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Classifying Associative Quadratic Algebras of Characteristic not Two as Lie Algebras

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Abstract. We present an alternative to existing classifications (Bröcker, L., Kinematische Räume, Geom. Dedicata 1 (1973), 241–268, Karzel, H., Kinematic spaces, Symposia Mathematica 11 (1973), 413–439) of those quadratic algebras (in the sense of Osborn) which are associative. The alternative consists in studying them as Lie algebras. This generalizes Plebański, J. F. and M. Przanowski, Generalizations of the quaternion algebra and Lie algebras, J. Math. Phys. 29 (1988), 529–535, where only algebras over the real and the complex numbers are considered, to algebras over arbitrary fields of characteristic not two; at the same time, considerable simplifications are obtained. The method is not suitable, however, for characteristic two.
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1. Introduction

A (unitary) algebra A over a commutative field F is called *quadratic* (in the sense of Osborn [Os]) if for every element $a \in A$ the linear subspace $Fa + F \cdot 1$ is a subalgebra. This is equivalent to $a^2 \in Fa + F \cdot 1$, so that a satisfies a quadratic equation with coefficients in F, whence the name.

All *associative* quadratic algebras were classified in 1973 independently by L. Bröcker [Br] and H. Karzel [Ka73], [Ka74] under the name of kinematic algebras. They had a geometric goal, namely the classification of kinematic spaces.

Here, we present a different approach to the classification, which was proposed by J. F. Plebański and M. Przanowski [PP] in 1988 for the special case that F is the field of real or complex numbers. Apparently they did not know about the general theory of quadratic algebras; associative quadratic algebras were termed quaternionlike algebras by them. Their method consists in considering the Lie algebras associated with associative quadratic algebras, which turn out to be very special Lie algebras, indeed. We show how to make this method work over arbitrary fields of characteristic not two; in characteristic two it is not appropriate. Incidentally, our effort of generalization has rendered the classification very simple.

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Where applicable, this way thus seems simpler than the methods of Bröcker and of Karzel, as well; however, the latter also cover the characteristic two case, in contrast to the approach here.

In Section 2, we present basic information about quadratic algebras whose characteristic is different from two; the latter will always be tacitly assumed in this paper. In such a quadratic algebra A, the elements which are not scalar multiples of 1, but whose squares are, are the non-zero elements of a linear subspace V of codimension 1. These elements together with 0 are called vectors. If A is associative, V is a Lie subalgebra of the Lie algebra associated with A; this Lie subalgebra will be called the vector algebra of A. Together with a certain symmetric bilinear form on V, the vector algebra determines the whole algebra A. In Section 3, the Lie algebras which are the vector algebras of associative quadratic algebras are characterized following Plebański and Przanowski, and a few direct consequences about their structure are drawn from this characterization. In Section 4, the vector algebras of associative quadratic algebras of characteristic not two are then classified. The quadratic algebras to which they belong will be made explicit in Section 5.

Much of the material of this paper has been developed in the diploma thesis [We] of the second author under the supervision of the first. Some arguments are taken from the diploma thesis [He] of S. Heinz, in which the results of Plebański and Przanowski for associative quadratic algebras over the real and the complex numbers are expounded in the framework of the general theory of quadratic algebras and in a coordinate free style.

2. Vectors in quadratic algebras

In what follows, A shall always be a quadratic algebra over a commutative field F of characteristic not two.

2.1 Definition. An element $v \in A$ is called a *vector* if $v^2 \in F \cdot 1$ and $v \notin F \cdot 1$ or $v = 0$. In the sequel, let V be the set of vectors.

We first recall some well-known basic facts about the set of vectors of a quadratic algebra, see [Di, Lemma 1 p.113], [Os, 2.1 p.202]. They can also be found in [Eb, Chap.8 §2.1 p.227] under the name of "Frobenius's lemma" for $F = \mathbb{R}$, but the proofs given there are valid in general.

2.2 Proposition. *Every element of A can be written uniquely as a sum of a scalar multiple of 1 and a vector. For all vectors u, v, the symmetric product $uv + vu$ is a scalar multiple of 1. The set V of vectors is a linear subspace of A, which, by the uniqueness of the scalar-vector decomposition, has codimension 1.*

These assertions are not true in general for quadratic algebras of characteristic two, even if they are associative, as can be seen from the explicit description of these algebras in [Br] or [Ka74].

The following might be folklore, but it seems that it is not well documented in the literature. The proof is taken from [He].

2.3 Proposition. *If the quadratic algebra A is associative, then for two vectors u, v their Lie bracket $uv - vu$ is again a vector, so that the space V of vectors is a Lie subalgebra of the Lie algebra associated with A. For vectors $u, v \in V$, the scalar-vector decomposition of the product uv therefore is*

$$uv = \frac{1}{2}(uv + vu) + \frac{1}{2}(uv - vu)$$

(note that the first summand on the right hand side is a scalar multiple of 1 by 2.2).

Proof. $(uv - vu)^2 = (uv)^2 + (vu)^2 - uv^2u - vu^2v = (uv + vu)^2 - 2uv^2u - 2vu^2v$, and this is a scalar multiple of 1, since u^2, v^2 and $(uv + vu)^2$ are. But $uv - vu = uv + vu - 2vu$ itself is a scalar multiple of 1 only if it is 0, for then $vu, uv \in F \cdot 1$, as well, so that $(uv)v = v(uv) = (vu)v$ and hence $uv = vu$. ■

2.4 The vector algebra of an associative quadratic algebra and its bilinear form. From now on, let A be an associative quadratic algebra over a commutative field F of characteristic not two, and V the space of vectors of A. According to 2.3, the operation $\times$ defined on V by

$$u \times v := \frac{1}{2}(uv - vu) \in V \quad \text{for} \quad u, v \in V$$

makes V into a Lie algebra, which will be called the *vector algebra* of A. Upon identification of $F \cdot 1 \subseteq A$ with F, the symmetric product $uv + vu$ of two vectors $u, v \in V$ may be considered as an element of the base field F, so that

$$(u, v) := \frac{1}{2}(uv + vu) \in F \quad \text{for} \quad u, v \in V$$

defines a symmetric bilinear form $(\ ,\ )$ on V. By 2.3, this symmetric bilinear form and the operation $\times$ determine the product of vectors according to the following multiplication rule:

$$uv = (u, v) \cdot 1 + u \times v \quad \text{for} \quad u, v \in V\ , \tag{$*$}$$

in particular

$$u^2 = (u, u) \cdot 1\ .$$

3. A characterization of vector algebras among Lie algebras

Characterization Theorem 3.1. *A bilinear antisymmetric multiplication $\times$ on a vector space V of characteristic not two makes V into a Lie algebra which is the vector algebra of an associative quadratic algebra if and only if there is a symmetric bilinear form $(\ ,\ )$ on V such that the following two identities hold for all $u, v, w \in V$:*

$$(v \times u, w) = -(u, v \times w) \tag{1}$$

$$(w \times u) \times v = (u, v)w - (v, w)u\ . \tag{2}$$

Remark. A bilinear form on a Lie algebra is commonly called *invariant* if identity (1) is satisfied.

Proof. The assertion is implicit in [PP]. For completeness' sake, we give a coordinate-free proof.

First we observe that if identity (2) is satisfied then the Jacobi identity holds so that we may assume right away that $(V,+,\times)$ is a Lie algebra. Indeed, the right hand side of (2) and the two expressions obtained from it by two-fold cyclic permutation of the variables u,v,w sum up to 0.

We may use this Lie algebra and the symmetric bilinear form to define an algebra multiplication on $A = F \oplus V$ via the multiplication rule 2.4($*$). This algebra is quadratic, since for $a \in A$ there is a vector v in the hyperplane V such that $Fa + F \cdot 1 = Fv + F \cdot 1$, and the latter subspace is obviously a subalgebra, as $v^2 = (v,v) \cdot 1 \in F \cdot 1$.

Thus, we have to show that equations (1) and (2) are equivalent to associativity. Since the equation expressing associativity is trilinear, and since every product of three factors one of which belongs to $F \cdot 1$ satisfies the associativity law, it suffices to consider products of three vectors from V. Now $(uv)w = (u,v)w + (u \times v, w) \cdot 1 + (u \times v) \times w$ and $u(vw) = (v,w)u + (u, v \times w) \cdot 1 + u \times (v \times w)$. Comparing the scalar and vector parts of these expressions, one obtains that associativity is equivalent to the validity of identity (1) together with the identity $(u,v)w + (u \times v) \times w = (v,w)u + u \times (v \times w)$. The latter is equivalent to $(u,v)w - (v,w)u = -(v \times w) \times u - (u \times v) \times w$. The right-hand side of this equation equals $(w \times u) \times v$ by the Jacobi identity, which gives us identity (2). ∎

In the sequel, we make a few observations about the structure of such Lie algebras. These observations are quite simple due to the strength of identity (2), but nevertheless very effective in classifying these Lie algebras.

First a remark about the case $\dim V = 1$. The corresponding algebras are 2-dimensional. Now all 2-dimensional unitary algebras over a field F are quadratic and associative, and they are all known (quadratic field extensions, the algebra $F(\varepsilon)$ of dual numbers with $\varepsilon^2 = 0$, and the algebra $F \times F$ with componentwise multiplication).

Thus, we may assume from now on that $\dim V \geq 2$.

3.2 Corollary. If the conditions of the Classification Theorem 3.1 are satisfied and if $\dim V \geq 2$, then the symmetric bilinear form $(\ ,\ )$ is uniquely determined by the Lie algebra V.

Indeed, for $u,v \in V$, identity 3.1(2) determines (u,v) if one chooses w to be linearly independent of u.

Consequently, an associative quadratic algebra of dimension at least 3 over a field of characteristic not two is uniquely determined by its vector algebra, in view of the multiplication rule 2.4($*$).

3.3 The radical. In the following structural results, we use the radical

$$R := \operatorname{rad}(\ ,\ )$$

of the symmetric bilinear form $(\ ,\ )$ on V. It should not be confused with the solvable radical of the Lie algebra V. In many cases, it is rather the Killing radical of V. Indeed, it can be proved ([He, 2.2]) that if the conditions of the

Characterization Theorem 3.1 are met and if the dimension of V is finite, then the Killing form of V is $\dim V - 1$ times the symmetric bilinear form $(\ , \ )$. Thus, if the characteristic is 0 or does not divide $\dim V - 1$, then R coincides with the Killing radical.

One is tempted to use this relation with the Killing form for the structural investigation (as was done implicitly via the Cartan criterion for solvability in [PP]), but because of the dimensional restrictions which this would bring about, the exceptions in positive characteristic and the great simplicity of the arguments below this is not advisable in the end.

3.4 Proposition. *Let* $(V, +, \times)$ *be a Lie algebra together with a symmetric bilinear form* $(\ , \ )$ *such that the conditions of the Characterisation Theorem* 3.1 *are satisfied. Let* V' *be the commutator algebra of* V*, and let* R *be the radical of* $(\ , \ )$*. Furthermore assume that* $\dim V \geq 2$*. Then the following assertions hold.*

(i) $V' \times R = 0$.

(ii) $V' \neq V$ *if and only if* $V' \subseteq R$.

(iii) If $u, w \in V$ *are linearly independent and if* $u \times w = 0$*, then* $u, w \in R$.

Proof. (i) follows directly from identity 3.1(2) if one chooses $v \in R$.

Concerning assertion (ii), note that for $w \in V'$ the left-hand side and the first term of the right-hand side of identity 3.1(2) belong to V'. Hence, if one may choose $u \in V \setminus V'$, the second term of the right-hand side shows that $(v, w) = 0$ for all $v \in V$, so that $w \in R$ and $V' \subseteq R$. — Conversely, assume that $V' \subseteq R$. If $V' = V$, then also $R = V$ and thus V would be abelian by (i), so that $V' = \{0\}$, a contradiction.

Assertion (iii) is immediate from identity 3.1(2). ∎

3.5 Corollary. *Under the assumptions of Proposition* 3.4*, the following assertions are equivalent.*

(i) *The center of* V *is non-trivial.*

(ii) $R = V$, *i.e.* $(\ , \ )$ *is identically zero.*

(iii) V *is abelian or nilpotent of class* 2.

Proof. (i) implies (ii): For a non-zero element c of the center of V and $v \in V \setminus Kc$ one has $c \times v = 0$; it follows from 3.4(iii) that $c, v \in R$.

(ii) implies (iii) by 3.4(i), and it is clear that (iii) implies (i). ∎

4. Classification of the vector algebras of associative quadratic algebras

The classification will proceed by the codimension of the commutator algebra.

4.1 Classification Theorem: Codimension at least two. *Let* V *be a Lie algebra of characteristic not two and assume that* $\dim V \geq 2$*. Then* V *is the vector algebra of an associative quadratic algebra and* $\operatorname{codim} V' \geq 2$ *if and only if* V *is abelian or nilpotent of class* 2.

Proof. Assume that V is the vector algebra of an associative quadratic algebra and that $\operatorname{codim} V' \geq 2$. Write V as the direct sum of V' and a linear subspace W of dimension at least 2. For linearly independent elements $u, w \in W$ and arbitrary $v \in V$, it follows from identity 3.1(2) that $(u,v)w - (v,w)u \in V' \cap W = \{0\}$. Hence $(u,v) = 0 = (v,w)$ for all $v \in V$, so that $W \subseteq R$. By 3.4(ii), we have $V' \subseteq R$. It follows that $V = R$, so that V is abelian or nilpotent of class 2 according to 3.5.

Conversely, assume that V is abelian or nilpotent of class 2. Then V' is contained in the center of V and hence has codimension at least 2. If V is endowed with the symmetric bilinear form which is identically 0, the conditions of the Characterization Theorem 3.1 are satisfied, since the left-hand side of identity 3.1(2) is 0. Thus, V is the vector algebra of an associative quadratic algebra. ∎

Remark. An explicit classification of all nilpotent Lie algebras of class 2 is hard and even a wild problem for higher dimensions. Of course, it is easy to give a construction yielding all such algebras; the difficult point is the classification up to isomorphism. This problem has been tackled systematically by Gauger [Ga]. Together with a wealth of general results, he has obtained a complete classification over an algebraically closed field of characteristic not two for Lie algebras of dimension at most 7, and far-reaching results for dimension 8. Recently, M. Stroppel [St] achieved a complete classification in low dimensions over arbitrary fields using properties of the Klein quadric. The classifications by Bröcker and Karzel of associative quadratic algebras do not help here, since at this point they are not more detailed than our result 4.1, see 5.1.

4.2 Classification Theorem: Codimension one. *Let V be a Lie algebra over a field of characteristic not two and assume* $\dim V \geq 2$. *Then V is the vector algebra of an associative quadratic algebra and* $\operatorname{codim} V' = 1$ *if and only if V' is abelian and for $e \in V \setminus V'$ the restriction of* $\operatorname{ad} e$ *to V' satisfies* $(\operatorname{ad} e|V')^2 = \lambda \cdot \mathrm{id}$ *for some $\lambda \in F \setminus \{0\}$. Such a Lie algebra is solvable, but not nilpotent.*

Remark. The Lie algebras of this type can be described explicitly, see 4.3 below.

Proof. Assume that V is the vector algebra of an associative quadratic algebra and that $\operatorname{codim} V' = 1$. By 3.4(ii), $V' \subseteq R$. Now R cannot fill V entirely, for else 3.5 and 4.1 would imply $\operatorname{codim} V' \geq 2$, contrary to our assumption. Thus $V' = R$. By 3.4(i), the operation $\times$ is trivial on $V' = R$. For $e \in V \setminus V'$, it follows that

$$\lambda := (e,e) \neq 0$$

since $R \neq V$. One infers from the multiplication rule 2.4($*$) that

$$e^2 = \lambda \cdot 1, \quad ev = e \times v = -v \times e = -ve \quad \text{for } v \in V' = R\,,$$

and $(\operatorname{ad} e)^2(v) = e \times (e \times v) = e(ev) = e^2 v = \lambda v$. Thus, V conforms to the asserted description.

Conversely, we have to show that such a Lie algebra is always the vector algebra of an associative quadratic algebra. Fix $e \in V \setminus V'$. A symmetric bilinear form $(\ ,\ )$ on V is uniquely defined if we decree V' to be its radical and $(e,e) = \lambda$.

We have to verify that the conditions of the Characterization Theorem 3.1 are met. Identity 3.1(1) is trivial since V' is the radical of $(\ ,\ )$. As to identity 3.1(2), let $u = \alpha e + u'$, $v = \beta e + v'$, and $w = \gamma e + w'$ for $u', v', w' \in V'$. Since the Lie operation on V' is trivial, we obtain that
$(w \times u) \times v = (\gamma e \times u' + \alpha w' \times e) \times (\beta e + v') = \beta\gamma(e \times u') \times e + \alpha\beta(w' \times e) \times e = -\beta\gamma(u' \times e) \times e + \alpha\beta(w' \times e) \times e = -\beta\gamma\lambda u' + \alpha\beta\lambda w'$.
On the other hand, as V' is the radical of $(\ ,\ )$,
$(u, v)w - (v, w)u = \alpha\beta\lambda\gamma e + \alpha\beta\lambda w' - \beta\gamma\lambda\alpha e - \beta\gamma\lambda u'$.
Thus, identity 3.1(2) holds, and V is the vector algebra of an associative quadratic algebra by the Characterization Theorem 3.1.

It is clear that the Lie algebra V is solvable, but not nilpotent since $e \times (e \times V') = (\mathrm{ad}\, e)^2(V') = V'$. ■

4.3 Addendum. The Lie algebras of the Classification Theorem 4.2 shall now be described explicitly. As stated there, the commutator algebra V' of such a Lie algebra V is an abelian subalgebra of codimension 1. Hence, in order to determine V completely, it suffices to fix an element $e \in V \setminus V'$ and to specify $e \times v$ for $v \in V'$ in such a way that $(\mathrm{ad}\, e|V')^2 = \lambda \cdot \mathrm{id}$ for some $\lambda \in F \setminus \{0\}$.

1) First let λ be a square in F. Replacing e by a scalar multiple, we may assume that $\lambda = 1$. Then $\mathrm{ad}\, e|V'$ is to be an involutory linear automorphism of V'. Decomposition of V' into the eigenspaces gives two linear subspaces V'_+ and V'_- such that

$$V' = V'_+ + V'_-, \quad V'_+ \cap V'_- = \{0\}$$

and

$$e \times v = \pm v \quad \text{for } v \in V'_\pm\ .$$

2) Now assume that λ is not a square in F. In this case, V is most easily described if one considers the corresponding quadratic algebra A of which V is the vector algebra. As λ is not a square, the polynomial $x^2 - \lambda$ is irreducible and is the minimal polynomial of e over F. Hence the subalgebra $F(e) := Fe + F \cdot 1$ of A is a quadratic field extension of F, the commutator algebra V' is a left vector space over $F(e)$, and the restriction of $\mathrm{ad}\, e$ to V' is just scalar multiplication by e.

Conversely, if one defines $\mathrm{ad}\, e$ on V' according to case 1) or 2) then it is clear that V satisfies the conditions of Theorem 4.2. In the proof there, it was shown that this implies the conditions of the Characterization Theorem 3.1, whereby V is a Lie algebra which is the vector algebra of an associative quadratic algebra.

Remark. As to possible isomorphisms between such Lie algebras, it is straightforward that a Lie algebra of type 1) cannot be isomorphic to a Lie algebra of type 2). It is not difficult to see that a Lie algebra of type 1) is determined up to isomorphism by the set of the two cardinalities of bases of V'_+ and V'_-, and that a Lie algebra of type 2) is determined up to isomorphism by the cardinality of a basis of V' and the coset $\lambda \cdot \{\alpha^2;\ 0 \neq \alpha \in F\}$.

4.4 Classification Theorem: Codimension zero. *Let V be a vector space over a field F of characteristic not two, and let $\times$ be a bilinear antisymmetric multiplication on V. Then the following assertions are equivalent.*

(i) *V is the vector algebra of an associative quadratic algebra, and $V' = V$.*

(ii) *There is a basis e, f, g of V and there are $\alpha, \beta \in F \setminus \{0\}$ such that*

$$e \times f = g, \;\; f \times g = \alpha e, \;\; g \times e = \beta f \; .$$

(iii) *V is a simple Lie algebra of dimension 3.*

The symmetric bilinear form of the vector algebra V then is given by

$$(e,e) = -\beta, \quad (f,f) = -\alpha, \quad (g,g) = -\alpha\beta, \quad (e,f) = (f,g) = (g,e) = 0 \; .$$

Proof. (i) implies (ii) and (iii): By 3.4(i), the radical R is contained in the center of V. Hence $R = \{0\}$, else the center would be non-trivial, and 3.5 would imply that V is abelian or nilpotent, in contradiction with $V' = V$.

In order to prove that V is simple, let $I \neq V$ be an ideal of V; we have to show that $I = \{0\}$. We use identity 3.1(2) for $u \in I$ and $w \in V \setminus I$. Since the left-hand side and the second term of the right-hand side lie in I, it follows that $(u, v) = 0$ for all $v \in V$. Hence u belongs to the radical R, which is zero, so that $u = 0$.

Now we produce a basis as asserted in (ii). Since $(x+y, x+y) = (x,x) + 2(x,y) + (y,y)$ and $R = \{0\}$, there is an element $e \in V$ such that $(e,e) \neq 0$. Likewise, the orthogonal space $e^{\perp} = \{x \in V; (e,x) = 0\}$ contains an element f such that $(f,f) \neq 0$ (or else $e^{\perp}$ would be contained in the radical R). Then by 3.4(iii), since $R = \{0\}$, we have that $g := e \times f \neq 0$. Let $\beta := -(e,e)$, $\alpha := -(f,f)$. Using identity 3.1(2), we now evaluate $g \times e = (e \times f) \times e = (f,e)e - (e,e)f = \beta f$, $f \times g = -g \times f = (f \times e) \times f = (e,f)f - (f,f)e = \alpha e$. In particular, e, f, and g are linearly independent, else g would be a linear combination $g = \gamma e + \delta f$ and $g \times g = \gamma(e \times g) + \delta(f \times g) = -\gamma\beta f + \delta\alpha e \neq 0$, which is nonsense.

Next we show that $I := Fe + Ff + Fg$ is an ideal of V. It then follows from simplicity that $V = I$ has dimension 3, and that e, f, g is a basis of V. For $v \in V$, using again identity 3.1(2), we obtain that $e \times v = \frac{1}{\alpha}((f \times g) \times v) = \frac{1}{\alpha}((g,v)f - (v,f)g) \in I$ and similarly $f \times v \in I$ and $g \times v \in I$.

(ii) implies (i): In order to show that V is the vector algebra of an associative quadratic algebra, we verify that with the symmetric bilinear form specified in the assertion the conditions of the Characterization Theorem 3.1 are satisfied.

As to identity 3.1(1), it suffices to consider it for $u, v, w \in \{e, f, g\}$, since the expressions in this identity are trilinear. If at least two of such elements u, v, w coincide, both sides of the identity are zero. The identity is invariant if u and w are interchanged. Hence, verification for $(u, v, w) \in \{(e,f,g), (e,g,f), (f,e,g)\}$ suffices. Now,
$(f \times e, g) = -(g,g) = \alpha\beta, \quad -(e, f \times g) = -\alpha(e,e) = \alpha\beta,$
$(g \times e, f) = \beta(f,f) = -\alpha\beta, \quad -(e, g \times f) = \alpha(e,e) = -\alpha\beta,$
$(e \times f, g) = (g,g) = -\alpha\beta, \quad -(f, e \times g) = \beta(f,f) = -\alpha\beta.$

In the same way, it suffices to verify identity 3.1(2) for $u, v, w \in \{e, f, g\}$. Then, if u, v, w are all different, or if $w = u$, both sides of the identity are zero.

Moreover, the identity is invariant if u and w are interchanged. Hence, it suffices to consider the cases $u \neq v = w$ for $u, v \in \{e, f, g\}$, for which (u, v, w) is one of the triples (e, f, f), (e, g, g), (f, e, e), (f, g, g), (g, e, e), (g, f, f). Now,
$(f \times e) \times f = -g \times f = \alpha e = (e, f)f - (f, f)e$,
$(g \times e) \times g = \beta f \times g = \alpha\beta e = (e, g)g - (g, g)e$,
$(e \times f) \times e = g \times e = \beta f = (f, e)e - (e, e)f$,
$(g \times f) \times g = -\alpha e \times g = \alpha\beta f = (f, g)g - (g, g)f$,
$(e \times g) \times e = -\beta f \times e = \beta g = (g, e)e - (e, e)g$,
$(f \times g) \times f = \alpha e \times f = \alpha g = (g, f)f - (f, f)g$.

Finally, it is immediate that $V' = V$.

(iii) implies (ii) according to [Ja, p.13]. ■

The following reflects the well-known special rôle played by the Lie algebra $\mathrm{sl}_2 F$ of 2×2-matrices over F having trace 0 among the 3-dimensional simple Lie algebras, in terms of vector algebras.

4.5 Addendum. *For a Lie algebra V over a field F of characteristic not two, the following assertions are equivalent.*

(i) *V satisfies the hypotheses of 4.4, and there is a non-zero vector in V which is isotropic with respect to the symmetric bilinear form $(\ ,\ )$ on V.*

(ii) *V is isomorphic to the Lie algebra $\mathrm{sl}_2 F$.*

$\mathrm{sl}_2 F$ is the vector algebra of the quadratic algebra $F^{2\times 2}$ of all 2×2-matrices over F.

Proof. Assume (i), and let $u \in V \setminus \{0\}$ such that $(u, u) = 0$. At the beginning of the proof of 4.4 it was shown that the radical R is trivial; hence there is $v \in V$ such that $(u, v) = 1$. For $\delta \in F$, we have $(v + \delta u, v + \delta u) = (v, v) + 2\delta$; replacing v by $v - 1/2(v, v)u$, we may therefore assume that $(v, v) = 0$. Let

$$w := u \times v\ .$$

By identity 3.1(2) then, $w \times u = (v, u)u - (u, u)v = u$ and $v \times w = -(u \times v) \times v = -(v, v)u + (v, u)v = v$.

Now, u, v are linearly independent as $(u, v) \neq 0 = (u, u)$, and w is linearly independent of u, v, for else $u = w \times u$ would belong to $F(v \times u) = Fw$, and $u = w \times u = 0$, which is a contradiction. Thus, u, v, w is a basis of V, and the Lie algebra V is uniquely determined up to isomorphism by the given products.

In $\mathrm{sl}_2 F$, the matrices

$$u = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix},\ v = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix},\ w = \begin{pmatrix} 1 & -2 \\ 0 & -1 \end{pmatrix}$$

constitute such a basis satisfying $w = u \times v = 1/2(uv - vu)$, $v = v \times w = 1/2(vw - wv)$, $u = w \times u = 1/2(wu - uw)$.

Every 2×2-matrix M is annulled by its characteristic polynomial so that $M^2 - \mathrm{tr}\, M \cdot M + \det M \cdot E = 0$ where E is the unit matrix. This shows at the same time that $F^{2\times 2}$ is a quadratic algebra and that its vector algebra consists of the matrices of trace 0.

It can be read off the above basis that sl_2F coincides with its commutator algebra. Thus, version (i) of the hypotheses of 4.4 is met. (Of course, it is also well known that sl_2F is simple, in keeping with 4.4(iii).) Finally, $u^2 = 0$ shows that $(u,u) = 0$. ■

We mention in passing that for simple vector algebras the close relation between the symmetric bilinear form $(\ ,\ )$ and the Killing form asserted in 3.3 is a special case of the well known fact that on a simple Lie algebra there is only one invariant symmetric bilinear form up to a constant. On sl_2F, the trace form coincides with $(\ ,\ )$. Note that the usual Killing form of sl_2F is four times the Killing form of the vector algebra sl_2F as used here, since by definition the operation $\times$ is half the usual Lie bracket of matrices.

5. Classification of the associative quadratic algebras

In this section, we specify the associative quadratic algebras belonging to the vector algebras which have been classified in Section 4.

5.1 Associative quadratic algebras with nilpotent vector algebras. Let A be an associative quadratic algebra of characteristic not two whose vector algebra V is nilpotent and at least 2-dimensional. According to Classification Theorems 4.1, 4.2, and 4.4, V is abelian or nilpotent of class 2. By 3.5, the symmetric bilinear form $(\ ,\ )$ on V is identically zero. Hence

$$v^2 = 0 \quad \text{and} \quad uv \in V \quad \text{for} \quad u,v \in V \tag{3}$$

by the multiplication rule $2.4(*)$.

Conversely, if A is an associative quadratic algebra which satisfies (3), then $(v,v) = 0$ for all $v \in V$. Hence $(\ ,\ )$ is identically 0, so that by 3.5 the vector algebra V of A is nilpotent.

In the classification by Bröcker [Br], these algebras appear in Theorem 4 p.265.

5.2 Associative quadratic algebras with solvable not nilpotent vector algebras. Let A be an associative quadratic algebra A of characteristic not two whose vector algebra V is solvable but not nilpotent. By Classification Theorems 4.1, 4.2, and 4.4, the commutator algebra V' has codimension 1 in V and is abelian. In the proof of 4.2, it is shown that furthermore V' is the radical of $(\ ,\ )$, and that, hence, for $e \in V \setminus V'$ there is $\lambda \in F \setminus \{0\}$ such that for all $v \in V'$

$$e^2 = \lambda \cdot 1, \quad ev = e \times v = -ve, \quad \text{so that} \quad \mathrm{ad}\, e|V' = \lambda \cdot \mathrm{id}\,.$$

Moreover, it follows by the multiplication rule $2.4(*)$ that

$$uv = 0 \quad \text{for} \quad u,v \in V'\,. \tag{4}$$

According to 4.3, these algebras fall into two types.

1) The first type is obtained with $\lambda = 1$. Then $\operatorname{ad} e|V'$ is an involution. Consider the decomposition of V' into the eigenspaces V'_+ and V'_- of $\operatorname{ad} e|V'$ as in 4.3, and let $\pi : V' \to V'$ be the linear projection such that

$$\pi^2 = \pi, \quad \operatorname{im} \pi = V'_-, \quad \ker \pi = V'_+ \ .$$

The decomposition of $v \in V'$ is obtained as $v = (v - \pi(v)) + \pi(v)$, since $v - \pi(v) \in \ker \pi = V'_+$. Using π, the product ev can thus be written as $ev = e \times v = v - \pi(v) - \pi(v) = v - 2\pi(v)$. Now let

$$i := \frac{1}{2}(1 - e) \ .$$

As $e^2 = 1$, one has $i^2 = 1/4 \cdot (1 - 2e + e^2) = 1/2 \cdot (1 - e) = i$. Moreover, for $v \in V'$, we obtain $iv = 1/2 \cdot (v - ev) = 1/2 \cdot (v - v + 2\pi(v)) = \pi(v)$ and $vi = 1/2 \cdot (v - ve) = 1/2 \cdot (v + ev) = 1/2 \cdot (v + v - 2\pi(v)) = v - \pi(v) = (1 - i)v$. Since $F \cdot 1 + Fe = F \cdot 1 + Fi$, the algebra A can be described as $F \cdot 1 + Fi + V'$, and its multiplication is determined by (4) and the products

$$i^2 = i, \quad iv = \pi(v), \quad vi = (1 - i)v$$

which we have just obtained. It is in this form that the algebra A is described in Bröcker's classification [Br, p.249]. (Note that the multiplication table given there contains a misprint; of course, $B \cdot 1$ is B and not A).

2) For the second type, λ is not a square in F. Then, as explained in 4.3, $F(e) = Fe + F \cdot 1$ is a quadratic field extension of F and V' is a left vector space over F. The minimal polynomial $x^2 - \lambda$ of e over F has the roots $e, -e$, so that the map

$$a = (\alpha \cdot 1 + \beta \cdot e) \mapsto \widetilde{a} = (\alpha \cdot 1 - \beta \cdot e)$$

is an involutory field automorphism of $F(e)$. Since $ve = -ev$ for $v \in V'$, we obtain for $a \in F(e)$ that

$$va = \widetilde{a}v \ .$$

Up to isomorphism, this together with (4) determines the multiplication of $A = F(e) + V'$. In this form, the algebra is presented in Bröcker's classification [Br, p.263].

5.3 Associative quadratic algebras with simple vector algebras. Let A be an associative quadratic algebra A of characteristic not two whose vector algebra V is simple. We distinguish two cases depending on whether the symmetric bilinear form $(\ ,\ )$ on V is anisotropic or not.

1) If V contains non-zero isotropic vectors, then V is isomorphic to sl_2F, and A is isomorphic to the algebra $F^{2\times 2}$ of all 2×2-matrices by Addendum 4.5; note that by 3.2, A is uniquely determined by its vector algebra.

2) If $(\ ,\ )$ is anisotropic, A will turn out to be a quaternion skew field.

We first show that every non-zero element of A has a multiplicative inverse, so that A is a skew field. This is clear for non-zero elements of $F \cdot 1$ and for elements of $V \setminus \{0\}$, since for $v \in V \setminus \{0\}$ we have $v^2 = (v, v) \cdot 1 \neq 0$ by assumption so that $(1/(v, v) \cdot v)v = 1$. All other non-zero elements are of the form $\rho \cdot 1 + v$ for $v \in V \setminus \{0\}$ and $\rho \in F \setminus \{0\}$. Now, the set of invertible elements is closed under multiplication, hence it suffices to show that $\rho \cdot 1 + v$ is the product of two elements of V. Choose $u \in V \setminus \{0\}$ such that $(u, v) = 0$. Then by identities 3.1(1) and (2) we have $(v \times u, u) = -(u, v \times u)$, so that $(v \times u, u) = 0$, and therefore $(v \times u)u = (v \times u) \times u = (u, u)v - (u, v)u = (u, u)v$. Thus,

$$\left(\frac{\rho}{(u, u)} u + \frac{1}{(u, u)} v \times u\right) u = \rho \cdot 1 + v$$

is a product of two elements of V and hence is invertible, and A is indeed a skew field.

That A is a quaternion skew field can now be easily deduced from the information given in Classification Theorem 4.4 about the basis e, f, g of V. From this, the following multiplication table is readily obtained by the multiplication rule 2.4(∗):

$$ef = g = -fe, \quad fg = \alpha e = -gf, \quad ge = \beta f = -ge\ ,$$
$$e^2 = -\beta \cdot 1, \quad f^2 = -\alpha \cdot 1, \quad g^2 = -\alpha\beta \cdot 1\ .$$

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