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About the Relation between Multiplicity Free and Strong Multiplicity Free

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Abstract. Let G be a unimodular Lie group with finitely many connected components and let H be a closed unimodular subgroup of G. Let π be an irreducible unitary representation of G on $\mathcal{H}$ and τ one of H on V. Denote by $\mathrm{Hom}_H(\mathcal{H}_\infty, V)$ the vector space of continuous linear mappings $\mathcal{H}_\infty \to V$ that commute with the H-actions. Set $\mathrm{m}(\pi, \tau) = \dim \mathrm{Hom}_H(\mathcal{H}_\infty, V)$. The pair (G, H) is called a multiplicity free pair if $\mathrm{m}(\pi, \tau) \leq 1$ for all π and τ. We show: if every π has a distribution character, then (G, H) is a multiplicity free pair if and only if $(G \times H, \mathrm{diag}(H \times H))$ is a generalized Gelfand pair.
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1. Introduction

In a recent paper [5], we have defined the notions of multiplicity free pair and strong multiplicity free pair. The relation between the two notions remained unsolved. In this paper we show that, under mild conditions on the groups, the notions are equivalent. The setting is as follows.

Let G be a Lie group with finitely many connected components and let H be a closed subgroup of G. We shall assume that both G and H are unimodular. Let π be an irreducible unitary representation of G on a Hilbert space $\mathcal{H}$ and let τ be one of H on V. Denote by $\mathcal{H}_\infty$ the space of C^∞ vectors for π. Then $\mathcal{H}_\infty$ is a Fréchet space and G acts on $\mathcal{H}_\infty$ by restriction of each $\pi(g)$ $(g \in G)$ from $\mathcal{H}$ to $\mathcal{H}_\infty$. Denote by

$$\mathrm{Hom}_H(\mathcal{H}_\infty, V)$$

the space of continuous linear mappings $A : \mathcal{H}_\infty \to V$ commuting with the H-actions:

$$A\,\pi(h) = \tau(h)\,A \quad (h \in H),$$

and set

$$\mathrm{m}(\pi, \tau) = \dim \mathrm{Hom}_H(\mathcal{H}_\infty, V).$$

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The pair (G, H) is called a multiplicity free pair if $m(\pi, \tau) \leq 1$ for all π and all τ. Recall, see [5], that (G, H) is said to be a strong multiplicity free pair if the pair $(G \times H, \operatorname{diag}(H \times H))$ is a generalized Gelfand pair. If every π has a distribution character, then both definitions are equivalent. This is the main result of this paper. The proof follows closely the treatment of generalized Gelfand pairs in [6], Chapter 8. See also [4].

The class of groups G that satisfy the assumption (distribution characters exist) includes:

(a) compact and abelian groups,

(b) connected semi-simple Lie groups with finite center,

(c) connected, simply-connected, nilpotent Lie groups,

(d) semi-direct products of a compact subgroup and a closed abelian normal subgroup.

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2. Invariant Hilbert subspaces of $D^*(G, H; \tau)$

Let G be a real Lie group with finitely many connected components and let H be a closed subgroup of G. Throughout this paper we shall assume that G and H are unimodular: both G and H admit a bi-invariant Haar measure. Let us fix Haar measures dg on G, dh on H and a G-invariant measure dx on $X = G/H$ in such a way that $dg = dh\, dx$.

Let π be a (continuous) unitary representation of G on a Hilbert space $\mathcal{H}$ and let $\mathcal{H}_\infty$ denote the Fréchet space of C^∞ vectors for π. Clearly $\mathcal{H}_\infty$ is G-invariant. The corresponding representation of G on $\mathcal{H}_\infty$ is called π_∞.

Denote by $\mathcal{H}_{-\infty}$ the anti-dual of $\mathcal{H}_\infty$ endowed with the strong topology. The group G acts on $\mathcal{H}_{-\infty}$ as well. The corresponding representation is called $\pi_{-\infty}$.

We shall take all scalar products anti-linear in the first and linear in the second factor

The following remarkable lemma, due to Dixmier and Malliavin, see [2], is of great help in this paper. Let us write $D(G)$ for the space of C_c^∞ functions on G.

Lemma 2.1. (Decomposition Lemma)

(a) *Any function $\varphi \in D(G)$ can be written as a finite sum of functions of the form $\varphi_1 * \varphi_2$ where $\varphi_1, \varphi_2 \in D(G)$ and 'star' means convolution product.*

(b) *let π be a continuous representation of G on a Fréchet space $\mathcal{H}$. Then $\mathcal{H}_\infty$ is the linear span of the vectors $\pi(\varphi)\,\xi$ where $\varphi \in D(G)$, $\xi \in \mathcal{H}$.*

Fix an *irreducible* unitary representation τ of H on a Hilbert space V.

Denote by $D(G, V)$ the space of C_c^∞ mappings $\varphi : G \to V$, endowed with its usual topology, and by $D^*(G, V)$ its anti-dual. One calls $D^*(G, V)$ the space of V-valued distributions on G.

Denote by $D(G, H; \tau)$ the space of C^∞ mappings $\varphi : G \to V$ satisfying

(i) Supp φ is compact mod H,
(ii) $\varphi(gh) = \tau(h^{-1})\,\varphi(g) \quad (h \in H, g \in G)$.
The space $D(G,H;\tau)$ is an LF-space, an inductive limit of a strictly increasing sequence of Fréchet spaces. Let $D^*(G,H;\tau)$ be its anti-dual.

Set $\mathbf{P}$ for the 'projection' $D(G,V) \to D(G,H;\tau)$ defined by

$$(\mathbf{P}\varphi)(g) = \int_H \tau(h)\,\varphi(gh)\,dh \quad (\varphi \in D(G,V)).$$

Then $\mathbf{P}$ is a continuous, open and surjective mapping. Define $\mathbf{Q} = \mathbf{P}^*$. So $\mathbf{Q} : \mathbf{D}^*(\mathbf{G},\mathbf{H};\tau) \to \mathbf{D}^*(\mathbf{G},\mathbf{V})$. Then $\mathbf{Q}$ is injective and a topological linear isomorphism onto $\mathrm{Im}\,(\mathbf{Q})$. Clearly

$$\begin{aligned}\mathrm{Im}\,(\mathbf{Q}) \;&=\; \{T \in D^*(G,V) : \langle \varphi,\, r(h^{-1})\,T\rangle = \langle r(h)\varphi,\, T\rangle \\ &\quad = \langle \varphi,\, \tau(h)\,T\rangle \text{ for all } \varphi \in D(G,V) \text{ and } h \in H\},\end{aligned} \tag{1}$$

where $r(h)$ means right-translation by $h \in H$.

Denote by $L^2(G,H;\tau)$ the closure of $D(G,H;\tau)$ with respect to the norm

$$\|\varphi\| = (\int_X \|\varphi(x)\|^2\,dx)^{1/2} \quad (\varphi \in D(G,H;\tau)).$$

Notice that the function on G given by $g \mapsto \|\varphi(g)\|$ (norm taken in V) is actually a function on $X = G/H$. The space $L^2(G,H;\tau)$ is a Hilbert space. The group G acts on $L^2(G,H;\tau)$ by means of left-translations, which gives rise to a unitary representation of G. Since $D(G,H;\tau)$ is a dense subspace of $L^2(G,H;\tau)$, we have a natural G-equivariant continuous linear injection

$$L^2(G,H;\tau) \to D^*(G,H;\tau).$$

Denote by $\mathrm{Hom}(\mathcal{H}_\infty, V)$ the space of continuous linear mappings $L : \mathcal{H}_\infty \to V$, and by $\mathrm{Hom}_G\,(\mathcal{H}, D^*(G,V))$ the space of all continuous linear mappings $A : \mathcal{H} \to D^*(G,V)$ commuting with the G-actions. There is a canonical linear mapping

$$\Phi_0 : \mathrm{Hom}\,(\mathcal{H}_\infty, V) \to \mathrm{Hom}_G\,(\mathcal{H}, D^*(G,V)).$$

Indeed, if $L \in \mathrm{Hom}\,(\mathcal{H}_\infty, V)$, then $A = \Phi_0(L)$ is given by

$$\begin{aligned}(A\xi)(\varphi \otimes v) \;&=\; \langle \varphi \otimes v,\, A\xi\rangle \\ &=\; \langle v,\, L(\pi(\widetilde{\varphi})\xi)\rangle\end{aligned}$$

$(\varphi \in D(G),\, v \in V,\, \xi \in \mathcal{H})$.

Theorem 2.2. *The mapping Φ_0 is a (topological) isomorphism.*

Proof. We shall give the inverse of Φ_0, called Ψ_0.

Let $A \in \mathrm{Hom}_G\,(\mathcal{H}, D^*(G,V))$ and set

$$\langle v,\, \Psi_0(A)\left(\sum_{i=1}^n \pi(\varphi_i)\xi_i\right)\rangle = \sum_{i=1}^n \langle \widetilde{\varphi}_i \otimes v,\, A\xi_i\rangle,$$

where $\varphi_i \in D(G)$, $\xi_i \in \mathcal{H}$ $(i = 1, \dots, n)$ and $v \in V$.

This is a well-defined definition. Indeed, let $\sum_{i=1}^{n} \pi(\varphi_i)\xi_i = 0$. Take $\psi \in D(G)$. Then

$$\begin{aligned}\sum_{i=1}^{n} \langle (\widetilde{\varphi}_i * \psi) \otimes v,\, A\xi_i \rangle &= \sum_{i=1}^{n} \langle \ell(\widetilde{\varphi}_i)\varphi \otimes v,\, A\xi_i \rangle = \sum_{i=1}^{n} \langle \psi \otimes v,\, \ell(\varphi_i)\, A\xi_i \rangle \\ &= \sum_{i=1}^{n} \langle \psi \otimes v,\, A\pi(\varphi_i)\xi_i \rangle = 0.\end{aligned}$$

Here ℓ stands for the representation of G by means of left-translations. Now let ψ tend to the δ-function. ∎

Let us write $\mathrm{Hom}_H(\mathcal{H}_\infty, V)$ for the subspace of $\mathrm{Hom}(\mathcal{H}_\infty, V)$ consisting of mappings L satisfying

$$L\pi_\infty(h) = \tau(h)L \quad (h \in H).$$

Furthermore, let Φ denote the restriction of Φ_0 to $\mathrm{Hom}_H(\mathcal{H}_\infty, V)$. The following proposition is obvious.

Proposition 2.3. *The mapping Φ is a topological linear isomorphism of the space $\mathrm{Hom}_H(\mathcal{H}_\infty, V)$ onto the space $\mathrm{Hom}_G(\mathcal{H}, D^*(G, H; \tau))$.*

Clearly, $\mathrm{Hom}_H(\mathcal{H}_\infty, V)$ is anti-isomorphic to $\mathrm{Hom}_H(V, \mathcal{H}_{-\infty})$. Therefore, we have the following reciprocity.

Proposition 2.4. *The space $\mathrm{Hom}_G(\mathcal{H}, D^*(G, H; \tau))$ is anti-isomorphic to the space $\mathrm{Hom}_H(V, \mathcal{H}_{-\infty})$.*

We shall say that π can be realized on a Hilbert subspace of $D^*(G, H; \tau)$ if there is a continuous linear injection

$$j : \mathcal{H} \to D^*(G, H; \tau)$$

such that $j\pi(g) = \ell(g)j$ for all $g \in G$. The space $j(\mathcal{H})$ is called an *invariant Hilbert subspace* of $D^*(G, H; \tau)$.

To the linear injection j we can attach, by Proposition 2.3, an element $L_\pi \in \mathrm{Hom}_H(\mathcal{H}_\infty, V)$ such that

$$\langle \mathbf{P}(\varphi \otimes v),\, j(\xi) \rangle = \langle v,\, L_\pi(\pi(\widetilde{\varphi})\xi) \rangle$$

$(\varphi \in D(G),\, v \in V,\, \xi \in \mathcal{H})$. Hence,

$$\langle j^*(\mathbf{P}(\varphi \otimes v),\, \xi \rangle = \langle \pi_{-\infty}(\varphi) L_\pi^* v,\, \xi \rangle.$$

Or, briefly,

$$j^*\left(\mathbf{P}(\varphi \otimes v)\right) = \pi_{-\infty}(\varphi) L_\pi^* v \quad (\varphi \in D(G),\, v \in V). \tag{2}$$

Clearly, $j^* : D^*(G, H; \tau) \to \mathcal{H}$ has a dense image. We call L_π^* (or L_π) **cyclic** in this case. So L_π^* is cyclic if and only if

$$\mathrm{span}\,\{\pi_{-\infty}(\varphi) L_\pi^* v : \varphi \in D(G),\, v \in V\} \text{ is dense in } \mathcal{H}.$$

Theorem 2.5. *Let π be a unitary representation of G on the Hilbert space $\mathcal{H}$. Then π can be realized on a Hilbert subspace of $D^*(G,H;\tau)$ if and only if* $\mathrm{Hom}_H(V,\mathcal{H}_{-\infty})$ *contains cyclic elements. There is a one-to-one correspondence between the cyclic elements L^*_π in* $\mathrm{Hom}_H(V,\mathcal{H}_{-\infty})$ *and the continuous linear injections $j:\mathcal{H}\to D^*(G,H;\tau)$ satisfying $j\pi(g)=\ell(g)j$ $(g\in G)$. The correspondence is given by* (2).

Let again π be a unitary representation of G on a Hilbert space $\mathcal{H}$, realized on $D^*(G,H;\tau)$ and let j be the corresponding injection. Denote by L^*_π the cyclic element in $\mathrm{Hom}_H(V,\mathcal{H}_{-\infty})$, defined by Theorem 2.5. Consider

$$\langle j^*\mathbf{P}(\varphi\otimes v),\, j^*\mathbf{P}(\psi\otimes w)\rangle = \langle \pi_{-\infty}(\widetilde{\psi}*\varphi)\, L^*_\pi v,\, L^*_\pi w\rangle,$$

where $\varphi,\psi\in D(G)$ and $v,w\in V$.

The right-hand side is also equal to

$$\langle L_\pi\pi_{-\infty}(\widetilde{\psi}*\varphi)\, L^*_\pi v,\, w\rangle.$$

Proposition 2.6. *Set*

$$\langle T,\,\varphi\rangle = L_\pi\,\pi_{-\infty}(\varphi)\,L^*_\pi \quad (\varphi\in D(G)). \tag{3}$$

Then T is an End (V)*-valued linear distribution on G, where* End (V) *is the algebra of bounded linear operators on V provided with the weak topology. The distribution T has the following properties:*

(i) $\ell(h)\,r(h')\,T=\tau(h^{-1})\,T\,\tau(h')\quad (h,h'\in H)$.

(ii) *T is positive-definite, i.e.*

$$\sum_{i=1}^{n}\langle\langle T,\widetilde{\varphi}_i*\varphi_j\rangle\, v_i,\, v_j\rangle\ge 0$$

for all n-tuples $\varphi_1,\dots,\varphi_n\in D(G)$ and $v_1,\dots,v_n\in V$, and all $n\in\mathbb{N}$.

(iii) *The form*

$$(\mathbf{P}(\varphi\otimes v),\,\mathbf{P}(\psi\otimes w))\mapsto\langle\langle T,\,\widetilde{\psi}*\varphi\rangle\, v,\, w\rangle\quad(\varphi,\,\psi\in D(G); v,w\in V)$$

can be extended to a continuous sesqui-linear form on $D(G,H;\tau)$.

We call T the *reproducing distribution* of π (or of $(\pi,\mathcal{H},j)$).

Given an End (V)-valued distribution T, satisfying the properties (i), (ii), (iii) of Proposition 2.6, we can easily construct a G-invariant Hilbert subspace of $D^*(G,H;\tau)$ with T as reproducing distribution. Indeed, let B denote the sesqui-linear form on $D(G,H;\tau)$ as in property (iii). This form is positive-definite by (ii). Denote by $\mathcal{W}_0$ the subspace of $D(G,H;\tau)$ consisting of elements of length zero: $\mathcal{W}_0=\{\varphi : B(\varphi,\varphi)=0\}$ and define $\mathcal{H}$ as the completion of $D(G,H;\tau)/\mathcal{W}_0$

with respect to the norm $\varphi \mapsto B(\varphi, \varphi)^{1/2}$. Then $\mathcal{H}$ is a Hilbert space and G acts unitarily on $\mathcal{H}$. Let $j^* : D(G,H;\tau) \to \mathcal{H}$ be the natural projection. Then clearly

$$\|j^*\mathbf{P}(\varphi \otimes v)\|^2 = \langle\langle T, \widetilde{\varphi} * \varphi\rangle\, v,\, v\rangle$$

for $\varphi \in D(G)$ and $v \in V$.

Let π_1 and π_2 be two unitary representations on $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively, realized on $D^*(G,H;\tau)$ by means of the G-equivariant injections j_1 and j_2. Denote the associated reproducing distributions by T_1 and T_2. Assume that $T_1 = T_2$. Then we have $\|j_1^*\varphi\|^2 = \|j_2^*\varphi\|^2$ for all $\varphi \in D(G,H;\tau)$, thus U given by $U(j_1^*\varphi) = j_2^*\varphi$ is well-defined and can be extended to a unitary operator from $\mathcal{H}_1$ onto $\mathcal{H}_2$ commuting with the actions of G. Moreover, $j_1 = j_2 \circ U$. We say that the triples $(\pi_1, \mathcal{H}_1, j_1)$ and $(\pi_2, \mathcal{H}_2, j_2)$ are equivalent. We shall alternatively call the G-invariant Hilbert subspaces $j_1(\mathcal{H}_1)$ and $j_2(\mathcal{H}_2)$ equivalent.

Summarizing we have:

Proposition 2.7. *The correspondence that associates to a G-invariant Hilbert subspace of $D^*(G,H;\tau)$ its reproducing distribution is a bijection between the set of equivalence classes of G-invariant Hilbert subspaces of $D^*(G,H;\tau)$ and the set of* End (V)*-valued distributions on G satisfying the properties of* Proposition 2.6.

Denote by Γ_G^τ the cone of End (V)-valued distributions T on G satisfying the properties (i), (ii) and (iii) of Proposition 2.6. An element $T \in \Gamma_G^\tau$ is said to be *extremal* if for any $T_1 \in \Gamma_G^\tau$ with $T_1 \leq T$ one has $T_1 = \alpha T$ for some scalar $\alpha \geq 0$. Here $T_1 \leq T$ means that $T - T_1 \in \Gamma_G^\tau$. One has the following proposition.

Proposition 2.8. *Let π be a unitary representation realized on a Hilbert subspace $j(\mathcal{H})$ of $D^*(G,H;\tau)$ and let T be its reproducing distribution. Then the following two statements are equivalent:*

(i) *π is irreducible.*

(ii) *T is extremal.*

The proof is similar to case $\tau = id$, see [6], Proposition 8.2.3.

Denote by $\operatorname{ext}\Gamma_G^\tau$ the set of extremal points of Γ_G^τ. Then one has, similar to case $\tau = id$ (see [6], Proposition 8.2.4):

Proposition 2.9. *There is a parametrization $s \mapsto T_s$ of $\operatorname{ext}\Gamma_G^\tau$ where $s \in S$ (S being a Hausdorff space) with the following property. For every $T \in \Gamma_G^\tau$ there is a (not necessarily unique) Radon measure m on S such that*

$$\langle T, \varphi\rangle = \int_S \langle T_s, \varphi\rangle\, dm(s)$$

for all $\varphi \in D(G)$. Here convergence is meant with respect to the weak topology of End (V).

3. Multiplicity one theorems

We keep the notations of Section 2. One has, similar to the case $\tau = id$, see [6], Proposition 8.3.2:

Theorem 3.1. *The following statements are equivalent.*

(i) $\dim \mathrm{Hom}_H(V, \mathcal{H}_{-\infty}) \leq 1$ *for all irreducible unitary representations* π *of* G *on* $\mathcal{H}$.

(ii) *For any unitary representation* π *of* G *on a Hilbert space* $\mathcal{H}$*, that can be realized on a Hilbert subspace of* $D^*(G, H; \tau)$*, the commutant of* $\pi(G)$ *in* End $(\mathcal{H})$ *is abelian.*

(iii) *For every* $T \in \Gamma_G^\tau$ *there exists a unique Radon* m *measure on* S *such that*

$$\langle T, \varphi \rangle = \int_S \langle T_s, \varphi \rangle \, dm(s)$$

for all $\varphi \in D(G)$.

It is convenient to write

$$\mathrm{m}\,(\pi, \tau) = \dim \mathrm{Hom}_H(V, \mathcal{H}_{-\infty}). \tag{4}$$

Recall that a pair (G, H) is said to be a *generalized Gelfand pair* if $\mathrm{m}\,(\pi, id) \leq 1$ for all irreducible unitary representations π of G.

Definition 3.2. The pair (G, H) is called a multiplicity free pair if $\mathrm{m}\,(\pi, \tau) \leq 1$ for any irreducible unitary representation π of G and any irreducible unitary representation τ of H.

In a previous paper ([5]) we have defined the notion of *strong multiplicity free pair.* We repeat it.

Definition 3.3. The pair (G, H) is said to be a strong multiplicity free pair if the pair $(G \times H, \mathrm{diag}\,(H \times H))$ is a generalized Gelfand pair.

We shall investigate under which conditions on G and H both definitions are equivalent. We need some preparation. Recall that G is said to be a type I group if each factor representation of G (i.e., each unitary representation such that the von Neumann algebra generated by $\pi(G)$ is a factor) is a, possibly infinite, multiple of a uniquely determined irreducible representation. The same definition applies to H. Recall that every G (or H) for which any irreducible unitary representation has a distribution character, is a type I group (see [1], 13.9.4).

One has the following result (see [1], 13.1.8):

Lemma 3.4. *Let* G *or* H *be a type I group. Then any irreducible unitary representation of* $G \times H$ *is of the form* $\pi \,\widehat{\otimes}_2\, \tau$ *where* π *is an irreducible unitary representation of* G *and* τ *an irreducible unitary representation of* H.

Denote by $\overline{V}$ the Hilbert space V with the new scalar multiplication $\alpha\cdot v = \overline{\alpha}\, v$ ($\alpha \in \mathbb{C}, v \in V$). Let $v \mapsto \overline{v}$ be the natural anti-linear identity mapping $V \to \overline{V}$ and define the scalar product in $\overline{V}$ by $\langle \overline{v}, \overline{w}\rangle = \overline{\langle v, w\rangle}$ ($v, w \in V$). Set $\overline{\tau}(h)\overline{v} = \overline{\tau(h)v}$ ($v \in V, h \in H$). Then $\overline{\tau}$ is again an irreducible unitary representation of H. Actually $\overline{\tau}$ is equivalent to the contragredient of τ.

We now arrive at our main result.

Theorem 3.5.

(i) *Let (G,H) be a strong multiplicity free pair and assume that any irreducible unitary representation of G has a distribution character. Then (G,H) is a multiplicity free pair.*

(ii) *Let (G,H) be a multiplicity free pair and assume that G or H is a type I group. Then (G,H) is a strong multiplicity free pair.*

Proof. (i) Let π be an irreducible unitary representation of G on the Hilbert space $\mathcal{H}$ and let $j : \mathcal{H} \to D^*(G,H;\tau)$ be a continuous linear G-equivariant injection. Denote by T the $\mathrm{End}\,(V)$-valued reproducing distribution. Assume that $\mathrm{trace}\,\langle T, \varphi\rangle$ exists for all $\varphi \in D(G)$, which is certainly true if π has a distribution character. Indeed, apply (2) and the Decomposition Lemma. Then $\pi\,\widehat{\otimes}_2\,\overline{\tau}$ can be realized on $D^*(G \times H/\mathrm{diag}\,(H \times H))$ by means of the injection j_1, (formally) defined on simple tensors by

$$j_1(\xi \otimes \overline{v})(g,h) = \langle \tau(h^{-1})v,\, j(\xi)(g)\rangle \quad (\xi \in \mathcal{H},\ \overline{v} \in \overline{V}).$$

Its reproducing distribution is precisely $\mathrm{trace}\, T$. Indeed, let $\varphi \in D(G)$. Take $\chi \in D(G)$ such that $\int_H \chi(gh)\,dh = 1$ for g in a neighbourhood of $\mathrm{Supp}\,\varphi$. Then the function ψ defined by

$$\psi(g,h) = \varphi(gh^{-1})\,\chi(g) \quad (g \in G,\ h \in H),$$

is an element of $D(G \times H)$ with the property

$$\int_H \psi(gh_0, hh_0)\,dh_0 = \varphi(gh^{-1}) \quad (g \in G,\ h \in H).$$

Consider now

$$\int_{G\times H} \overline{\langle \tau(h^{-1})v,\, j(\xi)(g)\rangle}\,\varphi(gh^{-1})\,\chi(g)\,dg\,dh.$$

This expression is equal to

$$\begin{aligned} &\int_{G\times H} \langle j(\xi)(g),\, \tau(h^{-1})v\rangle\,\varphi(gh^{-1})\,\chi(g)\,dg\,dh \\ =\ &\int_G \langle j(\xi))(g),\, v\rangle\,\varphi(g)\,dg \\ =\ &\langle \xi,\, j^*\mathbf{P}(\varphi \otimes v)\rangle = \langle \xi,\, \pi_{-\infty}(\varphi)\,L_\pi^*\,v\rangle. \end{aligned}$$

Let $\{e_i\}$ be an orthonormal basis of $\mathcal{H}$ and $\{f_j\}$ one of V. Notice that both $\mathcal{H}$ and V are separable. Then we obtain

$$\begin{aligned} &\langle \xi,\, \pi_{-\infty}(\varphi)\,L_\pi^*\,v\rangle \\ =\ &\langle \xi \otimes \overline{v},\, \sum_{i,j} \langle e_i,\, \pi_{-\infty}(\varphi)\,L_\pi^*\,f_j\rangle\,(e_i \otimes \overline{f}_j)\rangle. \end{aligned}$$

In order that j_1^* (and hence j_1) is continuous, we have to check that

$$\sum_{i,j} |\langle e_i,\, \pi_{-\infty}(\varphi)\, L_\pi^*\, f_j\rangle|^2$$

is finite and depends continuously on φ. This is clearly the case since this expression equals

$$\sum_j \|\pi_{-\infty}(\varphi)\, L_\pi^*\, f_j\|^2, \tag{5}$$

which, in its turn, is equal to

$$\operatorname{trace} L_\pi\, \pi_{-\infty}(\widetilde{\varphi} * \varphi)\, L_\pi^* = \operatorname{trace} \langle T,\, \widetilde{\varphi} * \varphi\rangle.$$

Since, by assumption, j_1 is determined up to multiplication by scalars, the same then clearly holds for j. Hence (G,H) is a multiplicity free pair.

(ii) By Lemma 3.3 any irreducible unitary representation of $G \times H$ is of the form $\pi \mathbin{\widehat{\otimes}_2} \overline{\tau}$. Assume that $\pi \mathbin{\widehat{\otimes}_2} \overline{\tau}$ can be realized on $D^*(G \times H/\mathrm{diag}\,(H \times H))$ and let λ denote a non-zero $\mathrm{diag}\,(H \times H)$-fixed linear form on $(\mathcal{H} \mathbin{\widehat{\otimes}_2} \overline{V})_\infty$. Set for $\xi \in \mathcal{H}_\infty$,

$$A(\xi)(v) = \lambda(\xi \otimes \overline{v}) \quad (v \in V_\infty).$$

Then $A(\xi)$ is a continuous anti-linear form on V_∞, so an element of $V_{-\infty}$. We have

$$A : \mathcal{H}_\infty \to V_{-\infty},$$

a continuous linear mapping commuting with the H-actions. Then, by the Decomposition Lemma, $A(\mathcal{H}_\infty) \subset V_\infty$ and, by the closed graph theorem, $A : \mathcal{H}_\infty \to V_\infty \to V$ is continuous. Since A is determined up to multiplication by scalars, the same holds for λ. Therefore, (G,H) is a strong multiplicity free pair. ∎

Remark 3.6. Notice that Theorem 3.5 holds if H is compact, and that in that case no further conditions are needed. This follows easily from the proof of the theorem, since V is finite-dimensional in this case and hence (5) is a finite sum. We thus obtain a new proof of a known result, see [3].

The proof of Theorem 3.5 also easily implies the following proposition.

Proposition 3.7. *If the pair (G,H) is a strong multiplicity free pair, then for any irreducible unitary representation π of G for which its distribution character exists, one has*

$$\mathrm{m}\,(\pi,\, \tau) \leq 1$$

for any irreducible unitary representation τ of H.

Recall that if G is a type I group, then, roughly speaking, any irreducible unitary representation of G that occurs in the Plancherel formula of G, has a distribution character.

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