Journal of Lie Theory
Volume **19** (2009) 527–530

The Lattice Subgroups Conjecture

Amira Ghorbel and Hatem Hamrouni

Communicated by E. B. Vinberg

Abstract. It has been conjectured by L. Corwin and F. P. Greenleaf that if Γ is a lattice subgroup of a connected, simply connected nilpotent Lie group G then $\log(\Gamma)$ is a Lie ring. In this note we show that this conjecture holds.
Mathematics Subject Classification 2000: 22E40.
Key Words and Phrases: Nilpotent Lie group, discrete uniform subgroup, lattice subgroup, rational structure..

Let G be a connected, simply connected nilpotent Lie group with Lie algebra $\mathfrak{g}$. A discrete uniform subgroup Γ of G is called lattice subgroup if $\log(\Gamma)$ is an additive subgroup of $\mathfrak{g}$ ([6]). In 1976, L. Corwin and F. P. Greenleaf in [2, page 141] (see also [1, page 222]) proposed the following

Conjecture 1 (**The lattice subgroups conjecture**). If Γ is a lattice subgroup of a connected, simply connected nilpotent Lie group G then $\log(\Gamma)$ is a Lie ring (i.e., for all X, Y in $\log(\Gamma)$, we have $[X, Y] \in \log(\Gamma)$).

Using the Campbell-Baker-Hausdorff formula, it is clear that the conjecture holds when G is k-step connected, simply connected nilpotent Lie group for $k \leq 4$ ([1, page 222]). But the problem in general remained open. The main result of this paper is the following theorem.

Theorem 2. *The lattice subgroups conjecture holds.*

Before we begin with the proof we need the following definitions and lemmas.

Definition 3 ([1]). Let $\mathfrak{g}$ be a nilpotent Lie algebra and let $\mathscr{B} = (X_1, \dots, X_n)$ be a basis of $\mathfrak{g}$. We say that $\mathscr{B}$ is a strong Malcev basis for $\mathfrak{g}$ if

$$\mathfrak{g}_i = \mathbb{R}\text{-span}\,\{X_1, \dots, X_i\}$$

is an ideal of $\mathfrak{g}$ for each $1 \leq i \leq n$.

Let Γ be a discrete uniform subgroup of a connected, simply connected nilpotent Lie group G. A strong Malcev basis $(X_1, \dots, X_n)$ for $\mathfrak{g}$ is said to be

ISSN 0949–5932 / \$2.50

based on Γ if
$$\Gamma = \exp \mathbb{Z}X_1 \cdots \exp \mathbb{Z}X_n.$$
Such a basis always exists (see [1], [5]).

Let G be a connected, simply connected nilpotent Lie group and let $\mathfrak{g}$ be its Lie algebra. We say that $\mathfrak{g}$ (or G) has a *rational structure* if there is a Lie algebra $\mathfrak{g}_{\mathbb{Q}}$ over $\mathbb{Q}$ such that $\mathfrak{g} \cong \mathfrak{g}_{\mathbb{Q}} \otimes \mathbb{R}$. It is clear that $\mathfrak{g}$ has a rational structure if and only if $\mathfrak{g}$ has an $\mathbb{R}$-basis $(X_1, \dots, X_n)$ with rational structure constants.

Let $\mathfrak{g}$ have a fixed rational structure given by $\mathfrak{g}_{\mathbb{Q}}$ and let $\mathfrak{h}$ be an $\mathbb{R}$-subspace of $\mathfrak{g}$. Define $\mathfrak{h}_{\mathbb{Q}} = \mathfrak{h} \cap \mathfrak{g}_{\mathbb{Q}}$. We say that $\mathfrak{h}$ is *rational* if $\mathfrak{h} = \mathbb{R}\text{-span}\,\{\mathfrak{h}_{\mathbb{Q}}\}$, and that a connected, closed subgroup H of G is *rational* if its Lie algebra $\mathfrak{h}$ is rational.

If G has a discrete uniform subgroup Γ, then $\mathfrak{g}$ (hence G) has a rational structure such that $\mathfrak{g}_{\mathbb{Q}} = \mathbb{Q}\text{-span}\,\{\log(\Gamma)\}$. Conversely, if $\mathfrak{g}$ has a rational structure given by some $\mathbb{Q}$-algebra $\mathfrak{g}_{\mathbb{Q}} \subset \mathfrak{g}$, then G has a discrete uniform subgroup Γ such that $\log(\Gamma) \subset \mathfrak{g}_{\mathbb{Q}}$ (see [1] and [5]). If we endow G with the rational structure induced by a uniform subgroup Γ and if H is a Lie subgroup of G, then H is rational if and only if $H \cap \Gamma$ is a discrete uniform subgroup of H.

A proof of the next result can be found in Proposition 5.3.2 of [1].

Lemma 4. *Let G be a connected, simply connected nilpotent Lie group with Lie algebra $\mathfrak{g}$ and let Γ be a discrete uniform subgroup of G. Let $\mathfrak{h}_1 \subsetneq \mathfrak{h}_2 \subsetneq \dots \subsetneq \mathfrak{h}_k = \mathfrak{g}$ be rational ideals of $\mathfrak{g}$ with dim$(\mathfrak{h}_i) = m_i$ for $1 \le i \le k$. Then there exists a strong Malcev basis $(X_1, \dots, X_n)$ for $\mathfrak{g}$ based on Γ such that for any $i = 1, \dots, k$, we have*
$$\mathfrak{h}_i = \mathbb{R}\text{-span}\,\{X_1, \dots, X_{m_i}\}.$$

A basis satisfying Lemma 4 is called strong Malcev basis for $\mathfrak{g}$ based on Γ passing through $\mathfrak{h}_1, \mathfrak{h}_2, \dots, \mathfrak{h}_k$.

Lemma 5 ([2], Lemma 3.9)**.** *If Γ is a lattice subgroup of a connected, simply connected nilpotent Lie group $G = \exp \mathfrak{g}$ and $(X_1, \dots, X_n)$ is a strong Malcev basis of $\mathfrak{g}$ based on Γ, then $(X_1, \dots, X_n)$ is a $\mathbb{Z}$-basis for the additive lattice* $\log(\Gamma) \subseteq \mathfrak{g}$.

The next lemma is the rational version of Kirillov's lemma ([4, Lemma 4.1], [3, Lemma 1]).

Lemma 6. *Let $\mathfrak{g}$ be a nilpotent Lie algebra with one dimensional center $\mathfrak{z}(\mathfrak{g})$.*

(1) *Then there exists a decomposition $\mathfrak{g} = \mathbb{R}X \oplus \mathbb{R}Y \oplus \mathfrak{z}(\mathfrak{g}) \oplus \mathfrak{n}$, with $[X, Y] \in \mathfrak{z}(\mathfrak{g}) \setminus \{0\}$ and $\mathfrak{g}_0 = \mathbb{R}Y \oplus \mathfrak{z}(\mathfrak{g}) \oplus \mathfrak{n}$ being the centralizer of Y in $\mathfrak{g}$.*

(2) *Furthermore, if $G = \exp \mathfrak{g}$ has a discrete uniform subgroup Γ, we may choose Y in* $\log(\Gamma)$*, so that $\mathfrak{z}_0 = \mathbb{R}Y \oplus \mathfrak{z}(\mathfrak{g})$ and $\mathfrak{g}_0$ will be rational ideals of $\mathfrak{g}$.*

Proof. [Proof of Theorem 2] Let $G = \exp \mathfrak{g}$ be a connected, simply connected nilpotent Lie group and Γ a lattice subgroup of G. Let $\Lambda = \log(\Gamma)$. The proof is by induction on the dimension of $\mathfrak{g}$. If $\dim(\mathfrak{g}) = 1$ there is nothing to prove. Suppose then that the result has been proved for all the groups of dimensions less than n and that $\dim(\mathfrak{g}) = n+1$. Let $(X_1, \ldots, X_{n+1})$ be a strong Malcev basis for $\mathfrak{g}$ strongly based on Γ passing through the center $\mathfrak{z}(\mathfrak{g})$ of $\mathfrak{g}$ (see Lemma 4). We shall prove that $[X_i, X_j] \in \Lambda$ for every $i, j \in \{1, \ldots, n+1\}$ such that $i > j$.

There are two cases to consider.

Case 1: $\dim(\mathfrak{z}(\mathfrak{g})) \geq 2$. For $k = 1, 2$, let

$$\mathfrak{a}_k = \mathbb{R}\text{-span}\{X_k\} \quad \text{and} \quad A_k = \exp \mathfrak{a}_k.$$

Also let

$$p_k : G \longrightarrow G/A_k \quad \text{and} \quad dp_k : \mathfrak{g} \longrightarrow \mathfrak{g}/\mathfrak{a}_k,$$

the canonical projections. Since A_k is a rational normal subgroup of G, then $p_k(\Gamma)$ is a lattice subgroup of G/A_k ([1, Lemma 5.1.4]). Now the inductive hypothesis says that

$$dp_k([X, Y]) \in \log(p_k(\Gamma)), \qquad \forall k = 1, 2.$$

Consequently, there exist $\gamma_1, \gamma_2 \in \Lambda$ and $t_1, t_2 \in \mathbb{R}$ such that

$$[X_i, X_j] = \gamma_1 + t_1 X_1 = \gamma_2 + t_2 X_2.$$

On the other hand, we have $\Lambda = \mathbb{Z}X_1 + \ldots + \mathbb{Z}X_{n+1}$, by Lemma 5. Then $t_1, t_2 \in \mathbb{Z}$ and so $[X_i, X_j] \in \Lambda$.

Case 2: $\dim(\mathfrak{z}(\mathfrak{g})) = 1$. Using Lemma 6 we can suppose that the basis $(X_1, \ldots, X_{n+1})$ passes through $\mathfrak{z}_0$ and $\mathfrak{g}_0$. Let $\mathfrak{a}$ be the subalgebra of $\mathfrak{g}$ generated by X_i, X_j and $A = \exp \mathfrak{a}$. First suppose that $\mathfrak{a} \neq \mathfrak{g}$. As $\mathfrak{a}$ is rational, then $A \cap \Gamma$ is a lattice subgroup of A and therefore the induction hypothesis says that $[X_i, X_j] \in \Lambda \cap \mathfrak{a}$, in particular $[X_i, X_j] \in \Lambda$. Finally, we consider the case when $\mathfrak{a} = \mathfrak{g}$. In this situation, we have $(i, j) = (n+1, n)$. Let's write

$$[X_{n+1}, X_n] = \alpha X_{n-1} + \ldots + \beta X_1.$$

Since $\mathfrak{a} = \mathfrak{g}$ we conclude that $\alpha \neq 0$. On the other hand, as $X_2 \in \mathfrak{z}_0 = \mathbb{R}Y \oplus \mathfrak{z}(\mathfrak{g})$ (see Lemma 6) and $\mathfrak{z}(\mathfrak{g}) = \mathbb{R}X_1$, then there exists $(z, y) \in \mathbb{R} \times \mathbb{R}^*$ such that $X_2 = zX_1 + yY$. Therefore, it is easy to check that $\mathfrak{g}_0$ is the centralizer of X_2 in $\mathfrak{g}$. Consequently, there exists $a \in \mathbb{R}^*$ such that

$$[X_{n+1}, X_2] = aX_1.$$

Let the mapping $\phi : \mathfrak{g} \longrightarrow \mathfrak{g}$ be defined by

$$\begin{aligned}
\phi(X_{n+1}) &= X_{n+1} \\
\phi(X_n) &= X_n - \frac{\beta}{a} X_2 \\
\phi(X_{n-1}) &= X_{n-1} - \frac{\beta}{\alpha} X_1 \\
\phi(X_i) &= X_i \quad (1 \leq i \leq n-2).
\end{aligned}$$

Since $[X_2, \mathfrak{g}_0] = \{0\}$ then the mapping ϕ is a Lie algebra automorphism of $\mathfrak{g}$. Using ϕ we can suppose that $\beta = 0$ (if not, replacing Γ by $\Phi(\Gamma)$ where Φ is the unique group automorphism of G such that the derivative of Φ at identity is ϕ). Let

$$p : G \longrightarrow G/Z(G) \quad \text{and} \quad dp : \mathfrak{g} \longrightarrow \mathfrak{g}/\mathfrak{z}(\mathfrak{g})$$

be the canonical projections. As above, since $Z(G)$ is a normal rational subgroup of G then $p(\Gamma)$ is a lattice subgroup of $G/Z(G)$. We have by the induction hypothesis

$$dp([X_{n+1}, X_n]) \in \log(p(\Gamma)).$$

Consequently, there exist $\gamma \in \Lambda$ and $t \in \mathbb{R}$ such that

$$[X_{n+1}, X_n] = \gamma + tX_1. \tag{1}$$

Equation (1) and the condition $\beta = 0$ imply that $t \in \mathbb{Z}$ and therefore $[X_{n+1}, X_n] \in \Lambda$. This proves the theorem. ■

References

[1] Corwin L., and F. P. Greenleaf, "Representations of nilpotent Lie groups and their applications. Part 1: Basic theory and examples," Cambridge Studies in Adv. Math., Vol. **18**, Cambridge University Press, New York, 1989.

[2] —, *Character formulas and spectra of compact nilmanifolds*, J. Func. Analysis **21** (1976), 123–154.

[3] Howe R., *On Frobenius reciprocity for unipotent algebraic group over* $\mathbb{Q}$, Amer. J. Math. **93** (1971), 163–172.

[4] Kirillov A. A., *Représentations unitaires des groupes de Lie nilpotents*, Uspekhi Math. Nauk **17** (1962), 57–110.

[5] Malcev A. I., *On aclass of homogeneous spaces*, Amer. Math. Soc. Transl. **39** (1951).

[6] Moore C. C., *Decomposition of unitary representations defined by discrete subgroups of nilpotent Lie groups*, Ann. of Math. **82** (1965), 146–182.

A. Ghorbel and H. Hamrouni
Department of Mathematics
Faculty of Sciences at Sfax
Route Soukra, B.P. 1171. 3000 Sfax
Tunisia
Amira.Ghorbel@fss.rnu.tn
hatemhhamrouni@voila.fr

Received August 22, 2009
and in final form September 13, 2009