Journal of Lie Theory
Volume **19** (2009) 55–78

Buildings of Classical Groups and Centralizers of Lie Algebra Elements

P. Broussous and S. Stevens

Communicated by Alain Valette

Abstract. Let F_o be a non-archimedean locally compact field of residual characteristic not 2. Let G be a classical group over F_o (with no quaternionic algebra involved) which is not a general linear group. Let β be an element of the Lie algebra $\mathfrak{g}$ of G that we assume semisimple for simplicity. Let H be the centralizer of β in G and $\mathfrak{h}$ its Lie algebra. Let I and I^1_β denote the (enlarged) Bruhat–Tits buildings of G and H respectively. We prove that there is a natural set of maps $j_\beta : I^1_\beta \to I$ which enjoy the following properties: they are affine, H-equivariant, map any apartment of I^1_β into an apartment of I and are compatible with the Lie algebra filtrations of $\mathfrak{g}$ and $\mathfrak{h}$. In a particular case, where this set is reduced to one element, we prove that j_β is characterized by the last property in the list. We also prove a similar characterization result for the general linear group.

In this article, we work with Lie algebra filtrations defined by using lattice models of buildings. It is not clear that they coincide with the filtrations constructed by A. Moy and G. Prasad for a general reductive group. This fact is proved by B. Lemaire (see his article in this volume).
Mathematics Subject Classification 2000: 22F50.
Key Words and Phrases: Bruhat-Tits building, classical groups over p-adic fields, Moy and Prasad filtrations, functoriality of affine buildings.

Introduction

Let F_o be a locally compact non-archimedean local field, let G, H be connected reductive algebraic groups over F_o, and suppose we have a morphism $f : H \longrightarrow G$ (of algebraic groups over F_o). Let $I^1(G, F_o)$ and $I^1(H, F_o)$ denote the *enlarged* affine Bruhat–Tits buildings of G and H respectively. Bruhat and Tits showed that f induces a natural map $f_* : I^1(H, F_o) \longrightarrow I^1(G, F_o)$ in the following cases:

- f is the natural injection of a Levi subgroup H of G [BT];
- G is the restriction of scalars $\mathrm{Res}_{K/F_o} H$, for K/F_o a finite galois extension and f the canonical inclusion (see [Ti, §2.6]).

ISSN 0949–5932 / \$2.50

Landvogt showed the existence of an induced map f_* in all generality [La]. His maps are continuous, $H(F_o)$-equivariant and are isometries when f is injective. (Landvogt also asks for compatibility with the action of a Galois group, which we will not go into here.) However, these conditions of Landvogt are not sufficient to characterize the map f_*. The simplest example is the following: Suppose F_o has odd residual characteristic, $G = \mathrm{SL}_2(F_o)$ and $H = E^1$ is the groups of norm-1 elements in a totally ramified separable extension E/F_o. Then the (enlarged) affine building I^1 of G is a tree, while that of H is a single point. There are then an infinity of maps f_*, and choosing one comes down to fixing an H-stable point of I^1 – that is any point in a certain edge determined by H.

Recent constructions in the theory of types for the smooth complex representations of p-adic reductive groups indicate an additional condition to impose on the maps f_*. (See [BK3] for an introduction to the general theory of types.) In the same way that the theory of modular forms requires one to define congruence subgroups, the theory of smooth representations of p-adic groups requires one to construct filtrations on parahoric subgroups. The history of the construction of such filtrations is very long and we will not recall it here. Suffice to say that it culminates in the very general constructions of A. Moy and G. Prasad [MP]. To each point x of the enlarged affine building $I^1(\mathcal{G}, F_o)$ of a reductive F_o-group $\mathcal{G}$, they associate a filtration $(\mathcal{G}_{x,r})$ of the parahoric subgroup $\mathcal{G}_x$ associated to x, and a filtration by lattice $(\mathfrak{g}_{x,r})$ of the Lie algebra of $\mathcal{G}$. (These filtrations are respectively indexed by the set of non-negative real numbers, and the set of real numbers.)

These filtrations have had spectacular applications in the theory of types. For example, they allow one to define and prove the existence of unrefined minimal K-types for a general connected reductive group ([MP, Theorem 5.2]). They also provide Bushnell and Kutzko with a language to construct all types for $GL(N)$ (see [BK1, BK2] and the work of Broussous, Grabitz, Stevens and Sécherre for other classical groups). Note that Bushnell and Kutzko do not use the language of Bruhat and Tits but the equivalent language of lattice functions (see [BL] for the connection between the two points of view).

From the definition of the filtration $(\mathfrak{g}_{x,r})$, it is straightforward to see that the map $r \mapsto \mathfrak{g}_{x,r}$, which associates to each real number a lattice in the Lie algebra, completely characterizes the image of x in the non-enlarged building of $\mathcal{G}$. It is thus natural to ask that the maps f_* be not only $H(F_o)$-equivariant but also compatible with the Lie algebra filtrations:

(Fil) $$\mathfrak{g}_{f_*(x),r} \cap \mathfrak{h} = \mathfrak{h}_{x,r},\ x \in I^1(H, F_o),\ r \in \mathbb{R}\ .$$

In the counterexample of SL_2 given above, there is only one map f_* satisfying the conditions (Fil); its image is the midpoint of the edge of the tree I determined by H.

Now we turn to the results of this paper and specialize our notations. Suppose F_o has odd residual characteristic and let G be the group of rational points of a classical group defined over F_o (a symplectic, orthogonal or unitary group). Let β be an element of the Lie algebra of G which, for the sake of simplicity in this introduction, we assume semisimple. Let H denote the centralizer of β in

G, for the adjoint action. We denote by I^1 (respectively I^1_β) the enlarged affine Bruhat–Tits building of G (respectively H).

The purpose of this article is to show that the inclusion $H \subset G$ induces certain natural H-equivariant maps $j_\beta : I^1_\beta \longrightarrow I^1$. Moreover, they are affine, compatible with the Moy–Prasad filtrations and send an apartment into an apartment. These maps form a single orbit under the action of H-invariant automorphisms of I^1_β.

The Lie algebra of G has a natural representation in a matrix algebra A. In the special case where β generates a field in A, we show that there exists one and only one map $j_\beta : I^1_\beta \longrightarrow I^1$ which is compatible with the Moy–Prasad filtrations. In the general case, we make the following unicity conjecture:

Conjecture. Let Z_H be the centre of H. Modulo the action of H-equivariant automorphisms of the building I, there exists one and only one Z_H-equivariant map $j_\beta\colon I^1_\beta \longrightarrow I^1$ satisfying (Fil).

In the case where G is a general linear group and β is a semisimple element of the Lie algebra of G, the first author and B. Lemaire constructed a map $j_\beta : I_\beta \longrightarrow I$ (here we must use the non-enlarged building of H) which is H-equivariant, affine, compatible with the Moy–Prasad filtrations and sends an apartment into an apartment. We show here that this map too is completely determined by the property of compatibility with the Moy–Prasad filtrations.

This work already has applications to the construction of smooth representations of the group G (see [S1, S2] for more details). Here, the basic datum is a pair (β, x), where $\beta \in \mathfrak{g}$ is semisimple and $x \in I^1_\beta$. From this (and following the methods of Bushnell–Kutzko [BK1]) the second author constructs a subgroup $J = J(\beta, x)$ of G and a set of irreducible representations λ of J. Moreover, if $Z(H)$ is compact and x is a vertex then the induced representation $\mathrm{Ind}_J^G \lambda$ is irreducible and supercuspidal, and all irreducible supercuspidal representations arise in this way ([S2]). In these constructions, and especially in the delicate refinement process required in the proof of exhaustion, our embedding j_β and the property (Fil) play a pivotal role.

In this article, we use lattice models of affine buildings constructed by F. Bruhat and J. Tits ([BT1], [BT2]). We actually work with Lie algebra filtrations that naturally arise from these models. It is proved by B. Lemaire in [Le] that they coincide with the filtrations defined by A. Moy and G. Prasad. Lemaire's proof works in any residual characteristic. He also points out that the results of the article should hold without the restriction on the residual caracteristic.

The paper is organized as follows. In §2 we recall the structure of the maximal split tori of G. In §3,4, using ideas of Bruhat and Tits, we give a model of the affine building of G in terms of *self-dual lattice functions.* In §5 we study the centralizers in $\mathfrak{g}$ and G of the Lie algebra element β. The construction of the maps j_β is done in §6 and their properties are established in §7,8 and 9. In §10 we prove the uniqueness result for the general linear group and finally §11 is devoted to the uniqueness result in the classical group case.

We thank G. Henniart, G. Prasad and J.-K. Yu for stimulating discussions, and the referee for several suggestions which have improved the paper. We thank B. Lemaire for writing the proof that the filtrations coincide.

1. Notation

Here F_o is the ground field; it is assumed to be non-archimedean, locally compact and equipped with a discrete valuation v normalized in such a way that $v(F_o^\times)$ is the additive group of integers. We assume that the residual characteristic of F_o is not 2. We fix a Galois extension F/F_o such that $[F:F_o]\leqslant 2$ and set $\sigma_F=\mathrm{id}_F$ if $F=F_o$ and take σ_F to be the generator of $\mathrm{Gal}(F/F_o)$ in the other case. We still denote by v the unique extension of v to F. We fix $\varepsilon\in\{\pm1\}$ and a finite dimensional left F-vector space V. Recall that a σ_F-skew form h on V is a $\mathbb{Z}$-bilinear map $V\times V\to F$ such that

$$h(\lambda x,\mu y)=\lambda^{\sigma_F}\mu h(x,y)\ ,\ \ \lambda,\mu\in F,\ \ x,y\in V\ .$$

Such a form is called ε-hermitian if $h(y,x)=\varepsilon h(x,y)^{\sigma_F}$ for all x, $y\in V$. From now on we fix such an ε-hermitian form on V and we assume it is non-degenerate (the orthogonal of V is $\{0\}$).

For $a\in\mathrm{End}_F(V)$, we denote by $a^{\sigma_h}=a^\sigma$ the adjoint of a with respect to h, i.e. the unique F-endomorphism of V satisfying $h(ax,y)=h(x,a^\sigma y)$ for all x, $y\in V$.

We denote by $\boldsymbol{G}$ the simple algebraic F_o-group whose set of F_o-rational points G is formed of the $g\in\mathrm{GL}_F(V)$ satisfying $g.h=h$ (it is not necessarily connected). Here $g.h$ is the form given by $g.h(x,y)=h(gx,gy)$, x, $y\in V$.

We know ([Sch, (6.6), page 260]) that in the case $\sigma_F\neq\mathrm{id}_F$, we may reduce to the case $\varepsilon=1$. So we have three possibilities:

$\sigma_F=\mathrm{id}_F$ and $\varepsilon=1$, the orthogonal case;

$\sigma_F=\mathrm{id}_F$ and $\varepsilon=-1$, the symplectic case;

$\sigma_F\neq\mathrm{id}_F$ and $\varepsilon=1$, the unitary case.

We abbreviate $\tilde G=\mathrm{GL}_F(V)$ and $\tilde{\mathfrak{g}}=\mathrm{End}_F(V)$.

2. The maximal split tori of $\boldsymbol{G}$

Recall that a subspace $W\subset V$ is totally isotropic if $h(W,W)=0$ and that maximal such subspaces have the same dimension r, the Witt index of h. Set $I=\{\pm1,\pm2,\ldots,\pm r\}$ and $I_o=\{(0,k)\ ;\ k=1,\ldots,n-2r\}$. We fix a *Witt decomposition* of V, that is

- two maximal totally isotropic subspaces V_+ and V_-,
- bases $(e_i)_{i=1,\ldots,r}$, $(e_{-i})_{i=1,\ldots,r}$, $(e_i)_{i\in I_o}$ of V_+, V_- and $V_o:=(V_++V_-)^\perp$,

such that

$h(e_i, e_i) = 0,\ i \in I,$

$h(e_i, e_j) = 0$, for $i,\ j \in I$ with $j \neq -i$ or $i \in I,\ j \in I_o,$

$h(e_i, e_{-i}) = 1$, for $i \in I$ with $i > 0,$

$h(x, x) \neq 0$, for $x \in V_o$ and $x \neq 0$.

The Witt decomposition gives rise to a maximal F_o-split torus $\boldsymbol{S}$ whose group of F_o-rational points is

$$S = \{s \in G\ ;\ se_i \in F_o e_i\ ,\ i \in I \text{ and } (s - \mathrm{Id})V_o = 0\}\ .$$

It has dimension r, the F_o-rank of $\boldsymbol{G}$. Conversely any maximal F_o-split torus of $\boldsymbol{G}$ is obtained from a Witt decomposition as above. The centralizer $\boldsymbol{Z}$ of $\boldsymbol{S}$ in $\boldsymbol{G}$ has for F_o-rational points

$$Z = \{z \in G\ ;\ ze_i \in Fe_i\ ,\ i \in I \text{ and } zV_o = V_o\}\ .$$

For each i from the index set I we have a morphism of algebraic groups $a_i \colon \boldsymbol{Z} \to \mathrm{Res}_{F/F_o}(\boldsymbol{G}_m)$ given by $ze_i = a_i(z)e_i$. Note that $a_{-i}(z) = a_i(z)^{-\sigma}$. We also denote by $a_i\ :\ \boldsymbol{S} \to \boldsymbol{G}_{m\ /F_o}$ the character obtained by restriction. We have $a_i = -a_{-i}$ in $X^*(\boldsymbol{S})$, the $\mathbb{Z}$-module of rational characters of $\boldsymbol{S}$. The a_i, $i \in I$, $i > 0$, form a basis of $X^*(\boldsymbol{S})$.

The normalizer $\boldsymbol{N}$ of $\boldsymbol{Z}$ in $\boldsymbol{G}$ is the sub-algebraic group whose F_o-rational points are the elements of G which stabilize X_o and permute the lines $V_i = Fe_i$, $i \in I$. The group $N = \boldsymbol{N}(F_o)$ is the semidirect product of Z by the subgroup N' formed of the elements which permute the $\pm e_i$, $i \in I$.

3. MM-norms and self-dual lattice-functions

We keep the notation as in the previous sections. Recall that a *norm* on V is a map $\alpha\ :\ V \to \mathbb{R} \cup \{\infty\}$ satisfying:

(i) $\alpha(x + y) \geqslant \inf(\alpha(x), \alpha(y))$, for $x,\ y \in V$;

(ii) $\alpha(\lambda x) = v(\lambda) + \alpha(x)$, for $\lambda \in F,\ x \in V$;

(iii) $\alpha(x) = \infty$ if and only if $x = 0$.

We denote by $\mathrm{Norm}^1(V)$ the set of norms on V.

Definition 3.1 (cf. [BT2, (2.1)])**.** Let $\alpha \in \mathrm{Norm}^1(V)$. We say that α is dominated by h if

$$\alpha(x) + \alpha(y) \leqslant v(h(x, y)) \text{ for all } x,\ y \in V\ .$$

We say that α is an MM-norm for h (*maximinorante* in French), if α is a maximal element of the set of norms dominated by h.

In [BT2, (2.5)] an involution $^-$ is defined on $\mathrm{Norm}^1(V)$ in the following way. If $\alpha \in \mathrm{Norm}^1(V)$, then

$$\bar{\alpha}(x) = \inf_{y\in V}[v(h(x,y)) - \alpha(y)] \ , \ x \in V \ .$$

We then have

Proposition 3.2 (cf. [BT2, Prop. 2.5])**.** *An element* α *of* $\mathrm{Norm}^1(V)$ *is an* MM*-norm if and only if* $\bar{\alpha} = \alpha$.

We are going to describe the set $\mathrm{Norm}^1_h(V)$ of MM-norms in terms of self-dual lattice-functions. Recall [BL] that a lattice-function in V is a function Λ which maps a real number to an $\mathfrak{o}_F$-lattice in V and satisfies:

(i) $\Lambda(r) \subset \Lambda(s)$ for $r \geqslant s$, r, $s \in \mathbb{R}$;

(ii) $\Lambda(r + v(\pi_F)) = \mathfrak{p}_F\Lambda(r)$, $r \in \mathbb{R}$;

(iii) Λ is left-continuous.

Here $\mathfrak{o}_F$ denotes the ring of integers of F, $\mathfrak{p}_F$ the maximal ideal of $\mathfrak{o}_F$ and π_F a uniformizer of F. As in [BL], we denote by $\mathrm{Latt}^1_{\mathfrak{o}_F}(V)$ (or by $\mathrm{Latt}^1(V)$ when no confusion may occur) the set of $\mathfrak{o}_F$-lattice-functions in V.

Recall [BL] that $\mathrm{Norm}^1(V)$ and $\mathrm{Latt}^1(V)$ may be canonically identified in the following way. To $\alpha \in \mathrm{Norm}^1(V)$, we attach the function $\Lambda = \Lambda_\alpha$ given by

$$\Lambda(r) = \{x \in V \ ; \ \alpha(x) \geqslant r\} \ , \ r \in \mathbb{R} \ .$$

Conversely a lattice-function Λ corresponds to the norm α given by

$$\alpha(x) = \sup\{r \ ; \ x \in \Lambda(r)\} \ , \ x \in V \ .$$

For $\Lambda \in \mathrm{Latt}^1(V)$ and $r \in \mathbb{R}$, set

$$\Lambda(r+) = \bigcup_{s>r} \Lambda(s) \ .$$

For an $\mathfrak{o}_F$-lattice L in V, we define its dual $L^\sharp = L^{\sharp_h}$ by

$$L^\sharp = \{x \in V \ ; \ h(x, L) \subset \mathfrak{p}_F\} \ .$$

Finally, we define the dual $\Lambda^\sharp = \Lambda^{\sharp_h}$ of a lattice-function Λ by

$$\Lambda^\sharp(r) = [\Lambda((-r)+)]^\sharp \ , r \in \mathbb{R} \ .$$

We say that a lattice function Λ is self dual if $\Lambda^\sharp = \Lambda$ and we denote by $Latt^1_h(V)$ the corresponding set.

Proposition 3.3. *Given a norm* $\alpha \in \mathrm{Norm}^1(V)$*, we have* $\Lambda_{\bar{\alpha}} = \Lambda^\sharp_\alpha$.

Corollary 3.4. *Let α be a norm on V. Then α is an MM-norm if and only if the attached lattice-function Λ is self-dual.*

Proof of Proposition 3.3. Let $x \in V$ and $r \in \mathbb{R}$. Then the fact that $x \in \Lambda_{\bar{\alpha}}(r)\backslash\Lambda_{\bar{\alpha}}(r+)$ is equivalent to the following points:

(i) $\bar{\alpha}(x) = r$;

(ii) there exists $y \in V$ such that $v(h(x,y)) - \alpha(y) = r$, and for all $y \in V$, we have $v(h(x,y)) - \alpha(y) \geqslant r$;

(iii) there exists $y \in V$ such that $v(h(x,y)) = 0$ and $\alpha(y) = -r$, and for all $y \in V$ such that $\alpha(y) > -r$, we have $v(h(x,y)) > 0$ (scale by a suitable power of a uniformizer π_F);

(iv) there exists $y \in \Lambda_\alpha(-r)\backslash\Lambda_\alpha(-r+)$ such that $h(x,y) \in \mathfrak{o}_F\backslash\mathfrak{p}_F$, and for all $y \in \Lambda_\alpha(-r+)$ we have $h(x,y) \in \mathfrak{p}_F$;

(v) $x \in \Lambda^\sharp_\alpha(r)\backslash\Lambda^\sharp_\alpha(r+)$.

This proves that the two lattice-functions $\Lambda_{\bar{\alpha}}$ and $\Lambda^\sharp_\alpha$ share the same discontinuity points and that at those points they take the same values; so there are equal. ■

Let $\mathrm{Norm}^2\tilde{\mathfrak{g}}$ (resp. $\mathrm{Latt}^2\tilde{\mathfrak{g}}$) denote the $\tilde{G}$-set of square norms in $\tilde{\mathfrak{g}}$ (resp. of square lattice-functions in $\tilde{\mathfrak{g}}$; see [BT1] and [BL]). Recall that a lattice-function Λ^2 in the F-vector space $\tilde{\mathfrak{g}}$ is square if there exists $\Lambda \in \mathrm{Latt}^1(V)$ such that $\Lambda^2 = \mathrm{End}(\Lambda)$, where

$$\mathrm{End}(\Lambda)(r) = \{a \in \tilde{\mathfrak{g}}\ ;\ a\Lambda(s) \subset \Lambda(s+r),\ s \in \mathbb{R}\},\ r \in \mathbb{R}\ .$$

An additive norm on $\tilde{\mathfrak{g}}$ is square if the corresponding lattice function is square. Recall [BT1, ??] that $\mathrm{Norm}^1(V)$ and $\mathrm{Norm}^2\tilde{\mathfrak{g}}$ (and therefore $\mathrm{Latt}^1(V)$ and $\mathrm{Latt}^2\tilde{\mathfrak{g}}$ by transfer of structure) are endowed with affine structures : the barycenter of two points with positive weights is defined.

The involution σ acts on $\mathrm{Norm}^2\tilde{\mathfrak{g}}$ via

$$\alpha^\sigma(a) = \alpha(a^\sigma),\ a \in \tilde{\mathfrak{g}},\ \alpha \in \mathrm{Norm}^2\tilde{\mathfrak{g}}\ .$$

By transfer of structure, σ acts on $\mathrm{Latt}^2\tilde{\mathfrak{g}}$ via

$$\Lambda^\sigma(r) = [\Lambda(r)]^\sigma,\ \Lambda \in \mathrm{Latt}^2\tilde{\mathfrak{g}}, r \in \mathbb{R}\ .$$

A square norm α (resp. a square lattice function Λ) is said to be self-dual if $\alpha = \alpha^\sigma$ (resp. $\Lambda = \Lambda^\sigma$). We denote by $\mathrm{Norm}^2_\sigma\tilde{\mathfrak{g}}$ and $\mathrm{Latt}^2_\sigma\tilde{\mathfrak{g}}$ the corresponding sets.

Now, in terms of lattice functions, [BT2, Corollary 2, page 163] can be written :

Lemma 3.5. *The map $\Lambda \mapsto \mathrm{End}(\Lambda)$ induces a bijection from the set of self-dual lattice functions in V to the set of self-dual square lattice functions in $\tilde{\mathfrak{g}}$.*

In other words, for any $\Lambda \in \mathrm{Latt}^2_\sigma\tilde{\mathfrak{g}}$, there exists a unique $\Lambda^2 = \Lambda^2_h \in \mathrm{Latt}^1_h(V)$ such that $\mathrm{End}(\Lambda) = \Lambda^2$.

Note that the sets $\mathrm{Latt}^1_h(V)$, $\mathrm{Norm}^1_h(V)$, $\mathrm{Latt}^2_\sigma\tilde{\mathfrak{g}}$ and $\mathrm{Norm}^2_\sigma\tilde{\mathfrak{g}}$ are G-sets and that the various identifications among them are G-equivariant.

Let $u \in F^\times$ and assume that uh is still an ε-hermitian form with respect to σ_F. Then the involution σ of $\tilde{\mathfrak{g}}$ corresponding to uh remains the same and defines the same unitary group $G \subset \tilde{G}$. For $\Lambda \in \mathrm{Latt}^1(V)$ and $s \in \mathbb{R}$, we denote by $\Lambda + s$ the lattice function given by $(\Lambda + s)(r) = \Lambda(s + r)$, $r \in \mathbb{R}$.

Lemma 3.6. *Let $\Lambda^2 \in \mathrm{Latt}^2_\sigma\tilde{\mathfrak{g}}$ and Λ^2_h (resp. Λ^2_{uh}) be the unique element of $\mathrm{Latt}^1_h(V)$ (resp. of $\mathrm{Latt}^1_{uh}(V)$) satisfying $\mathrm{End}(\Lambda^2_h) = \Lambda^2$ (resp. $\mathrm{End}(\Lambda^2_{uh}) = \Lambda^2$). Then $\Lambda^2_{uh} = \Lambda^2_h - v(u)/2$, that is $\Lambda^2_{uh}(r) = \Lambda^2_h(r - v(u)/2)$, $r \in \mathbb{R}$.*

Proof. We easily check that for $\Lambda \in \mathrm{Latt}^1(V)$ and $s \in \mathbb{R}$, we have

$$\Lambda^{\sharp_{uh}} = u^{-\sigma}\Lambda^{\sharp_h} \text{ and } (\Lambda + s)^{\sharp_h} = \Lambda - s\ .$$

We certainly have $\mathrm{End}(\Lambda^2_h - v(u)/2) = \mathrm{End}(\Lambda^2_h) = \Lambda^2$. So by a unicity argument, we must prove that $\Lambda^2_h - v(u)/2 \in \mathrm{Latt}^1_{uh}(V)$. But

$$\begin{aligned}(\Lambda^2_h - v(u)/2)^{\sharp_{uh}} &= u^{-\sigma}(\Lambda^2_h - v(u)/2)^{\sharp_h} = u^{-\sigma}(\Lambda^2_h + v(u)/2)\\ &= \Lambda^2_h + v(u)/2 - v(u^\sigma) = \Lambda^2_h - v(u)/2\ ,\end{aligned}$$

as required. ∎

4. The building as a set of self-dual lattice-functions

Let I denote the building of the standard valuated root datum of G introduced in [BT2] and A denote the apartment of I attached to $\boldsymbol{S}$. Write $V^* = X^*(\boldsymbol{S}) \otimes \mathbb{R}$; this is an $\mathbb{R}$-vector space with basis $(a_i)_{i=1,\dots,r}$. Let V denote the linear dual of V^*. We identify A with V.

To a point $p \in A \simeq V$, we attach the norm α_p on V defined by

$$\alpha_p(\sum_{i\in I}\lambda_i e_i + x_o) = \inf[\omega(x_o), \inf_{i\in I}(v(\lambda_i) - a_i(p))],\ x_o \in V_o,\ \lambda_i \in F \text{ for } i \in I\ .$$

Here $\omega(x_o) = \frac{1}{2}v(h(x_o, x_o))$, $x_o \in V_o$.

Here are two important facts from [BT2].

Proposition 4.1 ([BT2, Prop. 2.9, 2.11(i)]). *The map $p \mapsto \alpha_p$ is a bijection from A to the set of MM-norms on V which split in the decomposition $V = \oplus_{i\in I} Fe_i \oplus V_o$. It is N-equivariant.*

For the notion of splitting for norms, see [BT1, (1.4)].

Proposition 4.2 ([BT2, (2.12)]). (i) *The map $p \mapsto \alpha_p$ extends in a unique way to a G-equivariant and affine bijection $j_h : I \to \mathrm{Norm}^1_h(V)$ (in particular $\mathrm{Norm}^1_h(V)$ is a convex subset of $\mathrm{Norm}^1(V)$).*

(ii) *The map* j_h *is the unique affine and* G*-equivariant map* $I \to \mathrm{Norm}^1_h(V)$.

From §3, we get a unique affine and G-equivariant map $I \to \mathrm{Latt}^1_h(V)$ that we still denote by j_h.

For $r \in \mathbb{R}$, let $\mathcal{V}_o^r$ be the lattice of V_o given by $\{x_o \in V_o \; ; \; \omega(x_o) \geqslant r\}$. For $x \in \mathbb{R}$, let $\lceil x \rceil$ denote the least integer greater than or equal to x. Then the map $j_h \;:\; I \to \mathrm{Latt}^1_h(V)$ is given on A by $j_h(p) = \Lambda_p$, where

$$\Lambda_p(r) = \mathcal{V}_o^r \oplus \bigoplus_{i \in I} \mathfrak{p}_F^{\lceil r + a_i(p) \rceil} e_i \;,\; r \in \mathbb{R} \;.$$

Let u be an element of $F^\times$ such that uh remains ε-hermitian with respect to σ_F. It follows from the proof of Lemma 3.6 that if $\Lambda \in \mathrm{Latt}^1(V)$, we have $\Lambda \in \mathrm{Latt}^1_h(V)$ if, and only if, $\Lambda - v(u)/2 \in \mathrm{Latt}^1_{uh}(V)$. Since $\mathrm{End}(\Lambda + s) = \mathrm{End}(\Lambda)$, for $\Lambda \in \mathrm{Latt}^1(V)$ and $s \in \mathbb{R}$, the bijective map $j_\sigma \;:\; I \to \mathrm{Latt}^2_\sigma(V)$, given by $j_\sigma = \mathrm{End} \circ j_h$, does not depend on the choice of the form h, the involution σ being fixed. By construction it is affine and G-equivariant. It is uniquely determined by these two properties. Indeed if $j'_\sigma \;:\; I \to \mathrm{Latt}^2_\sigma(V)$ is affine and G-equivariant, so is $(j'_\sigma)^{-1} \circ j_\sigma \;:\; I \to I$. But such a map must be the identity map.

We also recall here the description of the enlarged building I^1 of $\tilde{G} = \mathrm{GL}_F(V)$ in terms of lattice functions.

Proposition 4.3 ([BT1, (2.11)]). (i) *There is a* $\tilde{G}$*-equivariant and affine bijection* $j \;:\; I^1 \to \mathrm{Norm}^1(V)$.

(ii) *If we have another affine and* $\tilde{G}$*-equivariant map* $j' : I^1 \to \mathrm{Norm}^1(V)$ *then there exists* $r \in \mathbb{R}$ *such that, for all* $\alpha \in \mathrm{Norm}^1(V)$, $j'(\alpha) = j(\alpha) + r$.

From [BL, Proposition 2.4], for each j as in Proposition 4.3, we get an affine and $\tilde{G}$-equivariant map $I^1 \to \mathrm{Latt}^1(V)$ that we also denote by j.

5. Centralizers of Lie algebra elements

We denote by $\mathfrak{g}$ the Lie algebra of G:

$$\mathfrak{g} = \{a \in \tilde{\mathfrak{g}} \; ; \; a + a^\sigma = 0\} \;.$$

We consider an element β of $\mathfrak{g}$ satisfying

The F-algebra $E := F[\beta] \subset \tilde{\mathfrak{g}}$ is a direct sum of fields.

We write $\tilde{\mathfrak{h}}$ (resp. $\mathfrak{h}$) for the centralizer of β in $\tilde{\mathfrak{g}}$ (resp. in $\mathfrak{g}$) and $\tilde{H}$ (resp. H) for the stabilizer of β in $\tilde{G}$ (resp. in G) for the adjoint action.

Since $\sigma(\beta) = -\beta$, we have easily that $E \subset \tilde{\mathfrak{g}}$ is σ-stable. We write

$$E = \bigoplus_{i=1,\dots,t} (E_i \oplus E_{-i}) \;\oplus\; \bigoplus_{k=1,\dots,s} E_{(0,k)},$$

where, for each i in $J = \{\pm 1, \dots, \pm t\}$ or $J_o = \{(0,k) : k = 1, \dots, s\}$, E_i is a field extension of F, and we have labeled the components such that, for each $i \in J_o \cup J$,

$$\sigma(E_i) = E_{-i}, \tag{5.1}$$

with the understanding that $i = -i$, for $i \in J_o$. We remark that the torus $E \cap G$ in G is anisotropic (modulo the centre) if and only if $J = \emptyset$ and that every maximal anisotropic torus in G takes this form (see [Mor, Proposition 1.3]).

For each $i \in J_o$, we set $E_i^o = \{a \in E_i \; ; a = a^\sigma\}$, so that E_i/E_i^o is a Galois extension of degree $\leqslant 2$ and a generator of $\mathrm{Gal}(E_i/E_i^o)$ is $\sigma_{E_i} := \sigma_{|E_i}$. For $i \in J_o \cup J$, let $\mathbf{1}_i$ be the idempotent of E attached to E_i; from (5.1), we have $\sigma(\mathbf{1}_i) = \mathbf{1}_{-i}$. We have the decomposition

$$V = \bigoplus_{i \in J_o \cup J} V_i \; , V_i = \mathbf{1}_i V \; .$$

Note that, if $i \neq -k$, $v \in V_i$ and $w \in V_k$, we have $h(v,w) = h(\mathbf{1}_i v, w) = h(v, \mathbf{1}_i w) = 0$ so, for $i \in J_o \cup J$,

$$V_i^\perp = \bigoplus_{k \neq -i} V_k.$$

For $i \in J_o \cup J$, V_i is naturally an E_i-vector space and we have obvious isomorphisms of algebras and groups respectively:

$$\tilde{\mathfrak{h}} \simeq \prod_{i \in J_o \cup J} \mathrm{End}_{E_i} V_i \; ,$$

$$\tilde{H} \simeq \prod_{i \in J_o \cup J} \mathrm{Aut}_{E_i} V_i \; .$$

The involution σ stabilizes $\tilde{\mathfrak{h}} \subset \tilde{\mathfrak{g}}$ and, for each i, $\sigma(\mathrm{End}_{E_i} V_i) = \mathrm{End}_{E_{-i}} V_{-i}$. For $i \in J_o$, we write $\sigma_i = \sigma_{|\mathrm{End}_{E_i} V_i}$. Let us fix $i \in J_o$. The map σ_i is an involution of the central simple E_i-algebra $\mathrm{End}_{E_i} V_i$. By a classical theorem ([Inv, Theorem 4.2]), there exists $\varepsilon_i \in \{\pm 1\}$ and a non-degenerate ε_i-hermitian form h_i on V_i relative to σ_{E_i} such that σ_i is the involution attached to h_i. Of course h_i is only defined up to a scalar in $E_i^\times$. Let

$$H_i = \{g \in \mathrm{Aut}_{E_i} V_i \; ; \; g g^{\sigma_i} = 1\}$$

be the unitary group attached to h_i. On the other hand, for $i \in J$, we put

$$H_i = \mathrm{Aut}_{E_i} V_i,$$

so that $\sigma(H_i) = H_{-i}$ and H_i is isomorphic to $\{g \in H_i \times H_{-i} : g g^\sigma = 1\}$ by $h \mapsto (h, h^{-\sigma})$. Then, putting $J_+ = \{1, \dots, t\}$, we have a natural group isomorphism

$$H \simeq \prod_{i \in J_o \cup J_+} H_i \; .$$

We may actually require a compatibility relation between the forms h_i, $i \in J_o$ and the form h. Let us fix $i \in J_o$. Let $\lambda_i : E_i \to F$ be any σ-equivariant non-zero F-linear form. Such forms exist. Indeed choose a non-zero linear form $\lambda_i^o : E_i^o \to F_o$. If $F = F_o$ then we put $\lambda = \lambda_i^o \circ \mathrm{Tr}_{E/E_i^o}$. Otherwise $E_i = FE_i^o$ and we can extend λ_i^o by linearity to get the required map λ_i. In all cases we have:

$$\lambda_i^o \circ \mathrm{Tr}_{E_i/E_i^o} = \mathrm{Tr}_{F/F_o} \circ \lambda \ . \tag{5.2}$$

We still write h for the restriction of h to V_i.

Lemma 5.3. *Let $i \in J_o$. There exists a unique ε-hermitian form $h_i : V_i \times V_i \to E_i$ relative to σ_{E_i} such that*

$$h(v,w) = \lambda_i(h_i(v,w)), \quad \textit{for all } v,w \in V_i \ . \tag{5.4}$$

It is non-degenerate.

Proof. Since we have the orthogonal decomposition

$$V = V_i \perp \bigoplus_{k \neq i} V_k \ ,$$

the restriction $h_{|V_i}$ is non-degenerate.

The F-linear map $\mathrm{Hom}_{E_i}(V_i, E_i) \to \mathrm{Hom}_F(V_i, F)$, $\varphi \mapsto \lambda_i \circ \varphi$ is an isomorphism of F-vector space. Indeed if φ lies in the kernel, we have $\mathrm{Im}(\varphi) \subset \mathrm{Ker}(\lambda_i)$, a strict subspace of E_i, and φ must be trivial. Moreover the two dual spaces have the same F-dimension. For $v \in V_i$ let h_v be the element of $\mathrm{Hom}_F(V_i, F)$ given by $h_v(w) = h(v,w)$. There exists a unique $\varphi_w \in \mathrm{Hom}_{E_i}(V_i, E_i)$ such that $h_v = \lambda_i \circ \varphi_w$. It is now routine to check that $h_i(v,w) := \varphi_v(w)$, for $v, w \in V_i$, has the required properties. ■

We easily check that if h_i is as in the lemma, then the corresponding involution on $\mathrm{End}_{E_i} V_i$ is σ_i. In the following we assume that the forms h_i, $i \in J_o$, satisfy (5.4).

For technical reasons, we need one more assumption on the λ_i, $i \in J_o$. We fix i again. Let

$$\mathfrak{I} = \{ e \in E_i^o \ ; \ \lambda_i^o(e\mathfrak{o}_{E_i^o}) \subset \mathfrak{p}_{F_o} \} \ .$$

This is an $\mathfrak{o}_{E_i^o}$-lattice in E_i^o and must have the form $t\mathfrak{p}_{E_i^o}$, for some $t \in (E_i^o)^\times$. So replacing λ_i by $e \mapsto \lambda_i(tx)$, we may assume that $\mathfrak{I} = \mathfrak{p}_{E_i^o}$. In the following we assume that the linear forms λ_i, $i \in J_o$, have this property.

Lemma 5.5. *Fix $i \in J_o$. Let λ_i^1, $\lambda_i^2 : E_i \to F$ be two linear forms as above and let h_i^1, h_i^2 be the corresponding ε-hermitian forms on V_i (i.e. h_i^1 and h_i^2 satisfy* (5.4)*). Then there exists $u \in \mathfrak{o}_{E_i^o}^\times$ such that $h_i^2 = uh_i^1$.*

Proof. Since h_i^1 and h_i^2 induce the same involution on $\mathrm{End}_{E_i} V_i$, there exists $u \in E_i^\times$ such that $h_i^2 = uh_i^1$. The fact that h_i^1 and h_i^2 are both ε-hermitian with respect to σ_{E_i} implies that u lies in E_i^o. Condition (5.4) writes

$$h(v,w) = \lambda_i^1(h_i^1(v,w)) = \lambda_i^2(uh_i^1(v,w)) \ , \ v,w \in V_i \ .$$

So $\lambda_i^1(e) = \lambda_i^2(ue)$, $e \in E_i$. By applying Tr_{F/F_o} to this equality, we get $\lambda_i^{o,1}(e) = \lambda_i^{o,2}(ue)$, $e \in E_i^o$. Hence

$$\begin{aligned} \mathfrak{p}_{E_i^o} &= \{e \in E_i^o \ ; \ \lambda_i^{o,1}(e\mathfrak{o}_{E_i^o}) \subset \mathfrak{p}_{F_o}\} \\ &= \{e \in E_i^o \ ; \ \lambda_i^{o,2}(ue\mathfrak{o}_{E_i^o} \subset)\mathfrak{p}_{F_o}\} \ = \ u^{-1}\mathfrak{p}_{E_i^o} \ . \end{aligned}$$

So $u \in \mathfrak{o}_{E_i^o}^\times$ as required. ∎

Let us fix i. Let L be an $\mathfrak{o}_{E_i^o}$-lattice in V_i. Then L has a dual $L^\sharp$ relative to the form $h_{|V_i}$ and a dual $L^{\sharp_i}$ relative to the form h_i.

Lemma 5.6. *The lattices $L^\sharp$ and $L^{\sharp_i}$ coincide.*

Proof. We have

$$\begin{aligned} L^\sharp &= \{v \in V_i \ ; \ h(v,L) \subset \mathfrak{p}_F\} \\ &= \{v \in V_i \ ; \ \mathrm{Tr}_{F/F_o} h(v,L) \subset \mathfrak{p}_{F_o}\} \\ &= \{v \in V_i \ ; \ \lambda_o \circ \mathrm{Tr}_{E_i/E_i^o} h_i(v,L) \subset \mathfrak{p}_{F_o}\} \\ &= \{v \in V_i \ ; \ \mathrm{Tr}_{E_i/E_i^o} h_i(v,L) \subset \mathfrak{p}_{E_i^o}\} \\ &= \{v \in V_i \ ; \ f(v,L) \subset \mathfrak{p}_{E_i}\} \\ &= L^{\sharp_i}, \end{aligned}$$

where the second and fifth equalities hold because F/F_o and E_i/E_i^o are at worst tamely ramified. ∎

6. Embedding the building of the centralizer

We keep the notation as in the previous section. Assume for a moment that the extensions E_i/F, $i \in J_o \cup J$, are separable. Then the group H is naturally the group of rational points of a reductive F-group $\boldsymbol{H}$. Indeed each H_i, $i \in J_o \cup J$, is naturally the group of rational points of a classical E_i-group $\boldsymbol{H}_i$ (we do not need E_i/F-separable here) and

$$\boldsymbol{H} \simeq \prod_{i \in J_o \cup J_+} \mathrm{Res}_{E_i/F} \boldsymbol{H}_i \ .$$

The (enlarged) affine building of $\boldsymbol{H}$, $I_\beta^1 := I^1(\boldsymbol{H}, F)$, is the cartesian product of the (enlarged) affine buildings $I^1(\mathrm{Res}_{E_i/F}\boldsymbol{H}_i, F)$, $i \in J_o \cup J_+$. For all i, the (enlarged) buildings $I^1(\mathrm{Res}_{E_i/F}\boldsymbol{H}_i, F)$ and $I^1(\boldsymbol{H}_i, E_i)$ identify canonically. Note also that, for $i \in J_o$, the centre of $\boldsymbol{H}_i$ is compact so the enlarged building is also the non-enlarged building; in particular, if $J = \emptyset$ then all the buildings involved are non-enlarged.

Since we do not want any restriction on the extensions E_i/F, we shall take as a definition of the (enlarged) building I_β^1 attached to the group H:

$$I_\beta^1 := \prod_{i \in J_o \cup J_+} I^1(\boldsymbol{H}_i, E_i) \tag{6.1}$$

We abbreviate $I_i^1 = I^1(\boldsymbol{H}_i, E_i)$, $i \in J_o \cup J_+$.

We are going to construct a map $j_\beta : I_\beta^1 \to I$. We normalize the lattice-functions in $\mathrm{Latt}^1_{\mathfrak{o}_{E_i}}(V_i)$ by $\Lambda_i(r + v_i(\pi_i)) = \mathfrak{p}_{E_i}\Lambda_i(r)$, $r \in \mathbb{R}$, where, for each i, π_i denotes a uniformizer of E_i and v_i the unique extension of v to a valuation of E_i. It is straightforward that we have a well defined map

$$\tilde{j}_\beta : \prod_{i \in J_o \cup J} \mathrm{Latt}^1_{\mathfrak{o}_{E_i}}(V_i) \longrightarrow \mathrm{Latt}^1(V)$$

$$(\Lambda_i)_{i \in J_o \cup J} \mapsto \bigoplus_{i \in J_o \cup J} \Lambda_i$$

where $\left(\bigoplus_{i \in J_o \cup J} \Lambda_i\right)(r) = \bigoplus_{i \in J_o \cup J} \Lambda_i(r)$, for $r \in \mathbb{R}$. This map is clearly injective and equivariant for the action of the group $\prod_{i \in J_o \cup J} \mathrm{Aut}_{E_i} V_i \subset \mathrm{Aut}_F V$.

For $i \in J_o$, we denote by $\sharp_i$ the involution on $\mathrm{Latt}^1_{\mathfrak{o}_{E_i}}(V_i)$ attached to h_i, and by $\mathrm{Latt}^1_{\mathfrak{o}_{E_i}, h_i}(V_i) \subset \mathrm{Latt}^1_{\mathfrak{o}_{E_i}}(V_i)$ the set of fixed points. For $i \in J$, we denote be $\sharp_i$ the map $\mathrm{Latt}^1_{\mathfrak{o}_{E_i}}(V_i) \to \mathrm{Latt}^1_{\mathfrak{o}_{E_{-i}}}(V_{-i})$ given by

$$\Lambda_i^{\sharp_i}(r) = \{v \in V_{-i} \ ; \ h(v, \Lambda_i(-r+)) \subset \mathfrak{p}_F\} \ .$$

for $\Lambda_i \in \mathrm{Latt}^1_{\mathfrak{o}_{E_i}}(V_i)$.

We define an involution b on $\prod_{i \in J_o \cup J} \mathrm{Latt}^1_{\mathfrak{o}_{E_i}}(V_i)$ by

$$(\Lambda_i)^b_{i \in J_o \cup J} = \left(\Lambda_{-i}^{\sharp_{-i}}\right)_{i \in J_o \cup J} \ ,$$

Then we have a bijection

$$\iota_h : \prod_{i \in J_o} \mathrm{Latt}^1_{\mathfrak{o}_{E_i}, h_i}(V_i) \times \prod_{i \in J_+} \mathrm{Latt}^1_{\mathfrak{o}_{E_i}}(V_i) \to \left(\prod_{i \in J_o \cup J} \mathrm{Latt}^1_{\mathfrak{o}_{E_i}}(V_i)\right)^b ,$$

given by $(\Lambda_i)_{i \in J_o \cup J_+} \mapsto (\Lambda_i)_{i \in J_o \cup J}$, with $\Lambda_{-i} = \Lambda_i^{\sharp_i}$, for $i \in J_+$.

Lemma 6.2. *For* $x \in \prod_{i \in J_o \cup J} \mathrm{Latt}^1_{\mathfrak{o}_{E_i}}(V_i)$, *we have* $\tilde{j}_\beta(x^b) = \tilde{j}_\beta(x)^{\sharp_h}$. *In particular* $\tilde{j}_\beta \circ \iota_h$ *maps* $\prod_{i \in J_o} \mathrm{Latt}^1_{\mathfrak{o}_{E_i}, h_i}(V_i) \times \prod_{i \in J_+} \mathrm{Latt}^1_{\mathfrak{o}_{E_i}}(V_i)$ *into* $\mathrm{Latt}^1_h(V)$.

Proof. Fix $(\Lambda_i)_{i \in J_o \cup J} \in \prod_{i \in J_o \cup J} \mathrm{Latt}^1_{\mathfrak{o}_{E_i}} V_i$ and set $\Lambda = \tilde{j}_\beta\left((\Lambda_i)_{i \in J_o \cup J_+}\right)$. We have

$$\Lambda^{\sharp_h}(r) = \Lambda(-r+)^{\sharp_h} = \{v \in V \ ; \ h(v, \Lambda(-r+)) \subset \mathfrak{p}_F\} \ , \ r \in \mathbb{R} \ .$$

Fix $r \in \mathbb{R}$. We have

$$\Lambda(-r+) = \bigoplus_{i \in J_o \cup J} \Lambda_i(-r+) \ .$$

Let $v=\sum_{i\in J_o\cup J} v_i$, with $v_i\in V_i$, be an element of V. Since $V_i^\perp=\bigoplus_{k\neq -i} V_k$, we have $v\in\Lambda^{\sharp_h}(r)$ if and only if $h(v_{-i},\Lambda_i(-r+))\subset\mathfrak{p}_F$, for all i, that is if $v_{-i}\in\Lambda_i^{\sharp_i}(r)$, for all i (by Lemma 5.6 for $i\in J_o$ or by definition for $i\in J$); the lemma follows. ■

With the notation of §4, for each set $\{j_i\}_{i\in J_+}$ of maps $j_i : I_i^1\to \mathrm{Latt}^1_{\mathfrak{o}_{E_i}}(V_i)$ given by Proposition 4.3, we define a map $j_\beta : \prod_{i\in J_o\cup J_+} I_i^1\to I$ by

$$j_\beta = j_h^{-1}\circ\tilde{j}_\beta\circ\iota_h\circ\left(\prod_{i\in J_o} j_{h_i}\times\prod_{i\in J_+} j_i\right) .$$

These maps depend *a priori* on the forms h, and h_i, $i\in J_o$.

Theorem 6.3. *Each map j_β is injective and H-equivariant. The set of such maps (as $\{j_i\}_{i\in J_+}$ varies) depends only on the involution σ.*

In particular, if $J=\emptyset$ then there is a unique map j_β, depending only on the involution σ.

Proof. The first two properties are straightforward. Assume that $h'=uh$, $u\in F^\times$, is another ε-hermitian form on V, with respect to σ_F, defining the same involution σ on $\tilde{\mathfrak{g}}$. Then $u\in F_o$. For $i\in J_o$, let h_i' be an ε-hermitian form on V_i satisfying

$$uh(v,w)=\lambda_i'(h_i'(v,w))\ \ v,w\in V_i\ ,$$

where the $\lambda_i' : E_i\to F$ are linear forms as above. Then by Lemma 5.5, for all $i\in J_o$, there exists $u_i'\in\mathfrak{o}_{E_i^o}^\times$ such that $u^{-1}h_i'=u_i'h_i$, that is $h_i'=uu_i'h_i$.

Let $\{j_i\}_{i\in J_+}$ be as above; we will show that, for a suitable choice of $\{j_i'\}_{i\in J_+}$, we have

$$j_h^{-1}\circ\tilde{j}_\beta\circ\iota_h\circ j = j_{h'}^{-1}\circ\tilde{j}_\beta\circ\iota_{h'}\circ j',$$

and the result follows.

By Lemma 3.6, for $i\in J_+$, for all $x_i\in I_i^1$, we have

$$j_{h_i'}(x_i)=j_{h_i}(x_i)-v(uu_i')/2=j_{h_i}(x_i)-v(u)/2.$$

For $i\in J_+$, we choose j_i' such that $j_i'(x)=j_i(x)-v(u)/2$ for $x\in I_i^1$, that is $j_i'\circ j_i^{-1}(\Lambda_i)=\Lambda_i-v(u)/2$ for $\Lambda_i\in\mathrm{Latt}^1_{\mathfrak{o}_{E_i}}(V_i)$. We abbreviate

$$j=\prod_{i\in J_o} j_{h_i}\times\prod_{i\in J_+} j_i,\qquad j'=\prod_{i\in J_o} j_{h_i'}\times\prod_{i\in J_+} j_i';$$

then, for $(\Lambda_i)_{i\in J_o\cup J_+}\in\prod_{i\in J_o}\mathrm{Latt}^1_{\mathfrak{o}_{E_i},h_i}(V_i)\times\prod_{i\in J_+}\mathrm{Latt}^1_{\mathfrak{o}_{E_i}}(V_i)$, we have

$$j'\circ j^{-1}\left((\Lambda_i)_{i\in J_o\cup J_+}\right)=(\Lambda_i-v(u)/2)_{i\in J_o\cup J_+} .$$

It is also straightforward to check that

$$\iota_{h'}\left((\Lambda_i - v(u)/2)_{i\in J_o\cup J_+}\right) = \iota_h\left((\Lambda_i)_{i\in J_o\cup J_+}\right) - v(u)/2,$$

for $(\Lambda_i)_{i\in J_o\cup J_+} \in \prod_{i\in J_o} \mathrm{Latt}^1_{\mathfrak{o}_{E_i},h_i}(V_i) \times \prod_{i\in J_+} \mathrm{Latt}^1_{\mathfrak{o}_{E_i}}(V_i)$. Then we have

$$\begin{aligned}
\tilde{j}_\beta\circ\iota_{h'}\circ j'\circ j^{-1}\left((\Lambda_i)_{i\in J_o\cup J_+}\right) &= \tilde{j}_\beta\circ\iota_{h'}\left((\Lambda_i - v(u)/2)_{i\in J_o\cup J_+}\right)\\
&= \tilde{j}_\beta\left(\iota_h\left((\Lambda_i)_{i\in J_o\cup J_+}\right) - v(u)/2\right)\\
&= \tilde{j}_\beta\circ\iota_h\left((\Lambda_i)_{i\in J_o\cup J_+}\right) - v(u)/2.
\end{aligned}$$

By Lemma 3.6 again, we have $j_{h'}(x) = j_h(x) - v(u)/2$, $x\in I$, that is $\Lambda - v(u)/2 = j_{h'}\circ j_h^{-1}(\Lambda)$, $\Lambda\in\mathrm{Latt}^1_h(V)$. So

$$j_{h'}\circ j_h^{-1}\circ\tilde{j}_\beta\circ\iota_h = \tilde{j}_\beta\circ\iota_{h'}\circ j'\circ j^{-1}\ ,$$

as required. ∎

7. Affine structures

We keep the notation as in the previous sections. For $x = (x_i)_{i\in J_o\cup J_+}$, $y = (y_i)_{i\in J_o\cup J_+}$ in $I^1_\beta = \prod_{i\in J_o\cup J_+} I^1_i$ and $t\in[0,1]$, we define the barycenter $tx + (1-t)y$ to be

$$(tx_i + (1-t)y_i)_{i\in J_o\cup J_+}\ .$$

We define the barycenter of two points in $\prod_{i\in J_o\cup J_+}\mathrm{Latt}^1_{\mathfrak{o}_{E_i}}(V_i)$ in a similar way. Since, for $i\in J_o$, $\mathrm{Latt}^1_{\mathfrak{o}_{E_i},h_i}(V_i)$ is convex in $\mathrm{Latt}^1_{\mathfrak{o}_{E_i}}(V_i)$, the subset

$$\prod_{i\in J_o}\mathrm{Latt}^1_{\mathfrak{o}_{E_i},h_i}(V_i)\times\prod_{i\in J_+}\mathrm{Latt}^1_{\mathfrak{o}_{E_i}}(V_i)$$

of $\prod_{i\in J_o\cup J_+}\mathrm{Latt}^1_{\mathfrak{o}_{E_i}}(V_i)$ is convex also.

Proposition 7.1. *Let β be as in §5. Then each map j_β is affine: for all x, $y\in I^1_\beta$, $t\in[0,1]$, we have*

$$j_\beta(tx + (1-t)y) = tj_\beta(x) + (1-t)j_\beta(y)\ .$$

Proof. By construction it suffices to prove that the maps $\tilde{j}_\beta$ and ι_h are affine. We begin with $\tilde{j}_\beta$. Let $(\Lambda_i)_{i\in J_o\cup J}$, $(M_i)_{i\in J_o\cup J}$ be elements of $\prod_{i\in J_o\cup J}\mathrm{Latt}^1_{\mathfrak{o}_{E_i}}(V_i)$. We must prove that

$$\bigoplus_{i\in J_o\cup J}(t\Lambda_i + (1-t)M_i) = t\left(\bigoplus_{i\in J_o\cup J}\Lambda_i\right) + (1-t)\left(\bigoplus_{i\in J_o\cup J}M_i\right).$$

Let us recall the construction of the barycenter of two lattice functions (we do it for $\mathrm{Latt}^1(V)$). Let $\Lambda,\ M \in \mathrm{Latt}^1(V)$. There exists an F-basis $(e_1, \ldots, e_n)$ of V which splits both Λ and M : there exist constants $\lambda_1, \ldots, \lambda_n,\ \mu_1, \ldots, \mu_n$ in $\mathbb{R}$ such that

$$\Lambda(r) = \bigoplus_{k=1,\ldots,n} \mathfrak{p}_F^{\lceil r+\lambda_k \rceil} e_k\ ,\ M(r) = \bigoplus_{k=1,\ldots,n} \mathfrak{p}_F^{\lceil r+\mu_k \rceil} e_k\ ,\ r \in \mathbb{R}\ .$$

Then for $t \in [0,1]$, $t\Lambda + (1-t)M$ is given by

$$(t\Lambda + (1-t)M)(r) = \bigoplus_{k=1,\ldots,n} \mathfrak{p}_F^{\lceil r+t\lambda_k+(1-t)\mu_k \rceil} e_k\ ,\ r \in \mathbb{R}\ .$$

The proof that $\tilde{j}_\beta$ is affine is then to construct a common splitting basis for $\bigoplus_{i\in J_o\cup J} \Lambda_i$ and $\bigoplus_{i\in J_o\cup J} M_i$ from bases $\mathcal{B}_i$ of V_i, $i \in J_o \cup J$, where $\mathcal{B}_i$ splits Λ_i and M_i. We leave this easy exercise to the reader.

Now we turn to ι_h. Suppose $i \in J_+$ and $\Lambda_i \in \mathrm{Latt}^1_{\mathfrak{o}_{E_i}}(V_i)$, and let $(e_1, \ldots, e_n)$ be an E_i-basis of V_i which splits Λ_i. Let $(e_{-1}, \ldots, e_{-n})$ be the dual E_{-i}-basis of V_{-i}, such that $h(e_{-k}, e_l) = \delta_{kl}$, for $1 \le k, l \le n$. It is straightforward to check that this basis splits $\Lambda_i^{\sharp_i}$ and that,

$$\text{if}\quad \Lambda_i(r) = \bigoplus_{k=1,\ldots,n} \mathfrak{p}_{E_i}^{\lceil r+\lambda_k \rceil} e_k \quad\text{then}\quad \Lambda_i^{\sharp_i}(r) = \bigoplus_{k=1,\ldots,n} \mathfrak{p}_{E_{-i}}^{\lceil r-\lambda_k \rceil} e_{-k}. \tag{7.2}$$

To show that ι_h is affine, we just need to check that, for $i \in J_+$, $\Lambda_i, M_i \in \mathrm{Latt}^1_{\mathfrak{o}_{E_i}}(V_i)$ and $t \in [0,1]$, we have

$$(t\Lambda_i + (1-t)M_i)^{\sharp_i} = t\Lambda_i^{\sharp_i} + (1-t)M_i^{\sharp_i}.$$

The details of the proof – which is to choose an E_i-basis of V_i which splits both Λ_i and M_i, take its dual basis and then use (7.2) – are again left to the reader. ■

8. The image of an apartment

We keep the notation of the previous sections. We will show that the image of an apartment of I^1_β under each map j_β is contained in an apartment of I.

Given a Witt decomposition $V = V_+ \oplus V_o \oplus V_-$, with basis $(e_l)_{l=1,\ldots,r}$ of V_+ and the dual basis $(e_{-l})_{l=1,\ldots,r}$ of V_- (as in §2), we get a (self-dual) decomposition

$$V = \bigoplus_{l=1}^{r} V^l \oplus V_o \oplus \bigoplus_{l=1}^{r} V^{-l},$$

where $V^l = Fe_l = \left(\bigoplus_{k\neq -l} V^l \oplus V_o\right)^\perp$. Such a decomposition (which we will also call a Witt decomposition) corresponds to the choice of an apartment $\mathcal{A}$ in I: in terms of lattice functions, $j_h(\mathcal{A})$ is the set of self-dual lattice functions Λ such that

$$\Lambda(s) = \bigoplus_{l=1}^{r} (V^l \cap \Lambda(s)) \oplus (V_o \cap \Lambda(s)) \oplus \bigoplus_{l=1}^{r} (V^{-l} \cap \Lambda(s)), \qquad \text{for all } s \in \mathbb{R},$$

that is, Λ is *split* by the decomposition (cf. Proposition 4.1).

Similarly, the choice of an (enlarged) apartment $\mathcal{A}^1$ in $I^1_\beta = \prod_{i\in J_o\cup J_+} I^1_i$ is given by similar E_i-decompositions of V_i for $i\in J_o$ and (without the self-duality restriction) $i\in J_+$.

Proposition 8.1. *Let $\mathcal{A}^1$ be an (enlarged) apartment of I^1_β. Then there is an apartment $\mathcal{A}$ of I such that $j_\beta(\mathcal{A}^1)\subset\mathcal{A}$.*

Proof. We write $\mathcal{A}^1=\prod_{i\in J_o\cup J_+}\mathcal{A}^1_i$, with $\mathcal{A}^1_i$ an (enlarged) apartment in I^1_i.

As above, for each $i\in J_o$, the apartment $\mathcal{A}^1_i$ corresponds to a Witt E_i-decomposition of V^i

$$V_i=\bigoplus_{l=1}^{r_i} V_i^l\oplus V_{i,o}\oplus\bigoplus_{l=1}^{r_i} V_i^{-l},$$

with $V_i^l=\left(\bigoplus_{k\neq -l} V_i^l\oplus V_{i,o}\right)^\perp$, $\dim_{E_i} V_i^l=1$ and r_i the (E_i-)Witt index of V_i. We write $\mathrm{Latt}^{\mathcal{A}^1}_{\mathfrak{o}_{E_i}}(V_i)$ for the set of lattice functions split by this decomposition, and $\mathrm{Latt}^{\mathcal{A}^1}_{\mathfrak{o}_{E_i},h_i}(V_i)$ for the subset of self-dual lattice functions, so that $j_{h_i}(\mathcal{A}^1_i)=\mathrm{Latt}^{\mathcal{A}^1}_{\mathfrak{o}_{E_i},h_i}(V_i)$.

Also, for each $i\in J_+$, the apartment $\mathcal{A}^1_i$ corresponds to a decomposition of V_i as a sum of 1-dimensional E_i-subspaces,

$$V_i=\bigoplus_{l=1}^{r_i} V_i^l,$$

with $r_i=\dim_{E_i} V_i$. As above, $j_i(\mathcal{A}^1_i)=\mathrm{Latt}^{\mathcal{A}^1}_{\mathfrak{o}_{E_i}}(V_i)$, the set of lattice functions split by this decomposition.

We also take the dual splitting of V_{-i} as a sum of 1-dimensional E_{-i}-subspaces,

$$V_{-i}^l=\left(\bigoplus_{k\neq l} V_i^k\right)^\perp.$$

We remark that, if $\Lambda\in\mathrm{Latt}^{\mathcal{A}^1}_{\mathfrak{o}_{E_i}}(V_i)$ then $\Lambda_i^{\#_i}$ is split by this decomposition.

Now, for $i\in J_o\cup J_+$ and $1\le l\le r_i$, we decompose V_i^l as a sum of 1-dimensional F-subspaces as follows: fix $v\in V_i^l$, $v\neq 0$, and let $\mathcal{B}_i$ be an F-basis for E_i which splits the $\mathfrak{o}_F$-lattice sequence $s\mapsto\mathfrak{p}_{E_i}^{\lceil s/e(E_i/F)\rceil}$; then we take the decomposition

$$V_i^l=\bigoplus_{b\in\mathcal{B}_i} Fbv.$$

Note that any $\mathfrak{o}_{E_i}$-lattice sequence in V_i^l is split by this decomposition. For $i\in J_o$, we also take the dual decomposition of V_i^{-l} and, for $i\in J_+$, the dual decomposition of V_{-i}^l.

Now we need to decompose the anisotropic parts $W:=\oplus_{i\in J_o}V_{i,o}$ suitably. Let $\boldsymbol{G}_o$ denote the classical group associated to the restriction of the form h

to W and, for $i \in J_o$, let $\boldsymbol{H}_{i,o}$ denote the group associated to the restriction of the form h_i to $V_{i,o}$. Note that the groups $H_{i,o}$ are compact so the building $I^1_{\beta,o} := I^1(\boldsymbol{H}_{i,o}, E_i)$ is reduced to a point.

Now, our constructions in §6 give an embedding of $I^1_{\beta,o}$ in the building $I^1_o := I^1(\boldsymbol{G}_o, F)$ and the image is certainly contained in some apartment. Hence there is a Witt F-decomposition of W which splits the (unique) self-dual lattice sequence in W corresponding to $I^1_{\beta,o}$, and this is the decomposition we take.

Altogether, we have described a Witt F-decomposition of V, which corresponds to an apartment $\mathcal{A}$ of I. We denote by $\mathrm{Latt}^{\mathcal{A}}_{\mathfrak{o}_F,h}(V)$ the set of self-dual lattice functions in V which are split by this splitting, so that $j_h(\mathcal{A}) = \mathrm{Latt}^{\mathcal{A}}_{\mathfrak{o}_F,h}(V)$.

Finally, by construction it is clear that $\tilde{j}_\beta \circ \iota_h$ maps $\prod_{i\in J_o} \mathrm{Latt}^{\mathcal{A}^1}_{\mathfrak{o}_{E_i},h_i}(V_i) \times \prod_{i\in J_+} \mathrm{Latt}^{\mathcal{A}^1}_{\mathfrak{o}_{E_i}}(V_i)$ into $\mathrm{Latt}^{\mathcal{A}}_{\mathfrak{o}_F,h}(V)$ so $j_\beta(\mathcal{A}^1) \subset \mathcal{A}$, as required. ∎

9. Compatibility with Lie algebra filtrations

In this section, we fix H_k-equivariant identifications $j_k : I^1(H_k, E_k) \to \mathrm{Latt}^1_{\mathfrak{o}_{E_k}}(V_k)$, $k \in J^+$. They give rise to the map $j_\beta : I^1_\beta \to I(G, H)$ defined in §6.

Let $x \in I(G, F) = I^1(G, F)$, that we see as a self-dual lattice function Λ in $\mathrm{Latt}^1_h(V)$. To x we can associate a filtration $(\mathfrak{g}_{x,r})_{r\in\mathbb{R}}$ of the Lie algebra $\mathfrak{g}$ as follows. First x defines a filtration $(\tilde{\mathfrak{g}}_{x,r})_{r\in\mathbb{R}}$ of $\tilde{\mathfrak{g}}$ by

$$\tilde{\mathfrak{g}}_{x,r} = \{a \in \tilde{\mathfrak{g}} \ ; \ a\Lambda(s) \subset \Lambda(s + r), \ s \in \mathbb{R}\}, \ r \in \mathbb{R} \ .$$

We then define

$$\mathfrak{g}_{x,r} := \tilde{\mathfrak{g}}_{x,r} \cap \mathfrak{g} = \{a \in \mathfrak{g} \ ; \ a\Lambda(s) \subset \Lambda(s + r), \ s \in \mathbb{R}\}, \ r \in \mathbb{R} \ .$$

Similarly a point x of I^1_β defines a filtration $(\mathfrak{h}_{x,r})_{r\in\mathbb{R}}$ of $\mathfrak{h}$. Write $x = (x_k)_{k\in J\cup J_o}$, $x_k \in I^1(H_k, E_k)$; each x_k corresponding to a lattice function Λ_k of $\mathrm{Latt}_{\mathfrak{o}_{E_k}}(V_k)$ (with $\Lambda_k^{\sharp_k} = \Lambda_{-k}$, $k \in J \cup J_o$). We then define

$$\mathfrak{h}_{x,r} := \bigoplus_{k\in J^+\cup J_o} \mathfrak{h}^k_{x_k,r}, \ r \in \mathbb{R},$$

where

$$\mathfrak{h}^k_{x_k,r} = \{a \in \mathrm{Lie}(H_k) \ ; \ a\Lambda_k(s) \subset \Lambda_k(s + r), \ s \in \mathbb{R}\}, \ r \in \mathbb{R}, \ k \in J^+ \cup J_o \ .$$

The filtration $(\mathfrak{h}_{x,r})_{r\in\mathbb{R}}$ only depends on the image $\bar{x}$ of x in the non-enlarged building I_β. For $x \in I(G, F)$, $(\mathfrak{g}_{x,r})_{r\in\mathbb{R}}$ is in fact the filtration of $\mathfrak{g}$ attached to x defined by Moy and Prasad [MP]. Similarly, when β is semisimple and $x \in I^1(H, F)$, $(\mathfrak{h}_{x,r})_{r\in\mathbb{R}}$ is the filtration of $\mathfrak{h}$ attached to $\bar{x}$ defined in loc. cit. This is proved by B. Lemaire in [Le].

Lemma 9.1. *Let us see* $\mathfrak{h}$ *as being canonically embedded in* $\tilde{\mathfrak{h}} = \mathrm{End}_E V = \bigoplus_{k\in J\cup J_o} \mathrm{End}_{E_k} V_k$ *via*

$$(a_k)_{k\in J^+\cup J_o} \mapsto (b_k)_{k\in J\cap J_o}\ ,$$

where $b_k = a_k,\ k\in J_o$, *and* $b_{-k} = -a_k^\sigma,\ k\in J^+$. *Fix* $x\in I^1_\beta$ *as before and consider the* $\mathfrak{o}_F$*-lattice function in* V *given by*

$$\Lambda = \bigoplus_{k\in J\cup J_o} \Lambda_k \ \text{(notation of §6)}.$$

For $r\in\mathbb{R}$, *let*

$$\tilde{\mathfrak{h}}_{x,r} = \{a\in\tilde{\mathfrak{h}}\ ;\ a\Lambda(s)\subset\Lambda(s+r),\ s\in\mathbb{R}\},\ r\in\mathbb{R}\ .$$

Then we have $\mathfrak{h}_{x,r} = \tilde{\mathfrak{h}}_{x,r}\cap\mathfrak{h}$, $r\in\mathbb{R}$.

Proof. Indeed, for all $a=(a_k)_{k\in J\cup J_o}\in\mathrm{End}_E V$, we have $a\in\tilde{\mathfrak{h}}_{x,r}\cap\mathfrak{h}$ if and only if $a+a^\sigma=0$ and $a\Lambda(s)\subset\Lambda(s+r)$, $s\in\mathbb{R}$, i.e.

$$a_k\Lambda_k(s)\subset\Lambda_k(s+r),\ s\in\mathbb{R}, k\in J\cup J_o\ .$$

For $k\in J_o$, these conditions can be rewritten $a_k\in\mathrm{Lie}(H_k)$ and $a_k\Lambda_k(s)\subset\Lambda_k(s+r)$, $s\in\mathbb{R}$, that is $a_k\in\mathfrak{h}^k_{x,r}$, as required. For $k\in J$, these conditions can be rewritten $a_{-k}=-a_k^\sigma$ and

$$a_k\Lambda_k(s)\subset\Lambda_k(s+r),\ s\in\mathbb{R} \tag{a}$$

$$-a_k^\sigma\Lambda_k^{\sharp_k}(s)\subset\Lambda_k^{\sharp_k}(s+r),\ s\in\mathbb{R}\ . \tag{b}$$

So we must prove that conditions (a) and (b) are equivalent. By symmetry we only prove one implication. Applying the duality $\sharp_k$ on lattices of V_k to inclusion (b), we obtain

$$\Lambda_k((-s-r)+)\subset[a_k^\sigma\Lambda_k^{\sharp_k}(s)]^{\sharp_k},\ s\in\mathbb{R},$$

with

$$[a_k^\sigma\Lambda_k^{\sharp_k}(s)]^{\sharp_k} = \{v\in V_k\ ;\ a_k v\in\Lambda_k((-s)+)\},\ s\in\mathbb{R}\ .$$

So we have

$$a_k\Lambda_k((-s-r)+)\subset\Lambda_k((-s)+)\subset\Lambda_k(-s),\ s\in\mathbb{R}\ ,$$

that is

$$a_k\Lambda(s+)\subset\Lambda_k(s+r),\ s\in\mathbb{R}\ .$$

On each open interval (u,v) where Λ_k is constant, we have

$$a_k\Lambda_k(s+) = a_k\Lambda_k(s)\subset\Lambda_k(s+r)\ ,$$

and (a) is true for $s\in(u,v)$. Finally if s_o is a jump of Λ_k with Λ_k constant on $(t,s_o]$, we have

$$a_k\Lambda_k(s_o) = a_k\Lambda_k(s+)\subset\Lambda_k(s+r),\ s\in(t,s_o)\ .$$

So

$$a_k\Lambda_k(s_o)\subset\bigcap_{s\in(t,s_o)}\Lambda_k(s+r) = \Lambda_k(s_o+r)\ ,$$

Λ_k being left continuous, and (a) is then true for all $s\in\mathbb{R}$. ∎

Proposition 9.2. *Let $x \in I^1_\beta$. Then we have*

$$\mathfrak{g}_{j_\beta(x),r} \cap \mathfrak{h} = \mathfrak{h}_{x,r},\ \ r \in \mathbb{R}\ .$$

Proof. Indeed, with the notation of Lemma 9.1 and by definition of j_β, we easily see that

$$\tilde{\mathfrak{g}}_{j_\beta(x),r} \cap \tilde{\mathfrak{h}} = \tilde{\mathfrak{h}}_{x,r}\ .$$

So our result is now a corollary of Lemma 9.1 since $\mathfrak{h} = \mathfrak{g} \cap \tilde{\mathfrak{h}}$. ■

10. A unicity result for the general linear group

As in [BL, §I.2], we define an equivalence relation $\sim$ on $\mathrm{Latt}^1(V)$ by $\Lambda_1 \sim \Lambda_2$ if there exists $s \in \mathbb{R}$ such that $\Lambda_1(s) = \Lambda_2(r+s)$, $s \in \mathbb{R}$. Then $\sim$ is compatible with the $\tilde{G}$-action and the quotient $\mathrm{Latt}_{\mathfrak{o}_F}(V) := \mathrm{Latt}^1(V)/\sim$ is naturally a $\tilde{G}$-set. We shall denote by $\bar{\Lambda}$ an element of $\mathrm{Latt}_{\mathfrak{o}_F}(V)$, where Λ is a representative in $\mathrm{Latt}^1(V)$. As a consequence of [BL, §I.2] and [BT1, ??], there is a unique affine and $\tilde{G}$-equivariant map $j\ :\ \tilde{I} \to \mathrm{Latt}_{\mathfrak{o}_F}(V)$, where $\tilde{I}$ denotes the non-enlarged building of $\tilde{G}$.

We fix an element β of $\tilde{\mathfrak{g}}$ satisfying

$$E := F[\beta] \text{ is a field.}$$

As in §5 we denote by $\tilde{\mathfrak{h}} = \mathrm{End_E}V$ the centralizer of β in $\tilde{\mathfrak{g}}$ and by $\tilde{H} = \mathrm{Aut}_E V$ its centralizer in $\tilde{G}$. There is a canonical identification of the non-enlarged affine building $\tilde{I}_\beta$ of $\tilde{H}$ with the $\tilde{H}$-set $\mathrm{Latt}_{\mathfrak{o}_E}(V)$. Here we normalize the lattice functions of $\mathrm{Latt}^1_{\mathfrak{o}_E}(V)$ by the condition $\Lambda(s+v(\pi_E)) = \pi_E\Lambda(s)$, $s \in \mathbb{R}$, where π_E is a uniformizer of E.

Any $\bar{\Lambda} \in \mathrm{Latt}_{\mathfrak{o}_F}(V)$ defines a filtration $(\tilde{\mathfrak{g}}_{\bar{\Lambda},r})_{r\in\mathbb{R}}$ by

$$\tilde{\mathfrak{g}}_{\bar{\Lambda},r} = \{a \in \mathrm{End}_F V\ ;\ a\Lambda(s) \subset \Lambda(r+s),\ s \in \mathbb{R}\}\ .$$

Then the map $\mathrm{End}(\bar{\Lambda})\ :\ r \mapsto \tilde{\mathfrak{g}}_{\bar{\Lambda},r}$ is an element of $\mathrm{Latt}^1\tilde{\mathfrak{g}}$. The map $\bar{\Lambda} \mapsto \mathrm{End}(\bar{\Lambda})$, $\mathrm{Latt}_{\mathfrak{o}_F}V \to \mathrm{Latt}^1\tilde{\mathfrak{g}}$ is a $\tilde{G}$-equivariant injection (cf. [BL, §4]) for the action of G on $\mathrm{Latt}^1\tilde{\mathfrak{g}}$ by conjugation. Its image is $\mathrm{Latt}^2\tilde{\mathfrak{g}}$. From now on we shall canonically identify $\tilde{I}$ with $\mathrm{Latt}^2\tilde{\mathfrak{g}}$) (resp. $\tilde{I}_\beta$ with $\mathrm{Latt}^2\tilde{\mathfrak{h}}$).

Let us recall the main result of [BL].

Theorem 10.1. *There exists a unique affine and $\tilde{H}$-equivariant map $\tilde{j}_\beta\tilde{I}_\beta \to \tilde{I}$. It is injective, maps any apartment into an apartment and is compatible with the Lie algebra filtrations in the following sense:*

$$\tilde{\mathfrak{g}}_{\tilde{j}_\beta(x),r} \cap \tilde{\mathfrak{h}} = \tilde{\mathfrak{h}}_{x,r},\ \ x \in \tilde{I}_\beta,\ \ r \in \mathbb{R}\ . \tag{10.2}$$

Let us recall how $\tilde{j}_\beta$ is constructed. If $x \in \tilde{I}_\beta$ corresponds to $\mathrm{End}(\bar{\Lambda}) \in \mathrm{Latt}^2\tilde{\mathfrak{h}}$, then $\tilde{j}(x)$ simply corresponds to $\mathrm{End}(\bar{\Lambda})$, where Λ, an $\mathfrak{o}_E$-lattice function in V, is now considered as an $\mathfrak{o}_F$-lattice function.

Theorem 10.3. *Let $x \in \tilde{I}_\beta$ and $y \in \tilde{I}$ satisfying*

$$\tilde{\mathfrak{g}}_{y,r} \cap \tilde{\mathfrak{h}} \supset \tilde{\mathfrak{h}}_{x,r}, \ r \in \mathbb{R} \ .$$

Then $y = \tilde{j}_\beta(x)$. As a consequence the map $\tilde{j}_\beta$ is characterized by property (10.2).

Proof. Assume that x and y correspond to elements $\bar{\Lambda}_x$ and $\bar{\Lambda}_y$ of $\mathrm{Latt}_{\mathfrak{o}_E}(V)$ and $\mathrm{Latt}_{\mathfrak{o}_F}(V)$ respectively. We need the following lemma :

Lemma 10.4. *Under the assumption of* (10.2)*, Λ_y is an $\mathfrak{o}_E$-lattice function.*

Proof. To prove that Λ_y is an $\mathfrak{o}_E$-lattice function we must prove that it is normalized by $E^\times = \langle \pi_E \rangle \mathfrak{o}_E^\times$, or equivalently:

$$x\tilde{\mathfrak{g}}_{y,r}x^{-1} = \tilde{\mathfrak{g}}_{y,r}, \ x \in E^\times, \ r \in \mathbb{R}. \tag{10.5}$$

We first notice than $\mathfrak{o}_E \subset \tilde{\mathfrak{h}}_{x,0} \subset \tilde{\mathfrak{g}}_{y,0}$, so that $\mathfrak{o}_E^\times \subset \tilde{\mathfrak{g}}_{y,0}^\times$ and (10.5) is true for $x \in \mathfrak{o}_E^\times$. We are reduced to proving (10.5) when $x = \pi_E$.

We have $\pi_E \in \tilde{\mathfrak{h}}_{x,1/e} \subset \tilde{\mathfrak{g}}_{y,1/e}$ and $\pi_E^{-1} \subset \tilde{\mathfrak{h}}_{x,-1/e} \subset \tilde{\mathfrak{g}}_{y,-1/e}$, where $e = e(E/F)$. It follows that

$$\pi_E\tilde{\mathfrak{g}}_{y,r}\pi_E^{-1} \subset \tilde{\mathfrak{g}}_{y,1/e}\tilde{\mathfrak{g}}_{y,r}\tilde{\mathfrak{g}}_{y,-1/e} \subset \tilde{\mathfrak{g}}_{y,r}, \ r \in \mathbb{R}. \tag{10.6}$$

Consider the duality "$*$" on subsets of $\tilde{\mathfrak{g}}$ given by

$$S^* = \{a \in \tilde{\mathfrak{g}} \ ; \ \mathrm{Tr}(aS) \subset \mathfrak{p}_F\}, \ S \subset \tilde{\mathfrak{g}},$$

where Tr is the trace map. Recall from [BL, 6.3] that $(\tilde{\mathfrak{g}}_{y,r})^* = \tilde{\mathfrak{g}}_{y,(-r)+}$, for $r \in \mathbb{R}$. Using a well known property of the trace map, we observe that

$$(\pi_E\tilde{\mathfrak{g}}_{y,r}\pi_E^{-1})^* = \pi_E(\tilde{\mathfrak{g}}_{y,r})^*\pi_E^{-1}, \ r \in \mathbb{R}.$$

So applying the duality to (10.6), we obtain

$$\tilde{\mathfrak{g}}_{y,(-r)+} \subset \pi_E\tilde{\mathfrak{g}}_{y,(-r)+}\pi_E^{-1}, \ r \in \mathbb{R}.$$

We have proved that on each open interval (r_1, r_2) where the lattice function $(\tilde{\mathfrak{g}}_{y,r})_{r\in\mathbb{R}}$ is constant, we have both containments

$$\pi_E\tilde{\mathfrak{g}}_{y,r}\pi_E^{-1} \subset \tilde{\mathfrak{g}}_{y,r} \text{ and } \pi_E\tilde{\mathfrak{g}}_{y,r}\pi_E^{-1} \subset \tilde{\mathfrak{g}}_{y,r}, \ r \in \mathbb{R}.$$

So by continuity we have $\pi_E\tilde{\mathfrak{g}}_{y,r}\pi_E^{-1} = \tilde{\mathfrak{g}}_{y,r}$, for all r, as required. ■

We now return to the proof of Theorem 10.3. Since Λ_y is an $\mathfrak{o}_E$-lattice function, we have

$$\tilde{\mathfrak{g}}_{y,r} \cap \tilde{\mathfrak{h}} = \tilde{\mathfrak{h}}_{x',r}, \ r \in \mathbb{R},$$

where $x' \in \tilde{I}_\beta$ is attached to $\bar{\Lambda}_y$, Λ_y being seen as an $\mathfrak{o}_E$-lattice function. So by injectivity of the map $\mathrm{Latt}^1_{\mathfrak{o}_E}(V) \to \mathrm{Latt}^2\tilde{\mathfrak{h}}$, we have $\bar{\Lambda}_x = \bar{\Lambda}_y$ and $y = \tilde{j}_\beta(x)$ by definition. ■

11. A unicity result in the 1-block case and a conjecture

With the notation of §5, we consider an element $\beta \in \mathfrak{g}$ satisfying:

$$E := F[\beta] \subset \tilde{\mathfrak{g}} \text{ is a field and } \beta \neq 0\,. \tag{11.1}$$

We fix an ε-hermitian form h_E on the E-vector space V relative to σ_E and we assume that it satisfies (5.4) as well as the condition $\mathcal{J} = \mathfrak{p}_{E^o}$ of §5. This allows us to identify I^1_β with $\mathrm{Latt}^1_{h_E}(V)$. Identifying I with $\mathrm{Latt}_h(V)$, the map j_β of §6 is simply given by

$$j_\beta(\Lambda) = \Lambda,\ \Lambda \in \mathrm{Latt}^1_{h_E}(V),$$

where on the right hand side Λ is considered as an $\mathfrak{o}_F$-lattice function.

Theorem 11.2. *Under the assumption* (11.1), *let $x \in I^1_\beta$ and $y \in I$ satisfying*

$$\mathfrak{g}_{y,r} \cap \mathfrak{h} = \mathfrak{h}_{x,r},\ r \in \mathbb{R}\,. \tag{11.3}$$

Then $y = j_\beta(x)$. In particular the map j_β is characterized by compatibility with the Lie algebra filtrations.

Proof. The point x (resp. y) corresponds to a self-dual lattice function $\Lambda_x \in \mathrm{Latt}^1_{h_E}(V)$ (resp. $\Lambda_y \in \mathrm{Latt}^1_h(V)$). We may see x and y as points of $\mathrm{Latt}^1_{\mathfrak{o}_E}(V)$ and $\mathrm{Latt}^1_{\mathfrak{o}_F}(V)$ respectively and they give rise to filtrations of $\tilde{\mathfrak{h}}$ and $\tilde{\mathfrak{g}}$ as in §9: $(\tilde{\mathfrak{h}}_{x,r})_{r\in\mathbb{R}}$ and $(\tilde{\mathfrak{g}}_{y,r})_{r\in\mathbb{R}}$. Write

$$\mathfrak{g}^+_{y,r} = \{a \in \tilde{\mathfrak{g}}_{y,r}\ ;\ a = a^\sigma\},\ r \in \mathbb{R}$$

and

$$\mathfrak{h}^+_{x,r} = \{a \in \tilde{\mathfrak{h}}_{x,r}\ ;\ a = a^\sigma\},\ r \in \mathbb{R}$$

Since 2 is invertible in $\mathfrak{o}_F$, we have:

$$\tilde{\mathfrak{g}}_{y,r} = \mathfrak{g}_{y,r} \oplus \mathfrak{g}^+_{y,r} \text{ and } \tilde{\mathfrak{h}}_{y,r} = \mathfrak{h}_{x,r} \oplus \mathfrak{h}^+_{x,r},\ r \in \mathbb{R}\,.$$

Write

$$r_o = v_{\Lambda_x}(\beta) := \mathrm{Sup}\{r \in \mathbb{R}\ ;\ \beta \in \tilde{\mathfrak{h}}_{x,r}\}\,.$$

Since $\beta \in E^\times$, it normalizes Λ_x so that $\beta\tilde{\mathfrak{h}}_{x,r} = \tilde{\mathfrak{h}}_{x,r+r_o}$, $r \in \mathbb{R}$. Moreover since β is central in $\tilde{\mathfrak{h}}$, we easily have that $\mathfrak{h}^+_{x,r} = \beta\mathfrak{h}_{x,r-r_o}$, $r \in \mathbb{R}$. Hence, for $r \in \mathbb{R}$, we have

$$\mathfrak{h}^+_{x,r} = \beta(\mathfrak{g}_{y,r-r_o} \cap \mathfrak{h}) = \beta(\mathfrak{g}_{y,r-r_o} \cap \tilde{\mathfrak{h}}) \subset \mathfrak{g}_{y,r} \cap \tilde{\mathfrak{h}}\,.$$

It follows that, for $x \in \mathbb{R}$, we have:

$$\tilde{\mathfrak{h}}_{x,r} = \mathfrak{h}_{x,r} \oplus \mathfrak{h}^+_{x,r} \subset \mathfrak{g}_{y,r} \cap \tilde{\mathfrak{h}} \oplus \mathfrak{g}^+_{y,r} \cap \tilde{\mathfrak{h}} \subset \tilde{\mathfrak{g}}_{y,r} \cap \tilde{\mathfrak{h}}\,.$$

By applying (10.3), we obtain $\bar{\Lambda}_y = \tilde{j}_\beta(\bar{\Lambda}_x)$, that is $\bar{\Lambda}_y = \bar{\Lambda}_x$. In particular we have $\mathrm{End}(\Lambda_x) = \mathrm{End}(\Lambda_y) \in \mathrm{Latt}^2_\sigma\tilde{\mathfrak{h}}$. But by Lemma 3.5 we have $\Lambda_x = \Lambda_y$, as required. ∎

Let us give an example. Assume that $G = \mathrm{Sp}_2(F) = \mathrm{SL}(2, F)$ (here $F = F_o$) and take $\beta \in \mathfrak{g}$ such that E/F is quadratic and ramified. Then H is the group E^1 of norm 1 elements in E. The building of H is reduced to a point $\{x\}$. The group $E^\times$ fixes a unique chamber C of I and $H \subset E^\times$ fixes C pointwise. There are infinitely many maps $j : I^1_\beta \to I$ which are affine and G-equivariant; indeed $j(x)$ can be any point of C. On the other hand there is a unique map $j : I^1_\beta \to I$ which is compatible with the Lie algebra filtrations: it maps x to the isobarycenter of C.

We conjecture that when $J = \emptyset$ (notation of §5) then the map j_β of §6 is characterized by condition (11.3). We may address the more general (but more informal) question: Given two F-reductive groups $\boldsymbol{H}$ and $\boldsymbol{G}$, as well as a morphism of algebraic groups $\varphi : \boldsymbol{H} \to \boldsymbol{G}$, is there an affine and $\boldsymbol{H}(F)$-equivariant map $I(\boldsymbol{H}, F) \to I(\boldsymbol{G}, F)$ which is compatible with the Lie algebra filtrations defined by Moy and Prasad? When is it characterized by this last property?

References

[BK1] Bushnell, C. J., and P. C. Kutzko, "The admissible dual of GL(N) via compact open subgroups," Ann. of Math. Studies **129**, Princeton Univ. Press, Princeton, NJ, 1993.

[BK2] —. *Semisimple types in* GL_n, Comp. Math. **119** (1999), 53–97.

[BK3] —, *Smooth representations of reductive p-adic groups: structure theory via types*, Proc. London Math. Soc. (3) **77** (1998), 582–634.

[BL] Broussous, P., and B. Lemaire, *Building of* $\mathrm{GL}(m, D)$ *and centralizers*, Transform. Groups **7** (2002), 15–50.

[BT] Bruhat, F., and J. Tits, *Groupes réductifs sur un corps local: II. Schémas en groupes. Existence d'une donnée radicielle valuée*, Publications Mathématiques de l'IHES **60** (1984), p. 5-184

[BT1] —, *Schémas en groupes et immeubles des groupes classiques sur un corps local, 1ère partie: le groupe linéaire général*, Bull. Soc. Math. Fr. **112** (1984), 259–301.

[BT2] —, *Schémas en groupes et immeubles des groupes classiques sur un corps local, 2ème partie: groupes unitaires*, Bull. Soc. Math. Fr. **115** (1987), 141–195.

[Inv] Knus, M.-A., A. Merkurjev, M. Rost, and J.-P. Tignol, "The book of involutions", Amer. Math. Soc. Colloquium Publications **44**, 1998.

[La] Landvogt, E., *Some functorial properties of the Bruhat–Tits building*, J. Reine Angew. Math. **518** (2000), 213–241

[Le] Lemaire, B., *Comparison of lattice filtrations and Moy-Prasad filtrations*, J. Lie Theory **19** (2009), 29–54.

[Mor] Morris, L., *Some tamely ramified supercuspidal representations of symplectic groups*, Proc. London Math. Soc. (3) **63** (1991), 519–551.

[MP] Moy, A., and G. Prasad, *Unrefined minimal K-types for p-adic groups*, Invent. Math. **116** (1994), 393–408.

[Sch] Scharlau, W., "Quadratic and Hermitian forms," Grundlehren der Mathematischen Wissenschaften **270**, Springer-Verlag, Berlin, 1985.

[S1] Stevens, S., *Semisimple characters for p-adic classical groups*, Duke Math. J. **127** (2005), 123–173.

[S2] —, *The supercuspidal representations of p-adic classical groups*, Invent. Math., **172** (2008), 289–352.

[Ti] Tits, J., *Reductive groups over local fields*, Proceedings of Amer. Math. Soc. Symposia in Pure Math. **33** (Part 1) (1979), 29–69.

Paul Broussous
Université de Poitiers
Département de Mathématiques
UMR 6086 du CNRS
Bd Marie et Pierre Curie
Téléport 2 – BP 30179
86962 Futuroscope Chasseneuil Cedex
France
broussou@math.univ-poitiers.fr

Shaun Stevens
School of Mathematics
University of East Anglia
Norwich NR4 7TJ
United Kingdom
Shaun.Stevens@uea.ac.uk

Received May 21, 2007
and in final form March 4, 2009