^i \binom{h-2}{i} [v_{h+i+3} x^2 {v_1}^{h-2-i}] \\
&= (-1)^{h-2} [v_a xx]
\end{aligned}
$$

completes the proof. ∎

The relation $[v_a xx] = 0$ proved in Lemma 6.2 contradicts our assumption that the diamond in degree $a(q-1)+1$ has finite nonzero type. Hence a must be even, and assertion (1) of Theorem 5.1 is proved.

Now assume that T has a diamond of nonzero finite type in degree $a(q-1)+1$, with $a > 2$, and that T has diamonds of infinite type in all lower degrees congruent to 1 modulo $q-1$, with the exception of q. Assume also that $T_{(a+1)(q-1)+3} \neq \{0\}$. According to Lemma 6.2, a must be even. As we may, we assume that the type of the finite diamond is 1, and so the relation

$$
[v_a xx] = [v_a yx]
$$

holds. The discussion after the statement of Theorem 5.1 shows that $[v_a yy] = 0 = [v_a xy] + [v_a yx]$, that

$$
[v_a yx^h y] = 0 \quad \text{for} \quad 1 \leq h \leq q-3
$$

and, finally, that $[v_{a+1} yy] = 0 = [v_{a+1} xy] + [v_{a+1} yx]$. These relations imply that the (possibly fake) diamond $T_{(a+1)(q-1)+1}$ can be assigned a type. Now we show that this type can only be infinity (and, in particular, the diamond is not fake). This follows from the relations

$$0 = [v_{a-1}x[v_2xx]] = [v_{a-1}xv_2xx] = [v_{a-1}xyx^{2q-1}] = -[v_{a+1}xxx]$$

and

$$\begin{aligned}
0 &= [v_{a-1}y[v_2xx]] = [v_{a-1}y[yx^{2q-1}]] \\
&= [v_{a-1}yx^{q-2}yxx^q] + [v_{a-1}yx^{q-2}xyx^q] + \\
&\quad - [v_{a-1}yx^{2q-3}yxx] - [v_{a-1}yx^{2q-3}xyx] - [v_{a-1}yx^{2q-3}xxy] \\
&= -[v_ayxx^q] - [v_axyx^q] + [v_ax^{q-1}yxx] + [v_ax^{q-1}xyx] + [v_ax^{q-1}xxy] \\
&= -[v_{a+1}yxx] - [v_{a+1}xyx] - [v_{a+1}xxy] \\
&= -[v_{a+1}xxy].
\end{aligned}$$

In fact, these relations imply that $[v_{a+1}xx]$ is central, and so $[v_{a+1}xx] = 0$ because of the covering property. It also follows that $[v_{a+1}x]$ and $[v_{a+1}y]$ are not proportional, otherwise they would be central, and hence vanish, contradicting our assumption that $T_{(a+1)(q-1)+3} \neq \{0\}$. We conclude that $T_{(a+1)(q-1)+1}$ is a diamond of infinite type, as claimed.

The following lemma proves assertion (2) of Theorem 5.1.

Lemma 6.3. *Under the hypotheses of Theorem 5.1, if $a \not\equiv 1 \pmod p$ and the diamond in degree $a(q-1)+1$ is of nonzero finite type, then $T_{(a+1)(q-1)+3} = \{0\}$.*

Proof. We work by way of contradiction and assume that $T_{(a+1)(q-1)+3} \neq \{0\}$. We compute $[v_axxv_1]$ in two different ways starting from the obvious relation $[v_axxv_1] = -[v_1[v_axx]]$. According to Lemma 6.1 we have

$$[v_{a+1}yx] = [v_axxv_1] = -[v_1[v_axx]] = 2[v_1xv_ax] - [v_1xxv_a],$$

since $[v_1v_axx] = -[v_{a+1}xx] = 0$. The two Lie brackets at the right-hand side become

$$\begin{aligned}
[v_1xv_ax] &= [v_1x[v_2v_1^{a-2}]x] = -\sum_{i=1}^{a-2}(-1)^i\binom{a-2}{i}[v_2yv_1^{i-1}v_2v_1^{a-2-i}x] \\
&= -[v_{a-1}yv_2x] = [v_{a+1}yx]
\end{aligned}$$

and

$$\begin{aligned}
[v_1xxv_a] &= [v_1xx[v_2v_1^{a-2}]] = -\sum_{i=1}^{a-2}(-1)^i\binom{a-2}{i}[v_2yxv_1^{i-1}v_2v_1^{a-2-i}] \\
&= (a-2)[v_2yxv_1^{a-4}v_2v_1] - [v_2yxv_1^{a-3}v_2] \\
&= (a-2)[v_{a-2}yxv_2v_1] - [v_{a-1}yxv_2] \\
&= (a-2)[v_axxv_1] + 2[v_{a+1}yx] = a[v_{a+1}yx].
\end{aligned}$$

Besides the fact that a is even, here we have used repeatedly Lemma 6.1, supplemented by the relations

$$\begin{aligned} &[v_1xv_1] = -[v_2y], && [v_1xxv_1] = -[v_2yx], \\ &[v_1xv_2] = 0, && [v_1xxv_2] = 0, \\ &[v_{a-1}yv_2] = -[v_{a+1}y], && [v_{a-1}yxv_2] = -2[v_{a+1}yx], \end{aligned}$$

which can be verified similarly. We conclude that $[v_{a+1}yx] = (2-a)[v_{a+1}yx]$, whence $a \equiv 1 \pmod{p}$. This contradicts one of our hypotheses. ∎

The following lemma completes the proof of Theorem 5.1 by establishing assertion (3).

Lemma 6.4. *Under the hypotheses of Theorem 5.1, suppose that $a \equiv 1 \pmod{p}$ and that the diamond in degree $a(q-1)+1$ is of type 1. Write $a = 1 + np^s$, for some integer $s \geq 1$, with $n \not\equiv 0 \pmod{p}$. If $n > 1$ then $T_{(a+p^s)(q-1)+2} = \{0\}$.*

Proof. This proof will be in a terser style than the previous ones, as we have encountered several similar calculations before. In particular, see the comment which precedes Lemma 6.2.

Extending the recursive definition (6) we set

$$v_{a+k} = [v_{a+k-1}xyx^{q-3}] \quad \text{for} \quad 2 \leq k \leq p^s.$$

As we have mentioned at the end of the discussion which follows the statement of Theorem 5.1, the inductive argument described there can be extended past v_{a+1} to prove that for $k = 1, \ldots, p^s$ we have

$$\begin{aligned} &[v_{a+k}yy] = 0, \quad [v_{a+k}xy] + [v_{a+k}yx] = 0, \quad \text{and} \\ &[v_{a+k}yx^hy] = 0 \quad \text{for} \quad 1 \leq h \leq q-3, \end{aligned}$$

provided we can show that each component $\langle [v_{a+k}x], [v_{a+k}y] \rangle$ is actually two-dimensional, that is, a diamond. This follows from the relations

$$[v_{a+k}xx] = 0 \quad \text{for} \quad 1 \leq k \leq p^s, \tag{7}$$

which we prove now (and also show that those diamonds have infinite type). As usual, we do this inductively, assuming that all claimed relations in lower degrees hold. The case $k = 1$ was proved before Lemma 6.3.

Since $a = np^s + 1 > p^s + 1$ we have $[v_{k+1}xx] = 0$ for $k = 2, \ldots p^s$. Consequently, we have

$$\begin{aligned} 0 = [v_{a-1}[v_{k+1}xx]] &= [v_{a-1}[v_2v_1^{k-1}]xx] - 2[v_{a-1}x[v_2v_1^{k-1}]x] \\ &= [v_{a-1}v_2v_1^{k-1}xx] - (k-1)[v_av_2v_1^{k-2}xx] \\ &\quad - 2[v_{a-1}xv_2v_1^{k-1}x] + 2(k-1)[v_axv_2v_1^{k-2}x] \\ &= (1-k)[v_{a+k}xx]. \end{aligned}$$

Besides the inductive hypotheses we have used Lemma 6.1 repeatedly, supplemented in the last step by the additional relations

$$[v_{a-1}v_2] = -2\,v_{a+1}, \qquad [v_{a-1}xv_2] = -[v_{a+1}x], \tag{8}$$

$$[v_a v_2] = v_{a+2}, \qquad [v_a x v_2] = 0, \tag{9}$$

which can be verified similarly. We conclude that $[v_{a+k}xx] = 0$ as desired, except, perhaps, when $k \equiv 1 \pmod p$. To deal with these exceptions we write $k = hp^t + 1$ with $h \not\equiv 0 \pmod p$. Since $k + p^t < 2k \leq 2p^s < a$ we have

$$\begin{aligned} 0 &= [v_{a-p^t}[v_{k+p^t}xx]] = [v_{a-p^t}[v_2 v_1^{k+p^t-2}]xx] - 2[v_{a-p^t}x[v_2 v_1^{k+p^t-2}]x] \\ &= \binom{k+p^t-2}{p^t-1}[v_{a-1}v_2 v_1^{k-1}xx] - \binom{k+p^t-2}{p^t}[v_a v_2 v_1^{k-2}xx] \\ &\quad - 2\binom{k+p^t-2}{p^t-1}[v_{a-1}xv_2 v_1^{k-1}x] + 2\binom{k+p^t-2}{p^t}[v_a x v_2 v_1^{k-2}x] \\ &= -h[v_{a+k}xx]. \end{aligned}$$

and this completes a proof of (7).

In particular, we have proved that

$$[v_{a+p^s}xy] + [v_{a+p^s}yx] = 0 \quad [v_{a+p^s}yy] = 0 \quad [v_{a+p^s}xx] = 0,$$

and hence the component of T of degree $(a+p^s)(q-1)+2$ is spanned by $[v_{a+p^s}xy]$. To complete the proof we only need to show that this element vanishes, and we do so by expanding both sides of the equality

$$[v_{p^s}[v_a yx]] = [v_{p^s}[v_a xx]].$$

Using Lemma 6.1 and relations (8) and (9) one sees that the right-hand side,

$$[v_{p^s}[v_a xx]] = [v_{p^s}[v_2 v_1^{a-2}]xx] - 2[v_{p^s}x[v_2 v_1^{a-2}]x],$$

equals some multiple of $[v_{a+p^s}xx]$, the actual coefficient being immaterial, and hence vanishes. Consequently, we have

$$0 = [v_{p^s}[v_a yx]] = [v_{p^s}v_a yx] - [v_{p^s}xv_a y] - [v_{p^s}yv_a x] + [v_{p^s}xyv_a].$$

Now we evaluate, in turn, each of the four summands.

By means of Lemma 6.1 and relations (8) and (9) we obtain

$$\begin{aligned} [v_{p^s}v_a] &= [v_{p^s}[v_2 v_1^{a-2}]] = \binom{a-2}{a-1-p^s}[v_{a-1}v_2 v_1^{p^s-1}] - \binom{a-2}{a-p^s}[v_a v_2 v_1^{p^s-2}] \\ &= [v_{a-1}v_2 v_1^{p^s-1}] + [v_a v_2 v_1^{p^s-2}] = -2\,v_{a+p^s} + v_{a+p^s} = -v_{a+p^s}, \end{aligned}$$

and hence $[v_{p^s}v_a yx] = -[v_{a+p^s}yx]$. Similarly, we have

$$[v_{p^s}xv_a y] = [v_{a-1}xv_2 v_1^{p^s-1}y] + [v_a x v_2 v_1^{p^s-2}y] = -[v_{a+p^s}xy].$$

The third summand is

$$[v_{p^s} y v_a x] = [v_{a-1} y v_2 v_1^{p^s-1} x] + [v_a y v_2 v_1^{p^s-2} x] = -[v_{a+p^s} y x],$$

where we have used the additional relations

$$[v_{a-1} y v_2] = -[v_{a+1} y], \qquad [v_a y v_2] = 0.$$

Finally, the fourth summand is

$$[v_{p^s} x y v_a] = [v_{a-1} x y v_2 v_1^{p^s-1}] + [v_a x y v_2 v_1^{p^s-2}] = -2[v_{a+p^s} x y].$$

where we have used the further relations

$$[v_{a-1} x y v_2] = -2[v_{a+1} x y], \qquad [v_a x y v_2] = 0.$$

It follows that

$$0 = [v_{p^s}[v_a y x]] = -[v_{a+p^s} x y],$$

which concludes the proof. ■

We can extract from the proof of Lemma 6.4 some information also on the case where $a = p^s + 1$, namely that

$$[v_{a+k} x x] = 0 \quad \text{for} \quad 2 \le k < p^s,$$

but not for $k = p^s$. Hence in this case T, if large enough, has further diamonds of infinite type in all degrees $k(q-1)+1$ for $p^s + 1 < k < 2p^s + 1$. This is one step towards a proof of Theorem 4.1. A complete proof is given in [4].

7. Concluding remarks

As announced in the Introduction we outline our strategy for a classification of (-1)-algebras. Let L be an infinite-dimensional (-1)-algebra. To each diamond of L we can assign a type, according to Section 2 and [28]. If all the diamonds of L have infinite type then L is closely related to a graded Lie algebra of maximal class, as proved in [13] and [29]. Otherwise L has at least one (genuine) diamond of finite type. If the latter occurs in degree congruent to 1 modulo $q-1$, and L has diamonds in all the lower degrees congruent to 1 modulo $q-1$, with the necessary exception of q, then Theorem 1.1 applies to show that the degree of the first diamond of finite type is of the form $(p^s+1)(q-1)+1$ for some $s \ge 0$. In this case L is still uniquely determined according to Theorem 1.2, which depends on [15] for the case $s = 0$ and on [4] for the case $s > 0$. It remains to prove that the diamonds of L, up to the first diamond of finite type, occur at regular intervals of $q-1$ degrees apart.

Partial progress in this direction is achieved in [29], where it is shown that given an arbitrary (-1)-algebra there exists another (-1)-algebra with diamonds in the same degrees as the given one, but all of infinite type. The connection with graded Lie algebras of maximal class established in [13] then implies that the difference in the degrees of consecutive diamonds of a (-1)-algebra can take the positive values of the form $2q-1-p^r$, with r a nonnegative integer or $-\infty$. Thus, what separates us from a classification of (-1)-algebras is a proof that the occurrence of a diamond of finite type in a (-1)-algebra forces p^r to be q between preceding diamonds (with the necessary exception of $p^r=1$ between the first and second diamond).

References

[1] Avitabile, M., and G. Jurman, *Diamonds in thin Lie algebras*, Boll. Unione Mat. Ital. Sez. B Artic. Ric. Mat. (8) **4** (2001), 597–608.

[2] Avitabile, M., and S. Mattarei, *Thin Lie algebras with diamonds of finite and infinite type*, J. Algebra **293** (2005), 34–64.

[3] —, *Thin loop algebras of Albert-Zassenhaus algebras*, J. Algebra **315** (2007), 824–851.

[4] Avitabile, M., *Some loop algebras of Hamiltonian Lie algebras*, Ph.D. thesis. Trento, 1999.

[5] —, *Some loop algebras of Hamiltonian Lie algebras*, Internat. J. Algebra Comput. **12** (2002), 535–567.

[6] Brandl, R., A. Caranti and C. M. Scoppola, *Metabelian thin p-groups*, Quart. J. Math. Oxford Ser. (2) **43** (1992), 157–173.

[7] —, *The Dilworth number of subgroup lattices*, Arch. Math. (Basel) **50** (1988), 502–510.

[8] Camina, R., *The Nottingham group*, in: "New horizons in pro-p groups", Progr. Math. **184**, Birkhäuser Boston 2000, 205–221.

[9] Caranti, A., *Presenting the graded Lie algebra associated to the Nottingham group*, J. Algebra **198** (1997), 266–289.

[10] —, *Thin groups of prime-power order and thin Lie algebras: an addendum*, Quart. J. Math. Oxford Ser. (2) **49** (1998), 445–450.

[11] —, *Loop algebras of Zassenhaus algebras in characteristic three*, Israel J. Math. **110** (1999), 61–73.

[12] Caranti, A., and G. Jurman, *Quotients of maximal class of thin Lie algebras. The odd characteristic case*, Comm. Algebra **27** (1999), 5741–5748.

[13] Caranti, A., and S. Mattarei, *Some thin Lie algebras related to Albert-Frank algebras and algebras of maximal class*, J. Austral. Math. Soc. Ser. A **67** (1999), 157–184.

[14] —, *Nottingham Lie algebras with diamonds of finite type*, Internat. J. Algebra Comput. **14** (2004), 35–67.

[15] —, *Gradings of non-graded Hamiltonian Lie algebras*, J. Aust. Math. Soc. **79** (2005), 399–440.

[16] Caranti, A., S. Mattarei, and M. F. Newman, *Graded Lie algebras of maximal class*, Trans. Amer. Math. Soc. **349** (1997), 4021–4051.

[17] Caranti, A., S. Mattarei, M. F. Newman, and C. M. Scoppola, *Thin groups of prime-power order and thin Lie algebras*, Quart. J. Math. Oxford Ser. (2) **47** (1996), 279–296.

[18] Caranti, A., and M. F. Newman, *Graded Lie algebras of maximal class. II*, J. Algebra **229** (2000), 750–784.

[19] Caranti, A., and M. R. Vaughan-Lee, *Graded Lie algebras of maximal class. IV*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **29** (2000), 269–312.

[20] Dickson, L. E., "History of the theory of numbers. Vol. I: Divisibility and primality.," Chelsea Publishing Co., 1966.

[21] Havas, G., M. F. Newman, and E. A. O'Brien, *ANU p-Quotient Program* (version 1.4), written in `C`, available as a share library with GAP and as part of `Magma`, or from `http://wwwmaths.anu.edu.au/services/ftp.html`, School of Mathematical Sciences, Australian National University, Canberra, 1997.

[22] Jurman, G., *Graded Lie algebras of maximal class. III*, J. Algebra **284** (2005), 435–461.

[23] Klaas, G., C. R. Leedham-Green, and W. Plesken, "Linear pro-p-groups of finite width," Lecture Notes in Mathematics **1674**, Springer-Verlag 1997.

[24] Leedham-Green, C. R., *The structure of finite p-groups*, J. London Math. Soc. (2) **50** (1994), 49–67.

[25] Leedham-Green, C. R., and S. McKay, "The structure of groups of prime power order," London Mathematical Society Monographs. New Series **27**, Oxford University Press, 2002.

[26] Leedham-Green, C. R., and M. F. Newman, *Space groups and groups of prime-power order. I*, Arch. Math. (Basel) **35** (1980), 193–202.

[27] Loday, J.-L., *Une version non commutative des algèbres de Lie: les algèbres de Leibniz*, Enseign. Math. (2) **39** (1993), 269–293.

[28] Mattarei, S., *Constituents of graded Lie algebras of maximal class and chain lengths of thin Lie algebras*, in preparation.

[29] —, *Deformations of thin Lie algebras*, in preparation.

[30] —, *Some thin pro-p-groups*, J. Algebra **220** (1999), 56–72.

[31] Shalev, A., *Simple Lie algebras and Lie algebras of maximal class*, Arch. Math. (Basel) **63** (1994), 297–301.

[32] —, *The structure of finite p-groups: effective proof of the coclass conjectures*, Invent. Math. **115** (1994), 315–345.

[33] Shalev, A., and E. I. Zelmanov, *Pro-p groups of finite coclass*, Math. Proc. Cambridge Philos. Soc. **111** (1992), 417–421.

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