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# Angle Measures of some Cones Associated with Finite Reflection Groups

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**Abstract.** We give a generalization of „Curious Identity“ of De Concini and Procesi. Our proof is based on the recent result of Waldspurger about the decomposition of the cone dual to the fundamental chamber of a finite reflection group as a disjoint union of some subcones.
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Let $V$ be a Euclidean vector space of dimension $n$ with the inner product $(\cdot,\cdot)$. For a convex polyhedral cone $C$ we denote by $C^\circ$ its open kernel, by $\langle C\rangle$ its linear span, and by $C^*$ its *dual cone*, i.e. $C^* = \{v \in V : (v,u) \geqslant 0 \ \forall\, u \in C\}$. Denote by $\sigma(C)$ *the relative angle measure of* $C$, i.e. $\sigma(C) = \operatorname{vol}(C \cap B)/\operatorname{vol}(B)$, where $B \subset \langle C\rangle$ is the unit ball centered at the origin. Let $F$ be a $k$-dimensional face of some solid cone $C$. By $C/F$ we denote the orthogonal projection of $C$ to the subspace $\langle F\rangle^\perp$. We note that $(C/F)^*$ is an $(n-k)$-dimensional face of $C^*$ that is orthogonal to $F$.

Let now $W \subset O(V)$ be a finite reflection group. Set $W^k = \{w \in W : \dim\ker(1-w) = k\}$ and denote $W^{\mathrm{reg}} := W^0$. For a subspace $U \subset V$ we denote by $W_U$ the subgroup of $W$ that fixes $U$ pointwise.

The aim of this paper is to prove the following theorem conjectured by the first author in [1].

**Theorem 1.** *For a fundamental cone $C$ of a finite reflection group $W$ and for each $k = 0,\dots,n$ we have*

$$\sum_{F\subset C,\ \dim F = k} \sigma(F)\cdot\sigma((C/F)^*) = |W^k|/|W|,$$

*where $F$ runs over the $k$-dimensional faces of $C$.*

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The following two results are crucial for the proof of Theorem 1. The first one is the fundamental result of Waldspurger [7], for the simplest proof of which we refer the reader to [2, 3]. The second one is the so-called „Curious Identity“of De Concini and Procesi [4] (see also [1, 5]).

**Theorem 2** (Walsdpurger). $C^* = \bigsqcup_{w\in W} (1-w)C^\circ$.

**Theorem 3** (Curious Identity). $\sigma(C^*) = |W^{\mathrm{reg}}|/|W|$.

**Proof.** Theorem 2 implies that $wC^* = \bigcup_{w'\in W^{\mathrm{reg}}}(1-w')wC$. Thus we obtain

$$|W|\sigma(C^*) = \sum_{w\in W} \sigma(wC^*) = \sum_{w'\in W^{\mathrm{reg}}} \sum_{w\in W} \sigma((1-w')wC) = \\ = \sum_{w'\in W^{\mathrm{reg}}} \sigma((1-w')V) = |W^{\mathrm{reg}}|. \qquad \blacksquare$$

**Remark 4.** It is easy to see from the previous proof that the number of cones $wC^*$ covering a generic point of $V$ is equal to $|W^{\mathrm{reg}}|$.

**Proof of Theorem 1**. Consider the sum $S := \sum_{(C,F)} \sigma(F)\cdot\sigma((C/F)^*)$ over all pairs $(C,F)$, where $C$ is a fundamental cone of $W$ and $F$ is a $k$-dimensional face of $C$. This sum is equal to the left hand side of the required formula multiplied by $|W|$. We shall calculate $S$ in a different way.

Let us recall that any two fundamental cones of $W$ with a common face $F$ are conjugate by a unique element of the reflection subgroup $W_F$ that fixes $F$ pointwise. We also note that $C/F$ is a fundamental cone for the action of $W_F$ on $\langle F\rangle^\perp$. By Theorem 3 we have $\sigma((C/F)^*) = |W_F^{\mathrm{reg}}|/|W_F|$. If we take the sum of $\sigma((C/F)^*)$ over all $C$ that contain a fixed face $F$, we get $|W_F^{\mathrm{reg}}|$. The group $W_F$ and the measure $\sigma((C/F)^*)$ depend only on the subspace $U := \langle F\rangle$ that is an intersection of reflection hyperplanes. Let us take the sum of $\sigma(F)\cdot\sigma((C/F)^*)$ over all pairs $(C,F)$ such that $F\subset U$ for a fixed $k$-dimensional space $U$. We get $|W_U^{\mathrm{reg}}|$ multiplied by the total measure of faces $F\subset U$, which is equal to one, since the faces $F$ decompose $U$. Taking the sum over all subspaces $U$ we obtain $S = |W^k|$. $\blacksquare$

**Remark 5.** Given a cone $C$ and its face $F$, we define the cone $F\oplus(C/F)^*$. It follows from the previous proof that for any $k$-dimensional subspace $U$, which is an intersection of reflection hyperplanes, a generic point of $V$ is covered by $|W_U^{\mathrm{reg}}|$ cones $F\oplus(C/F)^*$ with $F\subset U$.

**Remark 6.** The sums from Theorem 1 can be expressed in terms of the exponents $m_1,\dots,m_n$ of $W$ with a help of the Solomon formula [6]: $\sum_{k=0}^{n} |W^{n-k}|t^k = \prod_{k=0}^{n}(1+m_kt)$.

**Definition 7.** Let $C$ be a fundamental cone of $W$. We say that two $k$-dimensional faces $F$ and $F'$ of $C$ are equivalent if there exists an element $w \in W$ such that $w\langle F\rangle = \langle F'\rangle$.

Denote by $N_F$ the subgroup of $W$ normalizing $\langle F\rangle$. Consider the sum $(|W|/|W_F|) \cdot \sum_{F'\sim F} \sigma(F')$ of relative angle measures for the $W$-translates of the $k$-dimensional faces $F' \subset C$ that are equivalent to $F$. It is the same as the total measure of the $W$-translates of $F' \subset \langle F\rangle$. The latter is equal to $|W|/|N_F|$ multiplied by the total measure of the faces $F' \subset \langle F\rangle$ which is equal to $1$. Thus we get $\sum_{F'\sim F} \sigma(F') = |W_F|/|N_F|$. Using Theorems 1, 3 and applying the previous equality we get:

$$\sum_F |W_F^{\mathrm{reg}}|/|N_F| = \sum_F \Big(\sum_{F'\sim F} \sigma(F')\Big) \cdot \sigma((C/F)^*) = |W^k|/|W|,$$

where the first sum is taken over representatives of all equivalence classes of $k$-dimensional faces $F \subset C$.

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