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Complex Structures on Quasi-filiform Lie Algebras

Lucia García Vergnolle * and Elisabeth Remm

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Abstract. The aim of this article is to classify quasi-filiform nilpotent Lie algebras $\mathfrak{g}$, that is with nilindex $\dim \mathfrak{g} - 2$, admitting a complex structure. Note that the non-existence of complex structures over nilpotent Lie algebras of maximal class, also called filiform, has already been proved in [7].
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1. Complex Structures on Lie algebras

Let $\mathfrak{g}$ be a real, even-dimensional Lie algebra.

Definition 1.1. A complex structure on $\mathfrak{g}$ is an endomorphism $J : \mathfrak{g} \to \mathfrak{g}$ such that:

1. $J^2 = -Id$,
2. $N(J)(X,Y) = [J(X), J(Y)] - [X,Y] - J([J(X),Y]) - J([X,J(Y)]) = 0$ for all $X, Y \in \mathfrak{g}$ (Nijenhuis condition).

Any such structure defines an invariant complex structure on a real Lie group G whose real Lie algebra is $\mathfrak{g}$. It should be mentioned that nilpotent Lie algebras provided with a complex structure in dimension less than or equal to 6 have been completely classified ([9] and [10]). The main general result concerns filiform Lie algebras. We recall that $\mathfrak{g}$ is a filiform Lie algebra if its nilindex is equal to $\dim(\mathfrak{g}) - 1$.

Proposition 1.2. *[7] If $\mathfrak{g}$ is an even-dimensional filiform Lie algebra, then $\mathfrak{g}$ admit no complex structure.*

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In [7], the authors have shown at first the non-existence of complex structures on the filiform Lie algebra $\mathfrak{L}_{2n}$ $(n \geq 2)$ defined in the basis $\{X_0, X_1, \dots, X_{2n}\}$ by the brackets:

$$\begin{cases} [X_0, X_i] = X_{i+1}, & 1 \leq i \leq 2n-1, \\ [X_i, X_j] = 0, & i, j \neq 0. \end{cases}$$

Therefore, they generalized this result to all $2n$-dimensional filiform Lie algebras by noting that the existence of a complex structure implies the decomposition $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ where $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are n-dimensional complex subalgebras. Such a decomposition is impossible on $\mathfrak{L}_{2n}$ and consequently on any deformation of this algebra. As every $2n$-dimensional filiform Lie algebras is a deformation of $\mathfrak{L}_{2n}$, they deduce the final result.

This proposition has also been proved in [4] from a completely different point of view based on the notion of generalized complex structures. Using this new approach, we will determine the existence of complex structures on other classes of nilpotent Lie algebras.

2. Generalized complex structures on Lie algebras

2.1. Definitions and link with complex structures.

Generalized complex structures can be defined in the general context of smooth manifolds, nevertheless, throughout this paper we study generalized complex structures on real Lie groups. We recall the definition in this context, which is purely algebraic.

Let $\mathfrak{g}$ be a real $2n$-dimensional Lie algebra and $\mathfrak{g}^*$ its dual space which can be identified with the space of left-invariant differential 1-forms on a connected Lie group with Lie algebra $\mathfrak{g}$. We consider the coboundary operator d mapping $\mathfrak{g}^*$ onto $\Lambda^2(\mathfrak{g}^*)$ and defined by $d\alpha(X,Y) = -\alpha[X,Y]$ where $\alpha \in \mathfrak{g}^*$ and $[,]$ is the Lie bracket of $\mathfrak{g}$. We define a bracket on $\mathfrak{g} \oplus \mathfrak{g}^*$ called the Courant bracket which is generally written:

$$[X+\xi, Y+\eta]_c = [X,Y] + \mathcal{L}_X\eta - \mathcal{L}_Y\xi - \frac{1}{2}d(I_X\eta - I_Y\xi),$$

where $X, Y \in \mathfrak{g}$, $\xi, \eta \in \mathfrak{g}^*$ and $I_X\eta$ represents the inner product of X on η. As $I_X\eta - I_Y\xi$ is a constant, then $d(I_X\eta - I_Y\xi) = 0$.

This operation is skew-symmetric and satisfies Jacobi identity (remark that, in the context of smooth manifolds, this bracket is defined in the sum of tangent and cotangent bundles and does not satisfy Jacobi identity). Thereby $\mathfrak{g} \oplus \mathfrak{g}^*$ endowed with the Courant bracket is a $4n$-dimensional real Lie algebra. It is moreover a quadratic Lie algebra. Indeed, the space $\mathfrak{g} \oplus \mathfrak{g}^*$ admits a natural non-degenerate inner product of signature $(2n, 2n)$ defined by:

$$\langle X+\xi, Y+\eta \rangle = \frac{1}{2}(\xi(Y) + \eta(X)).$$

Definition 2.1. Let $\mathfrak{g}$ be a real Lie algebra of dimension $2n$. A generalized complex structure on $\mathfrak{g}$ is a linear endomorphism $\mathcal{J}$ of $\mathfrak{g} \oplus \mathfrak{g}^*$ such that:

1. $\mathcal{J}^2 = -Id$,
2. $\mathcal{J}$ is orthogonal with respect to the scalar product $\langle\ ,\ \rangle$, that is,
$$\langle \mathcal{J}(X+\xi), \mathcal{J}(Y+\eta)\rangle = \langle X+\xi, Y+\eta\rangle \quad \forall X, Y \in \mathfrak{g}, \forall \xi, \eta \in \mathfrak{g}^*,$$
3. The $+i$-eigenspace L of $\mathcal{J}$ is required to be involutive with respect to the Courant bracket, that is $[L, L]_c \subset L$.

The subalgebra L of $\mathfrak{g} \oplus \mathfrak{g}^*$ is an isotropic space, which means that
$$\langle X+\xi, Y+\eta\rangle = 0,$$
for all $X+\xi, Y+\eta \in L$. Since its dimension is $2n$, L is maximal isotropic. We consider the projection of L on $\mathfrak{g}$ and denote by k the codimension of this projection. It is clear that
$$0 \leq k \leq n.$$

Definition 2.2. The type of a generalized complex structure is the codimension of the projection of L on $\mathfrak{g}$.

Example 2.3. Let $\mathfrak{g}$ be a real Lie algebra of dimension $2n$ provided with a complex structure $J : \mathfrak{g} \to \mathfrak{g}$ satisfying $J^2 = -Id$ and the Nijenhuis condition $N(J)(X, Y) = 0$ for all $X, Y \in \mathfrak{g}$. We define a generalized complex structure associated to $\mathcal{J}$
$$\mathcal{J}_J : \mathfrak{g} \oplus \mathfrak{g}^* \to \mathfrak{g} \oplus \mathfrak{g}^*$$
by setting
$$\mathcal{J}_J(X+\xi) = -J(X) + J^*(\xi), \quad \forall X \in \mathfrak{g},\ \forall \xi \in \mathfrak{g}^*.$$
The conditions 1 and 2 of Definition 2.1 are easy to check. It remains to prove the condition 3. We denote by T_+ (resp. T_-) the eigenspace of J associated to the eigenvalue $+i$ (resp. $-i$) so the $+i$-eigenspace of $\mathcal{J}_J$ is given by:
$$L = T_- \oplus (T_+)^*.$$
Remark that the involutivity of L with respect to the Courant bracket means that T_- is a subalgebra of $\mathfrak{g}$. Therefore $\mathcal{J}_J$ is a generalized complex structure of type n.

Example 2.4. Let $\mathfrak{g}$ be a $2n$-dimensional real Lie algebra endowed with a symplectic structure ω, that is ω is a skew-symetric 2-form satisfying
$$\begin{cases} \omega^n = \omega \wedge \omega \cdots \wedge \omega \neq 0, \\ d\omega(X, Y, Z) = \omega([X, Y], Z) + \omega([Y, Z], X) + \omega([Z, X], Y) = 0. \end{cases} \tag{1}$$
This form can be identified to an isomorphism, also named ω:
$$\omega : \mathfrak{g} \longrightarrow \mathfrak{g}^*$$

and given by $\omega(X) = I_X\omega$. Thereby, we define the generalized complex structure

$$\mathcal{J}_\omega : \mathfrak{g} \oplus \mathfrak{g}^* \to \mathfrak{g} \oplus \mathfrak{g}^*$$

by putting

$$\mathcal{J}_\omega(X + \xi) = \omega(X) + \omega^{-1}(\xi), \quad \forall X \in \mathfrak{g},\ \forall \xi \in \mathfrak{g}^*.$$

This generalized complex structure is of type 0 since its $+i$-eigenspace is

$$L = \{X - i\, I_X\omega\,,\ X \in \mathfrak{g} \otimes \mathbb{C}\}.$$

In the two previous examples, complex and symplectic geometry appear as extremal cases of generalized complex structures. According to [4], any generalized complex structure of type k can be written as a direct sum of a complex structure of dimension k and a symplectic structure of dimension $2n - 2k$. We deduce that every generalized complex structure of the type 0 arises from a complex structure and every generalized complex structure of the type n is given by a symplectic form.

2.2. Spinorial approach.

Let T be the tensor algebra of $\mathfrak{g} \oplus \mathfrak{g}^*$ and I the ideal generated by $\{X + \xi \otimes X + \xi - \langle X + \xi, X + \xi\rangle \cdot 1\ ,\ X + \xi \in \mathfrak{g} \oplus \mathfrak{g}^*\}$. The factor algebra $C = T/I$ is called the Clifford algebra of $\mathfrak{g} \oplus \mathfrak{g}^*$ associated to the scalar product $\langle\ ,\ \rangle$. As C is a simple associative algebra, all its simple representations are equivalent ([1]). A representation $\phi : C \to End_{\mathbb{R}}(S)$ of the Clifford algebra on the vector space S is called a spin representation if it is simple. In this case, S is the space of spinors. Henceforth, we will consider $S = \wedge\mathfrak{g}^*$ with the spin representation given by the Clifford action:

$$\begin{array}{llll} \circ : & \mathfrak{g} \oplus \mathfrak{g}^* \times \wedge\mathfrak{g}^* & \to & \wedge\mathfrak{g}^* \\ & (X + \xi, \rho) & \mapsto & (X + \xi) \circ \rho = i_X\rho + \xi \wedge \rho. \end{array}$$

If $\rho \in \wedge\mathfrak{g}^*$ a nonzero spinor, we define its null space $L_\rho \subset \mathfrak{g} \oplus \mathfrak{g}^*$ as follows:

$$L_\rho = \{X + \xi \in \mathfrak{g} \oplus \mathfrak{g}^* :\ (X + \xi) \circ \rho = 0\}.$$

It is clear that L_ρ is an isotropic space. We say that ρ is a pure spinor when L_ρ is maximal isotropic. Conversely, if L is a maximal isotropic space, we can consider the set U_L of pure spinors ρ such that $L = L_\rho$. If L is the $+i$-eigenspace of a generalized complex structure, it is proved that the set of pure spinors U_L is a line generated by:

$$\rho = \Omega\ e^{B + i\omega}$$

with B, ω real 2-forms and $\Omega = \theta_1 \wedge \cdots \wedge \theta_k$, where $\theta_1, \ldots, \theta_k$ are complex forms. Moreover, we deduce from (Proposition III.2.3) that $L \cap \overline{L} = \{0\}$ if and only if:

$$\omega^{2n-2k} \wedge \Omega \wedge \overline{\Omega} \neq 0, \tag{2}$$

where L is the $+i$-eigenspace of the generalized complex structure. It is also proved that the involutivity condition among L is equivalent to the following integrability condition:

$$\exists X + \xi \in \mathfrak{g} \oplus \mathfrak{g}^* \ /\ d\rho = (X + \xi) \circ \rho. \tag{3}$$

2.3. Application to nilpotent Lie algebras.

Let us consider a real nilpotent Lie algebra $\mathfrak{g}$ of even dimension. The central descending serie is defined by:

$$\begin{cases} \mathfrak{g}^0 & = \mathfrak{g}, \\ \mathfrak{g}^i & = [\mathfrak{g}^{i-1}, \mathfrak{g}] . \end{cases}$$

We denote by m the nilpotency index of $\mathfrak{g}$. In $\mathfrak{g}^*$, we consider now the increasing serie of subspaces V_i where V_i is the annihilator of $\mathfrak{g}^i$, that is to say:

$$\begin{cases} V_0 = \{0\} , \\ V_i = \left\{\varphi \in \mathfrak{g}^* \text{ such that } \forall \mathrm{X} \in \mathfrak{g}^{\mathrm{i}}, \varphi(\mathrm{X}) = 0\right\} . \end{cases}$$

It is clear that $V_m = \mathfrak{g}^*$. Those subspaces can also be expressed as

$$V_i = \{\varphi \in \mathfrak{g}^* \text{ such that } \forall \mathrm{X} \in \mathfrak{g}, \mathrm{I_X d}\varphi \in \mathrm{V_{i-1}}\} .$$

Definition 2.5. Let α be p-form of $\mathfrak{g}$. The nilpotent degree of α, denoted by $\mathrm{nil}(\alpha)$, is the smallest integer i such that $\alpha \in \wedge^p V_i$.

Suppose that $\mathfrak{g}$ admits a generalized complex structure of type k. We shall do a special choice of the forms $\{\theta_1, \dots, \theta_k\}$, at first we order these forms according to their nilpotent degree and we choose them in such a way that $\{\theta_j : \mathrm{nil}(\theta_j) > i\}$ are linearly independent modulo V_i, then the decomposition $\Omega = \theta_1 \wedge \cdots \wedge \theta_k$ satisfies

a) $\mathrm{nil}(\theta_i) \leq \mathrm{nil}(\theta_j)$ for $i < j$,

b) for each i, the forms $\{\theta_j : \mathrm{nil}(\theta_j) > i\}$ are linearly independent modulo V_i.

Such a decomposition will be called appropriate.

Theorem 2.6. *If $\mathfrak{g}$ is a nilpotent Lie algebra provided with a generalized complex structure, the corresponding pure spinor must be a closed differential form.*

Corollary 2.7. *If we choose an appropriate decomposition of Ω, then*

a) $d\theta_i \in \mathcal{I}(\{\theta_j : nil(\theta_j) < nil(\theta_i)\})$. In particular

$$d\theta_i \in \mathcal{I}(\theta_1 \dots \theta_{i-1}).$$

b) If $\dim\left(\frac{V_{j+1}}{V_j}\right) = 1$ then, either there exists θ_i with nilpotent degree j, or no θ_i has nilpotent degree $j+1$.

Remark 2.8. Supppose there exists a $j > 0$ such that:

$$\dim\left(\frac{V_{i+1}}{V_i}\right) = 1, \quad \forall i \geq j;$$

It is a consequence of the previous corollary that if none θ_i has nilpotent degree $s \geq j$ then there can be none with nilpotent degree upper than s. Using this fact, we can find an upper bound for all the nilpotent degrees. From Corollary 2.7, we deduce that $\mathrm{nil}(\theta_1) = 1$. If $j > 1$, we can see that $\mathrm{nil}(\theta_2) \leq j$. Otherwise it would not exist any θ_i of nilpotent degree j and neither of upper degree. This leads to a contradiction with $\mathrm{nil}(\theta_2) > j$. Likewise, we prove by induction that $\mathrm{nil}(\theta_i) \leq j + i - 2$. For $j = 1$, we see in the same way that $\mathrm{nil}(\theta_i) \leq i$.

Theorem 2.9. *Let $\mathfrak{g}$ be a real nilpotent Lie algebra of dimension $2n$ endowed with a generalized complex structure of type $k > 1$. If there exists $j > 0$ such that:*

$$dim\left(\frac{V_{i+1}}{V_i}\right) = 1, \quad \forall i \geq j,$$

then k is bounded above by:

$$k \leq \begin{cases} 2n - nil(\mathfrak{g}) + j - 2 & if \quad j > 1, \\ 2n - nil(\mathfrak{g}) & if \quad j = 1. \end{cases}$$

Proof. Suppose $j > 1$. According to the above remark, $\mathrm{nil}(\theta_k) \leq j + k - 2$ and thus all the $\theta_1, \dots, \theta_k$ belong to V_{j+k-2}. Since $\Omega \wedge \overline{\Omega} \neq 0$, we have

$$\dim V_{j+k-2} \geq 2k.$$

On the other hand, $\dim V_{j+k-2} = 2n - \dim\left(\frac{\mathfrak{g}^*}{V_{j+k-2}}\right)$ and

$$\frac{\mathfrak{g}^*}{V_{j+k-2}} \simeq \frac{V_{\mathrm{nil}(\mathfrak{g})}}{V_{\mathrm{nil}(\mathfrak{g})-1}} \oplus \cdots \oplus \frac{V_{j+k-1}}{V_{j+k-2}},$$

so the dimension of V_{j+k-2} is equal to $2n - \mathrm{nil}(\mathfrak{g}) + j + k - 2$. By replacing in the above inequality we finally obtain:

$$k \leq 2n - \mathrm{nil}(\mathfrak{g}) + j - 2.$$

For $j = 1$, we deduce the required result by using similar arguments and considering that $\mathrm{nil}(\theta_k) \leq k$. ∎

Remark 2.10. **Application to filiform Lie algebras**. The main result stated in [7] for filiform Lie algebras is a consequence of the previous theorem. In fact, by taking $m = 2n - 1$ and $j = 1$ we obtain that $k < 2$. Thereby, there are no generalized complex structures of type n excepted for $n = 1$ but in this case the algebra is abelian. In the next section, we are going to study the quasi-filiform case for which $m = 2n - 2$.

3. Complex structures on quasi-filiform Lie algebras

3.1. Classification of naturally graded quasi-filiform Lie algebras.

Let $\mathfrak{g}$ be nilpotent Lie algebra with nilindex m. This algebra is naturally filtered by the descending sequence of derived ideals:

$$\mathfrak{g}^0 = \mathfrak{g} \supset \mathfrak{g}^1 \supset \mathfrak{g}^2 \supset \cdots \supset \mathfrak{g}^k \supset \cdots \supset \mathfrak{g}^m = \{0\}.$$

We can associate a graded Lie algebra to $\mathfrak{g}$ which is usually denoted by $\mathrm{gr}(\mathfrak{g})$, and defined by:

$$\mathrm{gr}(\mathfrak{g}) \;=\; \sum_{i=1}^{m} \frac{\mathfrak{g}^{i-1}}{\mathfrak{g}^i} \;=\; \sum_{i=1}^{m} W_i,$$

with the brackets:

$$[X + \mathfrak{g}^i, Y + \mathfrak{g}^j] = [X, Y] + \mathfrak{g}^{i+j}, \quad \forall X \in \mathfrak{g}^{i-1},\ \forall Y \in \mathfrak{g}^{j-1}.$$

By definition, a Lie algebra is naturally graded if $\mathfrak{g}$ and $\mathrm{gr}(\mathfrak{g})$ are isomorphic Lie algebras. We say that the algebra $\mathfrak{g}$ has the form $\{p_1, \dots, p_m\}$ when $\dim W_i = p_i$. Clearly, the graded Lie algebra $\mathrm{gr}(\mathfrak{g})$ has the same form as $\mathfrak{g}$.
Note that a nilpotent Lie algebra is filiform if and only if it has the form $\{2, 1, 1, \dots, 1\}$. Therefore, the graded algebra of a filiform Lie algebra is also filiform.

Definition 3.1. Let $\mathfrak{g}$ be a nilpotent Lie algebra , $\mathfrak{g}$ is said quasi-filiform if its nilindex m is equal to $\dim\, \mathfrak{g} - 2$.

If $\mathfrak{g}$ is quasi-filiform, there are two possibilities:

1. either $\mathfrak{g}$ has the form $t_1 = \{p_1 = 3,\, p_2 = 1,\, p_3 = 1,\, \dots, p_m = 1\}$.
2. or $\mathfrak{g}$ has the form $t_r = \{p_1 = 2,\, p_2 = 1, \dots,\, p_{r-1} = 1,\, p_r = 2,\, p_{r+1} = 1, \dots,\, p_m = 1\}$ where $r \in \{2, \dots, m\}$.

Proposition 3.2. *Let $\mathfrak{g}$ be a quasi-filiform naturally graded Lie algebra of dimension $2n$. If $\mathfrak{g}$ has the form t_r with $r \in \{1, \dots, 2n-2\}$ then there exists a homogeneous basis $\{X_0, X_1, X_2, \dots, X_{2n-1}\}$ of $\mathfrak{g}$ in which X_0 and X_1 belong to W_1, $X_i \in W_i$ for $i \in \{2, \dots, 2n-2\}$ and $X_{2n-1} \in W_r$. Futhermore, $\mathfrak{g}$ is given in this basis by one of the algebras specified below.*

1. *If $\mathfrak{g}$ has the form t_1*
   $\mathfrak{L}_{2n-1} \oplus \mathbb{R} \quad (n \geq 2)$

$$\{[X_0, X_i] = X_{i+1}, \quad 1 \leq i \leq 2n-3.$$

2. *If $\mathfrak{g}$ has the form t_r with $r \in \{2, \dots, 2n-2\}$*

   (a) $\mathfrak{L}_{2n,r};\quad n \geq 3,\ r \text{ odd},\ 3 \leq r \leq 2n-3$

$$\begin{cases} [X_0, X_i] = X_{i+1}, & i = 1, \dots, 2n-3, \\ [X_i, X_{r-i}] = (-1)^{i-1} X_{2n-1}, & i = 1, \dots, \frac{r-1}{2}. \end{cases}$$

(b) $\mathfrak{T}_{2n,2n-3}$; $\quad n \geq 3$

$$\begin{cases} [X_0, X_i] = X_{i+1}, & i = 1, \dots, 2n-4, \\ [X_0, X_{2n-1}] = X_{2n-2}, & \\ [X_i, X_{2n-3-i}] = (-1)^{i-1} X_{2n-1}, & i = 1, \dots, n-2, \\ [X_i, X_{2n-2-i}] = (-1)^{i-1}(n-1-i) X_{2n-2}, & i = 1, \dots, n-2. \end{cases}$$

(c) $\mathfrak{N}_{6,3}$

$$\begin{cases} [X_0, X_i] = X_{i+1}, \quad i = 1, 2, 3, \\ [X_1, X_2] = X_5, \\ [X_1, X_5] = X_4, \end{cases}$$

The nonwritten brackets are zero, excepted those that follow from antisymmetry.

In order to obtain this classification, we have to revised the complex one ([3]). For example, if $\mathfrak{g}$ is a quasi-filiform Lie algebra of dimension 6 with the form t_3, then there exists a basis $\{X_0, X_1, \dots, X_5\}$ satisfying

$$\begin{cases} [X_0, X_i] = X_{i+1},\ i = 1, 2, 3, \\ [X_1, X_3] = bX_4, \\ [X_1, X_2] = bX_3 - X_5, \\ [X_5, X_1] = aX_4. \end{cases}$$

When $a = b = 0$, $\mathfrak{g}$ and $\mathfrak{L}_{6,3}$ are isomorphic algebras. In other way, by making the change of basis

$Y_0 = \alpha X_0$, $Y_1 = \beta X_1 + X_0$, $Y_2 = \alpha\beta X_2$, $Y_3 = \alpha^2\beta X_3$, $Y_4 = \alpha^3\beta X_4$, $Y_5 = -\alpha\beta^2 X_5$, with $\beta = \begin{cases} -\frac{1}{b - \sqrt{|a|}} & \text{if } \ b \neq \sqrt{|a|}, \\ -\frac{1}{2\sqrt{|a|}} & \text{if } \ b = \sqrt{|a|}, \end{cases}$ and $\alpha = b\beta + 1$, we obtain the brackets

$$\begin{cases} [Y_0, Y_i] = Y_{i+1},\ i = 1, 2, 3, \\ [Y_1, Y_3] = Y_4, \\ [Y_1, Y_2] = Y_3 + Y_5, \\ [Y_5, Y_1] = \delta Y_4,\ \delta = \pm 1. \end{cases}$$

With another change of basis, we can see that $\mathfrak{g}$ is isomorphic to the algebras $\mathfrak{T}_{6,3}$ for $\delta = 1$ and $\mathfrak{N}_{6,3}$ for $\delta = -1$. We remark that, those algebras $\mathfrak{T}_{6,3}$ and $\mathfrak{N}_{6,3}$ are isomorphic as complex algebras. Beyond dimension 6, the way of construction in the complex case gives the real classification.

Corollary 3.3. *Let $\mathfrak{g}$ be a quasi-filiform Lie algebra of dimension $2n$. Then there exists a basis $\{X_0, X_1, X_2, \dots, X_{2n-1}\}$ of $\mathfrak{g}$ such that:*

1. *If* $\mathrm{gr}(\mathfrak{g}) \simeq \mathfrak{L}_{2n-1} \oplus \mathbb{R} \quad (n \geq 2)$,

$$\begin{cases} [X_0, X_i] = X_{i+1}, & 1 \leq i \leq 2n-3, \\ [X_i, X_j] = \sum\limits_{k=i+j+1}^{2n-2} C_{i,j}^k X_k, & 1 \leq i < j \leq 2n-3-i, \\ [X_i, X_{2n-1}] = \sum\limits_{k=i+2}^{2n-2} C_{i,2n-1}^k X_k, & 1 \leq i \leq 2n-4. \end{cases}$$

2. *If* $\mathrm{gr}(\mathfrak{g}) \simeq \mathfrak{L}_{2n,r}$ $\quad n \geq 3,\ r : \text{odd},\ 3 \leq r \leq 2n-3$,

$$\begin{cases} [X_0, X_i] = X_{i+1}, & i = 1, \dots, 2n-3, \\ [X_0, X_{2n-1}] = \sum\limits_{k=r+2}^{2n-2} C_{0,2n-1}^k X_k, & \\ [X_i, X_j] = \sum\limits_{k=i+j+1}^{2n-1} C_{i,j}^k X_k, & 1 \leq i < j \leq r-i-1, \\ [X_i, X_j] = \sum\limits_{k=i+j+1}^{2n-2} C_{i,j}^k X_k, & 1 \leq i < j \leq 2n-3-i,\ r < i+j, \\ [X_i, X_{2n-1}] = \sum\limits_{k=r+i+1}^{2n-2} C_{i,2n-1}^k X_k, & 1 \leq i \leq 2n-3-r, \\ [X_1, X_{r-1}] = X_{2n-1}, & \\ [X_i, X_{r-i}] = (-1)^{(i-1)} X_{2n-1} + \sum\limits_{k=r+1}^{2n-2} C_{i,r-i}^k X_k, & 2 \leq i \leq \frac{r-1}{2}. \end{cases}$$

3. *If* $\mathrm{gr}(\mathfrak{g}) \simeq \mathfrak{T}_{2n,2n-3}$ $\quad n \geq 3$,

$$\begin{cases} [X_0, X_i] = X_{i+1}, & i = 1, \dots, 2n-4, \\ [X_0, X_{2n-1}] = X_{2n-2}, & \\ [X_i, X_j] = \sum\limits_{k=i+j+1}^{2n-1} C_{i,j}^k X_k, & 1 \leq i < j \leq 2n-4-i, \\ [X_1, X_{2n-4}] = X_{2n-1}, & \\ [X_i, X_{2n-3-i}] = (-1)^{(i-1)} X_{2n-1} + C_{i,2n-3-i}^{2n-2} X_{2n-2}, & 2 \leq i \leq n-2. \end{cases}$$

4. *If* $\mathrm{gr}(\mathfrak{g}) \simeq \mathfrak{N}_{6,3}$ *then* $\mathfrak{g} \simeq \mathfrak{N}_{6,3}$,

$$\begin{cases} [X_0, X_i] = X_{i+1}, \quad i = 1, 2, 3, \\ [X_1, X_2] = X_5, \\ [X_1, X_5] = X_4. \end{cases}$$

Such a basis $\{X_0, X_1, X_2, \dots, X_{2n-1}\}$ is called an adapted basis of $\mathfrak{g}$.

3.2. Complex structures on quasi-filiform Lie algebras.

The aim of this section is to find the quasi-filiform Lie algebras endowed with a complex structure or equivalently a generalized complex structure of type $k = n$. If $\mathfrak{g}$ has the form t_1, Theorem 2.9 says that $k = n = 2$ so the algebra $\mathfrak{g}$ is isomorphic to $\mathfrak{L}_3 \oplus \mathbb{R}$. We can check that this algebra admits a complex structure associated to the pure spinor

$$\Omega = (\omega_0 + i\omega_1) \wedge (\omega_2 + i\omega_3),$$

where $\{\omega_0, \omega_1, \omega_2, \omega_3\}$ denotes the dual basis corresponding to the homogeneous basis $\{X_0, X_1, X_2, X_3\}$ of Proposition 3.2. Assume that $\mathfrak{g}$ is a quasi-filiform Lie algebra the form t_r with $r \geq 3$. According to Theorem 2.9, $n = k \leq r$.

Lemma 3.4. *Let* $\mathfrak{g}$ *be a quasi-filiform Lie algebra of the form* t_r *with* $r \geq 3$. *If* $\mathfrak{g}$ *admits a generalized complex structure of the type* k, *then we can choose* *the complex forms $\theta_1, \dots, \theta_k$ corresponding to the generalized complex structure, satisfying either:*

$$nil(\theta_1) = 1,\ nil(\theta_2) = r,\ nil(\theta_3) = r+1, \dots, nil(\theta_k) = r+k-2$$

or

$$nil(\theta_1) = 1,\ nil(\theta_2) = r,\ nil(\theta_3) = r \dots, nil(\theta_k) = r+k-3$$

and in this case, $k < r$.

Proof. Let us consider an appropriate decomposition $\{\theta_1 \dots \theta_k\}$. From Corollary 2.7, we deduce that $\mathrm{nil}(\theta_1) = 1$ and $\mathrm{nil}(\theta_2) \in \{1, 2, r\}$. As $\dim V_1 = 2$ and $\dim V_2 = 3$, Condition (2) implies $\mathrm{nil}(\theta_2) = r$. According to Corollary 2.7:

$$\mathrm{nil}(\theta_{i-1}) \leq \mathrm{nil}(\theta_i) \leq r + i - 2 \quad i = 3, \dots, k.$$

Indeed, there are two possible values for $\mathrm{nil}(\theta_3)$:

1. $\mathrm{nil}(\theta_3) = r + 1$
   If we suppose that $\mathrm{nil}(\theta_4) = \mathrm{nil}(\theta_3) = r + 1$, the forms θ_4 and θ_3 belong to V_{r+1} and as they are linearly independent modulo V_r, this leeds to $\dim\left(\frac{V_{r+1}}{V_r}\right) \geq 2$ in contradiction with $\dim\left(\frac{V_{r+1}}{V_r}\right) = 1$. We deduce that $\mathrm{nil}(\theta_4) = r + 2$ and by the same way we obtain:

   $$\mathrm{nil}(\theta_i) = r + i - 2, \quad \text{for } i = 3, \dots, k.$$

2. $\mathrm{nil}(\theta_3) = r$
   By using similar arguments, we can prove that $\mathrm{nil}(\theta_i) = r + i - 3$ for $i = 3, \dots, k$. In this case, we remark that, when $k = r$, the nilindex of θ_r is equal to $2r - 3$ and then $\dim V_{2r-3} \geq 2r$. This is impossible because $\dim V_{2r-3} = 2r - 1$. Indeed $k < r$.

■

Example 3.5. Let us consider a quasi-filiform Lie algebra $\mathfrak{g}$ of dimension 6 defined in the basis $\{X_0, X_1, \dots, X_5\}$ by:

$$\begin{cases} [X_0, X_i] = X_{i+1}, & i = 1, 2, 3, \\ [X_1, X_2] = X_5, & \\ [X_1, X_5] = \delta X_4, & \delta \in \{0, 1, -1\}. \end{cases}$$

Let us suppose that $\mathfrak{g}$ admits a complex structure which determines a generalized complex structure of the type $k = 3$ and with the spinor:

$$\Omega = \theta_1 \wedge \theta_2 \wedge \theta_3,$$

where θ_1, θ_2 and θ_3 are complex forms. Note that this algebra has the form t_3 and according to the previous lemma the corresponding nilindices are given by:

$$\mathrm{nil}(\theta_1) = 1,\ \mathrm{nil}(\theta_2) = 3,\ \mathrm{nil}(\theta_3) = 4.$$

The complex forms θ_1, θ_2 and θ_3 can be written as:

$$\begin{aligned}
\theta_1 &= \lambda_0\omega_0 + \lambda_1\omega_1, \\
\theta_2 &= \beta_0\omega_0 + \beta_1\omega_1 + \beta_2\omega_2 + \beta_3\omega_3 + \beta_5\omega_5, \\
\theta_3 &= \gamma_0\omega_0 + \gamma_1\omega_1 + \gamma_2\omega_2 + \gamma_3\omega_3 + \gamma_4\omega_4 + \gamma_5\omega_5,
\end{aligned}$$

where $\omega_1, \cdots, \omega_5$ is the dual basis of $X_0, \cdots, X_6$, $\lambda_i, \beta_i, \gamma_i \in \mathbb{C}$, γ_4 is non-zero and β_3, β_5 can not be simultaneously zero. Moreover, the condition $\theta_1 \wedge \overline{\theta_1} \neq 0$ means that the imaginary part of $\lambda_0\overline{\lambda_1}$ is non-zero. Corollary 2.7 leads to:

$$\begin{cases} \theta_1 \wedge d\theta_2 = 0, \\ \theta_1 \wedge \theta_2 \wedge d\theta_3 = 0, \end{cases}$$

that is

$$\begin{cases} \beta_5\lambda_0 - \beta_3\lambda_1 & = 0, \\ -\gamma_3\beta_3\lambda_1 + \gamma_4\beta_2\lambda_1 + \gamma_5\beta_3\lambda_0 & = 0, \\ \gamma_4(\beta_5\lambda_1 + \delta\beta_3\lambda_0) & = 0, \\ -\gamma_3\beta_5\lambda_1 - \delta\gamma_4\beta_2\lambda_0 + \gamma_5\beta_5\lambda_0 & = 0. \end{cases}$$

From the first and third equations, we deduce:

$$\lambda_1^2 + \delta\lambda_0^2 = 0.$$

For $\delta = 0$, this is in contradiction with $\theta_1 \wedge \overline{\theta_1} \neq 0$. If $\delta = -1$, we deduce that $\lambda_1 = \pm\lambda_0$ and since the spinor is uniquely defined up to a multiplicative constant, we can take $\theta_1 = \omega_0 \pm \omega_1$ in contradiction with $\theta_1 \wedge \overline{\theta_1} \neq 0$. Finally, when $\delta = 1$, the spinor $\Omega = (\omega_0 + i\omega_1) \wedge (\omega_3 + i\omega_5) \wedge (\omega_2 + i\omega_4)$ is associated to a complex structure of $\mathfrak{g}$. We conclude that the Lie algebra $\mathfrak{g}$ admits a complex structure if and only if $\delta = 1$.

Theorem 3.6. *Let $\mathfrak{g}$ be a real quasi-filiform Lie algebra endowed with a complex structure. Then $\mathfrak{g}$ is isomorphic either to the four-dimensional algebra $\mathfrak{L}_3 \oplus \mathbb{R}$ or to the algebra $\mathfrak{n}_6^{10}$ of dimension 6.*

Proof. Let $\mathfrak{g}$ be a $2n$-dimensional real quasi-filiform Lie algebra of the form t_r with $r \in \{1, 3, \ldots, 2n-3\}$. Let us suppose that $\mathfrak{g}$ admits a complex structure which determines a generalized complex structure of the type $k = n$.
For $r = 1$, $\mathfrak{g}$ is isomorphic to $\mathfrak{L}_3 \oplus \mathbb{R}$ and this algebra admits a complex structure. Henceforth, we assume $r \in \{3, \ldots, 2n-3\}$. Applying Theorem 2.9 and the inequality

$$\mathrm{nil}(\theta_k) = \mathrm{nil}(\theta_n) \leq \mathrm{nil}(\mathfrak{g})$$

in each of the possibilities of Lemma 3.4, we deduce:

1. If $\mathrm{nil}(\theta_3) = r + 1$ then $\mathrm{nil}(\theta_k) = r + k - 2$ and:

$$n = k \leq r \leq n \;\Rightarrow\; r = n.$$

2. If $\mathrm{nil}(\theta_3) = r$ then $\mathrm{nil}(\theta_k) = r + k - 3$ and in this case, $k < r$ so

$$n = k < r \leq n + 1 \;\Rightarrow\; r = n + 1.$$

Moreover, since the graduate algebra $gr(\mathfrak{g})$ must be isomorphic to one of the algebras of Proposition 3.2 we obtain the following possibilities:

1. $gr(\mathfrak{g}) \sim \mathfrak{L}_{2n,r}$; $\quad n \geq 3,\ r$ odd, $3 \leq r \leq 2n-3$.

   (a) If $\mathrm{nil}(\theta_3) = r+1$ then $gr(\mathfrak{g}) \sim \mathfrak{L}_{2n,n}$ with $n \geq 3$ odd.
   For $n=3$, $\mathfrak{g}$ is the algebra of Example 3.5 with $\delta = 0$ and it does not admit a complex structure. Suppose $n > 3$. If $\{X_0, X_1, \dots, X_{2n-1}\}$ is an adapted basis of $\mathfrak{g}$ and $\{\omega_0, \omega_1, \dots, \omega_{2n-1}\}$ is its dual basis, then

$$\begin{cases} \theta_1 = \lambda_1^0 \omega_0 + \lambda_1^1 \omega_1, \\ \theta_2 = \sum\limits_{k=0}^{n} \lambda_2^k \omega_k + \lambda_2^{2n-1} \omega_{2n-1}. \end{cases}$$

   Since $\theta_1 \wedge d\theta_2 = 0$, grouping together $\omega_0 \wedge \omega_1 \wedge \omega_{n-1}$, $\omega_0 \wedge \omega_2 \wedge \omega_{n-2}$ and $\omega_0 \wedge \omega_3 \wedge \omega_{n-3}$ in $\theta_1 \wedge d\theta_2$, it results:

$$\lambda_2^n = \lambda_2^{2n-1} = 0.$$

   This is impossible since $\mathrm{nil}(\theta_2) = n$. Hence, there are no complex structures, excepted for $n = 3$. Suppose $\mathrm{nil}(\theta_3) = r$ then $gr(\mathfrak{g}) \sim \mathfrak{L}_{2n,n+1}$ with $n \geq 4$ even. We can write θ_1 and θ_2 as:

$$\begin{cases} \theta_1 = \lambda_1^0 \omega_0 + \lambda_1^1 \omega_1, \\ \theta_2 = \sum\limits_{k=0}^{n+1} \lambda_2^k \omega_k + \lambda_2^{2n-1} \omega_{2n-1}. \end{cases}$$

   where $\{\omega_0, \omega_1, \dots, \omega_{2n-1}\}$ is the dual basis of an adapted basis of $\mathfrak{g}$. In the equation $\theta_1 \wedge d\theta_2 = 0$, the coefficients of $\omega_0 \wedge \omega_1 \wedge \omega_n$ and $\omega_0 \wedge \omega_2 \wedge \omega_{n-1}$ give:

$$\begin{cases} \lambda_1^0 \lambda_2^{2n-1} - \lambda_1^1 \lambda_2^{n+1} = 0, \\ \lambda_1^1 \lambda_2^{2n-1} = 0. \end{cases}$$

   Since $\theta_1 \wedge \overline{\theta_1} \neq 0$, we deduce that $\lambda_2^{n+1} = \lambda_2^{2n-1} = 0$ contradicting $\mathrm{nil}(\theta_2) = n+1$.

2. $gr(\mathfrak{g}) \sim \mathfrak{T}_{2n,2n-3}$; $\quad n \geq 3$

   (a) If $\mathrm{nil}(\theta_3) = r+1$ then $gr(\mathfrak{g}) \sim \mathfrak{T}_{6,3}$. In this case $\mathfrak{g}$ is isomorphic to the algebra of Example 3.5 with $\delta = -1$ which do not admit any complex structure.

   (b) When $\mathrm{nil}(\theta_3) = r$, $gr(\mathfrak{g}) \sim \mathfrak{T}_{8,5}$ and there is an adapted basis $\{X_0, X_1, \dots, X_7\}$ of $\mathfrak{g}$ satisfying:

$$\begin{cases} [X_0, X_i] = X_{i+1}, & i = 1, \dots, 4, \\ [X_0, X_7] = X_6, & \\ [X_1, X_i] = \sum_{k=i+2}^{7} C_{1,i}^k X_k, & i = 2, 3, \\ [X_1, X_4] = X_7, & \\ [X_1, X_5] = 2X_6, & \\ [X_2, X_4] = -X_6, & \end{cases}$$

In the dual basis $\{\omega_0, \omega_1, \ldots, \omega_7\}$, we can write θ_1, θ_2 and θ_3 as:

$$\begin{cases} \theta_1 = \lambda_1^0 \omega_0 + \lambda_1^1 \omega_1, \\ \theta_2 = \sum\limits_{k=0}^{5} \lambda_2^k \omega_k + \lambda_2^7 \omega_7, \\ \theta_3 = \sum_{k=0}^{5} \lambda_3^k \omega_k + \lambda_3^7 \omega_7. \end{cases}$$

According to Corollary 2.7, $\theta_1 \wedge d\theta_2 = 0$ and $\theta_1 \wedge d\theta_3 = 0$. Thus:

$$\begin{cases} \lambda_1^0 \lambda_2^7 - \lambda_1^1 \lambda_2^5 & = 0, \\ \lambda_1^0 \lambda_3^7 - \lambda_1^1 \lambda_3^5 & = 0. \end{cases}$$

Assuming that $\lambda_1^1 = \lambda_2^7 = \lambda_3^7 = 1$, we deduce $\lambda_1^0 = \lambda_2^5 = \lambda_3^5$, in contradiction with the choice of θ_2 et θ_3 since they are linearly independent modulo V_4.

3. $gr(\mathfrak{g}) \sim \mathfrak{n}_6^{10}$. $\mathfrak{g}$ is isomorphic to the algebra $\mathfrak{n}_6^{10}$, which is the algebra of Example 3.5 with $\delta = 1$ admiting a complex structure.

■

The notion of characteristic sequence has been defined in [6]. This invariant is finer than the nilindex. More precisely

$$s(\mathfrak{g}) = \sup\{s(ad(X)) \,\text{such that}\, \mathrm{X} \in \mathfrak{g}, \mathrm{X} \notin \mathcal{D}(\mathfrak{g})\},$$

where $s(ad(X))$ is the ordered decreasing sequence of the dimension of Jordan blocks of the nilpotent operator $ad(X)$. The order relation is the lexicographic order. If $\mathfrak{g}$ is filiform then $s(\mathfrak{g}) = (2n-1, 1)$. Combining this with the result of [7] we obtain the following main result:

Corollary 3.7. *Let $\mathfrak{g}$ be a $2n$-dimensional real Lie algebra with $n \geq 4$. If $\mathfrak{g}$ is provided with a complex structure then its characteristic sequence $s(\mathfrak{g})$ satisfies $s(\mathfrak{g}) \leq (2n-2, 1, 1)$.*

4. Lie Algebra $\mathfrak{n}_6^{10}$

The mentioned paper [10] listed the 6-dimensional nilpotent Lie algebras endowed with a complex structure, we verify that $\mathfrak{n}_6^{10}$ is the only quasi-filiform Lie algebra in this classification. Our aim is now to write precisely all the complex structures on this algebra.

We say that two complex structures J_1 and J_2 on a real Lie algebra $\mathfrak{g}$ are equivalent if there exists an automorphism $\sigma \in Aut(\mathfrak{g})$ such that $\sigma J_1 = J_2 \sigma$.

Proposition 4.1. *The algebra $\mathfrak{n}_6^{10}$ has only two non-equivalent complex structures.*

This result has been proved by Magnin [8]. If we consider the commutation relations of the basis $\{X_0, X_1, \ldots, X_5\}$:

$$\left\{\begin{array}{l} [X_0, X_i] = X_{i+1}, \quad i = 1, 2, 3, \\ [X_1, X_2] = X_5, \\ [X_1, X_5] = X_4, \end{array}\right.$$

the non-equivalent complex strutures can be expressed by the matrix

$$J(\zeta) \sim \begin{pmatrix} 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \zeta & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & \zeta & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix} \qquad (4)$$

where $\zeta = \pm 1$.

The study of Kähler structures on a nilmanifold was initiated by [2] and completed by [5]. We can quickly look at this result for this special case.

In order to compute all the symplectic structures of the algebra $\mathfrak{n}_6^{10}$, we consider a skew-symmetric 2-form $\omega = \sum_{0 \le i < j \le 5} \lambda_{i,j} \omega_i \wedge \omega_j$ with $\{\omega_0, \omega_1, \ldots, \omega_5\}$ the dual of the preceding basis and we impose Conditions (1). We deduce that the symplectic structures of $\mathfrak{n}_6^{10}$ are given by:

$$\begin{aligned} \omega =& \lambda_{0,1}\omega_0 \wedge \omega_1 + \lambda_{0,2}\omega_0 \wedge \omega_2 + \lambda_{0,3}\omega_0 \wedge \omega_3 + \lambda_{0,4}\omega_0 \wedge \omega_4\omega_0 \wedge \omega_4 + \lambda_{0,5}\omega_0 \wedge \omega_5 \\ & + \lambda_{1,2}\omega_1 \wedge \omega_2 + \lambda_{0,5}\omega_1 \wedge \omega_3 + \lambda_{1,4}\omega_1 \wedge \omega_4 + \lambda_{1,5}\omega_1 \wedge \omega_5 - \lambda_{1,4}\omega_2 \wedge \omega_3 + \lambda_{0,4}\omega_2 \wedge \omega_5 \end{aligned}$$

with

$$\lambda_{0,3}\lambda_{0,4}\lambda_{1,4} + \lambda_{0,4}^2\lambda_{0,5} + \lambda_{0,4}\lambda_{1,4}\lambda_{1,5} - \lambda_{0,5}\lambda_{1,4}^2 \neq 0. \qquad (5)$$

A symplectic form ω of a Lie algebra $\mathfrak{g}$ is a Kähler structure if there exists a complex structure J which is compatible with ω, that is

$$\omega(JX, JY) = \omega(X, Y) \quad \forall X, Y \in \mathfrak{g}.$$

Note that if a symplectic structure ω is compatible with a complex structure J_1 and J_1 is equivalent to another complex structure J_2 then ω is also compatible with J_2. Thus, to determine the Kähler structures of $\mathfrak{n}_6^{10}$ is enough to compute the symplectic forms compatible with the complex structures $J(\pm 1)$ of (4). We obtain the extra conditions

$$\left\{\begin{array}{l} \lambda_{0,2} = \lambda_{0,4} = \lambda_{0,5} = \lambda_{1,2} = \lambda_{1,4} = 0, \\ \lambda_{0,3} = \lambda_{1,5}, \end{array}\right. \qquad (6)$$

which is in contradiction with (5). We conclude that there is no Kähler structure on $\mathfrak{n}_6^{10}$.

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Lucia García Vergnolle
Laboratoire de Mathématiques
Université de Haute Alsace
Faculté des Sciences et Techniques
4, rue des Frères Lumière
68093 Mulhouse cedex, France
lucigarcia@mat.ucm.es

Elisabeth.Remm@uha.fr,
Laboratoire de Mathématiques,
Université de Haute Alsace,
Faculté des Sciences et,Techniques,
4, rue des Frères Lumière,
68093 Mulhouse cedex, France

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