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Contraction of Discrete Series via Berezin Quantization

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Abstract. We establish and study a contraction of the holomorphic discrete series representations of a non-compact semi-simple Lie group to the unitary irreducible representations of a Heisenberg group by means of Berezin quantization.
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1. Introduction

Contractions of representations of Lie algebras have been intensively studied since the pioneering work of Inönü and Wigner [26], see for instance [14] and its references. Contractions of unitary representations of Lie groups have been not investigated as methodically. Some notable exceptions are the works of Mickelsson and Niederle [30] and of Dooley and Rice [17], [20], [21].

In the paper [30], a proper definition of the contraction of unitary representations of Lie groups was given for the first time. The non-zero mass representations of the Euclidean group $\mathbb{R}^{n+1} \rtimes SO(n+1)$ and the positive mass-squared representations of the Poincaré group $\mathbb{R}^{n+1} \rtimes SO_0(n,1)$ were obtained by contraction (i.e. as limits in the sense defined in [30]) of the principal series representations of $SO_0(n+1,1)$. More generally, in [21], Dooley and Rice established a contraction of the principal series representations of a semi-simple Lie group to some unitary irreducible representations of its Cartan motion group.

A contraction of Lie group representations provides a link between the Harmonic Analysis on two different Lie groups. In particular, contractions allow to recover some classical formulas of the theory of special functions [20], [33]. Contractions also permit to transfer results on L^p-multipliers from unitary groups to Heisenberg groups [19], [34].

In [17], Dooley suggested interpreting contractions of representations in the context of the Kirillov-Kostant method of orbits (see also the introduction of [21]) and, in [15], Cotton and Dooley showed how to obtain contraction results

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by using a notion of adapted functional calculus which was introduced in [5] and [6] (see also [10]). The basic idea is then to study the behavior in the contraction process of the symbols of the representation operators which are functions on the coadjoint orbits corresponding to the representations.

The approach of [15] is particularly efficient in the case when the coadjoint orbits have Kählerian structures (see [7], [8] and [9]). In this case, the representation spaces are reproducing kernel Hilbert spaces and the so-called Berezin calculus generally provides an adapted functional calculus on the corresponding coadjoint orbits (see [13]). For example, in [9], we established contractions of the discrete series representations of $SU(n,1)$ and of the unitary irreducible representations of $SU(n+1)$ to the unitary irreducible representations of the $(2n+1)$-dimensional Heisenberg group (see also [31], [33], [18], [7] and [8] for earlier results on contractions of discrete series representations of unitary groups). In order to extend the results of [9] to a more general situation, we studied in [11] a contraction of the unitary irreducible representations of a compact semi-simple Lie group to the unitary irreducible representations of a Heisenberg group. However, the results of [11] were not entirely satisfying: we have just obtained a contraction result for the coefficients of the representations, not a contraction of representations in the stronger sense of [30]. In fact, in the compact case, it is difficult to realize the unitary irreducible representations in compatible ways in spaces of holomorphic functions because these spaces have different finite dimensions. This leads us to consider the case of the holomorphic discrete series representations which is very closed to the compact case.

So the aim of the present paper is to extent the results of [9] concerning the contraction of the holomorphic discrete series of $SU(n,1)$ to a more general situation. Let G be a connected semi-simple non-compact Lie group with finite center and K be a maximal compact subgroup of G. Assume that the center of K has positive dimension. Suppose that π_χ is a discrete series representation of G which is holomorphically induced from a unitary character χ of K. Let G_0 be the Heisenberg group of dimension $\dim G - \dim K + 1$ and ρ be a non-degenerated irreducible unitary representation of G_0. The Hermitian symmetric space of the non-compact type G/K is then diffeomorphic to a bounded symmetric domain $\mathcal{D}$ which can be quantized by the general method of quantization introduced by Berezin [3]. The representation π_χ is usually realized on a Hilbert space of holomorphic functions on $\mathcal{D}$. In [12], we gave an explicit formula for the Berezin symbol of $\pi_\chi(g)$ for $g \in G$. Here, we use this formula in order to establish a contraction of the sequence $(\pi_{\chi^m})_{m\in\mathbb{N}\setminus(0)}$ to ρ.

This paper is organized as follows. In Section 2, we describe the unitary irreducible representations of G_0 and we introduce the Berezin calculus on the associated coadjoint orbits. In Section 3, we recall the results of [12]: we give a realization of π_χ on a reproducing kernel Hilbert space $\mathcal{H}_\chi$ of holomorphic functions on $\mathcal{D}$; we introduce the corresponding Berezin calculus and we give explicit formulas for the coherent states (i.e. for the reproducing kernel of $\mathcal{H}_\chi$) and for the Berezin symbols of $\pi_\chi(g)$. In Section 4, we introduce a contraction (in the generalized sense of [9]) of G to G_0. We relate the parametrizations of the coadjoint orbits associated with π_{χ^m} and ρ in this contraction. In Section 5, we relate the contraction of a sequence of operators acting on the spaces $\mathcal{H}_{\chi^m}$ to the simple convergence of their Berezin symbols. In Section 6, we establish our main results. We show that the coefficients of ρ are limits of the coefficients of the representations π_{χ^m} and that ρ is a contraction of (π_{χ^m}) in the sense of [30]. In particular, by using the fact that the decomposition of the action of K on the space of polynomial functions on $\mathcal{D}$ is well-understood (see [35] for instance), we obtain more simple proofs than in [9]. Finally, in Section 7, we obtain analogous contraction results for the derived representations.

2. Berezin quantization for the Heisenberg group

In this section, we recall some well-known facts on the Bargmann-Fock realization of the unitary irreducible representations of a $(2n+1)$-dimensional Heisenberg group and the corresponding Berezin calculus [1], [9], [23]. The material of this section is essentially taken from [9].

Let G_0 be the Heisenberg group of dimension $2n+1$ and $\mathfrak{g}_0$ its Lie algebra. Let $\{X_1,\cdots,X_n,Y_1,\ldots,Y_n,\tilde{Z}\}$ be a basis of $\mathfrak{g}_0$ in which the only non trivial brackets are $[X_k\,,\,Y_k]=\tilde{Z}$, $\quad k=1,2,\ldots,n$ and let

$$\{X_1^*,\cdots,X_n^*,Y_1^*,\cdots,Y_n^*,\tilde{Z}^*\}$$

be the corresponding dual basis of $\mathfrak{g}_0^*$.

For $a=(a_1,a_2,\ldots,a_n)\in\mathbb{R}^n$, $b=(b_1,b_2,\ldots,b_n)\in\mathbb{R}^n$ and $c\in\mathbb{R}$, we denote by $[a,b,c]$ the element $\exp_H(\sum_{k=1}^n a_kX_k+\sum_{k=1}^n b_kY_k+c\tilde{Z})$ of G_0.

Fix a real number $\gamma>0$ and denote by $\mathcal{O}_\gamma$ the orbit of the element $\xi_\gamma=\gamma\tilde{Z}^*$ of $\mathfrak{g}_0^*$ under the coadjoint action of G_0. By the Stone-von Neumann theorem, there exists a unique (up to unitary equivalence) representation ρ_γ of G_0 whose restriction to the center of G_0 is the character $[0,0,c]\to e^{i\gamma c}$ [23]. The representation ρ_γ is associated to the coadjoint orbit $\mathcal{O}_\gamma$ by the Kirillov-Kostant method of orbits [27]. Here we introduce the Bargmann-Fock realization of ρ_γ as follows.

Let $\mathcal{H}_\gamma$ be the Hilbert space of holomorphic functions on $\mathbb{C}^n$ such that

$$\|f\|_\gamma^2:=\int_{\mathbb{C}^n}|f(z)|^2\,d\mu_\gamma(z)<+\infty$$

where $d\mu_\gamma(z)=(2\pi\gamma)^{-n}\exp(-|z|^2/2\gamma)\,dx_1\,dy_1\ldots dx_n\,dy_n$. Here we use the notation $z=(z_1,z_2,\ldots,z_n)$ and $z_k=x_k+iy_k$, $\quad x_k,\,y_k\in\mathbb{R}$ for $k=1,2,\ldots n$. Then ρ_γ is the representation of G_0 on $\mathcal{H}_\gamma$ defined by

$$\rho_\gamma([a,b,c])\,f(z)=\exp(ic\gamma+\frac{1}{4}(b+ai)(2z+\gamma(-b+ai))^t)\,f(z+\gamma(-b+ai))\,.$$

The derived representation $d\rho_\gamma$ is given by

$$d\rho_\gamma(X)f(z)=\frac{1}{2}\sum_{k=1}^n(ia_k+b_k)z_k\,f(z)+\gamma\sum_{k=1}^n(ia_k-b_k)\frac{\partial f}{\partial z_k}+i\gamma cf(z) \qquad (2.1)$$

for $X = \sum_{k=1}^n a_k X_k + \sum_{k=1}^n b_k Y_k + c\tilde{Z} \in \mathfrak{g}_0$.

For $z \in \mathbb{C}^n$, introduce the coherent states $e_z^\gamma(w) = \exp(wz^*/2\gamma)$. We have the reproducing property

$$f(z) = \langle f, e_z^\gamma \rangle_\gamma \qquad (f \in \mathcal{H}_\gamma)$$

where $\langle \cdot, \cdot \rangle_\gamma$ denotes the inner product on $\mathcal{H}_\gamma$.

Consider now a bounded operator A on $\mathcal{H}_\gamma$. The Berezin (covariant) symbol of A is the function defined on $\mathbb{C}^n$ by

$$s_\gamma(A)(z) = \frac{\langle A\, e_z^\gamma\, ,\, e_z^\gamma \rangle_\gamma}{\langle e_z^\gamma\, ,\, e_z^\gamma \rangle_\gamma} \tag{2.2}$$

and the double Berezin symbol of A is the function defined by

$$S_\gamma(A)(z,w) = \frac{\langle A\, e_w^\gamma\, ,\, e_z^\gamma \rangle_\gamma}{\langle e_w^\gamma\, ,\, e_z^\gamma \rangle_\gamma} \tag{2.3}$$

for z, $w \in \mathbb{C}^n$ such that $\langle e_z^\gamma, e_w^\gamma \rangle_\gamma \neq 0$ (see [2], [3] and [4]). The function $S_\gamma(A)$ is holomorphic in the variable z and anti-holomorphic in the variable w. Moreover, by using the reproducing property, we see that we can reconstruct the operator A from its double symbol $S_\gamma(A)$ (see [13]):

$$A\, f\, (z) = \int_{\mathbb{C}^n} f(w)\, S_\gamma(A)(z,w) \langle e_w^\gamma\, ,\, e_z^\gamma \rangle_\gamma\, d\mu_\gamma\, (w). \tag{2.4}$$

From this, we deduce immediately an integral formula for $\langle Af, g \rangle_\gamma$ which we require later

$$\langle A\, f, g \rangle_\gamma = \int_{\mathbb{C}^n \times \mathbb{C}^n} f(w)\, \overline{g(z)}\, S_\gamma(A)(z,w) \langle e_w^\gamma\, ,\, e_z^\gamma \rangle_\gamma\, d\mu_\gamma(z)\, d\mu_\gamma(w)\,. \tag{2.5}$$

For $g \in G_0$, the Berezin symbol of the operator $\rho_\gamma(g)$ is easily obtained from the reproducing property:

$$S_\gamma(\rho_\gamma(g))(z,w) = \exp\left(i\gamma c - \frac{1}{4}\gamma(|a|^2 + |b|^2) + \frac{1}{2}(b+ia)z^t - \frac{1}{2}\overline{w}(b-ia)^t \right). \tag{2.6}$$

Moreover, for $X = \sum_{k=1}^n a_k X_k + \sum_{k=1}^n b_k Y_k + c\tilde{Z} \in \mathfrak{g}_0$, we have

$$S_\gamma(d\rho_\gamma(X))(z,w) = \frac{1}{2}(b+ia)z^t + \frac{1}{2}(ia-b)\overline{w}^t + i\gamma c. \tag{2.7}$$

Let us introduce the parametrization ψ_γ of the orbit $\mathcal{O}_\gamma$ defined by

$$\psi_\gamma(z) = (\operatorname{Re} z)\, X^* + (\operatorname{Im} z)\, Y^* + \gamma\, \tilde{Z}^* \tag{2.8}$$

with obvious notation. Then, for all $z \in \mathbb{C}^n$, $X \in \mathfrak{g}_0$, we have

$$S_\gamma\, (d\rho_\gamma\, (X))(z,z) = i\langle \psi_\gamma\, (z), X \rangle. \tag{2.9}$$

Hence the Berezin calculus provides an adapted functional calculus on $\mathcal{O}_\gamma$ in the sense of [6] and [10].

3. Berezin quantization for discrete series representations

In this section, we introduce the notation and we review some well-known facts on Hermitian symmetric spaces of the non-compact type and on holomorphic discrete series representations (see [24], Chapter VIII, [28], Chapter 6, [32], Chapter XII and [35], Chapter II).

Let G be a connected semi-simple non-compact real Lie group with finite center and let K be a maximal compact subgroup of G. We assume that the center of the Lie algebra of K is non-trivial. Then the homogeneous space G/K is a Hermitian symmetric space of the non-compact type.

Let $\mathfrak{g}$ and $\mathfrak{k}$ be the Lie algebras of G and K, respectively. Let $\mathfrak{g}^c$ and $\mathfrak{k}^c$ be the complexifications of $\mathfrak{g}$ and $\mathfrak{k}$ and G^c, K^c the corresponding complex Lie groups containing G and K, respectively. We denote by β the Killing form of $\mathfrak{g}^c$, that is, $\beta(X,Y)=\mathrm{Tr}(\mathrm{ad}\,X\,\mathrm{ad}\,Y)$ for X, $Y\in\mathfrak{g}^c$. Let $\mathfrak{p}$ be the ortho-complement of $\mathfrak{k}$ in $\mathfrak{g}$ with respect to β. Then $\mathfrak{g}=\mathfrak{k}\oplus\mathfrak{p}$ is a Cartan decomposition of $\mathfrak{g}$.

We fix a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{k}$. Then $\mathfrak{h}$ is also a Cartan subalgebra of $\mathfrak{g}$. We denote by $\mathfrak{h}^c$ the complexification of $\mathfrak{h}$. Let H be the connected subgroup of K with Lie algebra $\mathfrak{h}$. Let Δ be the root system of $\mathfrak{g}^c$ relative to $\mathfrak{h}^c$ and let $\mathfrak{g}^c=\mathfrak{h}^c\oplus\sum_{\alpha\in\Delta}\mathfrak{g}_\alpha$ be the root space decomposition of $\mathfrak{g}^c$. Then we have the direct decompositions $\mathfrak{k}^c=\mathfrak{h}^c\oplus\sum_{\alpha\in\Delta_c}\mathfrak{g}_\alpha$ and $\mathfrak{p}^c=\sum_{\alpha\in\Delta_n}\mathfrak{g}_\alpha$ where $\mathfrak{p}^c$ denotes the complexification of $\mathfrak{p}$ and Δ_c (resp. Δ_n) denotes the set of compact (resp. non-compact) roots. We choose an ordering on Δ as in [24], p. 384 and we denote by Δ^+, Δ_c^+ and Δ_n^+ the corresponding sets of positive roots, positive compact roots and positive non-compact roots, respectively. We set $\mathfrak{p}^+=\sum_{\alpha\in\Delta_n^+}\mathfrak{g}_\alpha$ and $\mathfrak{p}^-=\sum_{\alpha\in\Delta_n^+}\mathfrak{g}_{-\alpha}$. Then we have $[\mathfrak{k}^c,\mathfrak{p}^\pm]\subset\mathfrak{p}^\pm$ and $\mathfrak{p}^+$ and $\mathfrak{p}^-$ are abelian subspaces [24], Proposition 7.2. We denote by P^+ and P^- be the analytic subgroups of G^c with Lie algebras $\mathfrak{p}^+$ and $\mathfrak{p}^-$, respectively.

For each $\mu\in(\mathfrak{h}^c)^*$, we denote by H_μ the element of $\mathfrak{h}^c$ satisfying $\beta(H,H_\mu)=\mu(H)$ for all $H\in\mathfrak{h}^c$. Note that if μ is real-valued on $i\mathfrak{h}$ then $iH_\mu\in\mathfrak{g}$. For μ, $\nu\in(\mathfrak{h}^c)^*$, we set $(\mu,\nu):=\beta(H_\mu,H_\nu)$.

Let θ denotes conjugation over the real form $\mathfrak{g}$ of $\mathfrak{g}^c$. For $X\in\mathfrak{g}^c$, we set $X^*=-\theta(X)$. We denote by $g\to g^*$ the involutive anti-automorphism of G^c which is obtained by exponentiating $X\to X^*$ to G^c. Recall that the multiplication map $(z,k,y)\to zky$ is a diffeomorphism from $P^+\times K^c\times P^-$ onto an open submanifold of G^c containing G [24], Lemma 7.9. Following [29], we introduce the projections $\kappa: P^+K^cP^-\to K^c$ and $\zeta: P^+K^cP^-\to P^+$. Then the map $gK\to\log\zeta(g)$ from G/K to $\mathfrak{p}^+$ induces a diffeomorphism from G/K onto a bounded domain $\mathcal{D}\subset\mathfrak{p}^+$ [24], p. 392. The natural action of G on G/K corresponds to the action of G on $\mathcal{D}$ given by $g\cdot Z=\log\zeta(g\exp Z)$. The G-invariant measure on $\mathcal{D}$ is $d\mu(Z)=\chi_0(\kappa(\exp Z^*\exp Z))\,d\mu_L(Z)$ where χ_0 is the character on K^c defined by $\chi_0(k)=\mathrm{Det}_{\mathfrak{p}^+}(\mathrm{Ad}\,k)$ and $d\mu_L(Z)$ is a Lebesgue measure on $\mathcal{D}$ [29]. To simplify the notation, we set $k(Z):=\kappa(\exp Z^*\exp Z)$ for $Z\in\mathcal{D}$.

Now we introduce the holomorphic discrete series representations of scalar type of G as follows. Let χ be a unitary character of K. We also denote by χ the extension of χ to K^c. Let us introduce the Hilbert space $\mathcal{H}_\chi$ of holomorphic functions on $\mathcal{D}$ such that

$$\|f\|_\chi^2 := \int_{\mathcal{D}} |f(Z)|^2 \chi(k(Z))\, c_\chi d\mu(Z) < +\infty$$

where the constant c_χ is defined by

$$c_\chi^{-1} = \int_{\mathcal{D}} (\chi{\cdot}\chi_0)(k(Z))\, d\mu_L(Z).$$

Note that $\chi(k(Z)) > 0$ for all $Z \in \mathcal{D}$. Indeed, for each $Z \in \mathcal{D}$ there exists $g_Z \in G$ such that $g_Z{\cdot}0 = Z$. Writing $g_Z = \exp Z k y$ with $k \in K^c$ and $y \in P^-$, we have $k(Z) = (k^*)^{-1}k^{-1}$ which gives $\chi(k(Z)) = \overline{\chi(k)}^{-1}\chi(k) = |\chi(g_Z^{-1}\exp Z)|^2 > 0$.

Proposition 3.1. [16], [28] *Let* $\lambda = d\chi|_{\mathfrak{h}^c}$ *and* $\delta = \frac{1}{2}\sum_{\alpha\in\Delta^+}\alpha$. *Then* $\mathcal{H}_\chi$ *is nonzero if and only if* $(\lambda+\delta,\alpha) < 0$ *for every non-compact positive root* α. *In that case,* $\mathcal{H}_\chi$ *contains all polynomials. Moreover, the action of* G *on* $\mathcal{H}_\chi$ *defined by*

$$\pi_\chi(g)f(Z) = \chi(\kappa(g^{-1}\exp Z))^{-1}\, f(g^{-1}\cdot Z)$$

is a unitary representation of G *which belongs to the holomorphic discrete series of* G.

In the rest of the paper, we assume that χ satisfies the condition of Proposition 3.1. Note that $\mathcal{H}_\chi$ is a reproducing kernel Hilbert space. More precisely, for $Z \in \mathcal{D}$, define the coherent state $e_Z \in \mathcal{H}_\chi$ by

$$e_Z(W) = \chi(\kappa(\exp Z^* \exp W))^{-1}.$$

Then we have the reproducing property $f(Z) = \langle f, e_Z\rangle_\chi$ for each $f \in \mathcal{H}_\chi$ (see [29] Chapter XII.2 for instance). Here $\langle\cdot,\cdot\rangle_\chi$ denotes the inner product on $\mathcal{H}_\chi$.

As in Section 2, we can define the Berezin symbol $s_\chi(A)(Z)$ and the double Berezin symbol $S_\chi(A)(Z,W)$ of a bounded operator A on $\mathcal{H}_\chi$. We also obtain an integral formula for $\langle Af, g\rangle_\chi$ analogous to that of Section 2:

$$\begin{aligned}\langle Af, g\rangle_\chi = \int_{\mathcal{D}\times\mathcal{D}} & f(W)\overline{g(Z)}S_\chi(A)(Z,W)\langle e_W^\chi, e_Z^\chi\rangle_\chi \\ & (\chi{\cdot}\chi_0)(k(Z))(\chi{\cdot}\chi_0)(k(W))c_\lambda^2\, d\mu_L(Z)d\mu_L(W).\end{aligned} \tag{3.1}$$

In [12], we give explicit expressions for the derived representation $d\pi_\chi$, for the Berezin symbols of $\pi_\chi(g)$ and $d\pi_\chi(X)$. In the rest of this section, we recall some results of [12].

If L is a Lie group and X is an element of the Lie algebra of L then we denote by X^+ the right invariant vector field on L generated by X, that is, $X^+(h) = \frac{d}{dt}(\exp tX)h|_{t=0}$ for $h \in L$.

Let $p_{\mathfrak{p}^+}$, $p_{\mathfrak{k}^c}$ and $p_{\mathfrak{p}^-}$ be the projections of $\mathfrak{g}^c$ onto $\mathfrak{p}^+$, $\mathfrak{k}^c$ and $\mathfrak{p}^-$ associated with the direct decomposition $\mathfrak{g}^c = \mathfrak{p}^+ \oplus \mathfrak{k}^c \oplus \mathfrak{p}^-$. The following proposition can be easily proved by differentiating the multiplication map from $P^+ \times K^c \times P^-$ onto $P^+K^cP^-$.

Lemma 3.2. [12] *Let* $X \in \mathfrak{g}^c$ *and* $g = z\,k\,y$ *where* $z \in P^+$, $k \in K^c$ *and* $y \in P^-$. *We have*
1) $d\zeta_g(X^+(g)) = (\mathrm{Ad}(z)\,p_{\mathfrak{p}^+}(\mathrm{Ad}(z^{-1})\,X))^+(z)$.
2) $d\kappa_g(X^+(g)) = (p_{\mathfrak{k}^c}(\mathrm{Ad}(z^{-1})\,X))^+(k)$.

From Lemma 3.2 we can deduce the following proposition (see [12]; for Point 2) see also [29], Proposition XII.2.1).

Proposition 3.3. [12]
1) Let $g \in G$. *We have*

$$S_\chi(\pi(g))(Z,W) = \chi\big(\kappa(\exp W^* g^{-1} \exp Z)^{-1} \kappa(\exp W^* \exp Z)\big).$$

2) For $X \in \mathfrak{g}^c$ *and* $f \in \mathcal{H}_\chi$, *we have*

$$d\pi_\chi(X)f(Z) = d\chi((p_{\mathfrak{k}^c}(\mathrm{Ad}((\exp Z)^{-1})\,X))\,f(Z) - (df)_Z\big(p_{\mathcal{P}^+}(e^{-\operatorname{ad} Z}\,X)\big).$$

3) Let $X \in \mathfrak{g}^c$. *We have*

$$S_\chi(d\pi_\chi(X))(Z,W) = \lambda\big(p_{\mathfrak{k}^c}(\mathrm{Ad}(\zeta(\exp W^* \exp Z)^{-1} \exp W^*)\,X\big).$$

We can write

$$S(d\pi_\chi(X))(Z,W) = i\beta(\tilde{\psi}_\chi(Z,W),X)$$

where

$$\tilde{\psi}_\chi(Z,W) := \mathrm{Ad}\big(\exp(-W^*)\,\zeta(\exp W^* \exp Z)\big)\,(-iH_\lambda).$$

Moreover, the map $\psi_\chi : Z \to \tilde{\psi}_\chi(Z,Z)$ *is a diffeomorphism from* $\mathcal{D}$ *onto the orbit* $\mathcal{O}_\chi$ *of* $-iH_\lambda \in \mathfrak{g}$ *for the adjoint action of* G.

4. Contraction of groups

We retain the notation from the previous sections. In particular, we fix a real number $\gamma > 0$ as in Section 2 and a unitary character χ of K as in Section 3.

Let us consider a Chevalley basis $(\tilde{E}_\alpha)_{\alpha\in\Delta} \cup (H_\alpha)_{\alpha\in\Delta_s}$ for $\mathfrak{g}^c$ (see for instance [27], Chapter 5). Here Δ_s denotes the set of simple roots corresponding to Δ^+. In particular, we have $[\tilde{E}_\alpha, \tilde{E}_{-\alpha}] = H_\alpha$ for $\alpha \in \Delta^+$. Note that $\mathfrak{g}$ is spanned by the elements $\tilde{E}_\alpha - \tilde{E}_{-\alpha}$, $i(\tilde{E}_\alpha + \tilde{E}_{-\alpha})$ for $\alpha \in \Delta_c^+$, $\tilde{E}_\alpha + \tilde{E}_{-\alpha}$, $i(\tilde{E}_\alpha - \tilde{E}_{-\alpha})$ for $\alpha \in \Delta_n^+$ and iH_α for $\alpha \in \Delta_s$. Then we have $\tilde{E}_\alpha^* = \tilde{E}_{-\alpha}$ for $\alpha \in \Delta_c$ and $\tilde{E}_\alpha^* = -\tilde{E}_{-\alpha}$ for $\alpha \in \Delta_n$. Now we introduce the basis $(E_\alpha)_{\alpha\in\Delta_n^+}$ for $\mathfrak{p}^+$ defined by $E_\alpha = \frac{1}{\sqrt{-(\lambda,\alpha)}}\tilde{E}_\alpha$ $(\alpha \in \Delta_n^+)$ and the corresponding Euclidean norm defined by $|Z| = (\sum_{\alpha\in\Delta_n^+} |z_\alpha|^2)^{1/2}$ for $Z = \sum_{\alpha\in\Delta_n^+} z_\alpha E_\alpha$.

Let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be an enumeration of Δ_n^+. For $k = 1, 2, \ldots, n$, we set $X_k = \frac{1}{2i}(E_{\alpha_k} - E_{-\alpha_k})$, $Y_k = \frac{1}{2}(E_{\alpha_k} + E_{-\alpha_k})$ and $\tilde{Z} = \frac{1}{2(\lambda,\lambda)} iH_\lambda$. Let $\mathfrak{g}_0$ be the subspace of $\mathfrak{g}$ generated by X_k, Y_k, $k = 1, 2, \ldots, n$ and $\tilde{Z}$. Let p_0 be the orthogonal projection of $\mathfrak{g}$ on the line generated by $\tilde{Z}$ with respect to the Killing form. For $r > 0$, we denote by C_r the linear isomorphism of $\mathfrak{g}$ defined by $C_r = r^2 p_0 + r(Id - p_0)$. We introduce the Lie bracket on $\mathfrak{g}$ defined by

$$[X, Y]_0 = \lim_{r \to 0} C_r^{-1}([C_r(X), C_r(Y)]).$$

Lemma 4.1.
1) For X and Y in $\mathfrak{g}$, we have $[X, Y]_0 = p_0([(Id - p_0)(X), (Id - p_0)(Y)])$.
2) The Lie algebra $(\mathfrak{g}_0, [\cdot, \cdot]_0)$ is a Heisenberg algebra. In the basis X_k, Y_k, $k = 1, 2, \ldots, n$, $\tilde{Z}$ of $\mathfrak{g}_0$, the only non trivial brackets are $[X_k, Y_k]_0 = \tilde{Z}$.
3) The Lie algebra $(\mathfrak{g}, [\cdot, \cdot]_0)$ is isomorphic as a Lie algebra to the product of the Heisenberg algebra $(\mathfrak{g}_0, [\cdot, \cdot]_0)$ by an abelian Lie algebra of dimension $\dim \mathfrak{g} - \dim \mathfrak{g}_0$.

Proof. Direct computation. ■

We denote also by C_r the restriction of C_r to $\mathfrak{g}_0$. Then we have

$$C_r \left(\sum_{k=1}^{n} a_k X_k + \sum_{k=1}^{n} b_k Y_k + c\tilde{Z} \right) = r \left(\sum_{k=1}^{n} a_k X_k + \sum_{k=1}^{n} b_k Y_k \right) + r^2 c\tilde{Z}$$

and it is immediate from Lemma 4.1 that that $C_r : \mathfrak{g}_0 \to \mathfrak{g}$ is a contraction of $\mathfrak{g}$ to $\mathfrak{g}_0$ in the sense of [9], Section 4. The corresponding contraction of G to the Heisenberg group G_0 with Lie algebra $\mathfrak{g}_0$ is the map $c_r : G_0 \to G$ defined by $c_r(\exp_{G_0}(X)) = \exp_G(C_r(X))$.

In the following proposition, we describe how the parametrizations of the orbits $\mathcal{O}_{\chi^m}$ and $\mathcal{O}_\gamma$ (see Section 2 and Section 3) are related in the contraction process. In the rest of the paper, we identify $\mathfrak{p}^+$ to $\mathbb{C}^n$ by means of the linear isomorphism $(z_1, z_2, \ldots, z_n) \to Z = \sum_{k=1}^n z_k E_{\alpha_k}$.

Proposition 4.2. *Let $r(m)$ be a sequence of $]0, 1]$ satisfying*

$$\lim_{m \to +\infty} m r(m)^2 = 2\gamma.$$

Then, for each $X \in \mathfrak{g}_0$ and each $Z \in \mathfrak{p}^+$, we have

$$\lim_{m \to +\infty} \beta \left(\psi_{\chi^m}(Z/\sqrt{2\gamma m}), C_{r(m)}(X) \right) = \langle \psi_\gamma(Z), X \rangle,$$

or, equivalently, for each $Z \in \mathfrak{p}^+$ we have

$$\lim_{m \to +\infty} C^*_{r(m)} \left(\psi_{\chi^m}(Z/\sqrt{2\gamma m}) \right) = \psi_\gamma(Z).$$

Proof. Assume that $X = X_k$ or $X = Y_k$. Then $C_{r(m)}(X) = r(m)X$. By Part 3) of Proposition 3.3, we have

$$\begin{aligned} &\beta \left(\psi_{\chi^m}(Z/\sqrt{2\gamma m}), C_{r(m)}(X) \right) = -imr(m) \\ &\times \beta \left(\zeta \left(\exp(Z^*/\sqrt{2\gamma m}) \exp(Z/\sqrt{2\gamma m}) \right) H_\lambda, \mathrm{Ad} \left(\exp(Z^*/\sqrt{2\gamma m}) \right) X \right). \end{aligned} \tag{4.1}$$

In order to study the behavior of this expression as $m \to +\infty$, we need a first-order Taylor formula for the function

$$F(t) = \beta\,(\zeta(\exp(tZ^*)\,\exp(tZ)\,H_\lambda\,,\,\mathrm{Ad}(\exp(tZ^*))X)$$

at $t = 0$. We have $F(0) = 0$ and by using Lemma 3.2 we find that $F'(0) = \beta(H_\lambda, [Z^* - Z, X])$. We then obtain

$$F(t) = \beta(H_\lambda, [Z^* - Z, X])\,t + o(t). \tag{4.2}$$

If $X = X_k$, then $\beta(H_\lambda, [Z^* - Z, X]) = -\frac{1}{2i}(z_k + \overline{z}_k)$ and (4.2) implies

$$\lim_{m\to+\infty} \beta\left(\psi_{\chi^m}(Z/\sqrt{2\gamma m}), C_{r(m)}(X)\right) = \frac{1}{2}(z_k + \overline{z}_k) = \mathrm{Re} z_k.$$

Similarly, if $X = Y_k$ then $\beta(H_\lambda, [Z^* - Z, X]) = \frac{1}{2}(z_k - \overline{z}_k)$ and

$$\lim_{m\to+\infty} \beta\left(\psi_{m\lambda}(Z/\sqrt{2\gamma m}), C_{r(m)}(X)\right) = -\frac{i}{2}(z_k - \overline{z}_k) = \mathrm{Im} z_k.$$

If $X = \tilde{Z}$ then we immediately obtain

$$\lim_{m\to+\infty} \beta\left(\psi_{m\lambda}(Z/\sqrt{2\gamma m}), C_{r(m)}(X)\right) = \gamma.$$

Comparing with (2.8), we get the desired result. ∎

Note that the condition $\lim_{m\to+\infty}(mr(m)^2) = 2\gamma$ is equivalent to the condition $\lim_{m\to+\infty} C^*_{r(m)}(-imH_\lambda) = \xi_\gamma$ in $\mathfrak{g}_0^*$ in the notation of Section 2 and Section 3. Then Proposition 4.2 says that the deformed orbits $C^*_{r(m)}(\mathcal{O}_{\chi^m})$ 'converge to' the orbit $\mathcal{O}_\gamma$, that is, each point ξ in $\mathcal{O}_\gamma$ is the limit of a sequence $\xi_m \in C^*_{r(m)}(\mathcal{O}_{\chi^m})$ (also see [17], Section 2). It is quite remarkable that, once the condition $\lim_{m\to+\infty}(mr(m)^2) = 2\gamma$ is fixed, all the rest follows, that is, we can establish convergence of Berezin symbols and contractions of representations (see the next sections).

5. Contraction of operators

In this section, we keep the notation of Section 3 and Section 4. We first establish an estimate for the function $\varphi_\chi(Z) = \chi^{-1}(k(Z))$ on $\mathcal{D}$. Recall that we have set $k(Z) := \kappa(\exp Z^* \exp Z)$.

Let us denote by $|\cdot|$ an arbitrary norm on $\mathfrak{g}^c$. Given a function ψ from a subspace of $\mathfrak{g}^c$ to $\mathfrak{g}^c$ and $p \in \mathbb{N}$, we write $\psi(X) = O(|X|^p)$ if there exist $M > 0$ and $\varepsilon > 0$ such that $|\psi(X)| \leq M\,|X|^p$ whenever $|X| < \varepsilon$.

Lemma 5.1.
1) For X in a sufficiently small neighborhood of 0 in $\mathfrak{g}^c$, write $\exp X = \exp Y \exp U \exp V$ *with $Y \in \mathfrak{p}^+$, $U \in \mathfrak{k}^c$ and $V \in \mathfrak{p}^-$. Then we have*

$$U = p_{\mathfrak{k}^c}(X) - \frac{1}{2} p_{\mathfrak{k}^c}([p_{\mathfrak{p}^+}(X), p_{\mathfrak{p}^-}(X)]) + O(|X|^3).$$

2) For Z and W in a sufficiently small neighborhood of 0 in $\mathfrak{p}^+$, write

$$\exp Z^* \exp W = \exp Y \exp U \exp V$$

with $Y \in \mathfrak{p}^+$, $U \in \mathfrak{k}^c$ and $V \in \mathfrak{p}^-$. Then we have

$$U = p_{\mathfrak{k}^c}([Z^*, W]) + O(\max(|Z|, |W|)^3).$$

Proof. 1) We first show that $Y, U, V = O(|X|)$. Set $U = u(X) := \log \kappa(\exp X)$. By Part 2) of Lemma 3.2, we have $du(0)(X) = p_{\mathfrak{k}^c}(X)$ for each $X \in \mathfrak{g}^c$. Then, by writing the first-order Taylor formula for u at 0, we obtain $u(X) = O(|X|)$. Similarly, we find $Y, V = O(|X|)$.

Now, by the Baker-Campbell-Hausdorff formula [36], we have

$$X = Y + U + V + \frac{1}{2}[Y, U] + \frac{1}{2}[Y, V] + \frac{1}{2}[U, V] + O(\max(|Y|, |U|, |V|)^3)$$

for X in a sufficiently small neighborhood of 0 in $\mathfrak{g}^c$. Since $[Y, U] \in \mathfrak{p}^+$ and $[U, V] \in \mathfrak{p}^-$, this gives

$$p_{\mathfrak{k}^c}(X) = U + \frac{1}{2} p_{\mathfrak{k}^c}([Y, V]) + O(|X|^3) \tag{5.1}$$

$$p_{\mathfrak{p}^+}(X) = Y + \frac{1}{2}[Y, U] + \frac{1}{2} p_{\mathfrak{p}^+}([Y, V]) + O(|X|^3) \tag{5.2}$$

$$p_{\mathfrak{p}^-}(X) = V + \frac{1}{2}[U, V] + \frac{1}{2} p_{\mathfrak{p}^-}([Y, V]) + O(|X|^3). \tag{5.3}$$

Equalities (5.2) and (5.3) implies

$$[p_{\mathfrak{p}^+}(X), p_{\mathfrak{p}^-}(X)] = [Y, V] + O(|X|^3). \tag{5.4}$$

Replacing (5.4) in (5.1) we then obtain the desired equality.
2) Using the Baker-Campbell-Hausdorff formula again, this is an immediate consequence of 1) . ∎

Lemma 5.2.
1) For $Z \in \mathfrak{p}^+$ close to 0, we have $\varphi_\chi(Z) = 1 + |Z|^2 + O(|Z|^3)$.
2) There exist a constant $C > 0$ and a constant $D > 0$ such that for each $Z \in \mathcal{D}$ we have

$$\varphi_\chi(Z) \geq 1 + C|Z|^2 \geq \frac{1}{1 - D|Z|^2}.$$

Proof. 1) By applying Part 2) of Lemma 5.1, we have

$$U := \log k(Z) = p_{\mathfrak{k}^c}([Z^*, Z]) + O(|Z|^3) = -\sum_{\alpha\in\Delta_n^+} \frac{1}{(\lambda,\alpha)}|z_\alpha|^2 H_\alpha + O(|Z|^3).$$

for $Z = \sum_{\alpha\in\Delta_n^+} z_\alpha E_\alpha$. Then $d\chi(U) = \lambda(U) = -|Z|^2 + O(|Z|^3)$. Hence

$$\varphi_\chi(Z) = e^{-\lambda(U)} = 1 + |Z|^2 + O(|Z|^3).$$

2) Note that the restriction of π_χ to H is given by

$$\pi_\chi(h)f(Z) = \chi(h)f(\mathrm{Ad}(h^{-1})Z)$$

for $h \in H$ and $Z \in \mathcal{D}$. For $Z = \sum_{k=1}^n z_k E_{\alpha_k} \in \mathcal{D}$, we set $f_0(Z) = 1$ and $f_k(Z) = z_k$ for $1 \le k \le n$. Then we see immediately that f_0 is a weight vector for the weight λ of π_χ and, for each $k = 1, 2, \ldots, n$, f_k is a weight vector for the weight $\lambda - \alpha_k$. Since the weight spaces for distinct weights are orthogonal, we have $\langle f_k, f_l\rangle_\chi = 0$ for $0 \le k \ne l \le n$.

On the other hand, it is well-known that for any orthonormal basis (g_k) for $\mathcal{H}_\chi$ we have $e_Z^\chi(W) = \sum_{k\ge 0} g_k(W)\overline{g_k(Z)}$. Taking (g_p) so that $g_k = \|f_k\|_\chi^{-1} f_k$ $(0 \le k \le n)$, we see that there exists a constant $C > 0$ such that $\varphi_\chi(Z) \ge 1 + C|Z|^2$ for each $Z \in \mathcal{D}$. Moreover, to obtain the second inequality, we have just to take the constant $D > 0$ such that $D^{-1} > 1/C + \sup_{Z\in\mathcal{D}} |Z|^2$.■

In order to simplify notation, in the rest of the paper, we write, π_m, $\mathcal{H}_m$, $\langle\cdot,\cdot\rangle_m$, e_Z^m, c_m, S_m instead of π_{χ^m}, $\mathcal{H}_{\chi^m}$, $\langle\cdot,\cdot\rangle_{\chi^m}$, $e_Z^{\chi^m}$, c_{χ^m}, S_{χ^m}, respectively. Moreover, we fix the Lebesgue measure $d\mu_L(Z)$ on $\mathcal{D}$ as follows. Writing $Z = \sum_{k=1}^n z_k E_{\alpha_k}$ and, for $k = 1, 2, \ldots, n$, $z_k = x_k + iy_k$, $x_k, y_k \in \mathbb{R}$, we take $d\mu_L(Z) = dx_1 dy_1 \ldots dx_n dy_n$.

Lemma 5.3.
1) For $Z \in \mathfrak{p}^+$, we have

$$\lim_{m\to+\infty} \chi^m\big(k(Z/\sqrt{2\gamma m})\big) = e^{-|Z|^2/2\gamma}.$$

2) For $Z, W \in \mathfrak{p}^+$, we have

$$\lim_{m\to+\infty} \langle e^m_{W/\sqrt{2\gamma m}}, e^m_{Z/\sqrt{2\gamma m}}\rangle_m = \langle e^\gamma_W, e^\gamma_Z\rangle_\gamma.$$

3) We have

$$\lim_{m\to+\infty} c_m^{-1}(2\gamma m)^n = \int_{\mathfrak{p}^+} e^{-|Z|^2/2\gamma}\, d\mu_L(Z) = (2\pi\gamma)^n.$$

Proof. 1) Fix $Z \in \mathfrak{p}^+$. By Part 1) of Lemma 5.2, we have

$$\begin{aligned}\log \chi^m\big(k(Z/\sqrt{2\gamma m})\big) &= -\, m \log\, \chi^{-1}\big(k(Z/\sqrt{2\gamma m})\big)\\ &= -\, m \log \varphi_\lambda(Z/\sqrt{2\gamma m})\\ &= -\, m\left(|Z|/\sqrt{2\gamma m}\right)^2 + O(1/\sqrt{m})\\ &= -\,(1/2\gamma)\,|Z|^2 + O(1/\sqrt{m}).\end{aligned}$$

2) See the proof of Proposition 6.1.
3) Recall that

$$c_m^{-1} = \int_{\mathcal{D}} (\chi^m.\chi_0)(k(Z))\, d\mu_L(Z).$$

Changing variables $Z \to Z/\sqrt{2\gamma m}$ in this integral, we get

$$c_m^{-1}(2\gamma m)^n = \int_{\sqrt{2\gamma m}\mathcal{D}} (\chi^m.\chi_0)(k(Z/\sqrt{2\gamma m}))\, d\mu_L(Z).$$

By 1), the integrand $I_m(Z) := (\chi^m.\chi_0)(k(Z/\sqrt{2\gamma m}))$ satisfies

$$\lim_{m\to+\infty} I_m(Z) = e^{-|Z|^2/2\gamma}.$$

In order to obtain 3), it suffices to verify that the Lebesgue dominated convergence theorem can be applied. But by Part 2) of Lemma 5.2 we have

$$\chi^m(k(Z)) \leq \left(1 - D\frac{|Z|^2}{2\gamma m}\right)^m \leq e^{-D/2\gamma|Z|^2}$$

for $Z \in \sqrt{2\gamma m}\mathcal{D}$. This ends the proof. ■

In the rest of the paper, we denote by $\mathcal{P}$ the space of all complex polynomials on $\mathfrak{p}^+$.

Proposition 5.4. *For each integer $m \geq 1$ let A_m be an operator of $\mathcal{H}_m$. Let A be a bounded operator of $\mathcal{H}_\gamma$. Assume that*
(i) The sequence $\|A_m\|_{op}$ is bounded;
(ii) we have $\lim_{m\to+\infty} S_m(A_m)(Z/\sqrt{2\gamma m}\,,\, W/\sqrt{2\gamma m}) = S_\gamma(A)(Z,W)$.
Then, for any complex polynomials P and Q, we have

$$\lim_{m\to+\infty} \langle A_m\, P(\sqrt{2\gamma m}\cdot), Q(\sqrt{2\gamma m}\cdot)\rangle_m = \langle A\, P, Q\rangle_\gamma.$$

Proof. First, note that we can assume that P and Q are homogeneous. By using Formula (3.1), we express $\langle A_m\, P(\sqrt{2\gamma m}\cdot), Q(\sqrt{2\gamma m}\cdot)\rangle_m$ as an integral. Changing variables $(Z,W) \to (Z/\sqrt{2\gamma m}\,,\, W/\sqrt{2\gamma m})$ in this integral, we get

$$\begin{aligned}&\langle A_m\, P(\sqrt{2\gamma m}\cdot), Q(\sqrt{2\gamma m}\cdot)\rangle_m\\ &\qquad = c_m^2(2\gamma m)^{-2n} \int_{(\sqrt{2\gamma m}\mathcal{D})^2} I_m(Z,W)\, d\mu_L(Z)\, d\mu_L(W)\end{aligned}$$

where the integrand $I_m(Z,W)$ is given by

$$I_m(Z,W) = P(W)\overline{Q(Z)}S_m(A_m)(Z/\sqrt{2\gamma m}\,,\,W/\sqrt{2\gamma m})$$
$$\langle e^m_{W/\sqrt{2\gamma m}}, e^m_{Z/\sqrt{2\gamma m}}\rangle_m (\chi^m.\chi_0)(k(Z/\sqrt{2\gamma m}))(\chi^m.\chi_0)(k(W/\sqrt{2\gamma m})).$$

By Lemma 5.3, we have $\lim_{m\to+\infty} c_m^2(2\gamma m)^{-2n} = (2\pi\gamma)^{-2n}$ and

$$\lim_{m\to+\infty} I_m(Z,W) = P(W)\overline{Q(Z)}S_\gamma(A)(Z,W)\langle e^\gamma_W, e^\gamma_Z\rangle_\gamma\, e^{-|Z|^2/2\gamma}\, e^{-|W|^2/2\gamma}.$$

Now, as in the proof of Part 3) of Lemma 5.3, we aim to apply the Lebesgue dominated convergence theorem. We write by using the Cauchy-Schwartz inequality

$$|\langle A_m\, e^m_{W/\sqrt{2\gamma m}}, e^m_{Z/\sqrt{2\gamma m}}\rangle_m| \le \|A_m\|_{op}\, \|e^m_{Z/\sqrt{2\gamma m}}\|_m\, \|e^m_{W/\sqrt{2\gamma m}}\|_m.$$

This implies that

$$|I_m(Z,W)| \le |P(W)|\,|Q(Z)|\,\|A_m\|_{op}\,\chi^{m/2}(k(Z/\sqrt{2\gamma m}))\chi^{m/2}(k(W/\sqrt{2\gamma m})).$$

Hence we conclude as in the proof of Part 3) of Lemma 5.3. ■

Corollary 5.5. *Let P and Q two homogeneous complex polynomials different to 0. Then, with the notation as in Proposition 5.4, we have*

$$\lim_{m\to+\infty}\langle A_m(\|P\|_m^{-1}\, P), \|Q\|_m^{-1}\, Q\rangle_m = \langle A(\|P\|_\gamma^{-1}\, P), \|Q\|_\gamma^{-1}\, Q\rangle_\gamma.$$

Proof. Since P and Q are homogeneous, one has

$$\langle A_m(\|P\|_m^{-1}\, P), \|Q\|_m^{-1}\, Q\rangle_m$$
$$= \|P(\sqrt{2\gamma m}\cdot)\|_m^{-1}\|Q(\sqrt{2\gamma m}\cdot)\|_m^{-1}\,\langle A_m\, P(\sqrt{2\gamma m}\cdot), Q\sqrt{2\gamma m}\cdot)\rangle_m.$$

But applying Proposition 5.4 to the particular case $A_m = Id$, $A = Id$, we get

$$\lim_{m\to+\infty}\langle P(\sqrt{2\gamma m}\cdot), Q(\sqrt{2\gamma m}\cdot)\rangle_m = \langle P, Q\rangle_\gamma$$

and then $\lim_{m\to+\infty}\|P(\sqrt{2\gamma m}\cdot)\|_m = \|P\|_\gamma$. The desired result then follows from Proposition 5.4. ■

Recall that $\mathcal{P}$ is a K-module for the action σ defined by $\sigma(k)P(Z) = P(\mathrm{Ad}(k^{-1})Z)$. By [22], [35] we have the Peter-Weyl decomposition

$$\mathcal{P} = \bigoplus_{s\in\mathbb{N}^r_+}\mathcal{P}_s$$

where $\mathbb{N}^r_+ = \{s = (s_1, s_2, \ldots, s_r) \in \mathbb{N}^r \,:\, s_1 \ge s_2 \ge \ldots \ge s_r \ge 0\}$ and, for each $s\in\mathbb{N}^r_+$, $\mathcal{P}_s$ is an irreducible K-module with highest weight $-\sum_{k=1}^r s_k\gamma_k$. Here $\gamma_1, \gamma_2, \ldots, \gamma_r$ denote the Harish-Chandra strongly orthogonal roots. The elements of $\mathcal{P}_s$ are homogeneous polynomials of degree $s_1 + s_2 + \cdots + s_r$. Moreover, for each $s\in\mathbb{N}^r_+$ and each integer $m\ge 1$, there exists a constant $c_{m,s} > 0$ such that

$$\langle P\,,\,Q\rangle_m = c_{m,s}^{-1}\langle P\,,\,Q\rangle_\gamma$$

for each $P, Q\in\mathcal{P}_s$. The constant $c_{m,s}$ can be expressed in terms of the Pochhammer symbol [35].

Now, for each $s\in\mathbb{N}^r_+$, we fix a basis $(f^\gamma_{s,j})_{1\le j\le\dim\mathcal{P}_s}$ for $\mathcal{P}_s$ which is orthonormal with respect to $\langle\cdot,\cdot\rangle_\gamma$. Then $(f^\gamma_{s,j})$ is an orthonormal basis for $\mathcal{H}_\gamma$ and $f^m_{s,j} := \sqrt{c_{m,s}}f^\gamma_{s,j}$ is an orthonormal basis for $\mathcal{H}_m$. We denote by B_m the unitary operator from $\mathcal{H}_\gamma$ onto $\mathcal{H}_m$ defined by $B_m f^\gamma_{s,j} = f^m_{s,j}$.

Proposition 5.6. *Let (A_m) and A as in Proposition 5.4.*
1) For $s, s' \in \mathbb{N}_+^r$, $1 \le j \le \dim \mathcal{P}_s$ and $1 \le j' \le \dim \mathcal{P}_s$ we have

$$\lim_{m\to+\infty} \langle A_m\, f^m_{s,j}\,,\, f^m_{s',j'}\rangle_m = \langle A\, f^\gamma_{s,j}\,,\, f^\gamma_{s',j'}\rangle_\gamma.$$

2) If moreover the operators A_m $(m \ge 1)$ and A are unitary then we have

$$\lim_{m\to+\infty} \|B_m^{-1} A_m B_m\, P - A\, P\|_\gamma = 0$$

for each $P \in \mathcal{P}$.

Proof. 1) Since the polynomials $f^\gamma_{s,j}$ are homogeneous, the result follows from Corollary 5.5.
2) By density and linearity we can assume that $P = f^\gamma_{s,j}$. Since B_m is unitary, for each (s', j'), we have

$$\langle B_m^{-1} A_m B_m\, f^\gamma_{s,j}, f^\gamma_{s',j'}\rangle_\gamma = \langle A_m\, f^m_{s,j}\,,\, f^m_{s',j'}\rangle_m \to \langle A\, f^\gamma_{s,j}\,,\, f^\gamma_{s',j'}\rangle_\gamma$$

as $m \to +\infty$. This implies that $(B_m^{-1} A_m B_m\, f^\gamma_{s,j})_m$ converges weakly to $A\, f^\gamma_{s,j}$ in $\mathcal{H}_\gamma$. Since $\|B_m^{-1} A_m B_m\, f^\gamma_{s,j}\|_\gamma = 1 = \|A\, f^\gamma_{s,j}\|_\gamma$, we conclude that the sequence $(B_m^{-1} A_m B_m\, f^\gamma_{s,j})_m$ converges strongly to $A\, f^\gamma_{s,j}$. ■

6. Contraction of representations

In this section, we use the results of Section 5 in order to establish our main results about the contraction of the sequence π_m to the representation ρ_γ of G_0. As in Section 5, we consider a sequence $r(m)$ such that $\lim_{m\to+\infty} mr(m)^2 = 2\gamma$.

Proposition 6.1. *For $g_0 \in G_0$ and Z, $W \in \mathfrak{p}^+$, we have*

$$\lim_{m\to+\infty} S_m(\pi_m(c_{r(m)}(g_0)))(Z/\sqrt{2\gamma m}\,,\, W/\sqrt{2\gamma m}) = S_\gamma(\rho_\gamma(g_0))(Z, W).$$

Proof. Let Z, $W \in \mathfrak{p}^+$ and $g_0 = \exp\bigl(\sum_{k=1}^n a_k X_k + \sum_{k=1}^n b_k Y_k + c\tilde{Z}\bigr) \in G_0$. Denoting by U_m the expression

$$\log \kappa\Bigl(\exp\bigl(W^*/\sqrt{2\gamma m}\bigr)\, \exp\Bigl(-r(m)\Bigl(\sum_{k=1}^n a_k X_k + \sum_{k=1}^n b_k Y_k\Bigr) - r(m)^2 c\tilde{Z}\Bigr) \times \exp\bigl(Z/\sqrt{2\gamma m}\bigr)\Bigr),$$

we have to study the behavior of the sequence $u_m := \chi^{-m}(\exp U_m)$ as $m \to +\infty$. From Lemma 5.1, we deduce that

$$U_m = -r(m)^2 c\tilde{Z} - \frac{r(m)^2}{2\gamma} p_{\mathfrak{k}^c}([W^*, \Sigma]) - \frac{r(m)^2}{2\gamma} p_{\mathfrak{k}^c}([\Sigma, Z]) + \frac{r(m)^2}{4\gamma^2} p_{\mathfrak{k}^c}([W^*, Z]) - \frac{1}{2} p_{\mathfrak{k}^c}([p_{\mathfrak{p}^+}(\Sigma), p_{\mathfrak{p}^-}(\Sigma)]) + O(r(m)^3)$$

where

$$\Sigma := \sum_{k=1}^{n}\Big(a_kX_k + b_kY_k\Big) = \sum_{k=1}^{n}\Big(\frac{1}{2i}(a_k+ib_k)E_{\alpha_k} + \frac{1}{2i}(-a_k+ib_k)E_{-\alpha_k}\Big).$$

Since we have

$$p_{\mathfrak{k}^c}([W^*,\Sigma]) = -\sum_{k=1}^{n} \frac{1}{2i}\overline{w}_k(a_k+ib_k)\,\frac{1}{(\lambda,\alpha_k)}H_{\alpha_k}$$
$$p_{\mathfrak{k}^c}([\Sigma,Z]) = -\sum_{k=1}^{n} \frac{1}{2i}z_k(a_k-ib_k)\,\frac{1}{(\lambda,\alpha_k)}H_{\alpha_k}$$
$$p_{\mathfrak{k}^c}([W^*,Z]) = -\sum_{k=1}^{n} \overline{w}_kz_k\frac{1}{(\lambda,\alpha_k)}H_{\alpha_k}$$

mod. $\sum_{\alpha\in\Delta_c}\mathfrak{g}_\alpha$, we find

$$\lim_{m\to+\infty}\log u_m = \lim_{m\to+\infty}(-m\lambda(U_m)) = i\gamma c + \sum_{k=1}^{n}\frac{1}{2}\overline{w}_k(ia_k-b_k)$$
$$+\sum_{k=1}^{n}\frac{1}{2}z_k(ia_k+b_k) + \frac{1}{2\gamma}\sum_{k=1}^{n}\overline{w}_kz_k - \frac{\gamma}{4}\sum_{k=1}^{n}(a_k^2+b_k^2).$$

The result then follows from Equality (2.6). ∎

Now we can apply the results of Section 5 to the operators $A_m = \pi_m(c_{r(m)})(g_0)$ and $A = \rho_\gamma(g_0)$ for $g_0 \in G_0$. Then we obtain immediately the following proposition.

Proposition 6.2.
1) For $P, Q \in \mathcal{P}$ and $g_0 \in G_0$, we have

$$\lim_{m\to+\infty}\langle\pi_m(c_{r(m)})(g_0)\,P(\sqrt{2\gamma m}\cdot),Q(\sqrt{2\gamma m}\cdot)\rangle_m = \langle\rho_\gamma(g_0)\,P,Q\rangle_\gamma.$$

2) For $g_0\in G_0$, $s,s'\in\mathbb{N}_+^r$, $1\le j\le\dim\mathcal{P}_s$ and $1\le j'\le\dim\mathcal{P}_s$ we have

$$\lim_{m\to+\infty}\langle\pi_m(c_{r(m)})(g_0)\,f^m_{s,j}\,,\,f^m_{s',j'}\rangle_m = \langle\rho_\gamma(g_0)\,f^\gamma_{s,j}\,,\,f^\gamma_{s',j'}\rangle_\gamma.$$

3) For $g_0\in G_0$ and $P\in\mathcal{P}$, we have

$$\lim_{m\to+\infty}\|B_m^{-1}\pi_m(c_{r(m)})(g_0)B_m\,P - \rho_\gamma(g_0)\,P\|_\gamma = 0.$$

In particular, we have $\lim_{m\to+\infty}(B_m^{-1}\pi_m(c_{r(m)})(g_0)B_m)\,P(Z) = \rho_\gamma(g_0)\,P(Z)$ for each $Z\in\mathcal{D}$.

Example. [29] Here we take $G=SU(p,q)$ and $K=S(U(p)\times U(q))$. Let $\mathfrak{h}$ be the abelian subalgebra of $\mathfrak{k}$ consisting of the matrices

$$\begin{pmatrix} iaI_p & 0\\ 0 & ibI_q\end{pmatrix}\qquad a,\,b\in\mathbb{R}\qquad pa+bq=0.$$

Then we have

$$\mathcal{D} \simeq \{Z \in M_{pq}(\mathbb{C}) \,:\, I_p - ZZ^\star > 0\} = \{Z \in M_{pq}(\mathbb{C}) \,:\, \|Z\|_{op} < 1\}$$

where $\star$ denotes conjugate-transposition. The action of G on $\mathcal{D}$ is given by

$$g \cdot Z = (AZ + B)(CZ + D)^{-1}, \qquad g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

We fix an integer m and we consider the unitary character χ_m of K defined by

$$\chi_m \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} = (\mathrm{Det}\, A)^m.$$

The condition of Proposition 3.1 is then equivalent to $m + p + q \leq 0$. The norm of the Hilbert space $\mathcal{H}_{\chi_m}$ is given by

$$\|f\|^2_{\chi_m} = \int_{\mathcal{D}} |f(Z)|^2 \,(\mathrm{Det}(I_q - Z^\star Z))^{-p-q-m}\, c_{\chi_m} d\mu_L(Z)$$

where the constant

$$c^{-1}_{\chi_m} = \int_{\mathcal{D}} (\mathrm{Det}(I_q - Z^\star Z))^{-p-q-m}\, d\mu_L(Z)$$

can be expressed in terms of the Gamma function [25], Theorem 2.2.1. The coherent states for $\mathcal{H}_{\chi_m}$ are given by

$$e^{\chi_m}_Z(W) = \chi_m(\kappa(\exp Z^* \exp W)^{-1}) = (\mathrm{Det}(I_q - Z^\star W))^m,$$

the representation π_{χ_m} is given by

$$(\pi_{\chi_m}(g)f)(Z) = (\mathrm{Det}(CZ{+}D))^m \, f((AZ{+}B)(CZ{+}D)^{-1}), \qquad g^{-1} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

and the Berezin symbol of $\pi_{\chi_m}(g)$ is

$$S_{\chi_m}(\pi_{\chi_m}(g))(W,Z) = \big(\mathrm{Det}(CW + D - Z^\star(AW + B))\big)^m \big(\mathrm{Det}(I_p - Z^\star W)\big)^{-m}.$$

We can apply Proposition 6.2 in order to obtain a contraction of the sequence π_{χ_m} to the unitary irreducible representation ρ_γ of the Heisenberg group of dimension $2pq + 1$. In [9], the particular case $p = 1$ was considered.

Let us mention that, in [31], a similar contraction of $SU(p,p)$ to $M_p(\mathbb{C}) \times S(U(p) \times U(p))$ was considered and extended to the infinite-dimensional versions of these groups. The calculations of [31] were based on the expansion of $(\mathrm{Det}(I_q - Z^\star W))^m$ as a power series in the variables $Z^\star$ and W.

7. Contraction of derived representations

In this section, we establish contraction results for the derived representations $d\pi_m$ and $d\rho_\gamma$ analogous to those of Section 6.

Proposition 7.1. *For $X\in\mathfrak{g}_0$ and $Z,W\in\mathfrak{p}^+$, we have*

$$\lim_{m\to+\infty} S_m(d\pi_m(C_{r(m)}(X)))(Z/\sqrt{2\gamma m}\,,\,W/\sqrt{2\gamma m})=S_\gamma(d\rho_\gamma(X))(Z,W).$$

Proof. Taking Equality (2.9) and Part 3) of Proposition 3.3 into account, the result follows from Proposition 4.2. ∎

Proposition 7.2.
1) For $P,Q\in\mathcal{P}$ and $X\in\mathfrak{g}_0$, we have

$$\lim_{m\to+\infty}\langle d\pi_m(C_{r(m)}(X))\,P(\sqrt{2\gamma m}\cdot),Q(\sqrt{2\gamma m}\cdot)\rangle_m=\langle d\rho_\gamma(X)\,P,Q\rangle_\gamma.$$

2) For $s,s'\in\mathbb{N}^r_+$, $1\leq j\leq\dim\mathcal{P}_s$, $1\leq j'\leq\dim\mathcal{P}_s$ and $X\in\mathfrak{g}_0$, we have

$$\lim_{m\to+\infty}\langle d\pi_m(C_{r(m)}(X))\,f^m_{s,j}\,,\,f^m_{s',j'}\rangle_m=\langle d\rho_\gamma(X)\,f^\gamma_{s,j}\,,\,f^\gamma_{s',j'}\rangle_\gamma.$$

3) For $P\in\mathcal{P}$ and $X\in\mathfrak{g}_0$, we have

$$\lim_{m\to+\infty}\|B_m^{-1}d\pi_m(C_{r(m)}(X))B_m\,P-\rho_\gamma(X)\,P\|_\gamma=0.$$

Proof. 1) We can assume that P is a homogeneous polynomial of degree p. Then we have

$$\begin{aligned}&\langle d\pi_m(C_{r(m)}(X))\,P(\sqrt{2\gamma m}\cdot),Q(\sqrt{2\gamma m}\cdot)\rangle_m\\&=\int_{\mathcal{D}}(\sqrt{2\gamma m})^p\left(d\pi_m(C_{r(m)}(X))P\right)(Z)\overline{Q(\sqrt{2\gamma m}Z)}\,c_m(\chi_m\cdot\chi_0)(k(Z))\,d\mu_L(Z)\\&=c_m(2\gamma m)^{-n}\int_{\sqrt{2\gamma m}\mathcal{D}}(\sqrt{2\gamma m})^p\left(d\pi_m(C_{r(m)}(X))P\right)(Z/\sqrt{2\gamma m})\overline{Q(Z)}\\&\times(\chi_m\cdot\chi_0)(k(Z/\sqrt{2\gamma m}))\,d\mu_L(Z)\end{aligned}$$

by the change of variables $Z\to Z/\sqrt{2\gamma m}$.

Now, by using the expression for $d\pi_m$ given in Proposition 3.3 (see also [12], Proposition 3.3 or, equivalently, [29], Proposition XII 2.1), we can easily verify that

$$\lim_{m\to+\infty}(\sqrt{2\gamma m})^p\left(d\pi_m(C_{r(m)}(X))P\right)(Z/\sqrt{2\gamma m})=d\rho_\gamma(X)P(Z)$$

and that there exists a polynomial $\tilde{P}$ independent of m such that

$$|(\sqrt{2\gamma m})^p \left(d\pi_m(C_{r(m)}(X))P\right)(Z/\sqrt{2\gamma m})| \leq \tilde{P}(|Z|)$$

for each $m \geq 1$ and each $Z \in \sqrt{2\gamma m}\mathcal{D}$. Then the result follows from the dominated convergence theorem in the same way as in the proof of 3) of Lemma 5.3.

2) From 1) we deduce that

$$\lim_{m\to+\infty} \langle d\pi_m(C_{r(m)}(X))(\|P\|_m^{-1} P), \|Q\|_m^{-1} Q\rangle_m = \langle d\rho_\gamma(X)(\|P\|_\gamma^{-1} P), \|Q\|_\gamma^{-1} Q\rangle_\gamma$$

for each homogeneous polynomials P and Q. The result then follows as in the proof of Part 1) of Proposition 5.6.

3) By density and linearity we can assume that $P = f^\gamma_{s,j}$. We see easily that

$$\|B_m^{-1} d\pi_m(C_{r(m)}(X))B_m\, f^\gamma_{s,j} - \rho_\gamma(X)\, f^\gamma_{s,j}\|^2_\gamma = \sum_{(s',j')} |d\pi_m(C_{r(m)})(X))\, f^m_{s,j}\,,\, f^m_{s',j'}\rangle_m - \langle d\rho_\gamma(X)\, f^\gamma_{s,j}\,,\, f^\gamma_{s',j'}\rangle_\gamma|^2.$$

Taking into account the expressions for $d\pi_m$ and $d\rho_\gamma$ given above (see Equality (2.1) and Part 2) of Proposition 3.3), we see that this sum is finite and the number of its nonzero terms does not depend on m. The results then follows from 1). ■

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